P443 HW #11
Due April 21, 2008

1. **Griffiths 9.1**. A hydrogen atom is placed in a (time-dependent) electric field $\mathbf{E} = E(t)\hat{k}$. Calculate all four matrix elements H'_{ij} of the perturbation $H' = eEz$ between the ground state ($n = 1$) and the (quadruply degenerate) first excited states ($n = 2$). Also show that $H'_{ii} = 0$ for all five states. $Note$: There is only one integral to be done here, if you exploit oddness with respect to z; only one of the $n = 2$ states is "accessible" from the ground state by a perturbation of this form, and therefore the system functions as a two-state configuration-assuming transitions to higher excited states can be ignored.

2. **Griffiths 9.11**. Calculate the lifetime in (in *seconds*) for each of the four $n = 2$ states of hydrogen. $Hint$: you ill need to evaluate matrix elements of the form $\langle \psi_{100} \mid x \mid \psi_{200} \rangle$, $\langle \psi_{100} \mid y \mid \psi_{211} \rangle$, and so on. Remember that $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r\cos\theta$. Most of these integrals are zero, so scan them before you start calculating. $Answer$: 2.6×10^{-9} seconds for all except ψ_{200}, which is infinite.

3. **Griffiths 9.14**. An electron in the $n = 3, l = 0, m = 0$ state of hydrogen decays by a sequence of (electric dipole) transitions to the ground state.

   (a) What decay routes are open to it? Specify them in the following way:
   $$\mid 300\rangle \rightarrow \mid nlm\rangle \rightarrow \mid n'l'm'\rangle \rightarrow ... \rightarrow \mid 100\rangle$$

   (b) If you had a bottle full of atoms in this state, what fraction of them would decay via each route?

   (c) What is the lifetime of this state? $Hint$: once it's made the first transition, it's no longer in the state $\mid 300\rangle$, so only the first step in each sequence is relevant in computing the lifetime. When there is more than one decay route open, the transition rates add.

4. **Griffiths 9.20**. A spin-1/2 particle with gyromagnetic ratio γ, at rest in a static magnetic field $B_0\hat{k}$, precesses at the Larmor frequency

$\omega_0 = \gamma B_0$. Now we turn on a small transverse radiofrequency (rf) field, $B_{rf}[\cos(\omega t)\hat{\imath} - \sin(\omega t)\hat{\jmath}]$, so that the total field is

$$\mathbf{B}(t) = B_{rf}\cos(\omega t)\hat{\imath} - B_{rf}\sin(\omega t)\hat{\jmath} + B_0\hat{k}.$$

(a) Construct the 2×2 Hamiltonian matrix for this system.

(b) If $\chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ is the spin state at time t, show that

$$\dot{a} = \frac{i}{2}\left(\Omega e^{i\omega t} b + \omega_0 a\right); \quad \dot{b} = \frac{i}{2}\left(\Omega e^{-i\omega t} a - \omega_0 b\right),$$

where $\Omega = \gamma B_{rf}$ is related to the strength of the rf field.

(c) Check that the general solution for $a(t)$ and $b(t)$, in terms of their initial values a_0 and b_0, is

$$\begin{aligned} a(t) &= \left\{a_0\cos(\omega' t/2) + \frac{i}{\omega'}[a_0(\omega_0 - \omega) + b_0\Omega]\sin(\omega' t/2)\right\} e^{i\omega t/2} \\ b(t) &= \left\{b_0\cos(\omega' t/2) + \frac{i}{\omega'}[b_0(\omega - \omega_0) + a_0\Omega]\sin(\omega' t/2)\right\} e^{-i\omega t/2} \end{aligned}$$

where

$$\omega' \equiv \sqrt{(\omega - \omega_0)^2 + \Omega^2}.$$

(d) If the particle starts out with spin up (i.e., $a_0 = 1$, $b_0 = 0$), find the probability of a transition to spin down, as a function of time. *Answer* : $P(t) = \{\Omega^2/[(\omega - \omega_0)^2 + \Omega^2]\}\sin^2(\omega' t/2)$.

(e) Sketch the **resonance curve**,

$$P(\omega) = \frac{\Omega^2}{(\omega - \omega_0)^2 + \Omega^2},$$

as a function of the driving frequency ω (for fixed ω_0 and Ω). Note that the maximum occurs at $\omega = \omega_0$. Find the "full width at half maximum," $\Delta\omega$.

(f) Since $\omega_0 = \gamma$, we can use the experimentally observed resonance to determine the magnetic dipole moment of the particle. In a **nuclear magnetic resonance** (nmr) experiment the g-factor of the proton is to be measured, using a static field of 10,000 gauss and an rf field of amplitude 0.01 gauss. What will the resonant frequency be? (See Griffiths Section 6.5 for the magnetic moment of the proton.) Find the width of the reonance curve. (Give your answers in Hz).

5. **Griffiths 9.21**. The true electric field of the light that is emitted or absorbed in an atomic transition would be

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t).$$

If the atom is centered at the origin, then $\mathbf{k} \cdot \mathbf{r} \ll 1$ over he relevant volume ($|\mathbf{k}| = 2\pi/\lambda$, so $\mathbf{k} \cdot \mathbf{r} \sim r/\lambda \ll 1$), and that's why we could afford to drop this term. Suppose we keep the first order correction:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0[\cos(\omega t) + (\mathbf{k} \cdot \mathbf{r}) \sin(\omega t)].$$

The first term gives rise to the **allowed (electric dipole)** transitions considered in the text; the second gives rise to the so-called **forbidden (magnetic dipole** and **electric quadrupole**) transitions (higher powers of $\mathbf{k} \cdot \mathbf{r}$ lead to even *more* forbidden transitions, associated with higher multipole moments).

(a) Obtain the spontaneous emission rate for forbidden transitions (don't bother to average over polarization and propagation directions, though this should really be done to coplete the calculation). *Answer* :

$$R_{b \to a} = \frac{q^2 \omega^5}{\pi \epsilon_0 \hbar c^5} |\left\langle a \mid (\hat{n} \cdot \mathbf{r})(\hat{k} \cdot \mathbf{r}) \mid b \right\rangle|^2$$

(b) Show that for a one-dimensional oscillator, the forbidden transitions go from level n to level $n-2$ and the transition rate (suitably averaged over $\hat{n}$ and $\hat{k}$) is

$$R = \frac{\hbar q^2 \omega^3 n(n-1)}{15 \pi \epsilon_0 m^2 c^5}.$$

Find the *ratio* of the "forbidden" rate to the "allowed" rate and comment on the terminology. (*Note* : ω is the frequency of the *photon*, not the *oscillator*.)

(c) Show that the $2S \to 1S$ transition in hydrogen is not possible even by a "forbidden" transition. (As it turns out, this is true for all the higher multipoles as well; the dominant decay is in fact by a two-photon emission, and the lifetime is about a tenth of a second.)