Physics 443 Prelim #1 with solutions
March 7, 2008

Each problem is worth 34 points.

1. **Harmonic Oscillator**
   Consider the Hamiltonian for a simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

(a) Use dimensional analysis to estimate the ground state energy and the characteristic size of the ground state wave function in terms of $m, \hbar,$and ω. (That is, determine the characteristic length l_0 and energy E_0.)
[Let $x = l_0 z$ and $E = \epsilon E_0$, where l_0 and E_0 have dimensions of length and energy respectively. Substitution into Schrodinger's equation gives

$$\left(-\frac{\hbar^2}{2ml_0^2}\frac{d^2}{dz^2} + \frac{1}{2}m\omega^2 l_0^2 z^2\right)\psi = \epsilon E_0 \psi$$

Multiply through by $2ml_0^2/\hbar^2$ to get

$$\left(-\frac{d^2}{dz^2} + m^2\omega^2\frac{l_0^4}{\hbar^2}z^2\right)\psi = \epsilon\frac{2ml_0^2}{\hbar^2}E_0\psi$$

Define l_0 so that the coefficient of $z^2\psi$ on the left hand side is 1 and E_0 so that the coefficient of $\epsilon\psi$ on the right hand side is 1. Then

$$l_0 = \sqrt{\frac{\hbar}{m\omega}} \text{ and } E_0 = \frac{1}{2}\hbar\omega.]$$

(b) At $t = 0$ a particle in the harmonic oscillator potential has as its wave function an even mixture of the first two stationary states with energies $\frac{1}{2}\hbar\omega$ and $\frac{3}{2}\hbar\omega$.

$$\Psi(x, 0) = A[\psi_0(x) + \psi_1(x)]$$

Compute A and find $\Psi(x, t)$ and $|\Psi(x, t)|^2$. Give your answers in terms of ψ_0 and ψ_1.
[A is the normilzation constant.

$$\begin{aligned}
\langle\psi \mid \psi\rangle &= 1 \\
&= |A|^2[\langle\psi_0 \mid \psi_0\rangle + \langle\psi_0 \mid \psi_1\rangle + \langle\psi_1 \mid \psi_0\rangle + \langle\psi_1 \mid \psi_1\rangle| \\
&= |A|^2[1 + 0 + 0 + 1] \\
\Rightarrow & \; A = \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\Psi(x,t) = \frac{1}{\sqrt{2}}[\psi_0(x)e^{-iE_0/\hbar t} + \psi_1(x)e^{-iE_1/\hbar t}] = \frac{1}{\sqrt{2}}[\psi_0(x)e^{-i\omega t/2} + \psi_1(x)e^{-3i\omega t/2}]$$

$$\begin{aligned}|\Psi(x,t)|^2 &= \frac{1}{2}[1 + 1 + \psi_0^*\psi_1 e^{-i\omega t} + \psi_1^*\psi_0 e^{\omega t}] \\ &= [1 + \psi_0^*\psi_1 \cos(\omega t)]\end{aligned}$$

(c) Compute $\langle x \rangle$ using $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$. What is the angular frequency of the oscillation?

$$\begin{aligned}[\langle x \rangle &= \langle \psi \mid x \mid \psi \rangle \\ &= \frac{1}{2}\left(\langle \psi_0 \mid x \mid \psi_0 \rangle + \langle \psi_1 \mid x \mid \psi_1 \rangle + \langle \psi_1 \mid x \mid \psi_0 \rangle e^{i\omega t} + \langle \psi_0 \mid x \mid \psi_1 \rangle e^{-i\omega t}\right) \\ &= \frac{1}{2}\left(\langle \psi_1 \mid x \mid \psi_0 \rangle e^{i\omega t} + \langle \psi_0 \mid x \mid \psi_1 \rangle e^{-i\omega t}\right) \\ &= (\langle \psi_1 \mid x \mid \psi_0 \rangle \cos(\omega t))\end{aligned}$$

The last step follows from the fact that the wave functions ψ_1 and ψ_0 are real and x is an Hermitian operator. Finally

$$\begin{aligned}\langle \psi_1 \mid x \mid \psi_0 \rangle &= \sqrt{\frac{\hbar}{2m\omega}} \langle \psi_1 \mid a_+ + a_- \mid \psi_0 \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} \langle \psi_1 \mid a_+ \mid \psi_0 \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}}\end{aligned}$$

The last step follows from the third equation for the one dimensional harmonic oscillator on the formula sheet. So

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t).$$

The angular frequency of the oscillation is ω.]

(d) If you measured the energy of this particle, what values might you get and what is the probability of getting each of them?

[A measurement of the energy of the particle would give either $\frac{1}{2}\hbar\omega$ or $\frac{3}{2}\hbar\omega$ with equal likelihood.]

(e) What is the expectation value of H?
$[\langle H \rangle = \frac{1}{2}[\langle \psi_0 \mid H \mid \psi_0 \rangle + \langle \psi_1 \mid H \mid \psi_1 \rangle] = \hbar\omega]$

2. **WKB**

Consider the infinite square well with a sloped floor

$$V(x) = \begin{cases} \infty & \text{for } (x < 0), \\ kx & \text{for } (0 < x < a), \\ \infty, & \text{for } (x > a) \end{cases}$$

(a) If the well is narrow (small a) and k is small, the turning points for the ground state will be at $x = 0$ and $x = a$. If the well is very broad, the right turning point for the ground state will occur along the floor, at $x < a$. Assuming that the turning points are at $x = 0$ and $x < a$, use the WKB approximation to find the energy of the ground state and the turning point.

[The turning point $x_t = E/k$. The quantization condition for a well with one infinite vertical wall is

$$\begin{aligned} (n - \frac{1}{4})\pi\hbar &= \int_{x_1}^{x_2} p dx \\ &= \int_0^{x_t} \sqrt{2m(E - kx)} dx \\ &= \sqrt{2mk} \int_0^{x_t} (\frac{E}{k} - x)^{\frac{1}{2}} dx \\ &= -\frac{2}{3}\sqrt{2mk}(\frac{E}{k} - x)^{3/2} \mid_0^{x_t} \\ &= \frac{2}{3}\sqrt{2mk}(\frac{E}{k})^{3/2} \\ \Rightarrow \quad E &= \left(\frac{3}{2}\frac{3}{4}\frac{\pi\hbar k}{\sqrt{2m}}\right)^{2/3}] \end{aligned}$$

(b) Now assume that the turning points are at $x = 0$ and $x = a$ and use the WKB approximation to write an expression that determines the energy levels of the system. (You do not need to solve for E_n.) [Now we use the quantization condition for two infinite vertical walls.

$$\begin{aligned}
\pi\hbar &= \int_{x_1}^{x_2} p dx \\
&= \int_0^a \sqrt{2m(E-kx)}dx \\
&= \sqrt{2mk}\int_0^{x_t} (\frac{E}{k}-x)^{\frac{1}{2}} dx \\
&= -\frac{2}{3}\sqrt{2mk}(\frac{E}{k}-x)^{3/2} |_0^a \\
\pi\hbar &= \frac{2}{3}\sqrt{2mk}\left(\left(\frac{E}{k}\right)^{3/2} - \left(\frac{E}{k}-a\right)^{3/2}\right)]
\end{aligned}$$

3. **Spin 1/2**

Suppose that a spin-1/2 particle is in the state

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix}.$$

where $a = \cos\alpha$ and $b = \sin\alpha$ are real and the state is normalized.

(a) What are the probabilities of getting $+\hbar/2$ and $-\hbar/2$, if you measure S_z and S_x?

[The probability for getting $+\hbar/2$ if you measure along the z direction is

$$|\chi_+^\dagger \chi|^2 = |\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}|^2 = |a|^2$$

and the probability for $-\hbar/2$ is $|b|^2$. The probability for getting $+\hbar/2$ if you measure along the x direction is

$$|\chi_+^{(x)\dagger} \chi|^2 = |\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}|^2 = \frac{1}{2}|a+b|^2.$$

$\chi_\pm^{(x)}$ are the eigenvectors of S_x with eigenvalues $\pm\frac{1}{2}\hbar$. The probability for $-\hbar/2$ is $\frac{1}{2}|a-b|^2$.]

(b) In a coordinate system rotated by an angle θ about the y-axis so that $z \rightarrow z'$ and $x \rightarrow x'$, what are the the probabilities of getting $+\hbar/2$ and $-\hbar/2$, if you measure $S_{z'}$ and $S_{x'}$?

[In the rotated coordinate system

$$\chi' = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a\cos(\theta/2) - b\sin(\theta/2) \\ b\cos(\theta/2) + a\sin(\theta/2) \end{pmatrix}$$

Now the probability of getting $+\hbar/2$ if you measure along the $+z'$ direction is $|a'|^2$ and the probability of getting $-\hbar/2$ is $|b'|^2$. The probability of getting $\pm\hbar/2$ if you measure along the $+x'$ axis is $\frac{1}{2}|a' \pm b'|^2$.

Formulae and Tables

- $$\begin{aligned} H\Psi &= i\hbar\frac{\partial \Psi}{\partial t} \\ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) &= i\hbar\frac{\partial}{\partial t}\Psi(x,t) \end{aligned}$$

- WKB

$$\int_{x_1}^{x_2} p(x)dx = \left(n - \frac{1}{2}\right)\pi\hbar \quad \text{(no infinite vertical walls)}$$

$$\int_{x_1}^{x_2} p(x)dx = \left(n - \frac{1}{4}\right)\pi\hbar \quad \text{(1 infinite vertical wall)}$$

$$\int_{x_1}^{x_2} p(x)dx = n\pi\hbar \quad \text{(2 infinite vertical walls)}$$

$$\psi(x) = \frac{A}{\sqrt{p}}\exp(\frac{i}{\hbar}\int^x p(x')dx') + \frac{B}{\sqrt{p}}\exp(-\frac{i}{\hbar}\int^x p(x')dx')$$

$$\psi(x) = \frac{C}{\sqrt{|p|}}\exp(\frac{1}{\hbar}\int^x |p(x')|dx') + \frac{D}{\sqrt{|p|}}\exp(-\frac{1}{\hbar}\int^x |p(x')|dx')$$

- One dimensional harmonic oscillator

$$\begin{aligned} H &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \\ a_\pm &= \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x}) \\ a_+\psi_n &= \sqrt{(n+1)}\psi_{n+1} \\ a_-\psi_n &= \sqrt{n}\psi_{n-1} \\ [a_-, a_+] &= 1 \\ H &= \hbar\omega(a_-a_+ - \frac{1}{2}) \\ H\psi_n &= \hbar\omega(n + \frac{1}{2})\psi_n \\ \psi_0 &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}, \quad \psi_1 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\sqrt{\frac{2m\omega}{\hbar}}xe^{-\frac{m\omega}{2\hbar}x^2} \end{aligned}$$

- Relativisitic energy momentum

$$E = \sqrt{p^2c^2 + (mc^2)^2}$$

- Time dependence of an expectation value

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \langle\frac{\partial Q}{\partial t}\rangle$$

- Three dimensional infinite cubical well

$$\psi_{n_x,n_y,n_z}(x,y,z) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x\pi}{a}x\right) \sin\left(\frac{n_y\pi}{a}y\right) \sin\left(\frac{n_z\pi}{a}z\right)$$

$$E^0_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2)$$

- Spin 1/2

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \left\langle \frac{1}{2}, \frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, \frac{1}{2}(z)\right\rangle & \left\langle \frac{1}{2}, \frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, -\frac{1}{2}(z)\right\rangle \\ \left\langle \frac{1}{2}, -\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, \frac{1}{2}(z)\right\rangle & \left\langle \frac{1}{2}, -\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2}, -\frac{1}{2}(z)\right\rangle \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

- Virial Theorem

  For stationary states

$$2\langle T\rangle = \langle \mathbf{r}\cdot\nabla V\rangle$$

- Generators

$$e^{i(\sigma\cdot\hat{n})\phi/2} = \cos(\phi/2) + i(\hat{n}\cdot\sigma)\sin(\phi/2)$$

- Boundary conditions for $V(x) = \alpha\delta(x)$

  $\psi(x)$ continuous,

  $\Delta\left(\frac{d\psi}{dx}\right) = \frac{2m\alpha}{\hbar^2}\psi(0)$