Name:____

P443 Prelim II
April 11, 2008

1. **Spin 1/2**

Consider a spin 1/2 particle with magnetic moment $\vec{\mu} = \frac{e}{m}\vec{S}$ in a magnetic field $\vec{B} = B_0\hat{z}$. The hamiltonian

$$H = -\vec{\mu}\cdot\vec{B} = \frac{e}{m}\vec{S}\cdot\vec{B}$$

where $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$. For convenience we define the constant $\gamma = \frac{e\hbar}{2m}$.

(a) Find the eigenvectors (wave functions) and the eigenvalues (energy levels) of the hamiltonian.
[The hamiltonian is

$$H = \frac{e}{m}S_z B_0 = \gamma\begin{pmatrix} B_0 & 0 \\ 0 & -B_0 \end{pmatrix}.$$

The eigenvectors are $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with eigenvalues $\pm\gamma B_0$.]
Now suppose we add a magnetic field in the x-direction such that $\vec{B} = B_0\hat{z} + B_x\hat{x}$ and $B_x \ll B_0$.

(b) Use perturbation theory to calculate the first order shift in the energy of each of the eigenstates due to $H' = -\vec{\mu}\cdot B_x\hat{x}$.
[The perturbation is

$$H' = \frac{e}{m}S_x B_x = \gamma\begin{pmatrix} 0 & B_x \\ B_x & 0 \end{pmatrix}$$

The first order shifts are

$$\begin{aligned} E^1_+ &= \langle\chi_+ \mid H' \mid \chi_+\rangle = (\,1 \quad 0\,)\,\gamma\begin{pmatrix} 0 & B_x \\ B_x & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \\ E^1_- &= \langle\chi_- \mid H' \mid \chi_-\rangle = (\,0 \quad 1\,)\,\gamma\begin{pmatrix} 0 & B_x \\ B_x & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0] \end{aligned}$$

(c) Use perturbation theory to compute the second order shift in the energy of each of the eigenstates.

[The second order energy shift

$$
\begin{aligned}
E_+^2 &= \frac{|\langle \chi_- \mid H' \mid \chi^+ \rangle|^2}{E_+^0 - E_-^0} = \frac{1}{2\gamma B_0} |\begin{pmatrix} 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 0 & B_x \\ B_x & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} |^2 = \frac{\gamma B_x^2}{2B_0} \\
E_-^2 &= \frac{|\langle \chi_+ \mid H' \mid \chi_- \rangle|^2}{E_- - E_+} = \frac{1}{-2\gamma B_0} |\begin{pmatrix} 1 & 0 \end{pmatrix} \gamma \begin{pmatrix} 0 & B_x \\ B_x & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} |^2 = -\frac{\gamma B_x^2}{2B_0}]
\end{aligned}
$$

(d) Use perturbation theory to compute the first order shift in the wave function of each of the eigenstates

[The first order shift in the wave functions is

$$
\begin{aligned}
\chi_+^2 &= \frac{\langle \chi_- \mid H' \mid \chi_+ \rangle}{E_+^0 - E_-^0} \chi_- = \frac{1}{2\gamma B_0} \begin{pmatrix} 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} 0 & B_x \\ B_x & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \chi_- = \frac{B_x}{2B_0} \chi_- \\
\chi_-^2 &= \frac{\langle \chi_+ \mid H' \mid \chi_- \rangle}{E_-^0 - E_+^0} \chi_+ = -\frac{1}{2\gamma B_0} \begin{pmatrix} 1 & 0 \end{pmatrix} \gamma \begin{pmatrix} 0 & B_x \\ B_x & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \chi_+ = -\frac{B_x}{2B_0} \chi_+]
\end{aligned}
$$

(e) Compute the exact energy levels of the system, expand to second order and compare with the result from perturbation theory.

[The exact hamiltonian is $H = \gamma \begin{pmatrix} B_0 & B_x \\ B_x & -B_0 \end{pmatrix}$. The exact energy levels are the eigenvalues of the hamiltonian, namely

$$
E_\pm = \pm \gamma \sqrt{B_0^2 + B_x^2} \sim \pm B_0 \left(1 + \frac{1}{2} \frac{B_z^2}{B_0^2} \right).
$$

The exact and perturbative solutions are in agreement to second order.]

(f) Compute the exact wave functions, expand to second order and compare with the result from perturbation theory. (You do not need to normalize.)

[The exact wave functions are the eigenvectors of the exact hamiltonian.

$$
\begin{aligned}
\gamma \begin{pmatrix} B_0 & B_x \\ B_x & -B_0 \end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} &= \gamma \sqrt{B_0^2 + B_x^2} \begin{pmatrix} 1 \\ a \end{pmatrix} \\
\rightarrow \begin{pmatrix} B_0 + aB_x \\ B_x - aB_0 \end{pmatrix} &= \sqrt{B_0^2 + B_x^2} \begin{pmatrix} 1 \\ a \end{pmatrix}
\end{aligned}
$$

$$\begin{aligned}
\rightarrow a &= \frac{\sqrt{B_0^2 + B_x^2}}{B_x} - \frac{B_0}{B_x} \\
&\sim \frac{1}{B_x}(B_0(1 + \frac{B_x^2}{2B_0^2}) - \frac{B_0}{B_x} \\
&\sim \frac{B_x}{2B_0}
\end{aligned}$$

Then the eigenvector is

$$\begin{pmatrix} 1 \\ \frac{B_x}{2B_0} \end{pmatrix} = \chi_+^0 + \frac{B_x}{2B_0}\chi_-^0$$

in agreement with the perturbation theory result in part (d). Similarly, to get the other eigenvector

$$\begin{aligned}
\gamma \begin{pmatrix} B_0 & B_x \\ B_x & -B_0 \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} &= -\gamma\sqrt{B_0^2 + B_x^2} \begin{pmatrix} a \\ 1 \end{pmatrix} \\
\rightarrow \begin{pmatrix} aB_0 + B_x \\ aB_x - B_0 \end{pmatrix} &= -\sqrt{B_0^2 + B_x^2} \begin{pmatrix} a \\ 1 \end{pmatrix} \\
\rightarrow a &= \frac{-\sqrt{B_0^2 + B_x^2}}{B_x} = \frac{B_0}{B_x} \\
&\sim -\frac{1}{B_x}(B_0(1 + \frac{B_x^2}{2B_0^2}) + \frac{B_0}{B_x} \\
&\sim -\frac{B_x}{2B_0}
\end{aligned}$$

Then the other eigenvector is

$$\begin{pmatrix} -\frac{B_x}{2B_0} \\ 1 \end{pmatrix} = \chi_-^0 - \frac{B_x}{2B_0}\chi_+^0]$$

2. **Fermi Energy**

(a) Calculate the Fermi energy for electrons in a two-dimensional infinite square well of area A. Give your answer in terms of the density of electrons, (the number of electrons per unit area).

[In a 2-d well, the energy

$$E_F = \frac{\hbar^2}{2m}\frac{\pi^2}{l^2}(n_x^2 + n_y^2)$$

The number of states with energy less than E_F is

$$N_{states} = \frac{1}{4}\pi(n_x^2 + n_y^2) = \frac{1}{4}\pi\frac{E_F 2ml^2}{\hbar^2\pi^2}$$

There are 2 electrons per state so the total number of electrons is

$$N_{ele} = \frac{1}{2}\pi\frac{E_F 2ml^2}{\hbar^2\pi^2}$$

The density of electrons (number/area) is

$$\frac{N_{ele}}{l^2} = \sigma = \frac{1}{2}\pi\frac{E_F 2m}{\hbar^2\pi^2}$$

Then

$$E_F = \frac{\sigma\hbar^2\pi}{m}]$$

(b) Calculate the total energy of the electrons

[The total energy is

$$\begin{aligned} E &= \int_0^{E_F} E\frac{dN}{dE}dE = \int_0^{E_F} E\frac{ml^2}{\hbar^2\pi}dE \\ &= \frac{1}{2}E_F^2\frac{ml^2}{\hbar^2\pi} = \frac{1}{2}\left(\frac{\sigma\hbar^2\pi}{m}\right)^2\frac{mA}{\hbar^2\pi} = \frac{1}{2}\frac{(N_{ele}/A)^2\hbar^2\pi A}{m} \\ &= \frac{1}{2}\frac{N_{ele}^2\hbar^2\pi}{mA} \end{aligned}$$

(c) Compute the pressure on the walls, $P = \frac{dE_{tot}}{dA}$. The pressure is

$$P = \frac{dE_{tot}}{dA} = -\frac{1}{2}\frac{N_{ele}^2\hbar^2\pi}{mA^2} = -\frac{E_{tot}}{A} = -\frac{1}{2}\frac{\sigma^2\hbar^2\pi}{m}]$$

3. **Three Identical Particles**

Imagine a situation in which there are three particles and only three states a, b, and c available to them. If the particles are distinguishable, there are a total of 27 distinct configurations of this system.

What is the total number of allowed distinct configurations for this system

(a) if the particles are bosons?

[If the particles are bosons there are 10 states:

$$\psi_a(1)\psi_a(2)\psi_a(3),\ \psi_b(1)\psi_b(2)\psi_b(3),\ \psi_c(1)\psi_c(2)\psi_c(3),$$
$$\psi_a(1)\psi_a(2)\psi_b(3),\ \psi_a(1)\psi_a(2)\psi_c(3),\ \psi_b(1)\psi_b(2)\psi_a(3),$$
$$\psi_b(1)\psi_b(2)\psi_c(3),\ \psi_c(1)\psi_c(2)\psi_a(3),\ \psi_c(1)\psi_c(2)\psi_b(3),$$
$$\psi_a(1)\psi_b(2)\psi_c(3)]$$

(b) if the particles are fermions? If the particles are fermions, there is only one state $\psi_a(1)\psi_b(2)\psi_c(3)$.]

Formulae

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+V\Psi=i\hbar\frac{\partial\Psi}{\partial t}$$

$$H\Psi=i\hbar\frac{\partial\Psi}{\partial t}$$

$$\hat{p}=-i\hbar\frac{\partial}{\partial x}$$

$$[x,p]=i\hbar$$

$$\sigma_x=\begin{pmatrix}0&1\\1&0\end{pmatrix};\quad \sigma_y=\begin{pmatrix}0&-i\\i&0\end{pmatrix};\quad \sigma_z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

- Time independent perturbation theory

$$E_n^1=\left\langle\psi_n^0\mid H'\mid\psi_n^0\right\rangle$$

$$\psi_n^1=\sum_{m\neq n}\frac{\langle\psi_m^0\mid H'\mid\psi_n^0\rangle}{(E_n^0-E_m^0)}\psi_m^0$$

$$E_n^2=\sum_{m\neq n}\frac{|\langle\psi_m^0\mid H'\mid\psi_n^0\rangle|^2}{(E_n^0-E_m^0)}$$

- One dimensional infinite square well

$$\psi_n(x)=\left(\frac{2}{a}\right)^{1/2}\sin\left(\frac{n\pi}{a}x\right)$$

$$E_n=\frac{\pi^2\hbar^2}{2ma^2}(n^2)$$

- Virial Theorem

$$2\langle T\rangle=\langle\mathbf{r}\cdot\nabla V\rangle$$

- Generators

$$e^{i(\sigma\cdot\hat{n})\phi/2}=\cos(\phi/2)+i(\hat{n}\cdot\sigma)\sin(\phi/2)$$

- Integral

$$\int_0^a\sin^2\frac{n\pi x}{a}dx=\frac{a}{2}$$