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Scattering

1 Time dependent perturbation theory

In our study of time dependent perturbation theory we determined the the transition probability from initial state ψ_a to final state ψ_b is given by the absolute square of the amplitude

$$c_b(t) = -\frac{i}{\hbar}\int_{-t/2}^{t/2} \langle \psi_b \mid H(t') \mid \psi_a \rangle \, e^{i\omega_{ab}t'} dt'$$

where $\omega_{ab} = \frac{E_a - E_b}{\hbar}$. To apply the theory to a scattering process we imagine that the perturbation $H(t')$ turns on at $-t/2$ and off at $t/2$ and while it is turned on it has constant value H. Then we can integrate and we get

$$c_b(t) = -\frac{i}{\hbar}\frac{1}{i\omega_{ab}}\left[e^{i\omega_{ab}t/2} - e^{-i\omega_{ab}t/2}\right] H_{ab} = -\frac{H_{ab}}{\hbar}\frac{2i \sin \omega_{ab}t/2}{\omega_{ab}}$$

and the transition probability is

$$|c_b|^2 = \frac{1}{\hbar^2}|H_{ab}|^2 \frac{4\sin^2 \omega_{ab}t/2}{\omega_{ab}^2} = \frac{1}{\hbar^2}|H_{ab}|^2 \left(\frac{\sin \omega_{ab}t/2}{\omega_{ab}t/2}\right)^2 t^2$$

Let's examine the ω_{ab} dependent piece

$$f(\omega) \equiv \left(\frac{\sin \omega t/2}{\omega t/2}\right)^2 t^2$$

The first zero of $f(\omega)$ occurs when $\omega = 2\pi/t$. Its maximum value (at $\omega = 0$) is t^2. In the limit of large t, $f(\omega) \rightarrow 2\pi t\delta(\omega)$. To check that assertion we integrate over all ω

$$\int_{-\infty}^{\infty} f(\omega)d\omega = \int \frac{\sin^2 x}{x^2} dx \frac{2}{t} t^2 = 2\pi t = \int 2\pi t \delta(\omega) d\omega$$

In terms of the energies of initial and final states,

$$2\pi t\delta(\omega) \rightarrow 2\pi t\hbar\delta(E_b - E_a)$$

and we can write

$$|c_b|^2 = \frac{1}{\hbar^2}|H_{ab}|^2 2\pi t\delta(E_b - E_z)$$

The transition rate is

$$R = \frac{2\pi}{\hbar}|H_{ab}|^2\delta(E_b - E_a)$$

In scattering experiments, the detector always has some finite acceptance. And what we measure is a sum over all final states consistent with that acceptance.

$$R = \frac{2\pi}{\hbar}\int |H_{ab}|^2\delta(E_b - E_a)\rho(E_b)dE_b$$

$\rho(E_b)$ is the density of final states, the number of final states per unit energy. Well, we have figured this out before. The number of states between k and $k + dk$ is

$$dN = \frac{Vk^2dkd\Omega}{8\pi^3} = \frac{Vp^2dpd\Omega}{(2\pi\hbar)^3}$$

Using $E = p^2/2m$ and $2pdp/2m = dE$ we have

$$dN = \frac{VpmdEd\Omega}{(2\pi\hbar)^3}$$

$$\frac{dN}{dE} = \rho(E) = \frac{Vpm}{(2\pi\hbar)^3}d\Omega$$

so

$$R = \frac{2\pi}{\hbar}\frac{1}{(2\pi\hbar)^3}Vpm|H_{ab}|^2d\Omega$$

Now how do we connect to the cross section? The incoming particle is represented by a plane wave

$$\psi_a = \frac{1}{\sqrt{V}}e^{i\mathbf{k_a}\cdot\mathbf{r}}$$

and the outgoing wave by

$$\psi_b = \frac{1}{\sqrt{V}}e^{i\mathbf{k_b}\cdot\mathbf{r}}$$

The $\sqrt{V}$ in the denominator is so that the wave function is normalized. The particle density in the incoming wave is $|\psi_a|^2 = 1/V$ and the flux of incoming particles is $\frac{v}{V} = \frac{p/m}{V}$. And

$$(d\sigma)\text{Flux} = dN = R$$

$$d\sigma = \frac{R}{\text{Flux}} = \frac{V^2 m^2}{(2\pi\hbar^2)^2} | \langle \psi_b \mid H \mid \psi_a \rangle |^2 d\Omega$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{Vm}{2\pi\hbar^2}\right)^2 |H_{ab}|^2$$

That means that

$$f(\theta) = -\frac{mV}{2\pi\hbar^2} \langle \psi_b \mid H \mid \psi_a \rangle$$

The negative sign is a convention. Suppose that $H = V(\vec{r})$. Then

$$\begin{aligned} f(\theta) &= -\frac{mV}{2\pi\hbar^2} \int \frac{1}{\sqrt{V}} e^{-i\vec{k}_b \cdot \vec{r}} V(\vec{r}) \frac{1}{\sqrt{V}} e^{i\vec{k}_a \cdot \vec{r}} d^3 r \\ &= -\frac{m}{2\pi\hbar^2} \int e^{i(\vec{k}_a - \vec{k}_b)\cdot \vec{r}} V(\vec{r}) d^3 r \end{aligned}$$

The result is equivalent to that of the Green's function analysis in the first Born approximation. (See Griffiths p. 413)