Linear potential
February 8, 2008

Linear Potential - Gravity

The Schrodinger equation for the wave function of a bouncing ball is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + mgx\psi = E\psi \tag{1}$$

where we assume a perfectly elastic collision of the ball with the floor. So $V(x) = \infty$ for $x < 0$, and $V(x) = mgx$ for $V > 0$. There is zero probability to find the ball at $x < 0$ so $\psi(0) = 0$. If we define a characteristic length,

$$l_0 = \left(\frac{\hbar^2}{2m^2 g}\right)^{\frac{1}{3}}$$

and energy

$$E_0 = \left(\frac{\hbar^2 m g^2}{2}\right)^{\frac{1}{3}}$$

then $x = yl_0$ and $E = \epsilon E_0$ where y and ϵ are dimensionless and substitution into Equation 1 gives

$$-\frac{d^2\psi}{dy^2} + y\psi = \epsilon\psi \tag{2}$$

Finally with the substitution of $z = y - \epsilon$, Equation 2 becomes

$$-\frac{d^2\psi}{dz^2} + z\psi = 0 \tag{3}$$

Equation 3 is Airy's equation and the solutions to it are Airy functions. (Some of the properties of Airy functions are summarized in Chapter 8 of Griffiths. For more details see http://dlmf.nist.gov/contents/AI/index.html)

The general solution to Airy's equation is

$$\psi(z) = aA_i(z) + bB_i(z)$$

where a and b are constants.

But $B_i(z)$ diverges for large z and it does not result in a normalizable wave function. Therefore the solution to Equation 1 is simply

Zeros of A_i	
1	-2.33810
2	-4.08794
3	-5.52055
4	-6.78670
5	-7.94413
6	-9.02265
7	-10.04017

$$\psi(z) = aA_i(z)$$

In order to satisfy the boundary condition at $x = 0$, namely that $\psi(0) = 0$, we must choose ϵ so that $A_i(z) = 0$, when $y = z + \epsilon = 0$. I found this table of zeros at the above web site.

We see that the boundary condition is satsified when $\epsilon = 2.33810, 408794, 5.52055$, etc. The energy eigenvalues for the first three states are

$$E_1 = (2.33810)\left(\frac{\hbar^2 mg^2}{2}\right)^{\frac{1}{3}}$$

$$E_2 = (4.08794)\left(\frac{\hbar^2 mg^2}{2}\right)^{\frac{1}{3}}$$

$$E_3 = (5.52055)\left(\frac{\hbar^2 mg^2}{2}\right)^{\frac{1}{3}}$$