Sample final questions.

1. Estimate the lifetime of an excited state of hydrogen. Give your answer in terms of fundamental constants.

2. A one-dimensional harmonic oscillator, originally in the ground state, is acted on by a force $f(t)$. Find the motion of the center of the wave packet ($\langle x \rangle$) by first-order perturbation theory.

3. Particle A with spin $\frac{1}{2}$ and particle B with spin 1 are in a total angular momentum state

$$| \frac{1}{2}\frac{1}{2}\rangle = \sqrt{\frac{2}{3}} \left(| 11\rangle | \frac{1}{2} - \frac{1}{2}\rangle - \sqrt{\frac{1}{3}} | 10\rangle | \frac{1}{2}\frac{1}{2}\rangle \right)$$

   What are $\langle J_{Bz} \rangle, \langle J_B^2 \rangle$ and $\langle J_{Az} \rangle$?

4. Consider the problem of a two-dimensional harmonic oscillator in Cartesian coordinates. The Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}k(x^2 + y^2)$$

   (a) Write H and L_z in terms of the creation and annihilation operators in two dimensions, $a_x, a_x^\dagger, a_y, a_y^\dagger$. Show, using the commutation relation of the a and $a^\dagger$ operators, that L_z commutes with H.

   (b) Consider the two eigenstates $| u_{01}\rangle$ and $| u_{10}\rangle$ belonging to the first excited state of the oscillator ($N = 1$). Are these eigenstates of L_z? What are the eigenvalues?

   (c) Demonstrate your answer to part (b) using the form of L_z derived in part (a) in terms of $a, a\dagger$ operators. Reconcile your answer with the conclusion reached in part (a).

5. What is the ground state wave function for two identical particles in a one dimensional box if the two particles are (a) fermions, (b) bosons ?

6. Two identical particles, each of mass m, move in one dimension in the potential

$$V(x_1, x_2) = \frac{1}{2}A(x_1^2 + x_2^2) + \frac{1}{2}B(x_1 - x_2)^2$$

   where A and B are positive constants and x_1 and x_2 denote the positions of the particles.

(a) Show that the Schrodinger equation is separable in the variables $x_1 + x_2$ and $x_1 - x_2$. Find the eigenvalues.

(b) What is the symmetry of the eigenfunctions with respect to particle exchange?

7. Explain why for two identical spin-$\frac{1}{2}$ fermions, the scattering amplitude should be written in the form

$$f = [f_s(\theta) + f_s(\pi - \theta)] + [f_t(\theta) - f_t(\pi - \theta)]$$

where the subscripts s and t refer to singlet and triplet spin states, respectively.

8. Given the matrix

$$H_0 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

and for the perturbation

$$H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

in the orthonormal basis $| \phi_1\rangle, | \phi_2\rangle$, and $| \phi_3\rangle$, determine the energy eigenvalues correct to second order in the perturbation.

9. Use the variational method for solving the Schrodinger equation for the truncated harmonic oscillator potential

$$\begin{aligned} V(x) &= \tfrac{1}{2}kx^2 \quad \text{for} \quad x > 0 \\ &= \infty \quad \text{for} \quad x < 0 \end{aligned}$$

Use the trial wave function $\psi = \exp(-bx)$, (b is the variational paramter) to calculate an approximate value for the ground state energy and compare with the exact result.

10. A particle of mass m is initially in the ground state of the (one-dimensional) infinite square well. At time $t = 0$ a "brick" is dropped into the well, so that the potential becomes

$$V(x) = \begin{cases} V_0, & \text{if} \quad 0 \le x \le a/2, \\ 0, & \text{if} \quad a/2 < x \le a, \\ \infty, & \text{otherwise,} \end{cases}$$

where $V_0 \ll E_1$. After a time T, the brick is removed, and the energy of the particle is measured. Find the probability (in first-order perturbation theory) that the result is now E_2.

11. The delta function well $V(x) = -\alpha\delta(x)$, supports a single bound state,

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2}; \quad E = -\frac{m\alpha}{2\hbar^2}.$$

calculate the grometric phase change when α gradually increases from α_1 to α_2. If the increase occurs at a constant rate ($d\alpha/dt = c$), what is the dynamic phase change for this process?

12. **Scattering**
The Yukawa potential is a phenomonological model for the force between a π and a He nucleus.

$$V(r) = \beta \frac{e^{-\mu r}}{r}$$

where β and μ are constants.

(a) Compute the scattering amplitude in the Born approximation for a π with energy E scattering off the He nucleus. You can assume that the He nucleus is fixed in space. Show the angular dependence explicitly. For spherically symmetric potentials the Born approximation for the scattering amplitude is

$$f(\theta) \sim -\frac{2m}{\hbar^2\kappa} \int_0^\infty rV(r)\sin(\kappa r)dr$$

and $\kappa = 2k\sin(\theta/2)$.

(b) Compute the total cross-section and express your answer as a function of E.

Formulae and Tables

- $$\begin{aligned} H\Psi &= i\hbar\frac{\partial\Psi}{\partial t} \\ -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi(x,t) &= i\hbar\frac{\partial}{\partial t}\Psi(x,t) \end{aligned}$$

- Born approximation

$$f(\theta,\phi) \sim -\frac{m}{2\pi\hbar^2}\int e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r_0}}V(\mathbf{r_0})d^3\mathbf{r_0}$$

- Born approximation for spherically symmetric potential.

$$f(\theta) \sim -\frac{2m}{\hbar^2\kappa}\int_0^\infty rV(r)\sin(\kappa r)dr$$

  Where $\kappa = 2k\sin(\theta/2)$.

- Spontaneous emission rate

$$A = \frac{\omega^3 |\langle\psi_f \mid q\vec{r} \mid \psi_i\rangle|^2}{3\pi\epsilon_0\hbar c^3}$$

- Stimulated emission and absorbtion rate

$$R_{b\to a} = \frac{\pi}{3\epsilon_o\hbar}|\langle\psi_f \mid q\vec{r} \mid \psi_i\rangle|^2\rho(\omega_0)$$

- Blackbody energy density of thermal radiation

$$\rho(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/K_B T}-1}$$

- One dimensional harmonic oscillator

$$\begin{aligned} H &= \frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2 \\ a_\pm &= \frac{1}{\sqrt{2m}}(\hat{p}\pm im\omega\hat{x}) \\ a_+\psi_n &= \sqrt{(n+1)\hbar\omega}\psi_{n+1} \end{aligned}$$

$$\begin{aligned} a_-\psi_n &= \sqrt{n\hbar\omega}\psi_{n-1} \\ [a_-, a_+] &= \hbar\omega \\ H &= a_-a_+ - \frac{1}{2}\hbar\omega \\ H\psi_n &= (n+\frac{1}{2})\hbar\omega \end{aligned}$$

- Time independent perturbation theory

$$E_n^1 = \left\langle \psi_n^0 \mid H' \mid \psi_n^0 \right\rangle$$

- Relativisitic energy momentum

$$E = \sqrt{p^2c^2 + (mc^2)^2}$$

- Time dependence of an expectation value

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle[H,Q]\rangle + \langle\frac{\partial Q}{\partial t}\rangle$$

- Three dimensional infinite cubical well

$$\psi_{n_x,n_y,n_z}(x,y,z) = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x\pi}{a}x\right)\sin\left(\frac{n_y\pi}{a}y\right)\sin\left(\frac{n_z\pi}{a}z\right)$$

$$E^0_{n_x,n_y,n_z} = \frac{\pi^2\hbar^2}{2ma^2}(n_x^2 + n_y^2 + n_z^2)$$

- Time dependent perturbation theory

$$c_b(t) = -\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'$$

- Spin 1/2

$$S_x = \frac{\hbar}{2}\sigma_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\sigma_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\sigma_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \left\langle \frac{1}{2},\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2},\frac{1}{2}(z)\right\rangle & \left\langle \frac{1}{2},\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2},-\frac{1}{2}(z)\right\rangle \\ \left\langle \frac{1}{2},-\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2},\frac{1}{2}(z)\right\rangle & \left\langle \frac{1}{2},-\frac{1}{2}(z') \mid R_y(\theta) \mid \frac{1}{2},-\frac{1}{2}(z)\right\rangle \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

- Berry phase

$$\gamma_n(T) = i\int_0^T \langle\psi_n \mid \frac{\partial\psi_n}{\partial t'}\rangle \cdot dt'$$

$$\gamma_n(T) = \oint \langle\psi_n \mid \nabla_R\psi_n\rangle \cdot d\mathbf{R}$$

- Time independent perturbation theory

$$E_n^1 = \left\langle \psi_n^0 \mid H' \mid \psi_n^0 \right\rangle$$

$$\psi_n^1 = \sum_{m\neq n} \frac{\langle\psi_m^0 \mid H' \mid \psi_n^0\rangle}{(E_n^0 - E_m^0)}\psi_m^0$$

$$E_n^2 = \sum_{m\neq n} \frac{\mid\langle\psi_m^0 \mid H' \mid \psi_n^0\rangle\mid^2}{(E_n^0 - E_m^0)}$$

- Dirac equation

$$i\hbar\frac{\partial\psi}{\partial t} = [c\alpha\cdot\mathbf{p} + \beta mc^2]\psi$$

where

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

and

$$\psi(\mathbf{r}, t) = \begin{pmatrix} \psi_1(\mathbf{r}, t) \\ \psi_2(\mathbf{r}, t) \end{pmatrix}$$

and ψ_1 and ψ_2 are each two component spinors. If we assume a time dependence $\psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-iEt/\hbar}$, then the Dirac equation becomes

$$E\psi = [c\alpha\cdot\mathbf{p} + \beta mc^2]\psi$$

- Hydrogen atom

$$E_n = -\frac{1}{n^2}\frac{1}{2}\alpha^2 mc^2, \quad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}, \quad a_0 = \frac{\hbar}{\alpha m_e c}$$