May 3, 2008

Electron in a rotating magnetic field

Rotating frame

Suppose we have an electron at rest in a magnetic field

$$\mathbf{B} = B_0(\sin\alpha\cos\omega t\hat{i} + \sin\alpha\sin\omega t\hat{j} + \cos\alpha\hat{k})$$

So the field vector points along the polar angle α and rotates about the z-axis with frequency ω. It is convenient to transform into the rotating frame. At $t = 0$, the magnetic field is $\vec{B} = B_0\cos\alpha\hat{k} + B_0\sin\alpha\hat{i}$. At a later time the field has rotated about the z-axis by an angle $\theta = \omega t$. If we rotate the spinor about the z-axis we can move to a frame in which the hamiltonian is independent of time. A rotation about z is accomplished with

$$R(\theta) = e^{i\theta\sigma_z/2} = e^{i(\omega t)\sigma_z/2} = \begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix}$$

And since

$$H = \frac{\omega_1\hbar}{2}B_0\begin{pmatrix} \cos\alpha & \sin\alpha e^{i\omega t} \\ \sin\alpha e^{-i\omega t} & -\cos\alpha \end{pmatrix} \tag{1}$$

then

$$\begin{aligned} R^{-1}HR &= H_0 \\ &= \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\frac{\omega_1\hbar}{2}\begin{pmatrix} \cos\alpha & \sin\alpha e^{i\omega t} \\ \sin\alpha e^{-i\omega t} & -\cos\alpha \end{pmatrix}\begin{pmatrix} e^{i\omega t/2} & 0 \\ 0 & e^{-i\omega t/2} \end{pmatrix} \\ &= \frac{\omega_1\hbar}{2}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}\begin{pmatrix} \cos\alpha e^{i\omega t/2} & \sin\alpha e^{i\omega t/2} \\ \sin\alpha e^{-i\omega t/2} & -\cos\alpha e^{-i\omega t/2} \end{pmatrix} \\ &= \frac{\omega_1\hbar}{2}\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \end{aligned}$$

and H_0 is the time independent hamiltonian in the rotating frame.

Then

$$H\chi = RH_0R^{-1}\chi = i\hbar\frac{\partial}{\partial t}\chi$$

and

$$H_0 R^{-1}\chi = i\hbar R^{-1}\frac{\partial}{\partial t}\chi$$

Since R^{-1} does not commute with $\frac{\partial}{\partial t}$ we need to be careful in the next step. We find that

$$\frac{\partial}{\partial t}(R^{-1}\chi) = \frac{\partial R^{-1}}{\partial t}\chi + R^{-1}\frac{\partial}{\partial t}\chi$$

and also

$$\frac{\partial R^{-1}}{\partial t} = -\frac{i\omega}{2}\begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & -e^{i\omega t/2} \end{pmatrix} = -i\frac{\omega}{2}\sigma_z R^{-1}$$

With this in mind we can write

$$H_0 R^{-1}\chi = i\hbar R^{-1}\frac{\partial}{\partial t}\chi = i\hbar\frac{\partial}{\partial t}(R^{-1}\chi) - \frac{\hbar\omega}{2}\sigma_z R^{-1}\chi$$

and

$$H'\chi' = i\hbar\frac{\partial}{\partial t}\chi' = i\hbar\frac{\partial}{\partial t}\chi'$$

where $\chi' = R^{-1}\chi$ and $H' = H_0 + \frac{\hbar\omega}{2}\sigma_z$. Now we have a time independent hamiltonian. To solve we can compute eigenvalues and eigenvectors to get $\chi'(t)$ and then transform back to the lab frame.

Another strategy is to construct the time translation operator $e^{-iH't/\hbar}$. First write $H'/\hbar$ in the form $\hat{n}\cdot\sigma\lambda/2$

$$H' = \frac{\hbar}{2}\begin{pmatrix} \omega_1\cos\alpha + \omega & \omega_1\sin\alpha \\ \omega_1\sin\alpha & -\omega_1\cos\alpha - \omega \end{pmatrix} = \frac{\hbar}{2}\hat{n}\cdot\sigma\lambda$$

where

$$\begin{aligned} n_z &= \frac{\omega_1\cos\alpha + \omega}{(\omega_1^2 + 2\omega_1\omega\cos\alpha + \omega^2)^{\frac{1}{2}}} \\ n_x &= \frac{\omega_1\sin\alpha}{(\omega_1^2 + 2\omega_1\omega\cos\alpha + \omega^2)^{\frac{1}{2}}} \\ n_y &= 0 \end{aligned}$$

and

$$\lambda = (\omega_1^2 + 2\omega_1\omega\cos\alpha + \omega^2)^{\frac{1}{2}}$$

Then

$$Q(t) = e^{-iH't/\hbar} = e^{-i\hat{n}\cdot\sigma\lambda t/2} = \cos(\lambda t/2) - i\hat{n}\cdot\sigma\sin(\lambda t/2)$$

$$= \begin{pmatrix} \cos(\lambda t/2) - i(\omega_1 \cos\alpha + \omega)\sin(\lambda t/2)/\lambda & -i\omega_1 \sin\alpha \sin(\lambda t/2)/\lambda \\ i\omega_1 \sin\alpha \sin(\lambda t/2)/\lambda & \cos(\lambda t/2) + i(\omega_1 \cos\alpha + \omega)\sin(\lambda t/2)/\lambda \end{pmatrix}$$

Finally

$$\chi(t) = R(t)\chi'(t) = R(t)Q(t)\chi'(t=0) = R(t)Q(t)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Adiabatic approximation

If we return to the original hamiltonian H, and define ψ_n such that

$$H(t)\psi_n(t) = E_n(t)\psi_n(t)$$

then

$$\psi_1(t) = \chi_+(t) = \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2)e^{-i\omega t} \end{pmatrix} \quad \text{and} \quad \psi_2(t) = \chi_-(t) = \begin{pmatrix} \sin(\alpha/2)e^{i\omega t} \\ -\cos(\alpha/2) \end{pmatrix}$$

are eigenvectors of the hamiltonian (Equation 1) with eigenvalues $E_\pm = \pm\hbar\omega_1/2$. If our initial state is $\chi_+ = \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{pmatrix}$, then

$$\begin{aligned}
\chi(t) &= R(t)Q(t)\chi_+ \\
&= R(t)\begin{pmatrix} \cos(\lambda t/2) - i(\omega_1 \cos\alpha + \omega)\sin(\lambda t/2)/\lambda & -i\omega_1 \sin\alpha \sin(\lambda t/2)/\lambda \\ -i\omega_1 \sin\alpha \sin(\lambda t/2)/\lambda & \cos(\lambda t/2) + i(\omega_1 \cos\alpha + \omega)\sin(\lambda t/2)/\lambda \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{pmatrix} \\
&= \begin{pmatrix} \left[(\cos(\lambda t/2) - i\frac{\omega_1+\omega}{\lambda}\sin(\lambda t/2))\cos(\alpha/2)\right]e^{i\omega t/2} \\ \left[(\cos(\lambda t/2) - i\frac{\omega_1-\omega}{\lambda}\sin(\lambda t/2))\sin(\alpha/2)\right]e^{-i\omega t/2} \end{pmatrix} \\
&= (\cos(\lambda t/2) - i\frac{\omega_1 + \omega\cos\alpha}{\lambda}\sin(\lambda t/2))e^{i\omega t/2}\chi_+(t) - i\frac{\omega\sin\alpha}{\lambda}\sin(\lambda t/2)e^{-i\omega t/2}\chi_-(t)
\end{aligned}$$

In the adiabatic limit, $\omega \ll \omega_1$

$$\begin{aligned}
\chi(t) &= (\cos(\lambda t/2) - i(1 - \frac{\omega}{\omega_1}\cos\alpha)\sin(\lambda t/2))e^{i\omega t/2}\chi_+(t) \\
&= (\cos(\lambda t/2) - i\sin(\lambda t/2))e^{i\omega t/2}\chi_+(t) \\
&= e^{-i\lambda t/2}e^{i\omega t/2}\chi_+(t)
\end{aligned}$$

and

$$\begin{aligned}\chi(t) &\rightarrow e^{-i\lambda t/2}e^{i\omega t/2}\chi_+(t) \\ &\sim e^{-i(\omega_1+\omega\cos\alpha)t/2}e^{i\omega t/2}\chi_+(t)\end{aligned}$$

Berry phase

The dynamic phase is $\theta_+ = -\omega_1 t/2$. The remaining phase is geometric $\gamma_+ = (\omega/2)(-\cos\alpha + 1)$. Unfortunately I started out rotating in the negative ϕ direction. If ω changes sign then $\gamma_+ = (\omega/2)(\cos\alpha - 1)$ and Berry's phase is $(\omega/2)\frac{2\pi}{\omega}(\cos\alpha - 1) = \pi(\cos\alpha - 1)$. 1/2 of the solid angle subtended by the tip of magnetic field vector.

We could also get the geometric phase by

$$\begin{aligned}\gamma &= i\int \langle \chi_+ \mid \frac{\partial \chi_+}{\partial t'}\rangle dt' \\ &= i\int \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2)e^{i\omega t'} \end{pmatrix} \begin{pmatrix} 0 \\ -i\omega\sin(\alpha/2)e^{-i\omega t'} \end{pmatrix} dt' \\ &= \int \omega \sin^2(\alpha/2)dt' \\ &= -\pi(1-\cos\alpha)\end{aligned}$$

That last $-$ comes from integrating backwards.

$$Q(t) \rightarrow= \begin{pmatrix} \cos(\lambda t/2) + i\cos\alpha\sin(\lambda t/2) & i\sin\alpha\sin(\lambda t/2) \\ i\sin\alpha\sin(\lambda t/2) & \cos(\lambda t/2) - i\cos\alpha\sin(\lambda t/2) \end{pmatrix}$$