Dirac Equation

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We begin with the notion that the energy and momentum operators are derivatives with respect to time and space, just as we did to construct the Schrodinger Equation. According to special relativity, space and time are on an equivalent footing, so we construct a wave equation that is first order in both space and time. And we require that it be consistent with the energy momentum relationship

$$E^2 = (pc)^2 + (mc^2)^2 \tag{1}$$

That linear combination that includes the mass is

$$H_D = c\alpha \cdot \mathbf{p} - \beta mc^2$$

where the energy operator $H_D = i\hbar\frac{\partial}{\partial t}$ and α and β are arbitrary constants. If we square both sides we get that

$$E^2 = c^2(\alpha \cdot \mathbf{p})^2 + c\alpha \cdot \mathbf{p}\beta mc^2 + c\beta(mc^2)\alpha \cdot \mathbf{p} + \beta^2(mc^2)^2 \tag{2}$$

Equation **??** is consistent with Equation eq:em if the following are true

$$p^2 = (\alpha \cdot \mathbf{p})^2 \tag{3}$$

$$0 = c(mc^2)(\alpha \cdot \mathbf{p}\beta + \beta\alpha \cdot \mathbf{p}) \tag{4}$$

$$1 = \beta^2 \tag{5}$$

And that means that α and β are not c numbers, but rather noncommuting matrices. In particular Equation 3 implies that

$$\begin{aligned} p^2 &= (\alpha \cdot \mathbf{p})^2 = (\alpha_x p_x + \alpha_y p_y + \alpha_z p_z)^2 \\ &= (\alpha_x^2 p_x^2 + \alpha_y^2 p_y^2 + \alpha_z^2 p_z^2) + p_x p_y(\alpha_x\alpha_y + \alpha_y\alpha_x) + p_y p_z(\alpha_y\alpha_z + \alpha_z\alpha_y) + p_z p_x(\alpha_z\alpha_x + \alpha_x\alpha_z) \\ &\rightarrow \{\alpha_i, \alpha_j\} = 2\delta_{ij} \end{aligned}$$

where the anticommutator $\{\alpha_i, \alpha_j\} \equiv \alpha_i\alpha_j + \alpha_j\alpha_i$. Equation 4 is true if $\{\alpha, \beta\} = 0$, and Equation 5 requires that $\beta^2 = 1$. The lowest dimension matrices that satisfy the above are 4X4. The following, although not unique, do the job

$$\alpha_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \alpha_y = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \alpha_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

A slightly more compact notation is

$$\alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

And so with these definitions of α and β we have the Dirac equation

$$\left(i\hbar\frac{\partial}{\partial t} - c\alpha \cdot \mathbf{p} - \beta mc^2 \right) \psi = 0$$

The Dirac Equation is the linear combination of time and space derivatives consistent with the energy momentum relationship

$$E = \sqrt{p^2c^2 + (mc^2)^2} \tag{6}$$

If we write $\psi e^{-iEt/\hbar} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} e^{-iEt/\hbar}$, then the Dirac equation is equivalent to the coupled equations

$$\begin{aligned} 0 &= E\psi_1 - c\sigma \cdot \mathbf{p}\psi_2 - mc^2\psi_1 \\ \rightarrow \psi_1 &= \frac{c\sigma \cdot \mathbf{p}}{E - mc^2}\psi_2 \qquad (7) \\ &\text{and} \\ 0 &= E\psi_2 - c\sigma \cdot \mathbf{p}\psi_1 + mc^2\psi_2 \\ \rightarrow \psi_2 &= \frac{c\sigma \cdot \mathbf{p}}{E + mc^2}\psi_1 \qquad (8) \end{aligned}$$

Non relativistic limit

We know that $E - mc^2 = E_S$, where E_S is the energy that appears in Schrodinger's equation. If we substitute the expression for ψ_2 in Equation 8 into Equation 7 and expand to first order in the nonrelativistic limit $pc \ll mc^2$, we get

$$\begin{aligned}
E_S\psi_1 &= \frac{(c\sigma\cdot\mathbf{p})(c\sigma\cdot\mathbf{p})}{E+mc^2}\psi_1 \\
&= \frac{c^2p^2}{2mc^2}(1-\frac{E_S}{2mc^2})\psi_1 \\
E_S(1+\left(\frac{cp}{2mc^2}\right)^2)\psi_1 &= \frac{c^2p^2}{2mc^2}\psi_1 \\
E_S\psi_1 &\sim \left(\frac{c^2p^2}{2mc^2}-\frac{c^4p^4}{8m^3c^6}+...\right)\psi_1 \\
E_S\psi_1 &\sim \left(\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+...\right)\psi_1
\end{aligned}$$

which looks like Schrodinger's equation with the relativistic correction.

Angular momentum

If the nonrelativistic hamiltonian is spherically symmetric, (the potential only depends on r), then the orbital angular momentum is a constant of the motion and $[H, \mathbf{L}] = 0$. That is not so for the Dirac Hamiltonian. Let us compute

$$\begin{aligned}
[L_x, H_D] &= [L_x, c\alpha\cdot\mathbf{p}+\beta mc^2] \\
&= [yp_z - zp_y, c\alpha\cdot\mathbf{p}] \\
&= [yp_z - zp_y, c(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z] \\
&= i\hbar c(\alpha_y p_z - \alpha_z p_y) \neq 0 \qquad (9)
\end{aligned}$$

Meanwhile we define the spin operator

$$\mathbf{S} = \frac{1}{2}\hbar\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix},$$

and evaluate

$$
\begin{aligned}
[S_x, H_D] &= [S_x, c\alpha\cdot\mathbf{p} + \beta mc^2] \\
&= \frac{1}{2}\hbar c\left[\begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}, \begin{pmatrix}0 & \sigma\\ \sigma & 0\end{pmatrix}\right]\cdot\mathbf{p} + \frac{1}{2}\hbar c\left[\begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}, \begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\right] mc^2 \\
&= \frac{1}{2}\hbar ic\begin{pmatrix}0 & p_y\sigma_z - p_z\sigma_y\\ p_y\sigma_z - p_z\sigma_y & 0\end{pmatrix} \\
&= \frac{1}{2}\hbar ic(p_y\alpha_z - p_z\alpha_y) \neq 0 \qquad (10)
\end{aligned}
$$

But we see from Equations 9 and 10 that the total angular momentum operator

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

does commute with H_D. Total angular momentum is conserved if the potential is sperically symmetric.

Coulomb Potential

We include $\mathbf{E}$ and bfB fields by substitution of canonical momentum and energy

$$\mathbf{p} \to \mathbf{p} - q\mathbf{A}, \quad i\hbar\frac{\partial}{\partial t} \to i\hbar\frac{\partial}{\partial t} - q\phi$$

Then the time independent Dirac Equation is

$$E\psi = \Big(c\alpha\cdot(\mathbf{p} - q\mathbf{A}) + \beta mc^2 + q\phi\Big)$$

In a Coulomb potential $\mathbf{A} = 0$ and we have

$$\begin{pmatrix}(E - mc^2 - q\phi)\psi_1\\ (E + mc^2 - q\phi)\psi_2\end{pmatrix} = c\begin{pmatrix}\sigma\cdot\mathbf{p}\psi_2\\ \sigma\cdot\mathbf{p}\psi_1\end{pmatrix}$$

From the second equation we get

$$\psi_2 = \frac{c\sigma\cdot\mathbf{p}}{(E + mc^2 - q\phi)}\psi_1 = \frac{c\sigma\cdot\mathbf{p}}{(E_S + 2mc^2 - q\phi)}\psi_1 = \frac{c\sigma\cdot\mathbf{p}}{2mc^2(1 + \frac{E_S - q\phi}{2mc^2})}\psi_1$$

Substitution into the first equation gives us

$$(E_S - q\phi)\psi_1 = (c\sigma\cdot\mathbf{p})\frac{1}{2mc^2(1 + \frac{E_S - q\phi}{2mc^2})}(c\sigma\cdot\mathbf{p})\psi_1 \qquad (11)$$

We expand to first order in the nonrelativistic limit, $(E_S - q\phi \ll mc^2)$ and have that

$$\begin{aligned} E_S\psi_1 &= \left(q\phi + (c\sigma\cdot\mathbf{p})\frac{1}{2mc^2}(1-\frac{E_S - q\phi}{2mc^2})(c\sigma\cdot\mathbf{p})\right)\psi_1 \\ &= \left(q\phi + \frac{(c\sigma\cdot\mathbf{p})^2}{2mc^2}(1-\frac{E_S}{2mc^2}) + (c\sigma\cdot\mathbf{p})\frac{q\phi}{(2mc^2)^2}(c\sigma\cdot\mathbf{p})\right)\psi_1 \quad (12) \end{aligned}$$

Among other things we need to evaluate the third term on the right hand side of Equation

$$T = (c\sigma\cdot\mathbf{p})\frac{q\phi}{(2mc^2)^2}(c\sigma\cdot\mathbf{p})$$

and we need to be careful because $\mathbf{p}$ is an operator. Note that

$$\mathbf{p}\phi\{\psi_1\} = -i\hbar(\nabla\phi + \phi\nabla)\{\psi_1\} = (-i\hbar\nabla\phi + \phi\mathbf{p})\{\psi_1\}$$

Then the expression T becomes

$$\begin{aligned} T &= \frac{qc^2}{(2mc^2)^2}\left((\phi\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{p}) + (-i\hbar\sigma\cdot\nabla\phi)(\sigma\cdot\mathbf{p})\right) \\ &= \frac{qc^2}{(2mc^2)^2}\Big((\phi p^2) + (-i\hbar\sigma\cdot\nabla\phi)(\sigma\cdot\mathbf{p})\Big) \\ &= \frac{qc^2}{(2mc^2)^2}\Big((\phi p^2) - i\hbar(\nabla\phi\cdot\mathbf{p} + i\sigma\cdot\nabla\phi\times\mathbf{p})\Big) \end{aligned}$$

$\nabla\phi = \hat{r}\frac{\partial\phi}{\partial r}$. In the last step we use the fact that for any vectors $\mathbf{A}$ and $\mathbf{B}$

$$(\sigma\cdot\mathbf{A})(\sigma\cdot\mathbf{B}) = i\sigma\cdot(\mathbf{A}\times\mathbf{B}) + \mathbf{A}\cdot\mathbf{B}$$

If the potential $\phi(r)$ is spherically symmetric (depends only on r) then

$$\begin{aligned} T &= \frac{q}{(2mc)^2}\left(\phi p^2 - i\hbar\frac{\partial\phi}{\partial r}\hat{r}\cdot\mathbf{p} + \hbar\frac{\partial\phi}{\partial r}\sigma\cdot\hat{r}\times\mathbf{p}\right) \\ &= \frac{q}{(2mc)^2}\left(\phi p^2 - i\hbar\frac{\partial\phi}{\partial r}\hat{r}\cdot\mathbf{p} + 2\frac{\partial\phi}{\partial r}\frac{\mathbf{S}\cdot\mathbf{L}}{r}\right) \end{aligned}$$

Substitution back into Equation gives

$$
\begin{aligned}
E_S\psi_1 &= \left(q\phi + \frac{(c\sigma\cdot\mathbf{p})^2}{2mc^2}(1-\frac{E_S}{2mc^2}) + \frac{q}{(2mc)^2}\left(\phi p^2 - i\hbar\frac{\partial\phi}{\partial r}\hat{r}\cdot\mathbf{p} + 2\frac{\partial\phi}{\partial r}\frac{\mathbf{S}\cdot\mathbf{L}}{r}\right)\right)\psi_1 \\
&= \left(q\phi + \frac{p^2}{2m}(1-\frac{E_S-q\phi}{2mc^2}) + \frac{q}{(2mc)^2}\left(i\hbar\frac{\partial\phi}{\partial r}\hat{r}\cdot\mathbf{p} - 2\frac{\partial\phi}{\partial r}\frac{\mathbf{S}\cdot\mathbf{L}}{r}\right)\right)\psi_1 \\
&= \left(q\phi + \frac{p^2}{2m}(1-\frac{E_S-q\phi}{2mc^2}) - \frac{q^2\hbar^2}{4m^2c^2}\frac{\partial\phi}{\partial r}\frac{\partial}{\partial r} + q\frac{\partial\phi}{\partial r}\frac{\mathbf{S}\cdot\mathbf{L}}{2m^2c^2r}\right)\psi_1 \\
&= \left(q\phi + \frac{p^2}{2m}(1-\frac{p^2}{(2m)^2c^2}) - \frac{q^2\hbar^2}{4m^2c^2}\frac{\partial\phi}{\partial r}\frac{\partial}{\partial r} + q\frac{\partial\phi}{\partial r}\frac{\mathbf{S}\cdot\mathbf{L}}{2m^2c^2r}\right)\psi_1 \\
&= \left(q\phi + \frac{p^2}{2m} - \frac{p^4}{4m^3c^2} - \frac{q^2\hbar^2}{4m^2c^2}\frac{\partial\phi}{\partial r}\frac{\partial}{\partial r} + q\frac{\partial\phi}{\partial r}\frac{\mathbf{S}\cdot\mathbf{L}}{2m^2c^2r}\right)\psi_1
\end{aligned}
$$

In that last step we note that in the nonrelativistic limit $E_S - q\phi = p^2/2m$ and make the substitution. Finally let's substitute $\phi = -\frac{e}{4\pi\epsilon_0 r}$ and we have

$$
E_s\psi_1 = \left(-\frac{e^2}{4\pi\epsilon_0 r} + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} - \frac{e^2\hbar^2}{16m^2c^2\pi\epsilon_0 r^2}\frac{\partial}{\partial r} - \frac{e^2}{4\pi\epsilon_0 r^3}\frac{\mathbf{S}\cdot\mathbf{L}}{2m^2c^2}\right)\psi_1
$$

which looks like the Schrodinger equation with relativistic correction (third term), and spin orbit coupling (fifth term). The fourth term is the so-called Darwin term. It is a relativistic correction to the potential. It has no classical analog.

Ultra-relativistic limit

In the ultra-relativistic limit $pc \gg mc^2$, Equations 7 and 8 become

$$
\psi_1 = \frac{c\sigma\cdot\mathbf{p}}{E}\psi_2 \qquad (13)
$$

$$
\psi_2 = \frac{c\sigma\cdot\mathbf{p}}{E}\psi_1 \qquad (14)
$$

If we define

$$
\psi_R = \psi_1 + \psi_2
$$

$$
\psi_L = \psi_1 - \psi_2
$$

Then we see that

$$\psi_R = \frac{c\sigma\cdot\mathbf{p}}{E}\psi_R \tag{15}$$

$$\psi_L = -\frac{c\sigma\cdot\mathbf{p}}{E}\psi_L \tag{16}$$

We next define the helicity operator $\tilde{h} = \frac{1}{2}\hbar\sigma\cdot\hat{p}$ to be the operator that measures the spin along the direction of motion. Then since in the ultra-relativistic limit $cp = E$, we get that

$$\begin{aligned}\psi_R &= \frac{2}{\hbar}\tilde{h}\psi_R \rightarrow \tilde{h}\psi_R = \frac{1}{2}\hbar\psi_R \\ \psi_L &= -\frac{2}{\hbar}\tilde{h}\psi_L \rightarrow \tilde{h}\psi_L = -\frac{1}{2}\hbar\psi_L\end{aligned}$$

ψ_R and ψ_L are eigenstates of helicity with eigenvalues $\pm\frac{1}{2}\hbar$. Like neutrinos, which are massless (or nearly so) and therefore always ultra-relativistic. The helicty is a Lorentz invariant. You can not transform into the rest frame of a massless partice.