Reading: Griffiths 2.1, 2.2 and review 1.1 and 1.2

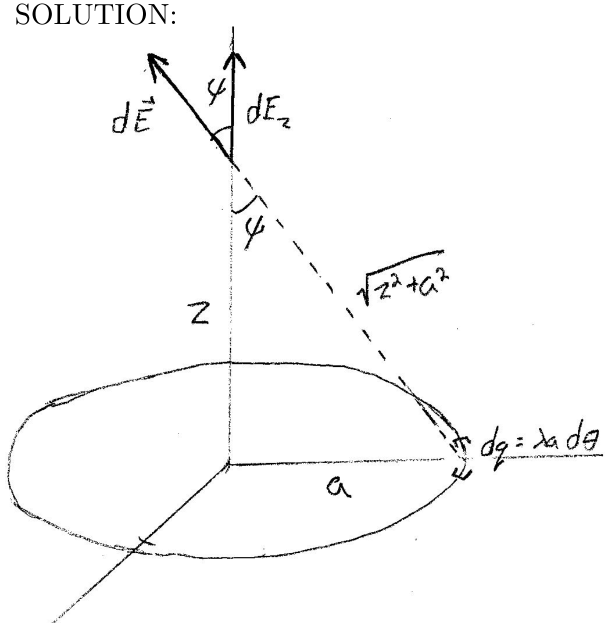
1. Ring of charge

A charge Q is distributed uniformly around a thin ring of radius a, which lies in the xy plane and is centered at the origin.

a) Locate the point on the positive z axis where the electric field is the strongest.

b) Estimate the axial distance from the ring beyond which the ring may be regarded as a point charge if the greatest admissible error in the magnitude of $\vec{E}$ is 0.05%.

SOLUTION:



By the symmetry of the situation, we can see that the electric field along the z axis must point in the $\hat{\mathbf{k}}$ direction.

a)First we need to find the electric field at each point z on the z axis. From the figure we can see that

$$E_z = \int dE_z = \int |d\vec{E}| \cos\psi. \tag{1.1}$$

We can get the quantity $|d\vec{E}|$ from Coulomb's law and $\cos\psi$ from the geometry.

$$|d\vec{E}| = \frac{1}{4\pi\epsilon_0}\frac{dq}{|\vec{r}-\vec{r'}|^2} = \frac{1}{4\pi\epsilon_0}\frac{\lambda a\, d\theta}{z^2+a^2}$$

$$\cos\psi = \frac{z}{\sqrt{z^2+a^2}}$$

Here θ parametrizes the ring, and $\lambda = \frac{Q}{2\pi}$ is the linear charge density of the ring. We can now calculate the electric field.

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda a z}{(z^2+a^2)^{3/2}}\hat{\mathbf{k}}\int_0^{2\pi} d\theta = \frac{\lambda a z}{2\epsilon_0\,(z^2+a^2)^{3/2}}\hat{\mathbf{k}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Qz}{(z^2+a^2)^{3/2}}\hat{\mathbf{k}} \tag{1.2}$$

Now we can take a derivative with respect to z to find the maximum. This formula will be a maximum when the function

$$f(z) = \frac{z}{(z^2+a^2)^{3/2}} \tag{1.3}$$

is a maximum.

$$0 = \frac{df}{dz} = \frac{(z^2+a^2)-3z^2}{(z^2+a^2)^{5/2}}$$

$$\Rightarrow \; 2z^2 = a^2$$
$$\Rightarrow \; z = \frac{a}{\sqrt{2}} \tag{1.4}$$

b)Let us consider points far from the ring so that $z \gg a$. We can use this to aproximate the function $f(z)$ given above. We want to see where $f(z)$ deviates from $\frac{1}{z^2}$ by less than $p = 0.05\%$

$$\frac{z}{(z^2+a^2)^{3/2}} = \frac{z}{z^3\left(1+\frac{a^2}{z^2}\right)^{3/2}} = \frac{1}{z^2}\left(1+\frac{a^2}{z^2}\right)^{-3/2}$$
$$\approx \frac{1}{z^2}\left(1-\frac{3a^2}{2z^2}\right) \tag{1.5}$$

We see that the fractional discrepancy is

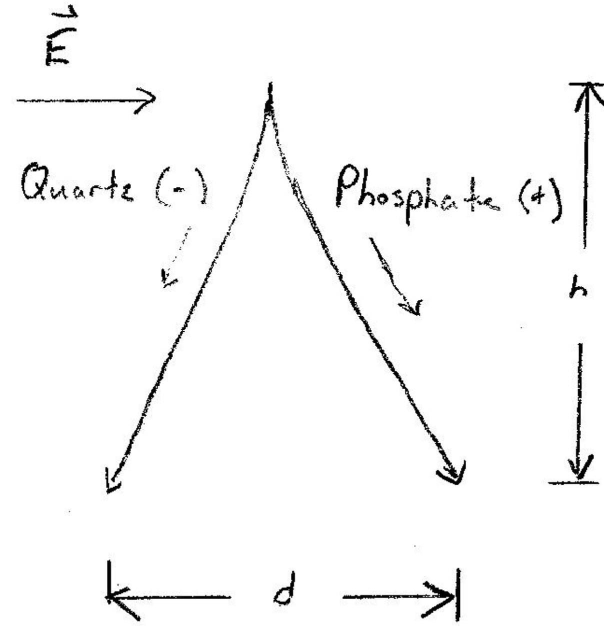
$$p = 5\times 10^{-4} \approx \frac{3a^2}{2z^2}.$$
$$\Rightarrow \; z \approx a\sqrt{\frac{3}{10}}\times 10^2 \approx 50a \tag{1.6}$$

This also validates our assumption that $z \gg a$.

2. Electrostatic separator

Crushed Florida phosphate ore consists of particles of quartz mixed with particles of phosphate rock. Vibrating the mixture results in positively charged phosphate and negatively charged quartz components. Pouring the mixture through a horizontally-directed electric field allows the phosphate to be separated out. Over what minimum distance must the particles fall if they must be separated horizontally by at least 0.2 m? The electric field in the separator has a magnitude of 4×10^5 volts/m, and the specific charge (the induced charge per unit mass) is 10^{-5} coulomb/kg.

SOLUTION:



Consider the figure. The quartz and the phosphate will move in opposite directions. Since the magnitude of the specific charge q/m is the same for both, they will both move the same distance horizontally but in opposite directions. The acceleration in the horizontal direction is given by:

$$F = ma_x = qE \Rightarrow \; a_x = \frac{q}{m}E \tag{2.1}$$

The rest of the problem is kinematics.

$$h = \frac{1}{2}gt^2$$
$$\frac{d}{2} = \frac{1}{2}a_xt^2 = \frac{1}{2}E\frac{q}{m}t^2$$
$$\frac{2h}{d} = \frac{g}{Eq/m}$$

$$h = \frac{dg}{2Eq/m} = 0.245\text{m} \tag{2.2}$$

3. Two charged rods

Consider a positively charged rod (uniform charge density λ) in the xy plane with endpoints at $(x, y) = (-1, 1)$ and $(1, 1)$. A second, negatively charged, rod (uniform charge density $-\lambda$) has endpoints $(-1, -1)$ and $(1, -1)$.

a) Calculate the magnitude and direction of the electric field along the z axis.

b) Show that for $|z| >> 1$, the field reduces to that of a dipole.

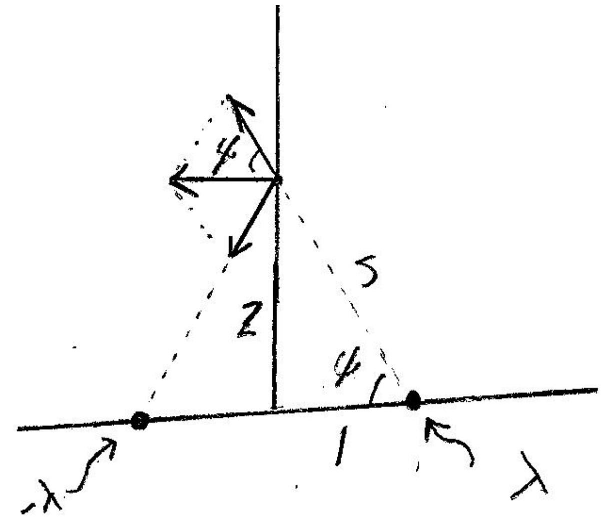
c) How far along the z axis must one go before the dipole field found in class accurately estimates this field to within 10%?

SOLUTION: For this problem, we will need the result from Example 2.1 on page 62 of Griffiths. It states that the electric field at a point at distance s directly above the midpoint of a line segment of charge of length $2L$ points directly away from the line and has magnitude:

$$E = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{s\sqrt{s^2 + L^2}} \tag{3.1}$$

In this problem $L = 1$

a)



The figure shows the situation viewed along the x axis. We can use the superposition principle and see from the symmetry that the electric field will point in the $-\hat{\mathbf{j}}$ direction. Its magnitude will be

$$E = \frac{2}{4\pi\epsilon_0} \frac{2\lambda}{s\sqrt{s^2 + 1}} \cos\psi. \tag{3.2}$$

Since $s = \sqrt{z^2 + 1}$ and $\cos\psi = 1/s$, we have

$$\vec{E} = -\frac{2}{4\pi\epsilon_0} \frac{2\lambda}{(z^2 + 1)\sqrt{z^2 + 2}} \hat{\mathbf{j}} = -\frac{\lambda}{\pi\epsilon_0 (z^2 + 1)\sqrt{z^2 + 2}} \hat{\mathbf{j}} \tag{3.3}$$

b)In the limit $|z| \gg 1$, this reduces to

$$\vec{E} = -\frac{\lambda}{\pi\epsilon_0 z^3} \hat{\mathbf{j}} \tag{3.4}$$

Far away from the charge distribution, we expect the electric field to look like a dipole with dipole moment $\vec{p} = 4\lambda\hat{\mathbf{j}}$. A dipole has electric field given by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}[3(\vec{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \vec{p}]. \tag{3.5}$$

In our case $\hat{\mathbf{r}} = \hat{\mathbf{k}}$ and $r = z$ so Eq. (**??**) becomes

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{[3(4\lambda\hat{\mathbf{j}}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}} - 4\lambda\hat{\mathbf{j}}]}{z^3} = -\frac{\lambda}{\pi\epsilon_0 z^3}\hat{\mathbf{j}} \tag{3.6}$$

in agreeance with Eq. (**??**).

c)We need to find where the values z where

$$\left|\frac{1}{(z^2+1)\sqrt{z^2+2}} - \frac{1}{z^3}\right| < \frac{0.1}{(z^2+1)\sqrt{z^2+2}}$$

This is true for $z > 4.50$. This number can be found by solving the above inequality, or more simply by plotting the two sides of the equation with a computer and finding where they meet.

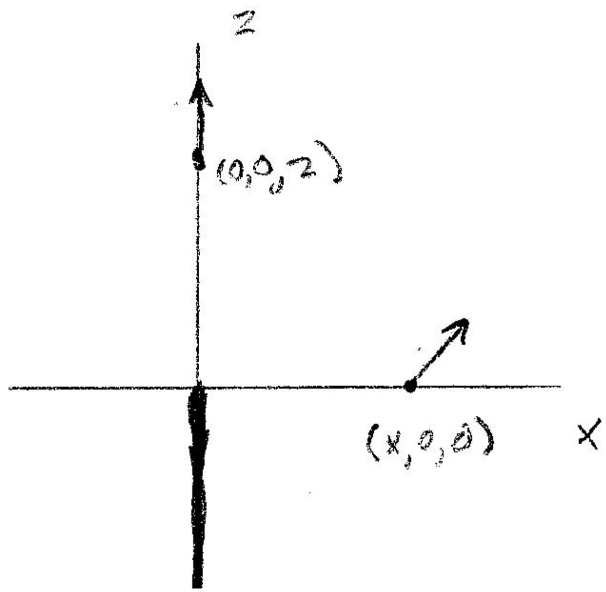
4. Semi-infinite wire

A semi-infinite wire lies on the negative z axis, that is from $z = 0$ to $z = -\infty$. It has a constant linear charge density λ.

a) **D**etermine $\vec{E}$ at a point $(0, 0, z)$ on the positive z axis.

b) **D**etermine $\vec{E}$ at a point $(x, 0, 0)$ on the positive x axis.

SOLUTION:



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{R^2}\hat{R} \tag{4.1}$$

(Here $\vec{R} = \vec{r} - \vec{r'}$)

a)

$$dq = \lambda dz'$$

$$\vec{R} = (z - z')\hat{\mathbf{k}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{0}\frac{\lambda\, dz'}{(z-z')^2}\hat{\mathbf{k}} = \frac{1}{4\pi\epsilon_0}\left[\frac{\lambda}{z-z'}\right]_{-\infty}^{0}\hat{\mathbf{k}}$$

$$\vec{E}(0,0,z) = \frac{\lambda\hat{\mathbf{k}}}{4\pi\epsilon_0 z} \tag{4.2}$$

b)

$$\vec{R} = x\hat{\mathbf{i}} - z'\hat{\mathbf{k}}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{0}\frac{\lambda\, dz'(x\hat{\mathbf{i}} - z'\hat{\mathbf{k}})}{(x^2+z'^2)^{3/2}}$$

For both the $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ integrals, use the same substitution $z' = x\tan\theta$, $dz' = x\sec^2\theta d\theta$, and make use of the identity $\tan^2\theta + 1 = \sec^2\theta$.

For the $\hat{\mathbf{i}}$ integral

$$\int_{-\infty}^{0} \frac{x\,dz'}{(x^2+z'^2)^{3/2}} = \int_{-\pi/2}^{0} \frac{x^2\sec^2\theta\,d\theta}{x^3\sec^3\theta} = \frac{1}{x}\int_{-\pi/2}^{0}\cos\theta = \frac{1}{x}$$

For the $\hat{\mathbf{k}}$ integral

$$\int_{-\infty}^{0} \frac{z'\,dz'}{(x^2+z'^2)^{3/2}} = \int_{-\pi/2}^{0} \frac{x^2\tan\theta\sec^2\theta\,d\theta}{x^3\sec^3\theta}$$

$$= \frac{1}{x}\int_{-\pi/2}^{0}\sin\theta = -\frac{1}{x}$$

Altogether then we have

$$\vec{E}(x,0,0) = \frac{\lambda}{4\pi\epsilon_0 x}(\hat{\mathbf{i}} + \hat{\mathbf{k}}) \tag{4.3}$$

5. Griffiths 2.15

SOLUTION: The spherical symmetry requires the electric field to point in the $\hat{\mathbf{r}}$ direction. We construct a spherical Gaussian surface of radius r centered at the center of the shell and use Gauss's law in integral form.

$$\frac{Q_{in}}{\epsilon_0} = \oint_S \vec{E}\cdot d\vec{A} = 4\pi r^2 E \Rightarrow E = \frac{1}{4\pi\epsilon_0}\frac{Q_{in}}{r^2} \tag{5.1}$$

It is now just a matter of calculating Q_{in} for values of r in the different regions.

For $r < a$, $Q_{in} = 0$ so $\vec{E} = 0$
For $a < r < b$, $Q_{in} = \int_a^r \frac{k}{r'^2} 4\pi r'^2\, dr' = 4\pi k(r-a)$
For $r > b$, $q_{in} = \int_a^b \frac{k}{r'^2} 4\pi r'^2\, dr' = 4\pi k(b-a)$

The electric field is thus

$$\vec{E} = \begin{cases} \vec{0} & r < a \\ \frac{k}{\epsilon_0 r^2}(r-a)\hat{\mathbf{r}} & a < r < b \\ \frac{k}{\epsilon_0 r^2}(b-a)\hat{\mathbf{r}} & r > b \end{cases} \tag{5.2}$$

6. Atmospheric electric field

The electric field in the atmosphere at the earth's surface is approximately 200 V/m, directed downward. At 1400m above the Earth's surface, the electric field in the atmosphere is only 20 V/m, again directed downward.

a) What is the atmosphere's average charge density below 1400m?

b) Does this consist predominantly of positive or negative ions?

SOLUTION:

a)We imagine the earth's atmosphere between ground level and 1400m to be a thin spherical shell covering the earth. The averge charge density in a region V is

$$\rho_{avg} = \frac{1}{V}\int_V \rho\, dV = \frac{\epsilon_0}{V}\int_V \nabla\cdot\vec{E}\, dV \tag{6.1}$$

with the last equality being a consequence of Gauss's law. Now apply the divergence theorem to the integral over the volume.

$$\rho_{avg} = \frac{\epsilon_0}{V}\oint_S \vec{E}\cdot d\vec{A} \tag{6.2}$$

There are two surfaces of approximately equal area (since 1400m $\ll r_{earth}$) to integrate over. On the outer surface (1400m), the outward normal is opposite the electric field, while on the inner surface (ground level) the outward normal is in the same direction. Thus we have

$$\rho_{avg} = \frac{\epsilon_0}{Ah}(E_g - E_a)A = \frac{\epsilon_0}{h}(E_g - E_a) \approx 1.14\,\mathrm{C/m}^3. \quad (6.3)$$

Here E_g is the electric field at ground level, E_a is the electric field at 1400m, A is the surface area of the earth, $h =$1400m and $V \approx Ah$ is the volume between these two heights.

b)Since the answer to part a) is positive, the charge density must consist mostly of positive ions