An Introduction to the Physics of High Energy Accelerators
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CHAPTER 6

Intensity Dependent Effects

Thus far, we have considered only the motion of a single particle, or of a beam of noninteracting particles, in the presence of external forces. Many of the interesting phenomena in accelerator physics involve the dynamic interplay between the beam and its surroundings through the electromagnetic fields initiated by the beam. Frequently—in fact, usually—the consequences are not benign, and there is a catalog of beam instabilities analogous to the more familiar instabilities of hydrodynamics.

In this chapter we introduce such effects for which the intensity of the beam is important. We begin with a discussion of space charge and its effects on betatron oscillation tunes. This will lead us to the introduction of coherent instabilities, in particular the so-called negative mass instability. The particular field used to motivate the treatment of the negative mass instability will be the longitudinal space charge field of the beam. Discussion of a general field is made possible through the introduction of impedance.

Having introduced impedance, we digress briefly with a section on the fields trailing a charge—the wake fields—and their relation with impedance. We return to the main theme, with macroparticle models of coherent instabilities that provide a simpler model for such effects as beam breakup in linacs and the head-tail effect.

A thorough treatment of coherent instabilities requires that we follow the evolution of the particle distribution function. We derive the Vlasov equation, obtain the dispersion relation for longitudinal stability of a coasting beam, and apply the result to the negative mass instability of a beam with momentum spread. We conclude with a brief general heuristic account of Landau damping.

6.1 SPACE CHARGE

Throughout most of this chapter we treat the particles constituting the beam as a continuum described by a space charge density distribution $\rho(\vec{r})$. That is, we ignore any statistical effects due to the fact that the beam is actually composed of many individual particles, and examine the motion of a single test particle under the influence of the surrounding "space charge." We should note in passing that the bulk of the present intense activity in this field lies in the low energy, high current regime where space charge forces are pervasive design considerations. For the domain of high energy accelerators it is usually possible to treat space charge effects as perturbations to the single particle motion, but their inclusion in the design process remains essential.

We will investigate transverse and longitudinal space charge effects separately. In the first case we will mainly look at examples of tune shift, essentially static field effects. Secondly, we will examine an example of a coherent instability brought on, in its simplest form, by longitudinal space charge fields.

6.1.1 The Transverse Space Charge Force

Here we begin by studying the static effects of space charge forces in unneutralized beams. We start this discussion by writing down the fields in some simple cases. First, consider a uniform cylindrical beam. If there are N particles per unit length in a beam of radius a, then, outside the beam, that is, for $r > a$, the electric and magnetic fields are

$$E = \frac{eN}{2\pi\epsilon_0 r}, \tag{6.1}$$

$$B = \frac{eNv}{2\pi\epsilon_0 rc^2}, \tag{6.2}$$

where v is the speed of the particles. Because the Coulomb repulsion is in the opposite direction to the magnetic attraction, the net outward force is

$$F = \frac{e^2N}{2\pi\epsilon_0 r\gamma^2}, \tag{6.3}$$

where γ is the Lorentz factor. On the other hand, inside the beam, that is, for $r < a$, the fields are

$$E = \frac{eN}{2\pi\epsilon_0 a^2} r, \tag{6.4}$$

$$B = \frac{eNv}{2\pi\epsilon_0 a^2 c^2} r, \tag{6.5}$$

from which

$$F = \frac{e^2 N}{2\pi\epsilon_0 a^2 \gamma^2} r. \tag{6.6}$$

Note that if the beams were partially neutralized, the space charge force could be much larger. Within the beam, the linear dependence of the force on the transverse position is reminiscent of a defocusing lens.

As another example, we next consider the case of a beam which is Gaussian in both transverse coordinates with standard deviations $\sigma_x = \sigma_y \equiv \sigma$; we still assume the density is independent of the longitudinal coordinate. In this case, the fields may be written as

$$E = \frac{eN}{2\pi\epsilon_0 r}\left(1 - e^{-r^2/2\sigma^2}\right), \tag{6.7}$$

$$B = \frac{eNv}{2\pi\epsilon_0 r c^2}\left(1 - e^{-r^2/2\sigma^2}\right), \tag{6.8}$$

and the force is given by

$$F = \frac{e^2 N}{2\pi\epsilon_0 \gamma^2 r}\left(1 - e^{-r^2/2\sigma^2}\right). \tag{6.9}$$

Obviously, for r large compared with σ, the force varies inversely with r as one would expect. On the other hand, the force at values of r small compared with σ is

$$F = \frac{e^2 N}{4\pi\epsilon_0 \gamma^2 \sigma^2} r \tag{6.10}$$

and so varies linearly with transverse displacement. This suggests that particles with small oscillation amplitudes will experience a force similar to the focusing forces of beam optics considered earlier, while particles with larger amplitudes will see less of this effect.

Field distributions can be written down for more complicated beam profiles, but the subsequent treatment becomes more complex, whereas the above simple cases will illustrate the physics.

6.1.2 Equation of Motion in the Presence of Space Charge

Our previous development of the equation of motion permits the inclusion of the space charge force:

$$x'' + K(s)x = \frac{1}{\gamma m v^2} \times (\text{space charge force}). \tag{6.11}$$

Putting in the space charge force for a uniformly charged round beam, we have

$$x'' + \left[K(s) - \frac{2r_0 N}{(v/c)^2 \gamma^3 a^2}\right]x = 0, \tag{6.12}$$

where r_0 is the *classical radius* of the particle,

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}. \tag{6.13}$$

(For the proton, $r_0 = 1.53 \times 10^{-18}$ m, and for the electron, $r_0 = 2.82 \times 10^{-15}$ m.) So, in a circular accelerator, the reduction in focusing strength will lead to a shift in the betatron oscillation tune. At low energies, however, the space charge force can vitiate, or indeed overcome, the external focusing force. As an example of the latter circumstance, we consider a focusing system with constant K. For low energy, where $\gamma \to 1$, the focusing force is exactly balanced by the space charge force when

$$\frac{2r_0 N}{(v/c)^2 a^2} = K \tag{6.14}$$

or, in terms of the beam current $I = eNv$,

$$\begin{aligned} I &= \frac{Kec(v/c)^3 a^2}{2r_0} \\ &= \frac{4\pi\epsilon_0 T a^2 B'}{m}, \end{aligned} \tag{6.15}$$

where T is the kinetic energy of the particle and the focusing force has been expressed in terms of the gradient B' of the external magnetic field.

For example, suppose we have a proton beam of 1 cm radius propagating through a field gradient of 1 T/m. Then, from the above, the ratio of the beam current to the kinetic energy must be below 1 A/MeV to ensure focusing.

6.1.3 Incoherent Tune Shift

Now let us look in more detail at the tune shift in a circular accelerator. From our discussion in Chapter 3, the change in the betatron oscillation tune due to a distribution of gradient errors is

$$\Delta\nu = \frac{1}{4\pi}\oint\frac{\beta(s)\Delta B'(s)}{(B\rho)}\,ds \to \frac{1}{4\pi}\oint\beta(s)\Delta K\,ds. \tag{6.16}$$

For our case, ΔK is

$$\Delta K = \frac{\Delta B'}{(B\rho)} \to \frac{F'}{ev(B\rho)} = \frac{F'}{pv} \tag{6.17}$$

where use has been made of the form of the Lorentz force due to a magnetic field to replace $\Delta B'$ in our formula with the gradient of a force in general. So, for a round Gaussian beam, and for small displacement compared with σ,

$$\begin{aligned}
\Delta\nu &= \oint\frac{1}{4\pi}\beta(s)\frac{F'(s)\,ds}{pv} \\
&= \frac{1}{4\pi}\frac{1}{pv}\frac{e^2N}{4\pi\epsilon_0\gamma^2}\oint\frac{\beta(s)}{\sigma^2(s)}\,ds \\
&= \frac{e^2}{4\pi\epsilon_0 mc^2}\frac{N}{(v/c)^2\gamma^3}\frac{1}{4}\oint\frac{\beta}{\pi\sigma^2}\,ds \\
&= \frac{r_0N}{4(v/c)\gamma^2}\oint\frac{ds}{\epsilon_N} \\
&= \frac{\pi N r_0 R}{2\epsilon_N(v/c)\gamma^2}.
\end{aligned} \tag{6.18}$$

where R is the average radius of the accelerator, and we have used $\epsilon_N = \pi\sigma^2(\gamma v/c)/\beta$ as our definition of normalized emittance. The tune is decreased due to the defocusing character of the space charge force.

As a numerical example, consider an unbunched beam entering the Fermilab booster synchrotron. Here $R = 75$ m, the injection kinetic energy is 200 MeV, the normalized emittance of the beam at entry is π mm mrad, and the number of particles per meter is 6×10^9. The tune shift for a particle undergoing infinitesimal transverse oscillations at the center of the beam is then 0.4. It is no wonder that emittance dilution and particle loss occur under these circumstances. The remarkable thing is that tune shifts of a significant fraction of unity can be sustained within the beam. The cures for beam loss and emittance dilution due to space charge are a higher injection energy and a smaller ring.

The tune shifts for particles having larger oscillation amplitudes are smaller than those for particles at the center of the beam. Thus the beam particles span a range of tunes, with the outer particles scarcely displaced from the single particle case.

Frequently, the tune is measured by observing a coherent motion of the beam centroid. The tune so measured is not necessarily any of the values within the span of the preceding paragraph. The tune shift that we calculated above is often called an incoherent tune shift for this reason.

6.1.4 The Beam-Beam Tune Shift

A similar incoherent tune shift occurs in a colliding beam accelerator. Each time the beams cross each other, the particles in one beam feel the electric and magnetic forces due to the particles in the other beam. Consider the case of two intersecting beams of particles with like charges. Because the velocity of the "test particle" in one beam is in the opposite direction of the velocity of the oncoming beam, the electric and magnetic forces do not cancel as in the previous section, but rather add, creating a net defocusing force. For particles undergoing infinitesimal betatron oscillations in a highly relativistic Gaussian beam, the net force would be

$$F = \frac{e^2 N}{2\pi\epsilon_0\sigma^2} r, \tag{6.19}$$

and the tune shift experienced by the particle would be

$$\begin{aligned} \Delta\nu &= \frac{1}{4\pi}\frac{1}{pc}\frac{e^2}{2\pi\epsilon_0}\oint\frac{N\beta(s)}{\sigma^2(s)}\,ds \\ &= \frac{r_0}{2\epsilon_N} \times \frac{1}{2}\int N\,ds. \end{aligned} \tag{6.20}$$

In this case, the force is felt only over the time that the two bunches are colliding. Only half of the integral over the presumed symmetric distribution is necessary, because the two beams are traveling in opposite directions. As soon as the "test particle" has traveled half the bunch length, the oncoming bunch has gone past. In terms of the total number of particles in a bunch, n, the beam-beam tune shift per collision is then

$$\Delta\nu = \frac{nr_0}{4\epsilon_N}. \tag{6.21}$$

In contrast to the somewhat larger space charge tune shift found in the previous section for a lower energy synchrotron, the beam-beam tune shift is typically rather small. For example, consider the shift for the Tevatron collider. Here, the typical numbers are $n = 6 \times 10^{10}$, $\epsilon_N = 3\pi$ mm mr. For one crossing, $\Delta\nu = 0.0025$. Up to this writing, the collider has operated with six bunches of protons and six bunches of antiprotons. This generates 12 interactions per revolution, which implies a possible tune shift of 0.03. The shift is positive due to the opposite charges involved. Unlike the case of the Booster synchrotron, where the injected beam remains in the presence of the space charge force for only a few milliseconds, the particles in the collider are stored for many hours. In view of the long-term sensitivity to resonances, one would expect that the allowed region in the tune diagram occupied by the beam particles would be quite limited. This is indeed the case.

6.1.5 Image Charge and Image Current Effects

So far we have ignored the interaction of the beam with its surroundings. Typically, a beam travels within a beam pipe made of some conducting material. So, in general, the electric field distribution will be influenced by the conducting boundary. Also, within a magnet, the magnetic field distribution will be shaped in part by the presence of magnetic materials. We take up the problem of static effects due to electric fields.

The method of images is well suited to problems of this type. Take, for example, the electric field distribution of a line charge near a perfectly conducting plane. The field lines are perpendicular to the plane everywhere, and the resulting distribution is the same as if the conducting plane were replaced by a second line distribution of opposite charge equidistant from the boundary, as shown in Figure 6.1. Therefore, the force on the line charge due to the conducting plane is the same as the force calculated due to the image.

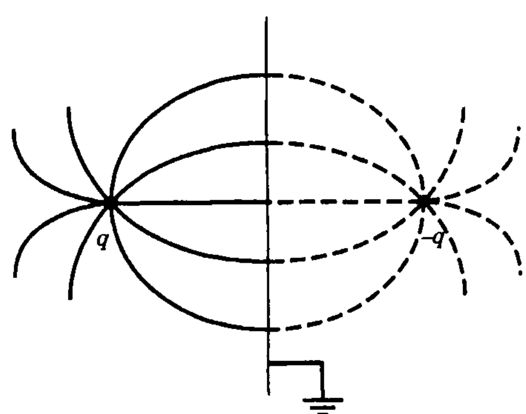

Figure 6.1. A charge near a conducting plane and its image.

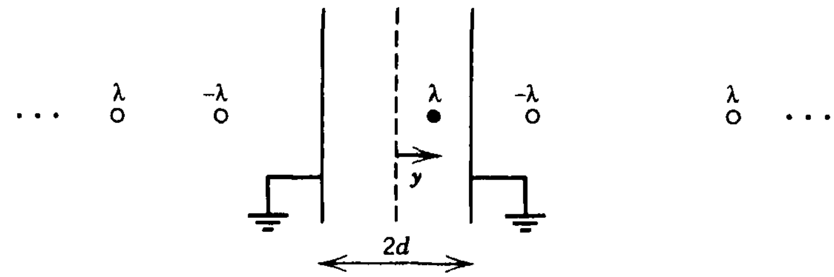

Figure 6.2. Line charge of density λ located within a rectangular beam pipe, approximated by two perfectly conducting parallel plates, and the associated images.

Of course, in a real accelerator, the beam chamber is closed. One common geometry is a chamber with rectangular cross section, where one side of the rectangle is much larger than the other. We can approximate this geometry by two parallel planes. As one would expect, in this case there are an infinite number of images. Suppose that we have a line charge with line density λ at a distance y from the center of the chamber as shown in Figure 6.2. Then at some field point y', the field due to the images alone is

$$E = \sum_i \frac{\lambda_i}{2\pi\epsilon_0 r_i} \tag{6.22}$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\frac{1}{2d-y-y'} - \frac{1}{2d+y+y'} + \frac{1}{4d-y+y'} - \frac{1}{4d+y-y'} + \cdots\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0}\left[\frac{2(y+y')}{4d^2-(y+y')^2} + \frac{2(y-y')}{16d^2-(y-y')^2} + \frac{2(y+y')}{36d^2-(y+y')^2+\cdots}\right]$$

$$\approx \frac{\lambda}{4\pi\epsilon_0 d^2}\left[\frac{y+y'}{1} + \frac{y-y'}{4} + \frac{y+y'}{9} + \frac{y-y'}{16} + \cdots\right]$$

$$= \frac{\lambda}{4\pi\epsilon_0 d^2}\left[\left(1+\frac{1}{4}+\frac{1}{9}+\cdots\right)y + \left(1-\frac{1}{4}+\frac{1}{9}-\cdots\right)y'\right],$$

$$= \frac{\lambda}{4\pi\epsilon_0 d^2}\left(\frac{\pi^2}{6}y + \frac{\pi^2}{12}y'\right)$$

$$= \frac{\pi\lambda}{24\epsilon_0 d^2}\left(y + \tfrac{1}{2}y'\right), \tag{6.23}$$

where we have assumed that y and y' are much less than d.

Note that the field gradient, $\partial E/\partial y'$, is independent of position y' and independent of the charge distribution. Thus the tune shift experienced by each particle in the beam is the same. This can be called a *coherent* tune shift, in contrast to the incoherent tune shift treated above.

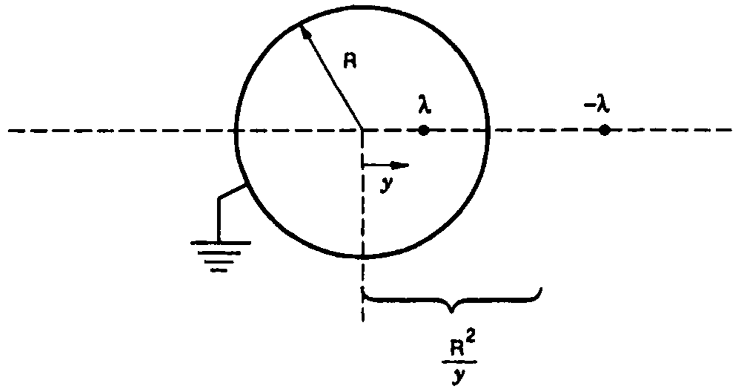

Figure 6.3. Line charge of density λ located within a round beam pipe, and the associated image.

Another simple and common geometry is that of a cylindrical beam pipe. In contrast to the parallel plate case, if a line charge is located at the center of the beam pipe, there will be no image field. But if the line charge is displaced a distance y from the center, an image line charge will appear at a distance R^2/y from the origin as shown in Figure 6.3.

Now at the field point y' the field gradient due to the image charge will be

$$\frac{\partial E}{\partial y'} = \frac{\lambda}{2\pi\epsilon_0} \frac{y^2}{R^4}, \tag{6.24}$$

where we assume y and y' are small compared with R. So the particles of a small beam will experience a coherent tune shift if there are, for example, orbit distortions or other regions in which the beam strays from the center of the vacuum chamber.

In the above, we have used electrostatic images as examples. Magnetic images can be treated in much the same way. For vacuum chambers composed of good conductors, the electric image is the simpler because the electric field lines terminate on the chamber walls and the charges that are treated in the image method flow rapidly to preserve this boundary condition. Magnetic field lines, however, penetrate the chamber wall, and the treatment of the magnetic images must include not only the external magnetic environment, but also time constants associated with field penetration through conducting materials.

6.2 THE NEGATIVE MASS INSTABILITY

Up to now we have assumed a charge density which is independent of the longitudinal coordinate. The symmetry of this situation implies that there is no longitudinal space charge force. The first coherent instability we will examine, the negative mass instability, is driven by longitudinal space charge forces that may arise as soon as this limitation is removed.

The origin of this instability can be seen as follows. Suppose two particles in a synchrotron are traveling close together, one behind the other. Ignore for the present any fields due to their environment. The two charges will repel each other. The charge in front will gain energy and the charge behind will lose energy. Above transition, the orbit period of the first charge will increase, and that of the second will decrease. So the charges will move closer together in the longitudinal coordinate. The circumstance that a repulsive force leads to the particles approaching each other accounts for the name "negative mass."

We will discuss this phenomenon in two stages. First, we will calculate the space charge force for an unbunched beam and, as a development of the argument in the preceding paragraph, indicate how a perturbation in the charge density can grow above transition. Second, we will present a quantitative treatment for a beam without momentum spread traveling in a synchrotron with a general longitudinal impedance.[1]

6.2.1 The Longitudinal Space Charge Field

Let's suppose that the beam has a linear charge density λ that uniformly populates a cylinder of radius a. Then, provided that the derivative of λ with respect to the longitudinal coordinate s is sufficiently small, the fields from Gauss's and Ampere's laws are

$$E_r = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}, \quad B_\phi = \frac{\mu_0 \lambda v}{2\pi}\frac{1}{r}, \qquad r \geq a; \tag{6.25}$$

$$E_r = \frac{\lambda}{2\pi\epsilon_0}\frac{r}{a^2}, \quad B_\phi = \frac{\mu_0 \lambda v}{2\pi}\frac{r}{a^2}, \qquad r \leq a. \tag{6.26}$$

We now find the electric field along the beam axis, as indicated in Figure 6.4. Using Faraday's law

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial}{\partial t}\int \vec{B} \cdot d\vec{A}, \tag{6.27}$$

[1]Adapted from A. Hofmann, "Single-Beam Collective Phenomena—Longitudinal," *Theoretical Aspects of the Behaviour of Beams in Accelerators and Storage Rings* (Proc. First Course of the International School of Particle Accelerators of the "Ettore Majorana" Centre for Scientific Culture, Erice, 1976), CERN 77-13, 1977.

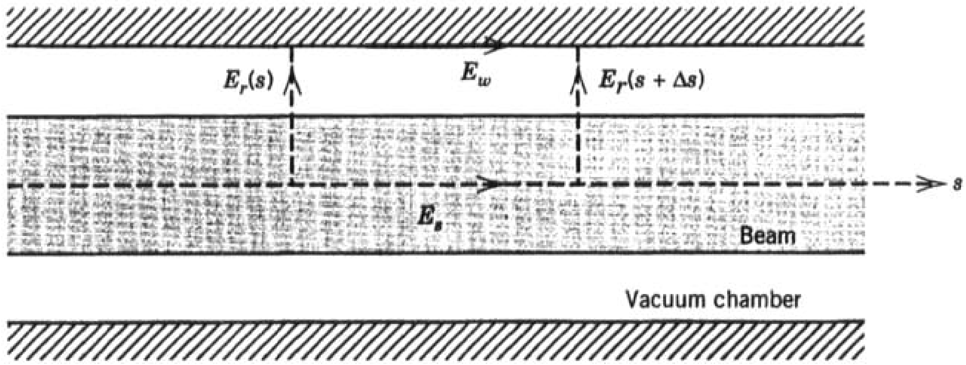

Figure 6.4. Beam passing through cylindrical beam pipe; the charge density may depend upon the longitudinal coordinate.

the left hand side is

$$\begin{aligned}(E_s - E_w)\Delta s &+ \left[\int_0^a \frac{r}{a^2}\,dr + \int_a^b \frac{1}{r}\,dr\right]\frac{\lambda(s+\Delta s)}{2\pi\epsilon_0} \\ &- \left[\int_0^a \frac{r}{a^2}\,dr + \int_a^b \frac{1}{r}\,dr\right]\frac{\lambda(s)}{2\pi\epsilon_0} \\ &= \left[(E_s - E_w) + \frac{\lambda'}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\right]\Delta s, \end{aligned}\tag{6.28}$$

where $\lambda' \equiv \partial\lambda/\partial s$. The right hand side of Equation 6.27 is

$$-\Delta s\left[\frac{1}{2} + \ln\frac{b}{a}\right]\frac{\mu_0 \dot{\lambda} v}{2\pi},\tag{6.29}$$

with $\dot{\lambda} \equiv \partial\lambda/\partial t$. Note that the rate of change of λ with time, as observed at a particular location in the ring, is opposite in sign to the rate of change of λ with respect to s; that is,

$$\dot{\lambda} = -\lambda'\frac{ds}{dt} = -\lambda' v^*,\tag{6.30}$$

where v^* is the phase velocity of the wave in the charge density. The right hand side of Equation 6.27 becomes

$$\Delta s\left[\frac{\lambda'}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\left(\frac{v}{c}\right)^2\right],\tag{6.31}$$

where we assume $v^* \approx v$.

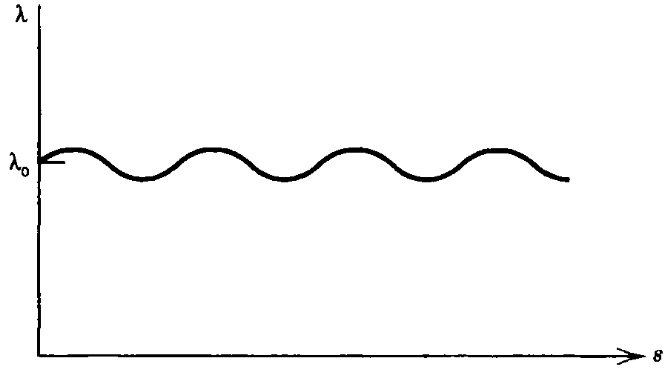

Figure 6.5. Perturbation in the line charge density λ as a function of s.

Putting the two sides together, we get

$$E_s - E_w = -\frac{\lambda'}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\left[1 - \left(\frac{v}{c}\right)^2\right] \equiv -\frac{\lambda' g_0}{4\pi\epsilon_0\gamma^2}. \quad (6.32)$$

In the case of a perfectly conducting wall, E_w will vanish. In this case, we have the pure space charge field

$$E_s = -\frac{g_0\lambda'}{4\pi\epsilon_0\gamma^2}. \quad (6.33)$$

Now suppose there is a small disturbance in the charge density as that sketched in Figure 6.5. In the regions where $\lambda' > 0$ the space charge field will be negative. For $\lambda' < 0$, the field is positive. So, below the transition energy, particles in the regions where $\lambda' < 0$ will speed up and increase their revolution frequency. They will move toward the trough in the wave. Similarly, particles in the regions where $\lambda' > 0$ will slow down and again fill in the trough. The disturbance is damped. But above transition, the reverse will be the case, and the disturbance will grow.

6.2.2 Perturbation of the Line Density

We now proceed with a standard instability treatment. We assume that there is a small perturbation superimposed upon the static charge distribution and look for the circumstance under which this perturbation will grow.

Suppose the line density is perturbed according to

$$\lambda(\theta,t) = \lambda_0 + \lambda_1 e^{i(\Omega t - n\theta)}, \quad (6.34)$$

where $0 < \theta < 2\pi$ is an angular coordinate describing the azimuthal location along the unbunched beam ($\theta = s/R$). The *mode number* is denoted by n, and a general perturbation could be described by a superposition of such modes. The angular frequency of the mode is Ω; observing the disturbance at one location on the perimeter of the ring, one would see the beam vibrate at frequency Ω, while the wave propagates at angular frequency $\omega = \Omega/n$. From the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0, \tag{6.35}$$

λ must satisfy

$$\frac{\partial \lambda}{\partial t} + \frac{\partial}{\partial \theta}(\lambda \omega) = 0, \tag{6.36}$$

where ω is the angular frequency of the distribution. If we write ω as

$$\omega(\theta, t) = \omega_0 + \omega_1 e^{i(\Omega t - n\theta)}, \tag{6.37}$$

then the continuity equation yields

$$\omega_1 = (\Omega - n\omega_0)\frac{\lambda_1}{n\lambda_0}, \tag{6.38}$$

where second order terms have been neglected. The local beam current $I(\theta, t)$ may be written as

$$\begin{aligned} I = R\omega\lambda &= I_0 + I_1 e^{i(\Omega t - n\theta)} \\ &= R\omega_0\lambda_0 + R(\omega_0\lambda_1 + \omega_1\lambda_0)e^{i(\Omega t - n\theta)}, \end{aligned} \tag{6.39}$$

from which we identify

$$I_0 = R\omega_0\lambda_0, \qquad I_1 = \frac{\Omega}{n}R\lambda_1. \tag{6.40}$$

We now consider an individual particle in the distribution. The rate of change of the particle's angular frequency is

$$\left(\frac{d\omega}{dt}\right)_p = \frac{\partial \omega}{\partial t} + \frac{\partial \omega}{\partial \theta}\left(\frac{d\theta}{dt}\right)_p. \tag{6.41}$$

The angular velocity of the particle is, to first order, $(d\theta/dt)_p = \omega_0$. Also, we learned in Chapter 2 that the revolution frequency of a particle is a function only of the particle's energy. That is,

$$\left(\frac{d\omega}{dt}\right)_p = \left(\frac{d\omega}{dE}\right)_p \frac{dE}{dt} = -\frac{\eta\omega_0}{(v/c)^2 E}\frac{dE}{dt}. \tag{6.42}$$

But the rate of change of the particle's energy may also be expressed in terms of an impedance to the flow of the charges in the longitudinal direction. With this concept of a longitudinal impedance, $Z_\parallel$, the rate of change of the particle's energy may be written as

$$\begin{aligned}\frac{dE}{dt} &= -e(\text{energy loss per unit charge/turn})(\text{no. of turns/sec}) \\ &= -e\left(I_1 Z_\parallel e^{i(\Omega t - n\theta)}\right)\left(\frac{\omega_0}{2\pi}\right).\end{aligned} \tag{6.43}$$

Therefore, the angular frequency must satisfy

$$\frac{\partial\omega}{\partial t} + \frac{\partial\omega}{\partial\theta}\omega_0 = \frac{e\eta I_1 Z_\parallel \omega_0^2}{2\pi(v/c)^2 E} e^{i(\Omega t - n\theta)}. \tag{6.44}$$

Substituting the original expression for ω into the above equation, we get

$$i\Omega\omega_1 - (in\omega_1)\omega_0 = \frac{e\eta I_1 Z_\parallel \omega_0^2}{2\pi(v/c)^2 E}, \tag{6.45}$$

which reduces to

$$(\Omega - n\omega_0)^2 = -i\frac{e\eta I_0 Z_\parallel \omega_0^2 n}{2\pi(v/c)^2 E}, \tag{6.46}$$

where we have made use of the fact that $R\Omega\lambda_0 \approx Rn\omega_0\lambda_0 = nI_0$.

We see immediately that if the longitudinal impedance is pure imaginary ($Z_\parallel = iZ_i$), then below transition ($\eta < 0$) the oscillations are stable if $Z_i < 0$ (i.e., if $Z_\parallel$ is capacitive). However, above transition the capacitive impedance will lead to instability. To conclude this section, we need only show that the space charge impedance is capacitive. Using Equation 6.33 for the space charge field E_s, we see that the energy loss per unit charge through one revolution is

$$-E_s 2\pi R = -\left(-\frac{g_0\lambda'}{4\pi\epsilon_0\gamma^2}\right)(2\pi R) = I_1 Z_\parallel e^{i(\Omega t - n\theta)}, \tag{6.47}$$

so that the longitudinal impedance is given by

$$Z_{\parallel} = \frac{g_0 \lambda'}{4\pi\epsilon_0\gamma^2} \frac{2\pi R}{I_1 e^{i(\Omega t - n\theta)}} = \frac{g_0 \lambda'}{4\pi\epsilon_0\gamma^2} \frac{2\pi R n}{R\Omega\lambda_1 e^{i(\Omega t - n\theta)}}. \tag{6.48}$$

But

$$\lambda' = \frac{d\lambda}{ds} = \frac{d\lambda}{d\theta}\frac{d\theta}{ds} = -in\lambda_1 e^{i(\Omega t - n\theta)}\frac{1}{R}, \tag{6.49}$$

so that the impedance becomes

$$Z_{\parallel} = -i\frac{n g_0}{2\epsilon_0 v \gamma^2}, \tag{6.50}$$

which is indeed capacitive and hence leads to instability above transition.

6.3 WAKE FIELDS AND IMPEDANCE

In the preceding section we discussed an instability which arises from the longitudinal space charge force generated by a round beam of infinite extent as it travels through a cylindrical beam pipe of constant radius. One would like to be able to describe coherent instabilities which may arise from more general beam distributions which may be traveling through more complex geometries. In a more common situation, for example one in which the beam is bunched, it is possible that particles in the tail end of the bunch feel forces which are generated by particles in the head of the bunch interacting with the environment. As was suggested by the introduction of a longitudinal impedance in the previous section, the leading particles will lose some amount of energy; the resulting electromagnetic fields, *wake fields*, can linger in the vacuum chamber to interact with the trailing particles.

The exact form of these high frequency wake fields and their response times depend heavily on the geometry of the problem as well as the materials in the vicinity of the beam. The fields are found, of course, by solving Maxwell's equations with the appropriate boundary conditions. Much of the work nowadays is done with the aid of a variety of computer codes, but for the most part, we will stay with analytical methods and simple situations. Our goal in this section is to introduce the formal definitions of wake functions and impedance. This will allow us to discuss some examples of common beam instabilities found in high energy linacs and synchrotrons using this by now standard language. Most of this section and the next has been adapted from Chao.[2]

[2]A. W. Chao, "Coherent Instabilities of a Relativistic Bunched Beam," *Physics of High Energy Particle Accelerators* (SLAC Summer School 1982), AIP Conference Proceedings No. 105, 1983.

6.3.1 Field of a Relativistic Charge in Vacuum

To begin with, recall how one obtains the electromagnetic field of a charged particle moving in a vacuum. Though this result will be of limited use in the present context, it is a basic starting point.

The fields in the rest frame of the particle are just given by Coulomb's law. In the laboratory, the fields are related to those in the rest frame (with the prime) according to

$$E_{\parallel} = E'_{\parallel}, \tag{6.51}$$

$$E_{\perp} = \gamma E'_{\perp}, \tag{6.52}$$

$$B_{\perp} = \gamma E'_{\perp} v/c^2, \tag{6.53}$$

where "$\parallel$" refers to the direction of motion of the particle, and "$\perp$" means radially outward for the electric field, and looping the particle trajectory for the magnetic field. Other field components vanish.

The coordinates in the rest and laboratory frames are related according to $x' = \gamma x$ and $y' = y$. So if our field point in the laboratory is characterized by r and ψ, the distance from the charge in the rest frame will be

$$\begin{aligned} r' &= \sqrt{\gamma^2 x^2 + y^2} \\ &= r\gamma \left(1 - \frac{v^2}{c^2}\sin^2\psi\right)^{1/2}. \end{aligned} \tag{6.54}$$

Using the coordinate and field transformation equations gives

$$E_{\parallel} = \frac{q}{4\pi\epsilon_0 r'^2}\frac{\gamma r\cos\psi}{r'}, \tag{6.55}$$

$$E_{\perp} = \frac{q}{4\pi\epsilon_0 r'^2}\frac{\gamma r\sin\psi}{r'}. \tag{6.56}$$

Therefore, in the laboratory the field is pointing directly away from the charge, just as it is in the rest frame. The electric field in vector notation is thus

$$\vec{E} = \frac{q}{4\pi\epsilon_0\gamma^2}\frac{\hat{r}}{r^2}\frac{1}{\left(1 - \dfrac{v^2}{c^2}\sin^2\psi\right)^{3/2}}. \tag{6.57}$$

As v approaches c, the field lines become concentrated perpendicular to the direction of motion, within an angular region of order $1/\gamma$. In the limit of highly relativistic motion, the fields are confined within a plane perpendicular to the direction of motion. Now that we know this result, the fields may be obtained directly from Gauss's and Ampere's laws:

$$E_r = \frac{q}{2\pi\epsilon_0 r}\delta(z - ct), \tag{6.58}$$

$$B_\theta = \frac{q}{2\pi\epsilon_0 cr}\delta(z - ct). \tag{6.59}$$

In principle, provided one knows where all the charges are, the fields can be obtained by integrating the expressions above over the charge distribution. One case where this can be done is the space charge field of a beam traveling along the axis of a perfectly conducting vacuum chamber. This problem was solved in the discussion of the negative mass instability by using Gauss's and Faraday's laws. Again, the radius of the beam is a and the radius of the vacuum chamber is b. If the charge per unit length in the beam is λ, then a surface charge $-\lambda$ is located on the inner surface of the vacuum chamber. We can find the total field by summing over the contributions of rings of charge as shown in Figure 6.6. The inner ring is a charge element of the beam, and the outer is its counterpart on the beam pipe. The amount of charge in radial range dy and in the longitudinal range dz is

$$d^2q = \lambda(z)\frac{2\pi y\,dy\,dz}{\pi a^2} = \frac{2\lambda}{a^2}y\,dy\,dz. \tag{6.60}$$

If the linear charge density λ is independent of z, then an integration over z will yield zero for the contribution to the field. So assume that the linear charge density has a first derivative with respect to z. The relevant charge density is then

$$d^2q = \frac{2\lambda'}{a^2}y\,dy\,z\,dz. \tag{6.61}$$

The contribution to the z-component of the electric field from the two rings

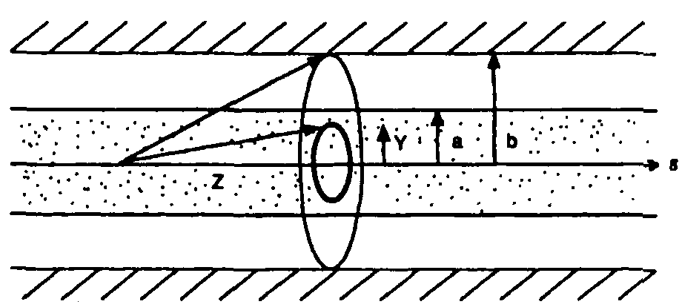

Figure 6.6. Calculation of space charge field by integrating over all the charges.

of charge is then

$$d^2E_z = -\frac{1}{4\pi\epsilon_0\gamma^2}\frac{2\lambda'}{a^2}y\,dy\left[\frac{z^2\,dz}{\left(z^2+\dfrac{y^2}{\gamma^2}\right)^{3/2}} - \frac{z^2\,dz}{\left(z^2+\dfrac{b^2}{\gamma^2}\right)^{3/2}}\right] \tag{6.62}$$

Integration over z followed by integration over y yields the same result as we obtained in the last section by a different argument:

$$E_z = -\frac{\lambda'[1+2\ln(b/a)]}{4\pi\epsilon_0\gamma^2}. \tag{6.63}$$

But generally one doesn't know where all the charges are; it then becomes necessary to solve the boundary value problem. The natural next step is to endow the beam pipe with a finite conductivity.

6.3.2 Wake Field for a Resistive Wall

There are two simple geometries that present themselves—a cylindrical beam pipe, and a rectangular chamber that is so wide that we may approximate it by two parallel plates. Let us take the former case, and immediately write down Maxwell's equations in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial(rE_r)}{\partial r} + \frac{1}{r}\frac{\partial E_\theta}{\partial \theta} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}, \tag{6.64}$$

$$\frac{1}{r}\frac{\partial B_z}{\partial \theta} - \frac{\partial B_\theta}{\partial z} = \mu_0 j_r + \frac{1}{c^2}\frac{\partial E_r}{\partial t}, \tag{6.65}$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \mu_0 j_\theta + \frac{1}{c^2}\frac{\partial E_\theta}{\partial t}, \tag{6.66}$$

$$\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \theta} = \mu_0 j_z + \frac{1}{c^2}\frac{\partial E_z}{\partial t}, \tag{6.67}$$

$$\frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0, \tag{6.68}$$

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} - \frac{\partial E_\theta}{\partial z} = -\frac{\partial B_r}{\partial t}, \tag{6.69}$$

$$\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = -\frac{\partial B_\theta}{\partial t}, \tag{6.70}$$

$$\frac{1}{r}\frac{\partial(rE_\theta)}{\partial r} - \frac{1}{r}\frac{\partial E_r}{\partial \theta} = -\frac{\partial B_z}{\partial t}. \tag{6.71}$$

For the case of a particle propagating down the axis of the cylindrical beam pipe, the charge and current densities are

$$\rho = \frac{q}{2\pi r}\delta(z - ct)\delta(r), \tag{6.72}$$

$$j_z = \frac{qc}{2\pi r}\delta(z - ct)\delta(r). \tag{6.73}$$

From the symmetry of the situation and our knowledge of the fields of the charge moving in a vacuum, we expect a solution with $B_r = 0$, $B_z = 0$, and $E_\theta = 0$. If we express the remaining field components in terms of their Fourier transforms according to

$$f(r, z, t) = \int_{-\infty}^{\infty} e^{ik(z-ct)}\tilde{f}\,dk, \tag{6.74}$$

where f is one of the field components, then within the beam pipe, transformation of Maxwell's equations yields

$$\frac{\partial \tilde{E}_r}{\partial r} + \frac{1}{r}\tilde{E}_r + ik\tilde{E}_z = \frac{q}{2\pi r \epsilon_0}\delta(r), \tag{6.75}$$

$$\tilde{B}_\theta = \frac{1}{c}\tilde{E}_r, \tag{6.76}$$

$$ik\tilde{E}_r - \frac{\partial \tilde{E}_z}{\partial r} = ikc\tilde{B}_\theta. \tag{6.77}$$

Only the transforms of the first, second, and seventh of the equations are shown. Four give no new information beyond that already conveyed by the symmetry argument, and the fourth equation is redundant.

Combination of the second and third of the set of three equations gives $\tilde{E}_z = A$, where A is a constant. Then, with this result, the solution of the first equation is

$$\tilde{E}_r = \frac{q}{2\pi\epsilon_0 r} - \tfrac{1}{2}ikAr. \tag{6.78}$$

Note that if $A = 0$, the result of Section 6.3.1 emerges. In order to find A in the present case, we have to determine the fields within the conducting wall and apply the boundary conditions. That is, we let $\vec{j} = \sigma\vec{E}$ in the material of the beam pipe, and we require E_z and B_θ to be continuous at the interface.

Again, we use the first, second, and seventh of Maxwell's equations. Application of the Fourier transformation yields

$$\frac{1}{r}\frac{\partial\left(r\tilde{E}_r\right)}{\partial r} + ik\tilde{E}_z = 0, \tag{6.79}$$

$$\tilde{B}_\theta = \frac{1}{c}\left(1 + i\frac{\mu_0 c\sigma}{k}\right)\tilde{E}_r, \tag{6.80}$$

$$ik\tilde{E}_r - \frac{\partial\tilde{E}_z}{\partial r} = ikc\tilde{B}_\theta, \tag{6.81}$$

from which we obtain for $\tilde{E}_z$ the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\tilde{E}_z}{\partial r}\right) + i\frac{k\sigma}{\epsilon_0 c}\tilde{E}_z = 0. \tag{6.82}$$

We anticipate that the fields do not penetrate far into the walls of the vacuum chamber. So in the two places that r appears in the equation above, we set $r = b$, where b is the radius of the beam pipe. Then, the solution of the equation for $\tilde{E}_z$ is of the form

$$\tilde{E}_z = Ae^{i\lambda(r-b)}. \tag{6.83}$$

Matching the fields at $r = b$ gives $\lambda^2 = ik\sigma/\epsilon_0 c$. In order that the fields in the material of the beam pipe remain finite for large r, the imaginary part of λ must be negative. Therefore, we can write

$$\lambda = \sqrt{\frac{|k|\sigma}{\epsilon_0 c}}\left(\frac{i + \operatorname{sgn} k}{2}\right). \tag{6.84}$$

Note that the skin depth is $\delta_s = 1/\operatorname{Im}\lambda$. To solve for A, we need to use the boundary condition on $\tilde{B}_\theta$.

Differentiating the second of our three Fourier transformed equations with respect to r and combining with the first gives an equation for $\tilde{B}_\theta$ in terms of $\tilde{E}_z$. Given our solution above for $\tilde{E}_z$, integration yields

$$\tilde{B}_\theta = -\frac{1}{c}\left(\frac{\lambda}{k} + \frac{k}{\lambda}\right)Ae^{i\lambda(r-b)}. \tag{6.85}$$

Then, matching the solutions at $r = b$ gives for A

$$A = \frac{q}{2\pi\epsilon_0 b\left[\frac{1}{2}ikb - \dfrac{k}{\lambda} - \dfrac{\lambda}{k}\right]}. \tag{6.86}$$

In order to proceed further analytically, let us make two approximations. First, assume that $|\lambda| \gg b^{-1}$. This is equivalent to saying that the pipe radius is large compared with the skin depth—generally a good approximation. Second, assume that $|kb| \ll |\lambda/k|$. This high frequency cutoff implies that we look at fields no closer than

$$|z - ct| \gg \left(\frac{\epsilon_0 b^2 c}{\sigma}\right)^{1/3}. \tag{6.87}$$

For typical parameters, this condition restricts us from looking closer than about 0.1 mm behind the charge responsible for the field.

Now, A reduces to

$$A = -\frac{qk}{2\pi\epsilon_0 b\lambda}, \tag{6.88}$$

and we can perform the inverse Fourier transforms to find the fields.[3] The results are

$$E_z = \sqrt{\frac{c}{2\pi\epsilon_0\sigma}}\,\frac{q}{b}\,\frac{1}{|z - ct|^{3/2}}[1 - H(z - ct)], \tag{6.89}$$

$$E_r = -\frac{3}{4}\sqrt{\frac{c}{2\pi\epsilon_0\sigma}}\,\frac{q}{b}\,\frac{r}{|z - ct|^{5/2}}[1 - H(z - ct)], \tag{6.90}$$

$$B_\theta = \frac{1}{c}E_r. \tag{6.91}$$

In the above, $H(s)$ is the Heaviside function,

$$\begin{aligned} H(s) &= 1 \quad \text{if } s > 0, \\ &= 0 \quad \text{if } s < 0, \end{aligned} \tag{6.92}$$

and so the solutions satisfy the requirements of causality. The fact that $E_z > 0$ in Equation 6.89 means that the field is in the accelerating direction.

[3] See, for example, M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge, 1958, Table 1.

Apparently, E_z will change sign for sufficiently small $|z - ct|$, otherwise a net acceleration would take place. To demonstrate this, we consider the limit in which $|kb| \gg |\lambda/k|$. Then

$$\tilde{E}_z = -i\frac{q}{\pi\epsilon_0 b^2}\frac{1}{k}. \tag{6.92}$$

For $\tilde{E}_r$, because there are two terms, it is necessary to choose the range of interest for r. Let's look at the field at the surface of the beam pipe. Then in order to find $\tilde{E}_r$, one has to use the complete expression for A because the large terms cancel in the numerator. What remains is

$$\tilde{E}_r = \frac{q}{\pi\epsilon_0 b^2}\sqrt{\frac{\sigma}{\epsilon_0 c}}\,|k|^{-3/2}(i\,\mathrm{sgn}\,k - 1). \tag{6.94}$$

Again making use of a table of Fourier transforms, the fields are

$$E_z = -\frac{2q}{\epsilon_0 b^2}(1 - H), \tag{6.95}$$

$$E_r = \frac{8q}{\epsilon_0 b^2}\sqrt{\frac{\sigma}{\epsilon_0 c}}\,|z - ct|^{1/2}(1 - H), \tag{6.96}$$

$$B_\theta = \frac{1}{c}E_r, \tag{6.97}$$

and one sees that, indeed, E_z changes sign close to the charge. An illustration of the fields is shown in Figure 6.7. Observe that the charges on the wall lag behind the particle, in contrast to the case for a perfectly conducting wall. At the wall, Poynting's vector is directed into the material, indicating that the particle loses energy by virtue of the finite conductivity of the beam pipe.

The solutions obtained thus far hold only for a beam traveling down the axis of a pipe of finite conductivity. One way of endowing the beam with an elementary shape is to use a charge distributed over a thin ring of radius a according to $\cos m\theta$. That is, the charge density representing a multipole of order $m > 0$ is

$$\rho = \frac{Q_m}{\pi a^{m+1}}\delta(z - ct)\delta(r - a)\cos m\theta, \tag{6.98}$$

where Q_m is the multipole coefficient. Note

$$Q_m = \int \rho r^m \cos m\theta \, r\,dr\,d\theta\,dz. \tag{6.99}$$

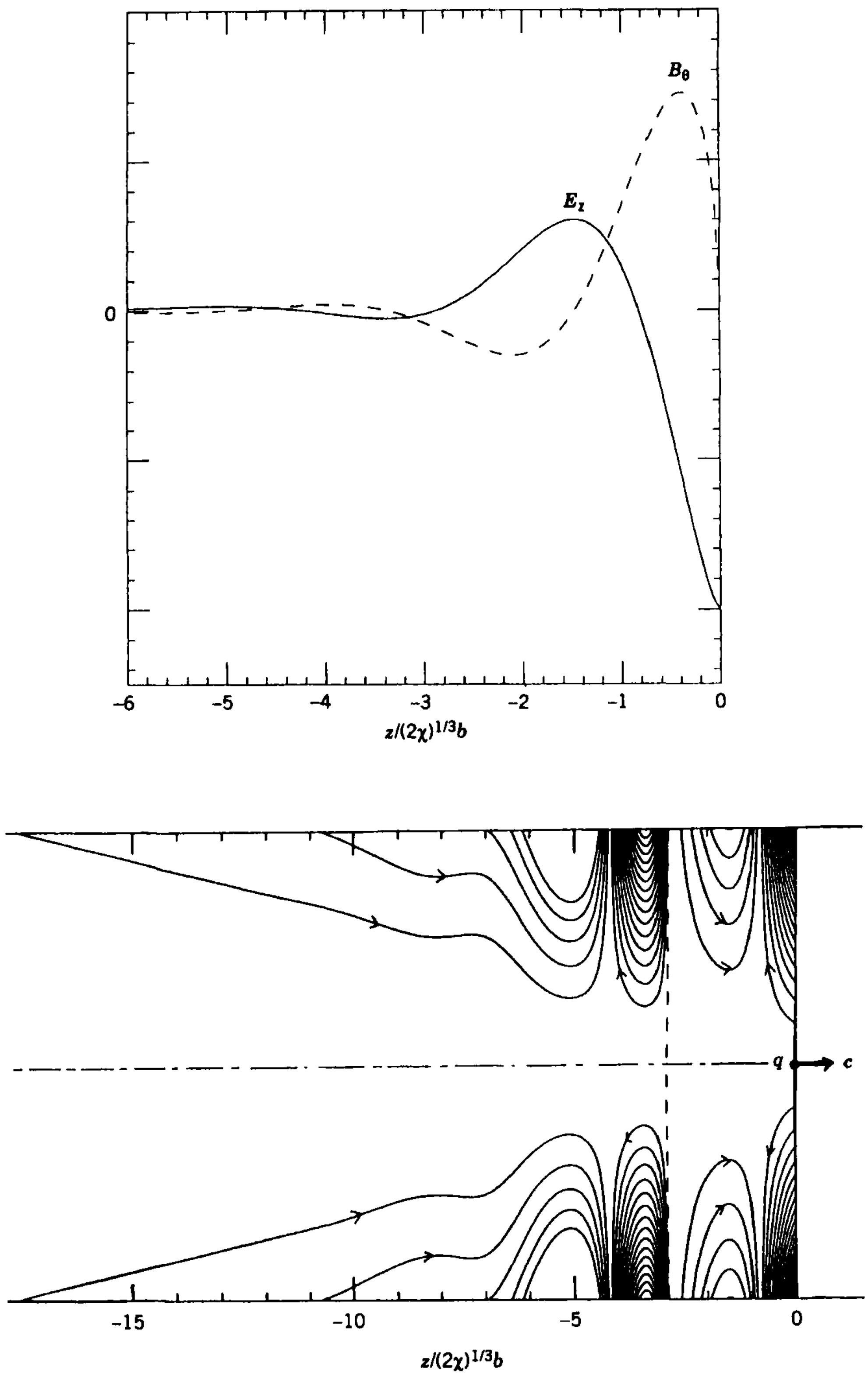

Figure 6.7. Wake fields generated by a relativistic charge traveling down the axis of a beam pipe with finite conductivity. (a) E_z and B_θ as functions of $z/(2\chi)^{1/3}b$, where $\chi = \epsilon_0 c/\sigma b$. (b) Wake electric field lines. The field line density to the left of the dashed line has been magnified by a factor of 40. (Courtesy K. Bane and A. Chao.)

For $m = 0$, the appropriate expression is

$$\rho = \frac{Q_0}{2\pi a}\delta(z - ct)\delta(r - a), \tag{6.100}$$

and hence $Q_0 = q$.

With this charge and associated current as sources, Maxwell's equations can be solved in a fashion analogous to that used for the centered charge above. Here, we will just reproduce the results from Chao within the beam pipe but not too close to the charge:

$$E_z = C_m r^m \cos m\theta \frac{1}{|z - ct|^{3/2}}, \tag{6.101}$$

$$E_r = -\tfrac{3}{4}C_m \frac{1}{m+1} r^{m-1} \cos m\theta \,(r^2 + b^2)\frac{1}{|z - ct|^{5/2}}, \tag{6.102}$$

$$E_\theta = -\tfrac{3}{4}C_m \frac{1}{m+1} r^{m-1} \sin m\theta \,(r^2 - b^2)\frac{1}{|z - ct|^{5/2}}, \tag{6.103}$$

$$B_z = -C_m r^m \sin m\theta \frac{1}{|z - ct|^{3/2}} \tag{6.104}$$

$$B_r = -E_\theta - 2C_m m r^{m-1} \sin m\theta \frac{1}{|z - ct|^{1/2}}, \tag{6.105}$$

$$B_\theta = E_r - 2C_m m r^{m-1} \cos m\theta \frac{1}{|z - ct|^{1/2}}, \tag{6.106}$$

where

$$C_m = \frac{Q_m}{\pi b^{2m+1}}\sqrt{\frac{2\pi c}{\epsilon_0 \sigma}}. \tag{6.107}$$

As in the preceding case, the fields vanish in front of the particle.

A couple of comments are appropriate. The radius of the ring, a, doesn't appear in these results, so for a given multipole order, the wake field is insensitive to the details of the charge distribution. For a test particle trailing the source of the wake, there will now be transverse deflecting fields at $r = 0$. And the transverse magnetic field has a long $|z - ct|^{-1/2}$ tail that will dominate at large distances.

6.3.3 Wake Functions

Instabilities develop sufficiently slowly, generally speaking, so that it is not necessary to examine the fields at each point along the trajectory; rather it is sufficient to average the fields along the trajectory. This average will depend

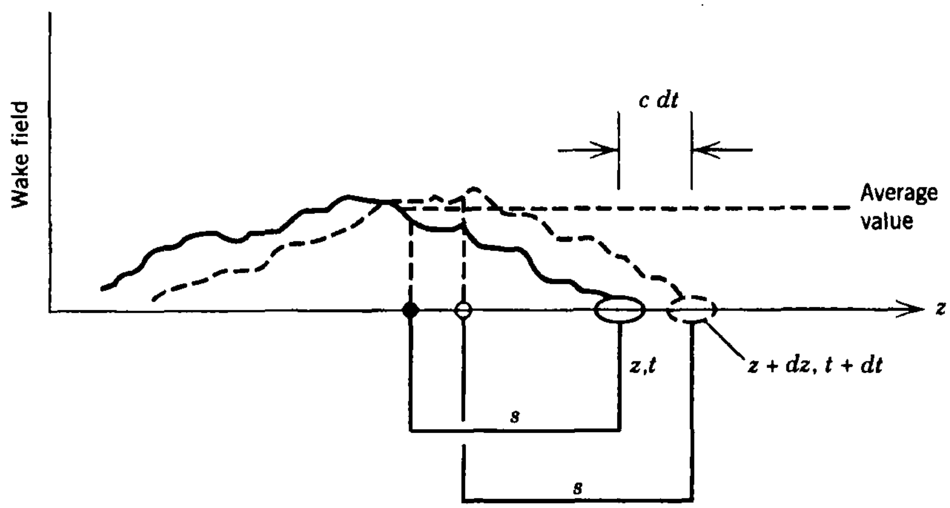

Figure 6.8. Wake fields set up by a source at longitudinal position z and time t, and at $z+dz$, $t+dt$. The repeat period of the hardware is L. A test particle trailing the source by a constant distance s will see the wake field "flutter" about an average value.

only on the separation between the source and test charges. It is, of course, the forces that we are interested in, and the equations for these forces undergo a surprising simplification as a result of this averaging.

To see this, consider a wake field which is set up by a charge distribution traveling through a structure of finite conductivity. The structure repeats itself after a length L as indicated in Figure 6.8. The wake field will have essentially the characteristics shown in Figure 6.7, but will have perturbations due to variations in the pipe structure. A test particle located a distance s behind the wake source will see an average value of the field, with time varying fluctuations. Since the fluctuations will be very fast compared to the growth rate of typical instabilities, we consider only the force on the particle due to the average fields, which will depend only upon the distance of separation

$$s \equiv ct - z \tag{6.108}$$

rather than on z and t independently. The average field components are thus functions of r, θ, and s; the time derivatives in Maxwell's equations for the average fields can be eliminated by noting that

$$\begin{aligned} \frac{df}{dt} &= \left(\frac{\partial f}{\partial t}\right)_z + \left(\frac{\partial f}{\partial z}\right)_t \frac{dz}{dt} = 0 \\ \Rightarrow \quad \frac{\partial f}{\partial t} &= -c\frac{\partial f}{\partial z}, \end{aligned} \tag{6.109}$$

where f stands for any field component, averaged over the repeat period.

Then, for example, the r-component of $\nabla \times \vec{E} = -\dot{\vec{B}}$ becomes

$$\frac{1}{r}\frac{\partial E_z}{\partial \theta} = \frac{\partial}{\partial z}(E_\theta + cB_r), \tag{6.110}$$

which, in terms of the forces, is

$$\frac{1}{r}\frac{\partial F_z}{\partial \theta} = \frac{\partial}{\partial z}F_\theta = -\frac{\partial}{\partial s}F_\theta. \tag{6.111}$$

We remind the reader that in these equations we are talking about the average fields and average forces. In an equivalent fashion, we can show that

$$ec\frac{\partial B_z}{\partial r} = \frac{\partial F_\theta}{\partial z} = \frac{1}{r}\frac{\partial F_z}{\partial \theta}, \tag{6.112}$$

$$-\frac{ec}{r}\frac{\partial B_z}{\partial \theta} = \frac{\partial F_r}{\partial z} = \frac{\partial F_z}{\partial r}. \tag{6.113}$$

The solutions which satisfy these equations may be written as

$$F_r = eQ_m m r^{m-1} \cos m\theta \, W_m(s), \tag{6.114}$$

$$F_\theta = -eQ_m m r^{m-1} \sin m\theta \, W_m(s), \tag{6.115}$$

$$F_z = -eQ_m r^m \cos m\theta \, W_m'(s), \tag{6.116}$$

$$ecB_z = Q_m r^m \sin m\theta \, W_m'(s), \tag{6.117}$$

where W is a function satisfying causality and W' is the derivative of W with respect to s.

The W's are called wake functions. Often, W and W' are called the transverse and longitudinal wake functions respectively, because of their role in the solutions above. These functions provide the building blocks for the description of the forces in the time domain. Wake functions can be computed for various beam distributions and hardware geometries, and from them the forces on beam particles can be readily calculated. What is more easily obtained in the laboratory setting is the impedance presented to the beam as it passes through certain devices, through the measurement of voltage drops and currents. We have already used the impedance language in our discussion of the negative mass instability. There is a direct relationship between impedance and wake functions, which we present next.

6.3.4 Impedance

In the frequency domain, the counterpart to the wake function is the impedance. Just as there are transverse and longitudinal wake functions, there are also transverse and longitudinal impedances. In order to identify the impedance, it is natural to use a beam which is a δ-function in frequency. So, for instance, let the beam current be described by the real part of

$$I_0(z,t) = \hat{I}_0 e^{ikz-i\omega t}, \tag{6.118}$$

where the subscript 0 implies that we are only talking about an $m = 0$ situation at present. From Equation 6.116 we can identify the longitudinal component of the electric field in terms of the wake functions W_0' due to a single point charge. Then adding up the contributions from all the charges preceding the test charge, the longitudinal electric field is

$$\begin{aligned} E_z(z,t) &= -\int dq\left(z, t - \frac{s'}{c}\right) W_0'(s') \\ &= -\int \frac{dq}{dt}\, dt\, W_0'(s') \\ &= -\int_0^{\infty} I_0\left(z, t - \frac{s'}{c}\right) W_0'(s') \frac{ds'}{c}, \end{aligned} \tag{6.119}$$

where s' is the distance from the source point to the field point. Using the above expression for I_0, the field may be rewritten as

$$E_z(z,t) = -I_0(z,t) \int_{-\infty}^{\infty} e^{i\omega s'/c} W_0'(s') \frac{ds'}{c}. \tag{6.120}$$

A particle traversing some length L of the structure will then experience an energy loss due to the voltage drop $E_z L$. It is reasonable to equate this drop to the product of the current and an impedance. According to the above, we have

$$V(z,t) = -I_0(z,t) Z_0^{\parallel}(\omega) \tag{6.121}$$

and hence may make the identification

$$\frac{Z_0^{\parallel}(\omega)}{L} = \frac{1}{c} \int_{-\infty}^{\infty} e^{i\omega s'/c} W_0'(s')\, ds'. \tag{6.122}$$

The quantity $Z_0^{\parallel}$ is called the longitudinal impedance and is just the Fourier transform of the wake field. An analogous expression may be written for the $m \neq 0$ multipole moments, where W_0 is replaced by W_m.

If we treat the transverse force in an equivalent fashion, we are led to the definition of a transverse impedance. Proceeding as above, the transverse force components may be written

$$F_r = ieQ_m m r^{m-1} \cos m\theta Z_m^{\perp}(\omega), \tag{6.123}$$

$$F_\theta = -ieQ_m m r^{m-1} \sin m\theta Z_m^{\perp}(\omega), \tag{6.124}$$

with the transverse impedance defined as

$$\frac{Z_m^{\perp}(\omega)}{L} = \frac{1}{ic}\int_{-\infty}^{\infty} e^{i\omega s'/c} W_m(s')\, ds'. \tag{6.125}$$

The factor of i has been included to reflect the fact that the transverse force tends to be 90° out of phase with the beam current; it is just a convention. While $Z_0^{\|}$ has units of ohms, $Z_1^{\perp}$ (the lowest order transverse impedance) has units of ohms per meter.

For given m, there is a relationship between the longitudinal and transverse impedances. We go back to the expressions for the force components in terms of the wake functions that were written down in the preceding section. By combining the transverse components, we have

$$\nabla_{\perp} F_z = \frac{\partial}{\partial s}\vec{F}_{\perp}, \tag{6.126}$$

which is referred to as the Panofsky-Wenzel theorem. In the frequency domain, this becomes

$$Z_m^{\|}(\omega) = \frac{\omega}{c} Z_m^{\perp}(\omega). \tag{6.127}$$

Though we will not calculate wake functions or impedances in the remainder of this chapter, the language we have developed will be used, in a qualitative fashion, to continue our introductory treatment of coherent instabilities.

6.4 MACROPARTICLE MODELS OF COHERENT INSTABILITIES

Interestingly enough, some insights can be gained into the physics at work in a number of collective instabilities by a very simple model: the entire bunch is replaced by two macroparticles. The leading macroparticle contains half of the particles of the bunch and creates the wake field that is experienced by the members of the second macroparticle. We will apply this technique to two frequently encountered instabilities in linacs and synchrotrons.

6.4.1 Beam Breakup in Linacs

The first instability we will study using the macroparticle model is that of beam breakup in a linear accelerator. Here, the beam is represented as two macroparticles, each containing $N/2$ particles, separated by a distance s, as depicted in Figure 6.9. In an electron linac, this distance will not change with time. The second macroparticle sees the transverse wake field of the first

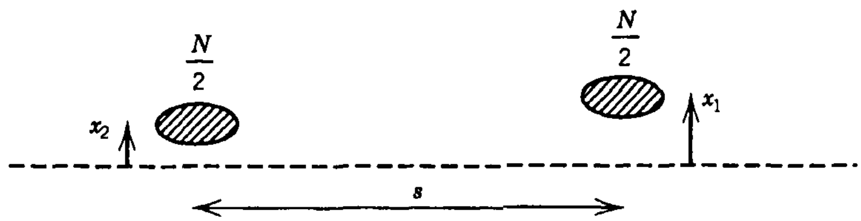

Figure 6.9. Two macroparticles, separated by a distance s, each containing half the particles of the bunch.

macroparticle which is undergoing betatron oscillations; the resulting force will drive the trailing particle's transverse oscillations.

To see this, consider the form of the transverse force due to a wake field of order $m = 1$ as presented earlier:

$$F_r = eQ_1W_1(s). \tag{6.128}$$

Here, motion has been restricted to one degree of freedom, and

$$\begin{aligned} Q_1 &= \int \rho r \cos\theta \, r\, dr\, d\theta\, dz \\ &= \int \frac{Ne}{2}\delta(x - x_1)\delta(y)\delta(z - ct)x\, dx\, dy\, dz \\ &= \frac{Ne}{2}x_1, \end{aligned} \tag{6.129}$$

where x_1 is the transverse coordinate for the leading macroparticle. If the leading macroparticle is executing betatron oscillations according to

$$x_1 = \hat{x}\cos\omega_\beta t, \tag{6.130}$$

then the equation of motion of the second macroparticle becomes

$$\begin{aligned} \ddot{x}_2 + \omega_\beta^2 x_2 &= \frac{Ne^2W_1}{2m\gamma}x_1 \\ &= \frac{Ne^2W_1}{2m\gamma}\hat{x}\cos\omega_\beta t. \end{aligned} \tag{6.131}$$

This is the equation of a driven oscillator, where the tail of the beam is driven exactly on resonance by the head of the beam. The solution to the above differential equation is

$$x_2(t) = \hat{x}_2 \cos \omega_\beta t + \hat{x}_1 \frac{Ne^2 W_1}{4\omega_\beta m\gamma} t \sin \omega_\beta t. \tag{6.132}$$

We see that on top of the free betatron oscillation, the amplitude of the motion grows linearly with time. If the two macroparticles had similar amplitudes initially, then at the end of the linac of length L, the amplitude of the oscillation in the tail will be a factor of

$$\frac{Ne^2 W_1 L}{4\omega_\beta m\gamma c} \tag{6.133}$$

larger than its initial value; thus intense beams can become quite distorted by the time they exit the linac.

6.4.2 The Strong Head-Tail Instability

The strong head-tail instability is basically the same as the beam breakup experienced by linac beams, but in this case the particles are undergoing synchrotron oscillations within a circular accelerator. If the synchrotron oscillations have a high enough frequency, the effect is to stabilize the beam against breakup. Above a certain intensity threshold, however, the beam can become unstable.

We begin by again considering a bunch to be composed of two macroparticles. During the first half synchrotron period, particle 1 leads particle 2 and thus particle 2 feels the effect of the wake generated by particle 1. During the second half of the synchrotron period, the roles are reversed. The equations of motion are thus

$$\left.\begin{aligned} \ddot{x}_1 + \omega_\beta^2 x_1 &= 0 \\ \ddot{x}_2 + \omega_\beta^2 x_2 &= \frac{Ne^2 W_1}{2m\gamma} x_1 \end{aligned}\right\} \quad 0 < t < T_s/2, \tag{6.134}$$

$$\left.\begin{aligned} \ddot{x}_1 + \omega_\beta^2 x_1 &= \frac{Ne^2 W_1}{2m\gamma} x_2 \\ \ddot{x}_2 + \omega_\beta^2 x_2 &= 0 \end{aligned}\right\} \quad T_s/2 < t < T_s. \tag{6.135}$$

It is assumed here that the wake function is a constant within the beam, but zero outside the beam.

Since we have learned how to analyze the stability of a system using 2×2 matrices, it is desirable to apply that method to the present problem. To do so, we notice that the solution to the unperturbed equation of motion,

$$x = x_0 \cos \omega_\beta t + \frac{\dot{x}_0}{\omega_\beta} \sin \omega_\beta t, \tag{6.136}$$

$$\dot{x} = \dot{x}_0 \cos \omega_\beta t - x_0 \omega_\beta \sin \omega_\beta t, \tag{6.137}$$

may be written in compact form as the phasor

$$\tilde{x}(t) = x + \frac{i}{\omega_\beta}\dot{x} = \tilde{x}(0) e^{-i\omega_\beta t}. \tag{6.138}$$

Thus, for $0 < t < T_s/2$, we have the solutions

$$x_1 = \hat{x}_1 \cos \omega_\beta t, \tag{6.139}$$

$$x_2 = \hat{x}_2 \cos \omega_\beta t + \frac{Ne^2 W_1}{4\omega_\beta m\gamma} \hat{x}_1 t \sin \omega_\beta t, \tag{6.140}$$

which can be written in phasor form as

$$\tilde{x}_2(t) = \tilde{x}_2(0) e^{-i\omega_\beta t} + i\frac{Ne^2 W_1}{4\omega_\beta m\gamma} \tilde{x}_1(0) t e^{-i\omega_\beta t} \tag{6.141}$$

if we note that the synchrotron period $T_s \gg 1/\omega_\beta$. We may then write the solution for propagation through the first half synchrotron period in matrix form as

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = e^{-i\omega_\beta T_s/2} \begin{pmatrix} 1 & 0 \\ i\eta_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix}, \tag{6.142}$$

where

$$\eta_1 = \frac{Ne^2 W_1 T_s}{8\omega_\beta m\gamma}. \tag{6.143}$$

For the second half of the synchrotron period, the two particles simply reverse roles. Particle 1 sees the wake field created by particle 2, and particle 2 undergoes a free betatron oscillation. Thus, the solution for $T_s/2 < t < T_s$ will be

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = e^{-i\omega_\beta T_s/2} \begin{pmatrix} 1 & i\eta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(T_s/2) \\ \tilde{x}_2(T_s/2) \end{pmatrix}, \tag{6.144}$$

and therefore, the motion for one complete synchrotron period is given by

$$\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = e^{-i\omega_\beta T_s} \begin{pmatrix} 1 - \eta_1^2 & i\eta_1 \\ i\eta_1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1(0) \\ \tilde{x}_2(0) \end{pmatrix}. \tag{6.145}$$

The motion over many synchrotron periods will be stable only if the absolute value of the trace of the matrix above is less than 2. That is, for stability, we must have

$$\frac{Ne^2 W_1 T_s}{8\omega_\beta m\gamma} \le 2. \tag{6.146}$$

The above criterion tells us that for low intensity beams, the motion is stable. Once the intensity reaches a certain threshold, instability arises. One of the stabilizing factors is the synchrotron period. If the synchrotron period is small, the head and tail of the bunch switch roles more frequently and the growth found in the linac case is stabilized. The beam breakup in a linac is just the special case where the synchrotron period is infinite.

6.4.3 The Head-Tail Instability

In the above treatment of two macroparticles undergoing synchrotron oscillations, an important feature has been omitted, namely, the variation of betatron oscillation frequency with momentum. We will find that inclusion of this phenomenon places a strict criterion on the chromaticity of the accelerator.

As before, we will study the motion of two macroparticles each of charge $Ne/2$. The betatron frequency of each macroparticle now depends upon its momentum deviation $\delta \equiv \Delta p/p_0$:

$$\begin{aligned} \omega_\beta(\delta) &= 2\pi\nu(\delta)f(\delta) \\ &= 2\pi\nu_0 f_0 + 2\pi f_0 \xi\delta - 2\pi f_0 \nu_0 \eta\delta \\ &\approx \omega_\beta + \omega_0 \xi\delta. \end{aligned} \tag{6.147}$$

We wish to use the longitudinal coordinate s as the independent variable. If we define Δt as the time interval between arrival of a particle and the arrival of the synchronous particle, then the synchrotron oscillations of the two macroparticles can be described by

$$\Delta t_1 = \Delta\hat{t}\sin(\omega_s s/c), \tag{6.148}$$

$$\Delta t_2 = -\Delta\hat{t}\sin(\omega_s s/c), \tag{6.149}$$

where ω_s is the synchrotron frequency. The accumulated betatron phase as a function of s is given by

$$\phi = \int \omega_\beta(\delta)\, dt = \int \frac{\omega_\beta\, ds}{c} + \omega_0 \xi \int \frac{\delta\, ds}{c} = \frac{\omega_\beta s}{c} - \frac{\omega_0 \xi\, \Delta t}{\eta}. \quad (6.150)$$

Therefore, we may write the solutions for free betatron oscillations as

$$x_1(s) = \hat{x}_1 \exp\left(-i\left[\frac{\omega_\beta s}{c} - \frac{\xi\omega_0}{\eta}\Delta\hat{t}\sin\frac{\omega_s s}{c}\right]\right), \quad (6.151)$$

$$x_2(s) = \hat{x}_2 \exp\left(-i\left[\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta\hat{t}\sin\frac{\omega_s s}{c}\right]\right). \quad (6.152)$$

For $0 < t < T_s/2$, particle 1 will obey the above equation, but particle 2 will have its equation of motion modified:

$$c^2\frac{d^2x_2}{ds^2} + \left[\omega_\beta + \frac{\xi\omega_0\,\Delta\hat{t}\omega_s}{\eta}\cos\left(\frac{\omega_s s}{c}\right)\right]^2 x_2 = \frac{Ne^2W_1}{2m\gamma}x_1, \quad (6.153)$$

where the quantity in brackets is the rate of change of the angular variable ϕ.

If we assume that x_2 is of the form given by the free oscillation, but that the amplitude $\hat{x}_2$ is allowed to change slowly with time, we may investigate the growth of the amplitude over time and look for conditions for stability. The first term in the equation above becomes

$$\begin{aligned} c^2\frac{d}{ds}&\left\{\frac{d\hat{x}_2}{ds}\exp\left(-i\left[\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta\hat{t}\sin\frac{\omega_s s}{c}\right]\right)\right\} \\ &\approx \left\{c^2\frac{d\hat{x}_2}{ds}\left(-2i\left[\frac{\omega_\beta}{c} + \frac{\xi\omega_0}{\eta}\Delta\hat{t}\frac{\omega_s}{c}\cos\frac{\omega_s s}{c}\right]\right) - c^2\hat{x}_2\left(\frac{d\phi}{ds}\right)^2\right\} \\ &\times \exp\left(-i\left[\frac{\omega_\beta s}{c} + \frac{\xi\omega_0}{\eta}\Delta\hat{t}\sin\frac{\omega_s s}{c}\right]\right), \end{aligned} \quad (6.154)$$

assuming $\omega_s \ll \omega_\beta$ and assuming $\xi\omega_0/\eta\,\Delta\hat{t}$ small. Substituting this back into the equation of motion and re-enforcing our approximations provides us with a relationship between the slowly varying amplitudes, namely

$$\frac{d\hat{x}_2}{ds} \approx \frac{iNe^2W_1}{4m\gamma\omega_\beta c}\hat{x}_1 \exp\left(2i\left[\frac{\xi\omega_0}{\eta}\Delta\hat{t}\sin\frac{\omega_s s}{c}\right]\right). \quad (6.155)$$

Already having assumed $\xi\omega_0/\eta\Delta\hat{t}$ to be small, we expand the right hand side of the above and integrate to obtain

$$\hat{x}_2 = \hat{x}_2(0) + \frac{iNe^2W_1}{4m\gamma\omega_\beta c}\hat{x}_1(0)\left[s + \frac{2i\xi\omega_0\,\Delta\hat{t}\,c}{\eta\omega_s}\left(1 - \cos\frac{\omega_s s}{c}\right)\right]. \tag{6.156}$$

Thus, we may obtain a set of equations relating the amplitudes of oscillations over the first half of the synchrotron period:

$$\begin{aligned}\hat{x}_2\left(\frac{\pi c}{\omega_s}\right) &= \hat{x}_2(0) + \frac{iNe^2W_1}{4m\gamma\omega_\beta c}\hat{x}_1\left(\frac{\pi c}{\omega_s} + \frac{4i\xi\omega_0\,\Delta\hat{t}\,c}{\eta\omega_s}\right)\\ &= \hat{x}_2(0) + i\frac{\pi Ne^2W_1}{4m\gamma\omega_\beta\omega_s}\left(1 + i\frac{4\xi\omega_0\,\Delta\hat{t}}{\pi\eta}\right)\hat{x}_1\\ &= \hat{x}_2(0) + i\eta_1\hat{x}_1. \end{aligned} \tag{6.157}$$

As before, this may be written in matrix form, the matrix for one complete synchrotron period may be obtained, and a stability criterion may be found by obtaining the eigenvalues of the 2×2 matrix. If the eigenvalues are

$$\lambda_\pm = e^{i\pm\mu}, \tag{6.158}$$

then for low intensity beams ($|\eta_1| \ll 1$), $2\cos\mu \approx 2 - \eta_1^2$, or $\mu \approx \eta_1$. The imaginary part of η_1 will then give the fractional growth per synchrotron period. The growth rate in terms of time is thus given by

$$\left(\frac{1}{\tau_{\text{growth}}}\right)_\pm = \mp\frac{Ne^2W_1}{2\pi m\gamma\omega_\beta}\cdot\frac{\xi\omega_0\,\Delta\hat{t}}{\eta}. \tag{6.159}$$

We see that as long as the chromaticity is not equal to zero, we have growth in one of the two eigenmodes, while motion in the other eigenmode will be damped. Stability can be guaranteed only for $\xi = 0$. The more complete analysis[4] using the Vlasov equation shows that the two-particle model overestimates the growth rate of the "−" mode. Thus, most synchrotron storage rings are operated with slightly positive chromaticities above transition.

6.5 EVOLUTION OF THE DISTRIBUTION FUNCTION

The simplified models of the previous section are often inadequate to describe some of the finer details of beam instabilities. For instance, descriptions of beam behavior for higher mode numbers are impossible to describe

[4] Chao, *op. cit.*

with a two-particle model. One could increase the number of macroparticles, but the analysis quickly becomes complicated. Though computer tracking codes may help, it is possible to do the bookkeeping on more than a few thousand particles, whereas real beams will contain 10^{10} particles per bunch or more. To proceed, one turns to a description of the beam in terms of a continuous particle density function. The basic relationship that this function must satisfy is known as the Vlasov equation.

6.5.1 The Vlasov Equation

Consider a small region of phase space, as shown in Figure 6.10. Let the number of particles in this region, n, be given in terms of a particle density function ψ:

$$n = \psi(x,p,t)\Delta x \Delta p. \tag{6.160}$$

Each particle in the phase space is moving according to the equations of motion of the system. After an infinitesimal time interval Δt, the number of particles present in the region becomes

$$n(t + \Delta t) = n(t) + \text{flow in} - \text{flow out}. \tag{6.161}$$

Consider, for a moment, only the flow in the x-direction. The number of particles entering the box from the left is

$$\psi(x, p, t)\Delta p \dot{x}(x, p, t)\Delta t, \tag{6.162}$$

whereas the number exiting the box on the right is

$$\psi(x + \Delta x, p, t)\Delta p \dot{x}(x + \Delta x, p, t)\Delta t. \tag{6.163}$$

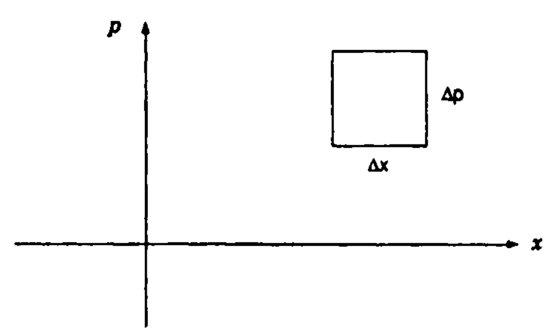

Figure 6.10. Infinitesimal area of phase space.

Therefore, the rate of change of particles in the region is given by

$$
\begin{aligned}
\frac{n(t+\Delta t)-n(t)}{\Delta t} &= \psi(x,p,t)\Delta p\dot{x}(x,p,t) \\
&\quad - \psi(x+\Delta x,p,t)\Delta p\dot{x}(x+\Delta x,p,t) \\
&= \psi(x,p,t)\Delta p\dot{x}(x,p,t) - \psi(x,p,t)\Delta p\dot{x}(x,p,t) \\
&\quad - \frac{\partial\psi}{\partial x}(x,p,t)\Delta x\Delta p\dot{x} \\
&= -\frac{\partial\psi}{\partial x}(x,p,t)\Delta x\Delta p\dot{x},
\end{aligned}
\tag{6.164}
$$

or

$$
\frac{\psi(t+\Delta t)-\psi(t)}{\Delta t}\Delta x\Delta p = -\frac{\partial\psi}{\partial x}(x,p,t)\Delta x\,\Delta p\dot{x}, \tag{6.165}
$$

from which

$$
\frac{\partial\psi}{\partial t} + \dot{x}\frac{\partial\psi}{\partial x} = 0. \tag{6.166}
$$

Performing the same argument for both x and p yields the Vlasov equation:

$$
\frac{\partial\psi}{\partial t} + \dot{x}\frac{\partial\psi}{\partial x} + \dot{p}\frac{\partial\psi}{\partial p} = 0. \tag{6.167}
$$

6.5.2 The Dispersion Relation

The solution to the Vlasov equation contains more information than we may be interested in knowing. Usually we just want to know if a situation is stable or unstable and, if unstable, what the growth rate of a perturbation would be. This determination can be expressed as a relationship between the unperturbed distribution and the perturbing forces. Expressions of this general type are called dispersion relations because they were first found in the analysis of the dispersive properties of optical materials; the name, of course, does not have this significance here.

We will use the Vlasov equation to develop a dispersion relation for particle beams with momentum spread. If $\theta \equiv s/R$ is the longitudinal coordinate and $\delta \equiv \Delta p/p$ is the conjugate momentum variable, then the Vlasov equation is

$$
\frac{\partial\psi}{\partial t} + \dot{\theta}\frac{\partial\psi}{\partial\theta} + \dot{\delta}\frac{\partial\psi}{\partial\delta} = 0, \tag{6.168}
$$

where

$$\dot{\theta} = \omega, \tag{6.169}$$

$$\dot{\delta} = \frac{1}{(v/c)^2}\frac{\Delta \dot{E}}{E}$$

$$= -\frac{e\omega_0 I_1 Z_{\parallel}}{2\pi (v/c)^2 E} e^{i(\Omega t - n\theta)}, \tag{6.170}$$

with Equation 6.43 being used in the last step.

If the unperturbed particle distribution is independent of θ (unbunched beam), then we may write ψ as

$$\psi(\delta, \theta, t) = \psi_0(\delta) + \psi_1(\delta) e^{i(\Omega t - n\theta)}. \tag{6.171}$$

So, to first order,

$$i(\Omega - n\omega)\psi_1 - \frac{\partial \psi_0}{\partial \delta}\frac{e\omega_0 I_1 Z_{\parallel}}{2\pi (v/c)^2 E} = 0. \tag{6.172}$$

But

$$\frac{\partial}{\partial \delta} = -\omega_0 \eta \frac{\partial}{\partial \omega}, \tag{6.173}$$

so that

$$\psi_1 = \frac{ie\omega_0^2 I_1 Z_{\parallel}\eta}{2\pi (v/c)^2 E}\frac{\partial \psi_0}{\partial \omega}\frac{1}{\Omega - n\omega}. \tag{6.174}$$

If we now integrate both sides over ω and then multiply by $ev/R = e\omega_0$, we obtain

$$e\omega_0 \int \psi_1(\omega)\, d\omega = -e\omega_0^2 \eta \int \psi_1(\delta)\, d\delta = -\omega_0 \eta I_1$$

$$= i\frac{e^2\omega_0^3 I_1 Z_{\parallel}\eta}{2\pi (v/c)^2 E}\int \frac{(\partial \psi_0/\partial \omega)}{\Omega - n\omega}\, d\omega, \tag{6.175}$$

so that we have the dispersion relation

$$1 = -i\frac{e^2\omega_0^2 Z_{\parallel}}{2\pi (v/c)^2 E}\int \frac{(\partial \psi_0/\partial \omega)}{\Omega - n\omega}\, d\omega. \tag{6.176}$$

6.5.3 Application to the Negative Mass Instability

Let's use the above dispersion relation to investigate the negative mass instability. First, we examine the case of an unbunched beam with no momentum spread as considered earlier. In this case, the phase space density is represented by a Dirac δ-function

$$\psi_0(\delta, \theta, t) = \frac{N}{2\pi}\delta(\delta). \tag{6.177}$$

where N is the total number of particles in the distribution. Written in terms of the angular frequency,

$$\psi_0(\omega, \theta) = -\frac{N\eta\omega_0}{2\pi}\delta(\omega - \omega_0). \tag{6.178}$$

We then obtain

$$\begin{aligned}\int \frac{\partial\psi_0/\partial\omega}{\Omega - n\omega}d\omega &= -\frac{N\eta\omega_0}{2\pi}\int\frac{\delta'(\omega - \omega_0)}{\Omega - n\omega}d\omega \\ &= \frac{N\eta\omega_0}{2\pi}\frac{n}{(\Omega - n\omega)^2},\end{aligned} \tag{6.179}$$

which, when substituted into the dispersion relation, gives us

$$1 = -i\frac{e^2\omega_0^2 Z_{\|}}{2\pi(v/c)^2 E}\frac{N\eta\omega_0}{2\pi}\frac{n}{(\Omega - n\omega)^2}, \tag{6.180}$$

or

$$(\Omega - n\omega)^2 = -i\frac{e\eta Z_{\|}\omega_0^2 n}{2\pi(v/c)^2 E}\left(\frac{eN\omega_0}{2\pi}\right) = -i\frac{e\eta I_0 Z_{\|}\omega_0^2 n}{2\pi(v/c)^2 E}, \tag{6.181}$$

which is the same result obtained previously.

To go one step further, we look at a more realistic beam, one in which there is a distribution in momentum space. As an example, consider an unbunched beam with a Gaussian distribution in momentum. The density function is of the form

$$\psi_0(\delta, \theta) = \frac{N}{(2\pi)^{3/2}\sigma}e^{-\delta^2/2\sigma^2}, \tag{6.182}$$

where $\sigma = (\Delta p/p)_{\text{rms}}$. In terms of the angular frequency,

$$\psi_0(\omega) = \frac{N}{(2\pi)^{3/2}\sigma} e^{-(\omega-\omega_0)^2/2(\eta\omega_0\sigma)^2}. \tag{6.183}$$

The dispersion integral becomes

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{\partial\psi_0}{\partial\omega} \frac{1}{\Omega - n\omega}\, d\omega &= -\frac{N}{(2\pi)^{3/2}\sigma} \frac{1}{(\eta\omega_0\sigma)^2} \\
&\quad \times \int_{-\infty}^{+\infty} \frac{\omega - \omega_0}{\Omega - n\omega} e^{-(\omega-\omega_0)^2/2(\eta\omega_0\sigma)^2}\, d\omega \\
&= \frac{N}{(2\pi)^{3/2}\sigma} \frac{1}{\eta\omega_0\sigma n} \\
&\quad \times \int_{-\infty}^{+\infty} \frac{\omega - \omega_0}{\omega - \Omega/n} e^{-(\omega-\omega_0)^2/2(\eta\omega_0\sigma)^2}\, d\left(\frac{\omega-\omega_0}{\eta\omega_0\sigma}\right) \\
&= \frac{N}{(2\pi)^{3/2}\eta\omega_0\sigma^2 n} \int_{-\infty}^{+\infty} \frac{u}{u - u_0} e^{-u^2/2}\, du,
\end{aligned} \tag{6.184}$$

where

$$u \equiv \frac{\omega - \omega_0}{\eta\omega_0\sigma}, \tag{6.185}$$

$$u_0 \equiv \frac{\Omega - n\omega_0}{\eta\omega_0\sigma n} = \frac{\Delta\Omega}{\eta\omega_0\sigma n}. \tag{6.186}$$

Then the dispersion relation becomes

$$1 = -i \frac{eI_0 Z_{\parallel}}{2\pi(v/c)^2 E\eta\sigma^2 n} I_D(u_0), \tag{6.187}$$

where

$$I_D(u_0) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{u}{u - u_0} e^{-u^2/2}\, du, \tag{6.188}$$

and we have identified $e\omega_0 N/2\pi$ as the current I_0.

We have seen that for a beam with zero momentum spread, a capacitive impedance ($Z_{\parallel} = iZ_i$, $Z_i < 0$) leads to instability above transition. We now use the above dispersion relation to look for a condition that will assure stability to a beam with a Gaussian momentum distribution.

Consider the case where $\Delta\Omega$ has a negative imaginary part, corresponding to an unstable solution. The dispersion integral can be found after noting that

$$\frac{1}{u-u_0} = -i\int_0^\infty e^{i(u-u_0)\alpha}\,d\alpha, \qquad (6.189)$$

if u_0 has a negative imaginary part. Then

$$\begin{aligned}
I_D(u_0) &= \frac{-i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} ue^{-u^2/2}\int_0^\infty e^{i(u-u_0)\alpha}\,d\alpha\,du \\
&= \frac{-i}{\sqrt{2\pi}}\int_0^\infty e^{-iu_0\alpha}\int_{-\infty}^{+\infty} ue^{-(u^2-2iu\alpha-\alpha^2)/2}e^{-\alpha^2/2}\,du\,d\alpha \\
&= \frac{-i}{\sqrt{2\pi}}\int_0^\infty e^{-iu_0\alpha}e^{-\alpha^2/2}\int_{-\infty}^{+\infty} ue^{-(u-i\alpha)^2/2}\,du\,d\alpha \\
&= -i\int_0^\infty e^{-iu_0\alpha}e^{-\alpha^2/2}i\alpha\,d\alpha \\
&= \int_0^\infty \alpha e^{-iu_0\alpha}e^{-\alpha^2/2}\,d\alpha. \qquad (6.190)
\end{aligned}$$

Notice that $I_D(0) = 1$. Therefore, for $\operatorname{Im} u_0 < 0$, $|I_D(u_0)| < 1$. So, from the dispersion relation, if in addition

$$\frac{eI_0|Z_\parallel|}{2\pi(v/c)^2E\eta\sigma^2 n} < 1, \qquad (6.191)$$

then the dispersion relation cannot be satisfied, implying that there will not be any unstable solutions. That is,

$$\sigma^2 > \frac{eI_0}{2\pi\eta(v/c)^2E}\left|\frac{Z_\parallel}{n}\right| \qquad (6.192)$$

is a sufficient condition for stability.

The general requirement

$$\left|\frac{Z_\parallel}{n}\right| < \mathscr{F}\frac{2\pi|\eta|(v/c)^2E\sigma^2}{eI_0} \qquad (6.193)$$

for stability, where $\mathscr{F}$ is a form factor of order unity which depends upon the particle distribution used, is known as the Keil-Schnell criterion.[5]

[5] E. Keil and W. Schnell, CERN Report CERN/ISR-TH-RF/69-48, 1969.

Finally, it is interesting to note that the integral $I_D(u_0)$ can be written in closed form as[6]

$$I_D(u_0) = e^{-u_0^2/4} D_{-2}(\pm i u_0), \tag{6.194}$$

where the plus sign is used for $\operatorname{Im} u_0 < 0$, the negative sign for $\operatorname{Im} u_0 > 0$. The function $D_{-2}(x)$ is a parabolic cylinder function and is described in Abramowitz and Stegun.[7]

6.6 LANDAU DAMPING

In the last section, we saw that there was a threshold for instability of a nondissipative system—nondissipative because in the example of space charge forces only, the impedance is reactive. For sufficiently small values of the impedance the system is stable. Buried within the mathematics of the dispersion relation is a stabilizing mechanism, which is called Landau damping. In this section we describe its origin.

Let's go back to the driven harmonic oscillator:

$$\ddot{x} + \omega_0^2 x = C \sin \omega t. \tag{6.195}$$

Consider first the situation on resonance, where $\omega = \omega_0$. For a particle starting from rest, the solution to the above differential equation is

$$x = \frac{C}{2\omega_0^2} \sin \omega_0 t - \frac{C}{2\omega_0} t \cos \omega_0 t. \tag{6.196}$$

Clearly, the envelope of the oscillation grows without bound. Off resonance,

$$\begin{aligned} x &= \frac{C}{\omega_0^2 - \omega^2}\left(\sin \omega t - \frac{\omega}{\omega_0} \sin \omega_0 t\right) \\ &= \frac{C}{(\omega_0 + \omega)\omega_0} \sin \omega_0 t \\ &\quad - \frac{C}{(\omega_0 + \omega)} t \left[\frac{\sin \frac{1}{2}\delta\omega\, t}{\frac{1}{2}\delta\omega\, t}\right] \cos\left(\frac{\omega + \omega_0}{2} t\right), \end{aligned} \tag{6.197}$$

[6] I. S. Gradshteyn, and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1980, p. 337, formula 3.462.

[7] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1970, p. 710. In this reference, the function $D_{-2}(x)$ is denoted by $U(1.5, x)$.

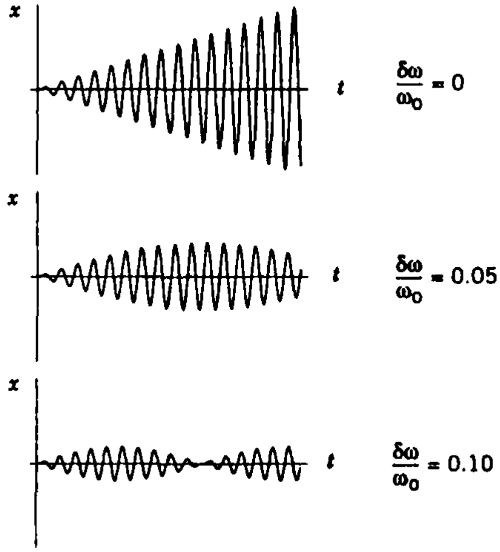

Figure 6.11. Solution to the driven oscillator equation for various relative frequency differences $(\omega - \omega_0)/\omega_0$.

where $\delta\omega \equiv \omega - \omega_0$. If $\delta\omega$ is small compared with ω_0, the two solutions are identical for t small. That is, the particle does not know initially whether or not it is on resonance. But at $t \approx 1/\delta\omega$ the solutions become distinct. The behavior is illustrated in Figure 6.11.

The circumstance that the fraction of the particles participating in secular growth diminishes with time suggests that the constant amplitude force does not produce an instability in the sense that we are looking for here. In contrast, let us consider a force that increases exponentially with time. Since we are looking for exponentially growing solutions it is reasonable to consider forces which grow exponentially as well. Suppose the equation of motion is of the form

$$\ddot{x} + \omega^2 x = Ae^{i\omega_0 t}e^{t/\tau}. \tag{6.198}$$

For $\omega = \omega_0$, the particular solution may be written as

$$x = \frac{A\tau}{2i\omega_0}e^{i\omega_0 t}e^{t/\tau} \tag{6.199}$$

for large τ. The off-resonance case ($\omega \neq \omega_0$) has the particular solution

$$x \approx \frac{A}{\omega^2 - \omega_0^2 + 2i\omega_0/\tau}e^{i\omega_0 t}e^{t/\tau}. \tag{6.200}$$

The amplitude of this solution, for $\omega \approx \omega_0$, is

$$\frac{A}{2\omega_0\delta\omega + 2i\omega_0/\tau} \tag{6.201}$$

where $\delta\omega = \omega - \omega_0$. If we consider an ensemble of particles with frequencies distributed about ω_0, then those particles within a frequency range $\delta\omega \approx 2/\tau$ will participate in the growth.

In the case of a coherent instability, the driving force is not external but is due to the deviation of the beam itself. We can extend the harmonic oscillator example above to contain such a driving force. For instance, assume that the force is proportional to the average displacement. Each member of the ensemble obeys an equation of the form

$$\ddot{x} + \omega^2 x = c\langle x \rangle. \tag{6.202}$$

Suppose $\Delta\omega$ characterizes the frequency width of the distribution. At some time, the fraction of the particles participating in the growth will be $2/\Delta\omega\,\tau$. Since these particles all will be at roughly the same displacement, $\langle x \rangle \sim (2/\Delta\omega\,\tau)x$. All of these particles have essentially the same frequency, ω_0. Thus, for one of these particles, the equation of motion is

$$\ddot{x} + \left(\omega_0^2 - \frac{2}{\Delta\omega\,\tau}c\right)x = 0. \tag{6.203}$$

If the particle's amplitude is to undergo resonant growth, the frequency ω, given by

$$\omega^2 = \omega_0^2 - \frac{2}{\Delta\omega\,\tau}c, \tag{6.204}$$

must have a negative imaginary part. That is,

$$\begin{aligned} \mathrm{Im}\,\omega &= \mathrm{Im}\sqrt{\omega_0^2 - \frac{2}{\Delta\omega\,\tau}c} \\ &= \mathrm{Im}\left\{\omega_0\left(1 - \frac{c}{\Delta\omega\,\omega_0^2\tau}\right)\right\} = -\frac{\mathrm{Im}\,c}{\Delta\omega\,\omega_0\tau} < 0. \end{aligned} \tag{6.205}$$

So for $\mathrm{Im}\,c > 0$, the oscillation amplitude grows like

$$\exp\left[\frac{(\mathrm{Im}\,c)t}{\Delta\omega\,\omega_0\tau}\right]. \tag{6.206}$$

But the argument of the exponential is just t/τ, and so for the onset of instability one must have

$$\mathrm{Im}\,c \approx \omega_0\Delta\omega. \tag{6.207}$$

Notice that the larger the frequency spread $\Delta\omega$, the larger the force coefficient must become in order to produce instability. In the preceding section, we found that the negative mass instability would be stabilized by sufficient momentum spread. Since particles of different momenta have different revolution frequencies, the conclusions of this section and of the preceding one are related. For sufficiently large frequency spread the feedback mechanism that is the potential source of instability is not strong enough to produce exponential growth. The stabilizing mechanism is called Landau damping.

PROBLEMS

1. For the round Gaussian beam, sketch as a function of r the space charge force and its derivative with respect to r. Calculate the values of r/σ for which F and dF/dr are maximum.

2. The dependence of tune on amplitude for the round Gaussian beam can be calculated analytically. In this problem, ignore the details of alternating gradient focusing and assume that a betatron oscillation in the absence of space charge is $x = a \cos \psi$, where ψ just advances linearly with azimuth, $\psi = \nu\theta$, $0 < \theta < 2\pi$.

   (a) Including space charge, show that the equation of motion is

   $$\frac{d^2u}{d\psi^2} + u = 4\frac{\Delta\nu}{\nu}\frac{1}{u}\left(1 - e^{u^2/2}\right),$$

   where $u \equiv x/\sigma$ and $\Delta\nu$ is the magnitude of the incoherent space charge tune shift as calculated in the text.

   (b) Next, assume that the solution of the equation of motion will be of the form

   $$u = u_0 \cos\left[\psi + \frac{\delta(u_0)}{\nu}\psi\right],$$

   where δ is the quantity that we are looking for. Use the method of phase averaging to extract this "DC" quantity. Substitute the trial solution into the equation of motion and average over ψ. Show that the amplitude dependence of tune is given by

   $$\delta(u_0) = -\frac{\Delta\nu}{(u_0/2)^2}\left\{1 - e^{-(u_0/2)^2} I_0\left[(u_0/2)^2\right]\right\}$$

   where I_0 is the modified Bessel function of order zero.

3. Compare the results of the phase averaging method in the preceding problem with the amplitude dependence of tune obtained by numerical integration of the equation of motion.

4. In the Fermilab booster, suppose that the injected beam is very rapidly neutralized; that is, there is quickly only a current and no charge density. Using the parameters in Section 6.1.3, calculate the tune shift for this completely neutralized beam.

5. In a proton-antiproton collider, the beam-beam tune shift parameter $\Delta\nu$ that was calculated in the text will be positive. So particles with small betatron oscillation amplitudes will have higher tunes than those with larger amplitudes, and the largest amplitude particles will have the tune prescribed by the accelerator lattice, ν. By choice of $\Delta\nu$ and ν, one can arrange that a low order resonance will be encountered at an intermediate amplitude. With the aid of a graphics terminal, develop the turn by turn mapping in one-degree-of-freedom phase space, and demonstrate the chain of four resonance islands that appear when a quarter integer resonance is within the beam. Note that the resonance does not lead to beam loss; for realistic parameters, no trajectories go to large amplitude.

6. Rewrite the expression for the luminosity of a collider found in Chapter 1 in terms of the normalized emittance (39%), and then eliminate the emittance in favor of the beam-beam tune shift parameter, $\Delta\,\nu_{bb}$. For the Tevatron operating at 1 TeV, with 5×10^{10} particles per bunch in each of six proton and six antiproton bunches, the tune shift parameter is near the upper limit of $\Delta\nu_{\mathrm{bb}} = 0.007$. Calculate the luminosity under these conditions if the value of the amplitude function at the interaction point is $\beta^* = 0.5$ m. (The radius of the collider is 1 km.)

7. In electron-positron colliders, at low beam current I the luminosity varies as I^2; then there is a discontinuity in slope beyond which the luminosity varies as I. Account for this behavior, under the assumption that the beam-beam effect is a limitation.

8. In colliders designed to have a short bunch spacing, it is desired that the beams be brought into collision with a crossing angle α so as to minimize the number of head-on beam-beam collisions. The intent is that the beam-beam tune shift will be kept in bounds. However, the bunches will still act upon each other at a distance, and so we have to keep the so-called long range beam-beam tune shift in mind. Suppose that the crossing angle is such that the two Gaussian bunches of equal total charge pass each other a distance d apart, where $d \gg \sigma$.
   (a) Compute the force on a particle in one bunch due to the other bunch. Show that there is a dipole term and a quadrupole term.
   (b) Let $\Delta\nu_{\mathrm{HO}}$ denote the beam-beam tune shift due to a single head-on collision between two bunches at the interaction point. Show that the

the long-range tune shift is

$$\Delta\nu_{\mathrm{LR}} = \frac{2\Delta\nu_{\mathrm{HO}}}{(d/\sigma)^2}.$$

(c) The head-on beam-beam force is defocusing in both degrees of freedom for like sign beams; what about the long range force?

(d) Suppose that the two beams collide at the interaction point, where the amplitude function has its minimum value β^*. Suppose that β^* is much smaller than the distance to the point at which long range passages take place. Show that

$$\left(\frac{d}{\sigma}\right)^2 = \frac{\pi\gamma\alpha^2\beta^*}{\epsilon_N},$$

where

$$\epsilon_N = \left(\gamma\frac{v}{c}\right)\pi\frac{\sigma^2}{\beta},$$

and is the phase space area enclosing some 39% of the beam in one transverse degree of freedom.

(e) If the bunch spacing is S_B, show that the total long range beam-beam tune shift across the region of length L is

$$\Delta\nu_{LR} = \frac{Nr_0L}{\pi\gamma\alpha^2\beta^*S_B}$$
$$= \Delta\nu_{\mathrm{HO}}\left(\frac{4L\epsilon_N}{\pi\gamma\alpha^2\beta^*S_B}\right).$$

(f) Calculate $\Delta\nu_{\mathrm{HO}}$ and $\Delta\nu_{\mathrm{LR}}$ for a 20 TeV collider with $\beta^* = 0.5$ m, a crossing angle, $\alpha = 0.075$ mrad, and $S_B = 5$ m, $N = 10^{10}$, $L = 180$ m.

(g) Estimate the steering error caused by the dipole component of the long range force for the parameters above.

9. It was stated in the text that if a line charge is displaced a distance y from the center of a beam pipe of radius R, an image charge equal in magnitude but opposite in sign will appear at a distance R^2/y. Prove that such is the case.

10. Estimate the coherent tune shift experienced by the particles in a bunch of 2×10^{10} protons traveling in a wide rectangular conducting vacuum chamber of height 5 cm. Assume that the bunch is 1 meter in length, the kinetic energy is 8 GeV, the average value of the amplitude function is 50 meters, and the orbit radius is 1000 meters. These parameters correspond to the Fermilab Main Ring at injection.

11. In the resistive wake, close to the charge, obtain the next order term by taking

$$A = \frac{q}{i\pi\epsilon_0 kb^2}\left(1 + \frac{2\lambda}{ik^2 b}\right).$$

Compare with the figures in the text.

12. For the resistive wake, find the wake functions W_0 and W_0'.

13. By Fourier transformation of W_0', find the longitudinal impedance $Z_{\|}$ for the resistive wake.

14. Estimate the longitudinal impedance presented to the beam passing through a bellows. One can proceed as follows. Calculate the flux "trapped" in the shaded area as though $\vec{B}$ were given by the Biot-Savart Law. If we call this flux Φ, then Faraday's law can be used to calculate the back emf on the beam. Show that the contribution of the bellows to Z/n is

$$iZ_0\left(\frac{(v/c)g}{2\pi R}\right)\ln\frac{d}{b},$$

where Z_0 is the impedance of free space.

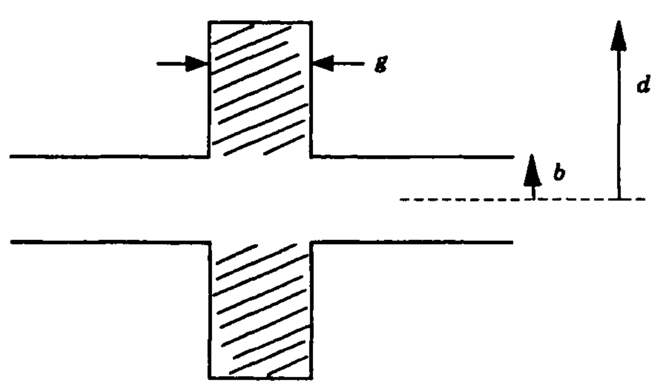

Figure 6-A.

15. Verify that in the "+" mode of the head-tail instability the two macroparticles are moving in phase, while in the "−" mode they are 180° out of phase.

16. Consider the Fermilab Main Ring at its maximum energy of 150 GeV. Assume $Z_{\|}/n = 10\ \Omega$. Suppose that 2×10^{13} protons are circulating in 1000 bunches, and that the peak instantaneous current I_0 is 10 times the average current. The smallest the mode number can be is thus $\sim 10^4$. Calculate the growth rate of the negative mass instability in this circumstance. What frequency would be detected for this motion? (For this reason the negative mass instability for a bunched beam is usually called the microwave instability.)

17. Find the intensity threshold N for the strong head-tail instability in the Tevatron at injection. Take $W_1 \approx 4 \times 10^{15}\ \mathrm{F}^{-1}\,\mathrm{m}^{-2}$, $T_s = 12$ msec, $\nu = 19.4$, and a revolution frequency of 47.7 kHz. Is this result realistic?

18. Show that Equation 6.202 leads to the dispersion relation

$$1 = \frac{c}{2\pi} \int \frac{\rho(\omega)\, d\omega}{\omega - \Omega}$$

where $\rho(\omega)$ is the distribution function of oscillator frequencies normalized to unity.

19. Show that the dispersion integral in Problem 18 becomes

$$\int_{-\infty}^{\infty} \frac{\rho(\omega)\, d\omega}{\omega - \Omega} = \mathrm{P.V.} \int_{-\infty}^{\infty} \frac{\rho(\omega)\, d\omega}{\omega - \Omega_r} - i\pi\rho(\Omega_r)$$

for Ω_i sufficiently small, that is, at threshold.

20. Apply the threshold condition of the preceding problem to a triangular frequency distribution. Let $\rho(\omega)$ be centered at ω_0, with width $2\,\Delta\omega$ at the base, and $\Delta\omega \ll \omega_0$. At the apex of the triangle, $\rho(\omega_0) = 1/\Delta\omega$.

(a) If $\mathrm{Re}\,\Omega = \omega_0$, show that the threshold for instability is

$$c = i\frac{2\omega_0\,\Delta\omega}{\pi}.$$

(b) If $\mathrm{Re}\,\Omega = \omega_0 + \Delta\omega$, show that $\mathrm{Im}\,c$ can be vanishingly small and the collective motion will still be unstable.

(c) Sketch the region of stability in $\mathrm{Im}\,c$ (y-axis) versus $\mathrm{Re}\,c$ coordinates.

21. The discussion connected with Equations 6.202 through 6.207 may be more persuasive if carried out in terms of energy. That is, compare the power required to produce amplitude growth with the power that the feedback mechanism is able to deliver. As before, take $\text{Re}\,\Omega = \omega_0$, and make the same assumptions concerning the fraction of the oscillators participating in the coherent motion. Remember that because energy involves the square of displacement or velocity, you must take the real part of displacement or velocity before squaring. Show that the threshold value for $\text{Im}\,c$ found by this method is essentially the same as that obtained by the argument in the text.

22. Imagine a distribution of particles of the form

$$\psi_0 = \frac{15N}{32\pi}\left[1 - \left(\frac{\delta}{\delta_{\max}}\right)^2\right]^2$$

where $\delta \equiv \Delta p/p$. Use the dispersion relation given by Equation 6.176 to find the locus of points in $\text{Re}\,Z_{\parallel}$, $\text{Im}\,Z_{\parallel}$ space at the threshold of instability.

23. The extraction kinetic energy of the Fermilab booster is 8 GeV. Suppose 2×10^{12} particles at that energy are circulating in an orbit whose frequency is 6×10^5 Hz. Suppose the bunching factor is 10. The beam size is roughly one-tenth the aperture at this energy. Calculate the momentum spread necessary to stabilize the microwave instability if it originates only from space charge. For this accelerator, $\gamma_t = 5.4$. Comment on the stability threshold near transition.