CHAPTER 4

Resonances and Transverse Nonlinear Motion

Despite our attempt to concentrate on linear behavior in the last chapter, we found ourselves compelled at the end to introduce nonlinear magnetic fields to compensate chromatic aberration. In fact, as accelerators have grown in energy, cost, and performance demands, it has become necessary to devote more and more attention to the effects of nonlinear fields on the single particle dynamics of synchrotrons, both those necessitated by the design and those arising from magnet imperfections.

In the first category are the present generation of synchrotron radiation sources. In these devices the emphasis on small emittance to produce high brightness beams results in very strong sextupole elements to compensate chromaticity. The attendant aberrations lead to a bound on the stable region in transverse phase space. The stable region is often called the *dynamic aperture*.

In the second category are large hadron colliders. Here, one is playing cost against the provision of the design magnetic field. Again, the presence of nonlinearities will lead to a finite dynamic aperture, and the designer must assure that performance goals are met without crossing the border into overdesign.

The introduction of a single nonlinear element can change dramatically the mappings in phase space, which in the last chapter were simple ellipses. The nonlinear equations of motion have the disadvantage that they cannot be solved in closed form. However, the iteration of the associated difference equations can demonstrate many of the essential features; all one needs is a home computer. For example, suppose there is a single thin sextupole installed in a synchrotron which has otherwise perfectly linear fields. For a

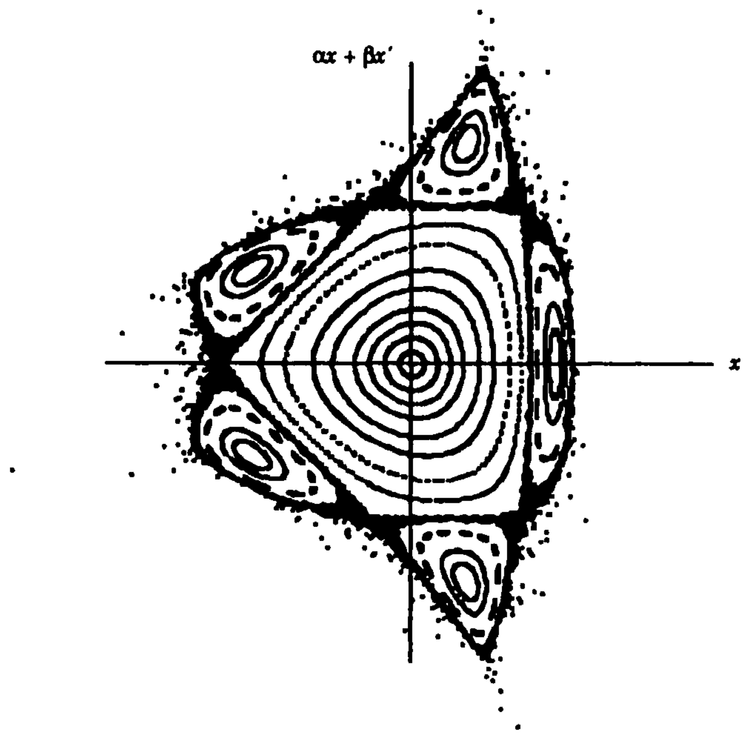

Figure 4.1. Phase space plot of particles circulating in a synchrotron lattice that is perfectly linear except for a single thin sextupole field introduced at the point of observation. For this plot, the tune of the accelerator is 0.42 plus an integer.

specific choice of parameters, the graphics output from a turn-by-turn iteration of the equations of motion is shown in Figure 4.1.

In the purely linear case, particle trajectories in this phase space lie on circles; for sufficiently small amplitudes this is still the case, as seen in the figure. However, as one moves to larger amplitudes, the nonlinearity manifests itself and the trajectories increasingly deviate from circles. At still larger radius, we encounter a set of five *islands*, isolated regions of stability. At slightly larger amplitude the motion becomes irregular, or *chaotic*. Finally, at large enough initial amplitudes the motion is completely unstable. If there were a clearly defined separatrix, as there was in the case of longitudinal dynamics, the definition of the dynamic aperture would be clear. But already, in this simple numerical example, we see that the richer phase space dynamics revealed here makes a straightforward definition of dynamic aperture difficult.

In this text we will not attempt to unravel all of the features of Figure 4.1; more extensive discussions may be found in the advanced references in the bibliography. Our purpose here will be to examine simple situations in which the equations of nonlinear motion are tractable. We will extend the discussion of resonances to include those of nonlinear origin, and interestingly enough these cases rather directly permit progress toward the solution of the equations of motion.

Acknowledging that the standard treatment of these matters is couched in the language of higher dynamics, we conclude with an introductory exposition of this approach.

4.1 TRANSVERSE RESONANCES

In the last chapter, we encountered two instances of resonant behavior, both stated in terms of field imperfections. If there are steering errors, however small, an integer value of the tune will lead to an oscillation growing in amplitude linearly from turn to turn without bound. The "without bound" is of course an artifact of our approximations, but because of the limited aperture in which the particle motion must be accommodated, the approximations are reasonable. In the second case, it was found that at a tune such that $2\nu = m$ where m is an integer, any quadrupole error will lead to amplitude growth. We did not prove it then, but the growth in this case can be faster than linear. In this section, we will attempt to generalize the discussion of resonances to include nonlinear field perturbations.

4.1.1 Floquet Transformation

We begin by completing the coördinate transformation so that the resulting motion is indeed just that of a simple harmonic oscillator. Start from the solution to the equation of motion:

$$x(s) = A\beta^{1/2}(s)\cos[\psi(s) + \delta]. \tag{4.1}$$

If we define a "reduced phase," ϕ, by $\phi = \psi/\nu$, then ϕ is a variable that increases by 2π for each turn. Even though ϕ is not a real polar angle measured from the center of a circle, it behaves like one. If now one defines a new dependent variable ζ according to

$$\zeta = \frac{x}{\beta^{1/2}}, \tag{4.2}$$

then

$$\zeta(\phi) = A\cos(\nu\phi + \delta) \tag{4.3}$$

and the free betatron oscillation reduces to simple harmonic motion, with ν oscillations for every advance of ϕ by 2π. The equation of motion for ζ is just

$$\frac{d^2\zeta}{d\phi^2} + \nu^2\zeta = 0. \tag{4.4}$$

The replacement of coordinates x, s by ζ, ϕ is called a Floquet transformation. A direct benefit of the Floquet transformation can be seen as follows. The equation of motion was developed for an accelerator without field errors. With field errors, one of the problems at the end of the chapter is to show that the right hand side of the above equation will no longer be zero; we will have instead

$$\frac{d^2\zeta}{d\phi^2} + \nu^2\zeta = -\nu^2\beta^{3/2}\frac{\Delta B(\zeta,\phi)}{(B\rho)}, \tag{4.5}$$

where ΔB represents all those fields not taken into account in setting up the design orbit.

Therefore, in coordinates ζ, ϕ, the full collection of mathematical methods for treating driven harmonic oscillations becomes available, and the notion of a resonance between some harmonic amplitude of the right hand driving term and the tune ν is just the same as in the case of a simple oscillator.

4.1.2 Multipole Expansion

The next task is to choose a method in which to express the field error ΔB. For many purposes, it is convenient to use a multipole expansion to go beyond the at most linear dependence on x to which we have limited ourselves thus far. But rather than take up the general case (which is left to Chapter 5), let us stay for the present in one degree of freedom and take

$$\Delta B = B_0(b_0 + b_1 x + b_2 x^2 + \cdots), \tag{4.6}$$

where B_0 is a reference field strength and the b_n are the multipole coefficients. In a bending magnet, for instance, B_0 would be the nominal bend field. For definiteness, we are taking the case of motion in the bending plane; ΔB is the variation in the vertical component of the field on the midplane, and, in "pole language" the b_n arise from normal multipole errors. That is to say, the field imperfections arise from pole distributions that do not have poles in the horizontal plane.

So b_0 is the dipole error, b_1 is the quadrupole error term, b_2 the sextupole term, and so on. The b_n are of course functions of s.

Some examples of how these errors arise might be useful. Suppose that all the bending magnets are intended to have a field of exactly 1 tesla, at a certain excitation current common to all the magnets. For a bending magnet such as that sketched in Figure 4.2, the field in the gap in the infinite permeability approximation is

$$B = \mu_0 NI/g, \tag{4.7}$$

where NI is the number of ampere-turns, g is the gap height, and the field

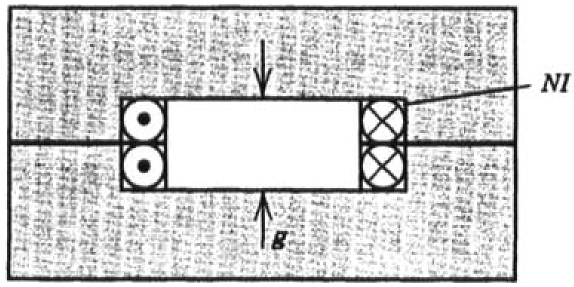

Figure 4.2. Dipole bending magnet with gap height g and N turns of conductor, each carrying current I.

will be a constant within the coil window. The laminations of which the steel yoke are typically made are produced by stamping them out of steel sheets. Such factors as wear of the die used in the stamping, or the use of multiple dies for a large production run, will lead to a variation in the gap height. If g is designed to be exactly 5 cm, then a change of 0.0025 cm (approximately one-thousandth of an inch) will give $b_0 = 5 \times 10^{-4}$, which is a significant field error.

Such magnets are often assembled from two "top and bottom" half cores, as suggested by the horizontal lines at the midplane in the sketch. If the cores meet perfectly on one side, but are separated by a small gap h on the other, it is easy to show that the resulting quadrupole term will be

$$b_1 = \frac{h}{gw}, \tag{4.8}$$

where w is the pole width. For $h = 0.0025$ cm in the same magnet, and $w = 10$ cm, then $b_1 = 0.5 \times 10^{-4}$/cm, again a significant error.

The most pernicious source of sextupole terms in both conventional and superconducting magnets is remanent magnetization, arising from finite remanence in ferromagnets and persistent currents in superconducting magnets. Rather than digress into a discussion of material properties, let us rather illustrate the appearance of sextupole terms in a simpler but equally common situation—that due to eddy currents in a conducting vacuum chamber.

In Figure 4.3, we have added a vacuum chamber to a dipole magnet. If the magnetic field is changing with time, according to Faraday's law there will be an emf induced in a loop characterized by positions $\pm x$ from the chamber center as illustrated in the right hand portion of the figure. The electric field is

$$E = \dot{B}x, \tag{4.9}$$

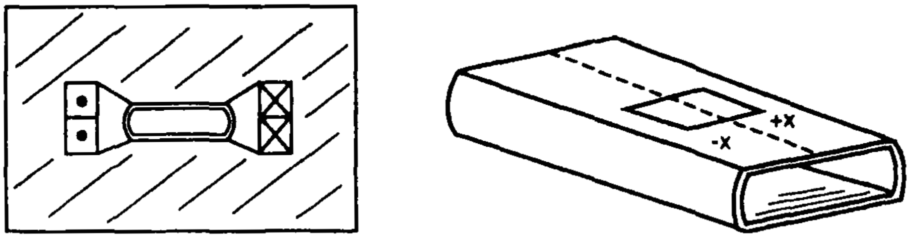

Figure 4.3. Dipole magnet with steel vacuum chamber.

and if the material has conductivity σ, the current density is

$$j = \sigma E = \sigma \dot{B} x. \tag{4.10}$$

The total current within the area between $x = 0$ and x is then

$$I = h\sigma \dot{B} x^2, \tag{4.11}$$

where h is the thickness of the chamber, and the contributions of both top and bottom have been included. Therefore, the field difference between the center of the magnet and x is

$$\Delta B = \mu_0 I/g = \mu_0 h \sigma \dot{B} x^2/g \tag{4.12}$$

which is a sextupole field, with moment

$$b_2 = \mu_0 \sigma \frac{h}{g} \frac{\dot{B}}{B}. \tag{4.13}$$

As an example, let's consider the Main Ring synchrotron at Fermilab. The vacuum chamber is stainless steel for which $\mu_0 \sigma \approx 2$ sec/m^2. For h/g, take $\frac{1}{30}$, and for $\dot{B}/B$ use 10/sec. Then $b_2 \approx \frac{2}{3}$ m^{-2}. Again, this is a significant field perturbation. To verify that the aforementioned field errors are indeed significant, the expressions for closed orbit distortions, tune shift, and chromaticity derived in the last chapter can be employed, using the table of Fermilab Main Ring parameters found in the Appendices.

4.1.3 The Driven Oscillator and Rational Numbers

We may now return to the equation of motion for a betatron oscillation as written after the Floquet transformation, Equation 4.5:

$$\frac{d^2\zeta}{d\phi^2} + \nu^2 \zeta = -\nu^2 \beta^{3/2} \frac{\Delta B(\zeta,\phi)}{(B\rho)}. \tag{4.14}$$

Writing ΔB in terms of the multipole expansion above, we obtain

$$\frac{d^2\zeta}{d\phi^2} + \nu^2\zeta = -\frac{\nu^2 B_0}{(B\rho)}\left[(\beta^{3/2}b_0) + (\beta^{4/2}b_1)\zeta + (\beta^{5/2}b_2)\zeta^2 + \cdots\right]. \tag{4.15}$$

A term on the right-hand side with the same frequency as the natural frequency, ν, of the oscillator would be cause for concern. The products of amplitude functions and multipole coefficients can be expressed as Fourier series in ϕ:

$$\left(\beta^{(n+3)/2}b_n\right) = \sum_k c_k e^{\pm ik\phi}, \tag{4.16}$$

while the solution to the inhomogeneous equation of motion can be written as

$$\zeta(\phi) = \zeta_0 e^{\pm i\nu\phi}. \tag{4.17}$$

Consider the first term; b_0 represents the dipole field error. If the product $\beta^{3/2}b_0$ has a nonvanishing kth harmonic at $k = \nu$, a resonant condition will exist. We already know that integral values of the tune must be avoided.

The next term contains the gradient errors, characterized by b_1. The kth harmonic of the factor $\beta^2 b_1$ can beat with the frequency ν presented by ζ to produce a driving frequency $k - \nu$. The resonance condition $k - \nu = \nu$ gives $k = 2\nu$; i.e., the tune shouldn't be a half integer. Again, we already know that. (The beat frequency with the plus sign, $k + \nu = \nu$, is a special case; the zeroth harmonic of $\beta^2 b_1$ is a tune shifting term and represents a "renormalization" of the left hand side of the equation of motion rather than a resonance.)

The third term represents the effect of sextupole moments. The factor ζ^2 can exhibit a frequency 2ν; when this is combined with the kth harmonic of $\beta^{5/2}b_2$, one can have the condition $k - 2\nu = \nu$, or $k = 3\nu$. That is, the tune should not be a third of an integer. The beat frequency $k + 2\nu$ just leads again to the condition that the tune should not be an integer.

In general, any tune of the form $\nu = k/n$ can resonate with some multipole moment. The integer in the denominator is called the order of the resonance; for instance, sextupoles can produce third order resonances, octopoles can produce fourth order resonances, and so on.

This argument can only suggest that problems can arise for tunes equal to some rational numbers. Both experience and more quantitative arguments indicate that low order resonances need to be avoided. How low is low depends on the application and resonance strength.

A distinction is made between driving terms that arise from random field errors and those that arise from systematic imperfections common to all of the magnets. The language used in the discussion above was in the spirit of the resonances having their origin in random field errors—construction tolerances result in each magnet being slightly different, and so all harmonics are represented in the resonance driving terms. The magnets may also possess systematic nonlinear multipoles; the overall symmetry of the ring will play a role in the presence or absence of particular harmonics. Suppose the overall lattice consists of P identical periods; the only harmonics arising from systematic field imperfections will be of the form Pk, $k = 1, 2, 3, \ldots$, and the resonant tunes will be Pk/n. The greater the symmetry of the ring, the easier it is to stay away from systematic resonances. For example, if a ring is made up of six identical superperiods and the bending magnets have a systematic sextupole field error, the systematic third order resonant tunes are the even integers. Highly symmetric synchrotrons are generally rather small. In very large accelerators, cost and operational considerations militate against the preservation of high symmetry. Therefore, corrector systems must undertake the role of compensation of unavoidable systematic field errors.

Even though we've worked in only one degree of freedom, we should at least quote the result for the complete transverse case. The resonances are lines in the ν_x, ν_y plane of the form

$$M\nu_x + N\nu_y = P, \tag{4.18}$$

where M, N, and P are integers all of the same sign (one of the pair M, N can be zero) for instability. Justification of this result will be provided in the next chapter. The sum of the absolute values of M and N is the order of the resonance, and the order can be related to a multipole term just as in the one degree of freedom case. If M and N are opposite in sign, the result is coupled but stable motion. Take the sextupole case again. There are four sum (hence, unstable) resonances:

$$3\nu_x = P, \tag{4.19}$$

$$2\nu_x + \nu_y = P, \tag{4.20}$$

$$\nu_x + 2\nu_y = P, \tag{4.21}$$

$$3\nu_y = P. \tag{4.22}$$

The first and third are driven by the sextupole term in the field expansion used above. The second and fourth are driven by a sextupole field, but one rotated by 30° with respect to the first to form a *skew* sextupole.

4.2 A THIRD-INTEGER RESONANCE

Having introduced the notion of nonlinear resonances in the previous section, the point now is to try to turn the purely qualitative approach into something that can claim to be a quantitative treatment. Let's take the case of a sextupole nonlinearity distributed in an arbitrary fashion around a circular accelerator. There are several approaches one may take to looking at the problem, including Fourier analysis, Hamiltonian perturbation theory, and computer simulation. For this discussion, we will treat the nonlinearity as a small perturbation of the linear motion. We will assume that the linear tune is very close to one-third of an integer, so that we may expect that the perturbation of the linear motion will be dominated by the sextupole field. On each turn, we add up the effects of the nonlinearities as though they were independent of each other, and take stock of the situation at one point on the ring after each turn.

4.2.1 Equation of Motion

We may write the linear oscillation as

$$x = a\left(\frac{\beta(s)}{\beta_0}\right)^{1/2} \cos\chi(s), \tag{4.23}$$

where the amplitude function, β_0, at the point of observation has been used explicitly so that the symbol a can denote a real amplitude at that point. The symbol for the phase has been switched to χ so that ψ can be reserved for the phase at the point of observation.

For the other coordinate of phase space, we use

$$p_x \equiv \beta(s)x' + \alpha(s)x = -a\left(\frac{\beta(s)}{\beta_0}\right)^{1/2} \sin\chi, \tag{4.24}$$

and the unperturbed motion at any point in the ring is just a circle (see Figure 4.4). (Note that p_x is not the transverse kinematic momentum.)

Assume a magnetic field $\Delta B(x, s)$, perpendicular to x and s, is introduced at s and extends over a length Δs. The sign of ΔB is positive if ΔB is

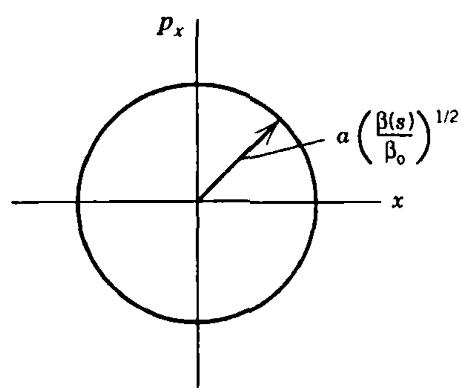

Figure 4.4. Phase space coordinates.

directed in the same sense as the main guide field. For sufficiently small Δs, x does not change as the particle traverses Δs, but the slope does, according to

$$\Delta x' = -\frac{\Delta B\,\Delta s}{(B\rho)}, \tag{4.25}$$

and so

$$\Delta p_x = -\beta(s)\frac{\Delta B\,\Delta s}{(B\rho)}. \tag{4.26}$$

As a result of the perturbation, the amplitude and phase have changed. From

$$\Delta x = \left(\frac{\beta}{\beta_0}\right)^{1/2}(\Delta a\cos\chi - a\sin\chi\cdot\Delta\chi) = 0, \tag{4.27}$$

$$\Delta p_x = -\left(\frac{\beta}{\beta_0}\right)^{1/2}(\Delta a\sin\chi + a\cos\chi\cdot\Delta\chi) = -\beta\frac{\Delta B\,\Delta s}{(B\rho)}, \tag{4.28}$$

one finds

$$\Delta a = \frac{\beta_0}{(B\rho)}\left(\frac{\beta}{\beta_0}\right)^{1/2}\Delta B\,\Delta s\sin\chi, \tag{4.29}$$

$$\Delta\chi = \frac{\beta_0}{(B\rho)}\left(\frac{\beta}{\beta_0}\right)^{1/2}\frac{\Delta B\,\Delta s}{a}\cos\chi. \tag{4.30}$$

Now we add up these perturbations over a turn. Suppose that at a given passage of our point of observation, the phase of the oscillation is ψ. Then on the succeeding turn, the unperturbed phase χ would develop according to

$$\chi(s) = \psi + \nu\phi(s), \tag{4.31}$$

where

$$\phi(s) \equiv \int\frac{ds}{\nu\beta(s)}. \tag{4.32}$$

To obtain the first order equations of motion, we assume that the changes in amplitude and phase can be found by adding up the individual perturbations.

For the amplitude we immediately obtain from Equation 4.29

$$\frac{da}{dn} = \frac{\beta_0}{(B\rho)} \oint \left(\frac{\beta}{\beta_0} \right)^{1/2} \Delta B(x,s) \sin[\psi + \nu\phi(s)]\, ds. \tag{4.33}$$

The change in phase at the observation point after passing through one turn is $2\pi\nu$ plus the accumulated phase change due to the sum of all the contributions typified by Equation 4.30:

$$\Delta\psi = 2\pi\nu + \frac{\beta_0}{(B\rho)} \oint \left(\frac{\beta}{\beta_0} \right)^{1/2} \frac{\Delta B(x,s)}{a} \cos[\psi + \nu\phi(s)]\, ds. \tag{4.34}$$

It is the difference between $\Delta\psi$ and $2\pi\nu$ that is small in the spirit of our perturbation calculation. Therefore, the differential equation for phase advance will be

$$\frac{d}{dn}(\psi - 2\pi\nu n) = \frac{\beta_0}{(B\rho)} \oint \left(\frac{\beta}{\beta_0} \right)^{1/2} \frac{\Delta B(x,s)}{a} \cos[\psi + \nu\phi(s)]\, ds. \tag{4.35}$$

4.2.2 Recognition of the Sextupole Resonance

So far nothing has been said about the variety of the nonlinearity represented by $\Delta B(x,s)$. Now we take the case of a sextupole distribution:

$$\Delta B(x,s) = \frac{B''(s)}{2} x^2. \tag{4.36}$$

Insertion of this form of the field into the equation of motion for the amplitude as well as Equation 4.23 yields, after some manipulation of the trigonometric functions,

$$\frac{da}{dn} = \tfrac{1}{4} a^2 \frac{\beta_0}{(B\rho)} \oint \left(\frac{\beta}{\beta_0} \right)^{3/2} \left(\frac{B''}{2} \right)$$
$$\times (\sin\psi \cos\nu\phi + \cos\psi \sin\nu\phi + \sin 3\psi \cos 3\nu\phi + \cos 3\psi \sin 3\nu\phi)\, ds. \tag{4.37}$$

We now look for terms that could be additive from turn to turn, that is, terms that could represent unstable motion. If the tune were close to an integer, the first two terms could be candidates. But if the tune is not near an integer (and we assume that it is not), $\sin\psi$ and $\cos\psi$ will change rapidly from turn to turn, and so the amplitude will not grow steadily. However, if 3ν were an integer, $\sin 3\psi$ and $\cos 3\psi$ would have constant values from turn to turn, and then the amplitude could exhibit growth. So we ignore the first two terms and retain the second pair.

Since we want to study the case where 3ν is not exactly an integer, but close, let $3\nu_0$ denote the integer of interest, with the tune difference, $\delta \equiv \nu - \nu_0$, small compared with unity. The equation of motion for the amplitude then can be written

$$\frac{da}{dn} = \tfrac{1}{4}a^2(A \sin 3\psi + B \cos 3\psi) \tag{4.38}$$

with

$$A \equiv \frac{\beta_0}{(B\rho)} \oint \left(\frac{\beta}{\beta_0}\right)^{3/2} \left(\frac{B''}{2}\right) \cos 3\nu_0 \phi \, ds, \tag{4.39}$$

$$B \equiv \frac{\beta_0}{(B\rho)} \oint \left(\frac{\beta}{\beta_0}\right)^{3/2} \left(\frac{B''}{2}\right) \sin 3\nu_0 \phi \, ds. \tag{4.40}$$

In defining A and B, we have used the proximity of ν to ν_0 so that A and B are true harmonic amplitudes.

The equation of motion for ψ is found by the same procedure, but with one modification. As a result of the unperturbed motion alone, the phase advances by $2\pi\nu$ in one turn, and so ψ by itself hardly qualifies as a continuous variable. This circumstance was already recognized in writing the left hand side of Equation 4.35. Now observe that the factors related to ψ that enter the right hand sides of the equations, $\cos 3\psi$ and $\sin 3\psi$, are insensitive to the replacement of ψ by $\psi - 2\pi\nu_0 n$. We introduce a new variable $\tilde{\psi} \equiv \psi - 2\pi\nu_0 n$, and obtain the equations of motion in terms of the new phase:

$$\frac{da}{dn} = \tfrac{1}{4}a^2\left(A \sin 3\tilde{\psi} + B \cos 3\tilde{\psi}\right), \tag{4.41}$$

$$\frac{d\tilde{\psi}}{dn} = \tfrac{1}{4}a\left(A \cos 3\tilde{\psi} - B \sin 3\tilde{\psi}\right) + 2\pi\delta. \tag{4.42}$$

With the foregoing redefinition, $\tilde{\psi}$ is not only a variable continuous in n, but the form of Equation 4.38 is preserved with the replacement of ψ by $\tilde{\psi}$. We have, in effect, made a transformation to rotating coordinates.

4.2.3 First Integral and the Separatrix

The equations of motion were developed in the phase-amplitude form because the characteristic of an unstable resonance is more readily identified there. The transformation back to (rotating) Cartesian coordinates $\tilde{x}, \tilde{p}_x$

follows from

$$\frac{d\tilde{x}}{dn} = \frac{\tilde{x}}{a}\left(\frac{da}{dn}\right) + \tilde{p}_x\left(\frac{d\tilde{\psi}}{dn}\right), \tag{4.43}$$

$$\frac{d\tilde{p}_x}{dn} = \frac{\tilde{p}_x}{a}\left(\frac{da}{dn}\right) - \tilde{x}\left(\frac{d\tilde{\psi}}{dn}\right), \tag{4.44}$$

and gives

$$\frac{d\tilde{x}}{dn} = \tfrac{1}{4}A(-2\tilde{x}\tilde{p}_x) + \tfrac{1}{4}B(\tilde{x}^2 - \tilde{p}_x^2) + 2\pi\delta\cdot\tilde{p}_x, \tag{4.45}$$

$$\frac{d\tilde{p}_x}{dn} = \tfrac{1}{4}A(\tilde{p}_x^2 - \tilde{x}^2) + \tfrac{1}{4}B(-2\tilde{x}\tilde{p}_x) - 2\pi\delta\cdot\tilde{x}. \tag{4.46}$$

For simplicity consider the case $B = 0$. A first integral of this system is

$$\left(\tilde{x} - \frac{4\pi\delta}{A}\right)\left[\tilde{p}_x^2 - \tfrac{1}{3}\left(\tilde{x} + \frac{8\pi\delta}{A}\right)^2\right] = \text{constant} \equiv k, \tag{4.47}$$

and different phase space trajectories are associated with different values of k. This first integral can be found by the usual mixture of technique and guesswork associated with solving differential equations. Or one might argue that the equations of motion in $\tilde{x}$ and $\tilde{p}_x$ are some version of Hamilton's equations, and so there ought to be a function $\mathscr{H}$ such that

$$\frac{d\tilde{x}}{dn} = \frac{\partial\mathscr{H}}{\partial\tilde{p}_x}, \tag{4.48}$$

$$\frac{d\tilde{p}_x}{dn} = -\frac{\partial\mathscr{H}}{\partial\tilde{x}}; \tag{4.49}$$

and the first integral can indeed by found by pursuing this course.

Notice that there are fixed points which satisfy $d\tilde{x}/dn = d\tilde{p}_x/dn = 0$ and are located at the points

$$\left(\tilde{x} = -\frac{8\pi\delta}{A}, \tilde{p}_x = 0\right), \tag{4.50}$$

and

$$\left(\tilde{x} = \frac{4\pi\delta}{A}, \tilde{p}_x = \pm\sqrt{3}\,\frac{4\pi\delta}{A}\right). \tag{4.51}$$

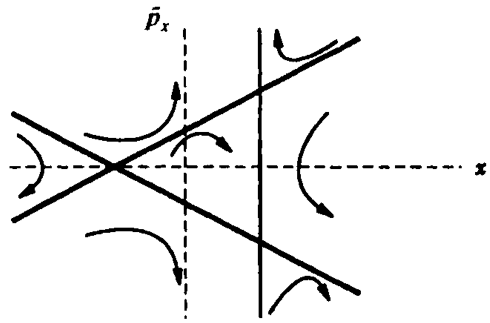

Figure 4.5. Separatrix when near the third-integer resonance. The arrows indicate direction of flow of phase space trajectories.

For these $\tilde{x}, \tilde{p}_x$, the constant k is zero and the associated figure in phase space—the separatrix—is just three intersecting straight lines. Motion within the triangle is bounded; motion outside the triangle is unbounded in this approximation (see Figure 4.5).

Along the vertical separatrix, the equation of motion becomes

$$\frac{d\tilde{p}_x}{dn} = \tfrac{1}{4}A\tilde{p}_x^2 - \tfrac{3}{4}A\left(\frac{4\pi\delta}{A}\right)^2, \qquad (4.52)$$

which is easily integrated to yield $\tilde{p}_x$ as a function of n. (See the problems at the end of the chapter.) Motion along or near the separatrix is important in resonant beam extraction as will be seen in the next subsection.

Finally, we can define the width of a nonlinear resonance, or at least present one of several similar definitions. There is a qualitative difference between the linear resonance produced, for example, by gradient errors and a nonlinear resonance. In the former case, the entire beam is either stable or unstable. In the latter case, the motion may be stable or unstable, depending on the oscillation amplitude. A nonlinear resonance doesn't produce a stopband—an essentially linear notion—but a range of tunes of the linear lattice throughout which varying fractions of the beam are unstable.

Suppose the beam has emittance $\epsilon = \pi\sigma^2/\beta_0$. If the tune is initially far from resonance and the resonance is approached sufficiently slowly, then it is reasonable to suppose (since the supposition can be easily checked by a simulation) that the phase space area will gradually deform from its originally circular boundary into the triangular shape characteristic of the resonance. When the tune difference is such that the stable area is equal to the beam emittance, one can say with some justification that 2δ is a reasonable definition of the width of the resonance. For the case that we have been considering, the resonance width is given by

$$2\delta = \frac{A}{2\pi}\left(\frac{2\beta_0\epsilon}{\sqrt{3}}\right)^{1/2}; \qquad (4.53)$$

here the emittance ϵ contains 39% of the particles. Obviously, there is some arbitrariness in such a definition. In the process of injection into a ring, the sudden rather than the adiabatic approximation would be reasonable.

4.2.4 Application to Resonant Extraction

Much of the foregoing discussion emphasized undesirable consequences of nonlinearities. The controlled introduction of nonlinearities may be used to advantage, however, and one important instance is the process of resonant extraction. It is easy to take the particles in an accelerator out in one turn. Unfortunately, that approach to extraction does not necessarily satisfy the needs of the experimental program. Rather, it is usually the case that particles are to be dribbled out on a time scale of a millisecond to many seconds. So we resort to the resonant extraction process, the groundwork for which has been established in the previous subsection.

In this process, the separatrix is made to gradually squeeze the phase space occupied by the beam. No matter how slowly this "squeeze" is carried out, motion near the unstable fixed points will fail to be adiabatic; for in the neighborhood of these points motion becomes arbitrarily slow in the static case. Particles depart the stable area at the fixed points and stream out along the outgoing arms of the separatrix. But the continuum in amplitudes of these outgoing particles leads to beam loss, since somewhere there has to be a device that is the start of the channel for departing particles. The partition, or septum, between "in" and "out" particles must be as thin as possible in order to obtain high extraction efficiency. At present, the thin septa are electrostatic. They are not strong enough to fling particles out of the ring, but they can establish enough of a gap between "in" and "out" particles so that a second, stronger magnetic septum can direct particles into an exit channel. The great virtue of the electrostatic septum is that the obstacle presented to the beam is very small, about 50 μm. For a proton accelerator with conventional magnets, it is already important that the losses off the primary extraction septum be minimized: the buildup of radioactivity in the ring is a major concern for operation and maintenance. For a proton accelerator using superconducting magnets, control of the septum loss is essential to avoid excessive energy deposition in the magnets with the attendant risk of superconducting to normal transition; that is, the extraction process brings with it the danger of a quench.

The third integer resonance can be used for slow extraction. The phase space at the first septum might look like the sketch in Figure 4.6(a), where the coordinates are the x and p_x in the unrotating frame. The septum is located a distance x_s from the center of the aperture. The *step size*, Δ, is the growth in x in three turns. The orientation of the figure is determined by the resonance driving terms, the azimuthal harmonics of the sextupole distribution. Only the outgoing arms of the separatrix are shown in this figure. The figure must be oriented in such a way that the septum kick transforms into a displacement at the second septum while preserving the distinction between

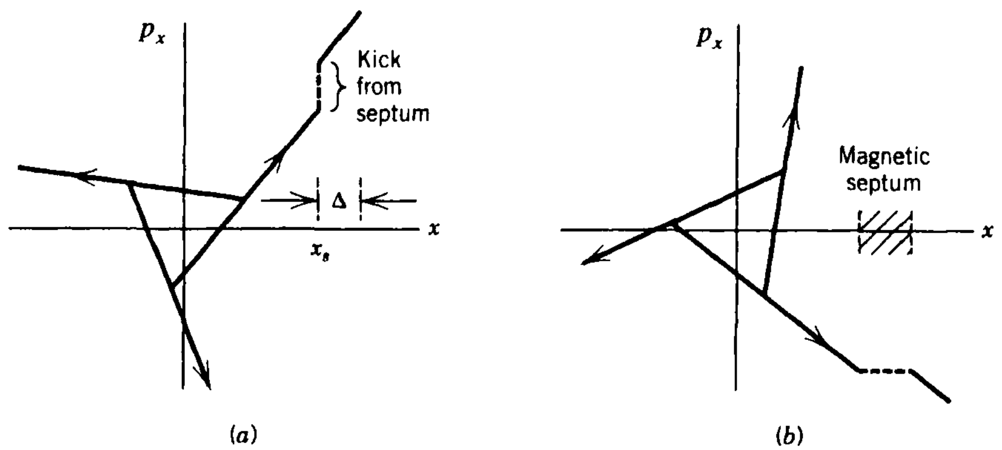

Figure 4.6. Phase space separatrices at (a) the location of the first (electrostatic) septum, and (b) the location of the second (magnetic) septum.

particles departing the aperture and those remaining in the accelerator. For instance, the phase space at the second septum might appear as shown in Figure 4.6(b). The 90° rotation between the two figures projects the kick of the first septum fully onto the second septum; the orientation of the first ensures that the extracted beam is well displaced from the circulating beam in the second.

All that remains to be specified are the strength of the resonance driving terms—the integrals A and B above—and the tune difference, $\delta\nu$, from resonance. Since the ratio of A and B is already set by the orientation of the separatrices at one of the septa, only two quantities remain to be determined. There are two conditions to be satisfied. The step size Δ should span the septum aperture for efficiency, and the stable area should correspond to the emittance of the beam at the onset of extraction. The relationships between the step size and stable area on the one hand and the driving terms and tune difference on the other were exhibited in the last section. A bit of geometry needs to be added to that discussion, having to do with the projection of motion along the separatrix onto the x-axis. The steps are not hard to carry out, if the critical reader wants to pursue them.

More in line with the focus of this discussion is the subject of extraction efficiency. As noted in the preceding paragraph, it was evident from the last section how to produce a slow spill beam. But what fraction of the particles strike the first septum?

In order to estimate the extraction inefficiency, let us suppose that the extraction process proceeds so slowly that it may be considered a static process. Then, the particle density distribution along an outgoing separatrix or along the projection of the separatrix onto a coordinate axis varies inversely as the rate of change of position along that coordinate. To convince oneself of this, let $F(x_1)\Delta x_1$ be the number of particles in an interval Δx_1 at x_1. After some time interval T has elapsed, the particles find themselves in

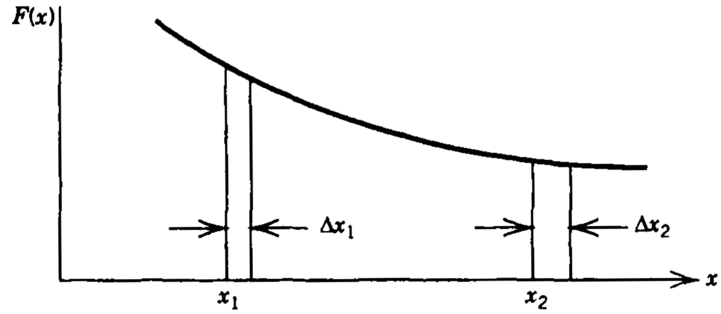

Figure 4.7. *Particle distribution function F(x).*

Δx_2 at x_2. The number of particles is the same, so

$$F(x_2)\Delta x_2 = F(x_1)\Delta x_1. \tag{4.54}$$

From

$$T = \int_{x_1}^{x_2} \frac{dx}{dx/dt} = \int_{x_1+\Delta x_1}^{x_2+\Delta x_2} \frac{dx}{dx/dt} \tag{4.55}$$

it follows that

$$\int_{x_1}^{x_1+\Delta x_1} \frac{dx}{dx/dt} = \int_{x_2}^{x_2+\Delta x_2} \frac{dx}{dx/dt}, \tag{4.56}$$

or

$$\frac{\Delta x_1}{(dx/dt)_1} = \frac{\Delta x_2}{(dx/dt)_2} \tag{4.57}$$

where Figure 4.7 may be helpful in identifying the quantities.

So Δx varies directly as dx/dt; therefore $F(x)$ varies inversely as dx/dt. It will be more convenient to use the *turn number*, n, as the independent variable; that is, we take the spatial dependence of F to be of the form

$$F \propto \frac{1}{dx/dn}. \tag{4.58}$$

If a septum of thickness w in the x-coordinate is located at a distance x_s from the central orbit, then the inefficiency e, defined as the fraction of particles that strike the first septum, is

$$e = \left[\int_{x_s}^{x_s+w} \frac{dx}{dx/dn}\right] \Bigg/ \left[\int_{x_s}^{\infty} \frac{dx}{dx/dn}\right] \tag{4.59}$$

$$= \left[\int_{x_s}^{x_s+w} \frac{dx}{dx/dn}\right] \Bigg/ \left[\int_{x_s}^{x_s+\Delta} \frac{dx}{dx/dn}\right]. \tag{4.60}$$

The second form above acknowledges, in the denominator, that the particle density distribution cuts off at a distance $x_s + \Delta$, with Δ being the *step size*, that is, the growth during x in the number of turns, N, between successive encounters with the septum at the proper phase for exit from the ring. In the third-integer case, $N = 3$.

The septum thickness w is small compared with x_s, and the integral in the numerator can be replaced by $w/(dx/dn)$ evaluated at x_s. The integral in the denominator is just N. So for the inefficiency we can use either of the following forms:

$$e = \frac{w}{(dx/dn)_{x_s}} \frac{1}{\int_{x_s}^{x_s+\Delta} dx/(dx/dn)} = \frac{1}{N}\frac{w}{(dx/dn)_{x_s}}. \tag{4.61}$$

Note that the flatter the distribution F, the better the efficiency. This circumstance favors the choice of low order multipoles to generate the step size.

Let us estimate the inefficiency for third-integer extraction in the limit of vanishing stable phase space; the algebra is simplified by going to this limit, but all the principles remain the same. The situation is illustrated in Figure 4.8. Projected on the x-axis, the equation of motion for a particle traveling outward on the separatrix is

$$\frac{dx}{dn} = \frac{1}{4}\frac{\text{resonance driving term}}{\cos\theta}x^2. \tag{4.62}$$

The expression for the inefficiency immediately gives

$$e = w\frac{x_s + \Delta}{x_s\Delta}. \tag{4.63}$$

The expression in the numerator is related to the maximum displacement to

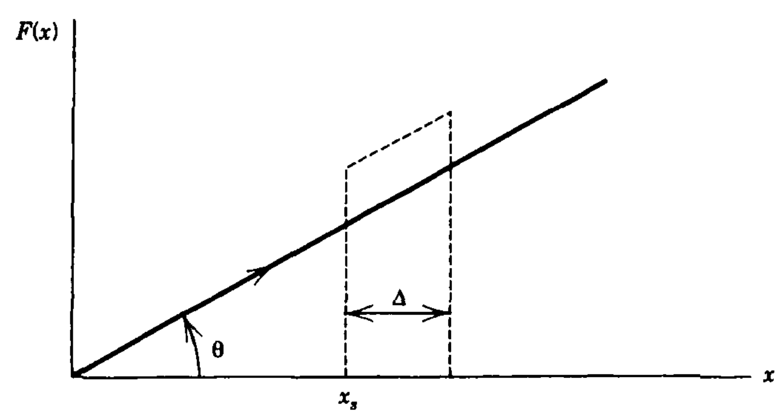

Figure 4.8. Step size across the septum.

be tolerated elsewhere in the ring, according to

$$x_s + \Delta = x_{\max}\left(\frac{\beta_s}{\beta_0}\right)^{1/2} \cos\theta, \tag{4.64}$$

where β_s is the amplitude function at the septum, and β_0 is the maximum value of the amplitude function in the standard cells of the ring where the aperture limitations are to be found. The angle θ has already been set by the arguments concerning the orientation of the separatrices at the two septa. For fixed $x_s + \Delta$, the mimimum in the inefficiency occurs for $x_s = \Delta$, and so

$$e_{\min} = \frac{4w}{x_s + \Delta}. \tag{4.65}$$

As a numerical example, let's take the superconducting synchrotron at Fermilab. The maximum oscillation amplitude in the arcs of the ring was fixed at 20 mm, after extensive simulation of particle motion in the fields provided by the superconducting main magnets. The lattice insertions for extraction devices were designed with β_s a factor of 2.3 larger than the maximum amplitude function in the standard cells. With $\theta = 45°$ and a septum thickness of 0.1 mm, the minimum inefficiency is 1.9%.

The accelerator in this example actually uses half-integer extraction; instead of sextupoles, the ring contains appropriate quadrupoles and octopoles. The distinction between the two approaches need not concern us here. The analogous calculation to that of the preceding paragraph leads to an inefficiency of 1.7%. It is interesting that inefficiency at this level implies generation of secondaries in the septum with a flux almost two orders of magnitude higher than would be tolerated by the superconducting magnets located downstream, requiring additional protective measures to be taken.

4.2.5 Comments on Correction Systems

In Chapter 3, we mentioned two types of correction magnets—those to compensate steering errors or make steering adjustments, and those to adjust the tune. These are elements of the correction magnet system. More properly this collection of elements should be called the correction and adjustment system, since it actually performs both functions.

Steering correction is conceptually simple. Tune correction brings with it the additional complexity of amplitude function perturbation, or equivalently, half-integer resonance excitation. Chromaticity compensation through the use of sextupole magnets can excite any of the sextupole driven resonances; or, if we wished to excite the third-integer resonance for slow extraction, we certainly would not wish to affect the chromaticity of the accelerator. That is, correction systems should perform specific functions cleanly without the introduction of possibly undesirable side effects.

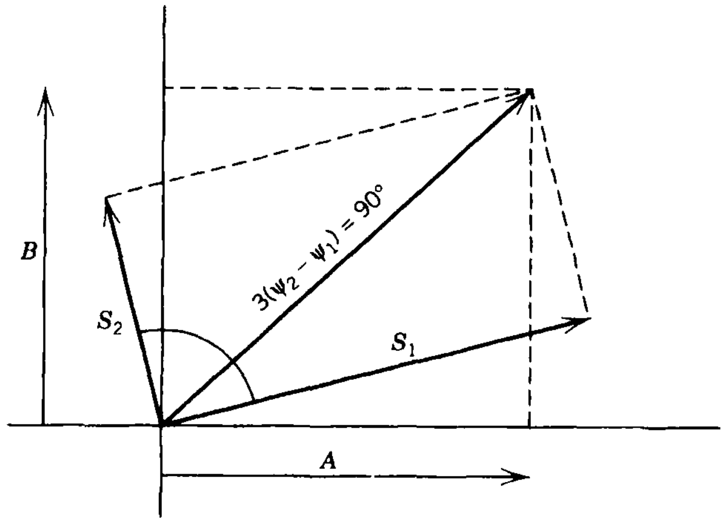

Figure 4.9. Two sextupoles of strengths S_1 and S_2 separated in phase by 90° on the $3\nu_0$ harmonic, where ν_0 is the third-integer resonance tune at which compensation is to be performed.

Since we have been discussing sextupole effects, let's concentrate on this case. For definiteness, suppose that we wish to compensate the one-degree-of-freedom third-integer resonance arising from field imperfections related to nonvanishing b_2 in the bending magnets of a synchrotron. This description is completely equivalent to the generation of resonance driving terms for third-integer extraction. In general, there will be two driving terms A and B, as given in Equations 4.39 and 4.40, generated by the errors. So we need two "knobs" which can be adjusted to compensate these driving terms. Note that the driving terms A and B are 90° out of phase with each other on the $3\nu_0$ harmonic.

The simplest way of effecting this correction might appear to be the introduction of two correction sextupoles at locations of equal values of the amplitude function β, differing in phase from one another by 90° on this harmonic; that is, the two correctors could be some odd multiple of 30° in betatron phase apart. We could represent these two sextupoles in a phasor diagram as shown in Figure 4.9, where the phasor amplitude is proportional to the strength of the sextupole, $S = B''l/2(B\rho)$.

The summation of all the sextupole field errors around the ring also can be represented by a phasor in this diagram. By a suitable choice of strengths and polarities of the two correction sextupoles, their resultant phasor can be made to cancel the phasor due to the errors, and hence the resonance driving terms given by Equations 4.39 and 4.40 can be brought to zero.

But unless we are so lucky that $\beta_1 S_1 D_1 = -\beta_2 S_2 D_2$, where D is the dispersion function, then the chromaticity would be changed by the correction. So, for pure resonance correction, we usually are led to a somewhat more complicated scheme. Suppose all of our harmonic compensation set are

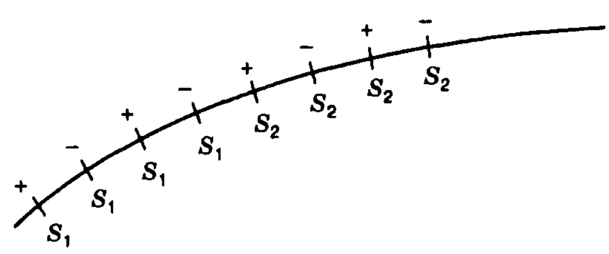

Figure 4.10. The $\nu_x = 19\frac{1}{3}$ resonance compensation circuit of the Tevatron contains sets of eight sextupoles, each located at a standard focusing quadrupole location in the FODO arc lattice, with the polarities shown. The sextupoles labeled S_1 are powered together, as are the ones labeled S_2.

located at equal values of dispersion as well. Then a scheme in which each sextupole is paired with a sextupole of equal strength but opposite sign 180° out of phase on the $3\nu_0$ harmonic will produce the desired results.

Let's carry this discussion to a more complicated case where the phase advance is not ideally suited to the introduction of sextupoles for resonance compensation. This is the circumstance, for instance, of the Tevatron. The phase advance per cell is 68°, and the tune is somewhat above $19\frac{1}{3}$. For the $19\frac{1}{3}$ resonance, the harmonic of interest is $k = 58$. Because the Tevatron is basically twofold symmetric, placement of equal strength sextupoles diametrically opposite one another will guarantee that only even harmonic resonances will be driven (or compensated).

The distribution of sextupoles to drive or compensate the $\nu_x = 19\frac{1}{3}$ resonance is shown in Figure 4.10. The amplitude function is the same at each sextupole, and the resulting phasor diagram is shown in Figure 4.11.

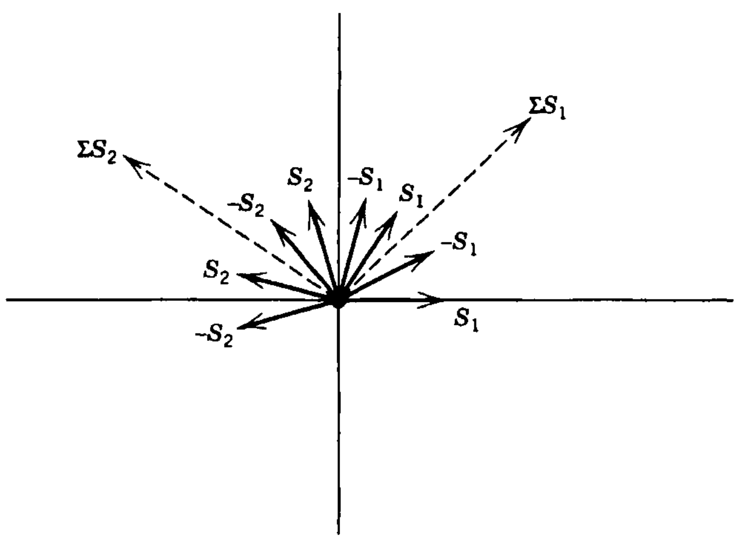

Figure 4.11. Phasor diagram resulting from set of eight sextupoles used to correct the $\nu_x = 19\frac{1}{3}$ resonance in the Tevatron.

The two families of sextupoles produce resultants which are approximately, though not exactly, 90° in phase apart on the $3\nu_0$ harmonic. Figure 4.10 shows two neighboring families of four sextupoles each, and in the Tevatron this pattern in replicated three additional times. The necessity for 32 sextupole magnets as opposed to, say, four is dictated by the need to produce a desired compensation strength while only limited space was available for an individual element.

It is probably clear that there is not a unique design approach for correction and adjustment systems. The approach reflects both the needs of the accelerator and the predelictions of the designer.

4.3 THE HAMILTONIAN FORMALISM

Only the most basic methods of dynamics have been used thus far, because we feel that the physics at work is most transparently illustrated in that way. But much of accelerator physics makes use of one form or another of higher dynamics. The Hamiltonian approach is the method most frequently encountered in the literature.[1]

In this section, we review the Hamiltonian form of dynamics, and then recast much of the material of the earlier discussions in this language.[2] No new physics is introduced, but the generality obtained may be helpful to the reader who wishes to pursue this approach further.

4.3.1 Review of Hamiltonian Dynamics

For a system with n degrees of freedom, there is a function $H(q, p, t)$ called the Hamiltonian. The variables are n generalized coordinates, their n conjugate momenta, and the time t. For the present, we are suppressing the subscripts on the variables, but we will include them when clarity demands it. The $2n$ equations of motion—Hamilton's equations—are then

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}. \tag{4.66}$$

The Hamiltonian approach focuses from the outset on motion in the $2n$-dimensional phase space of the dynamical variables p and q. At the beginning of our discussion, the variables will indeed resemble the momenta and coordinates of elementary mechanics. But that resemblance will fade as

[1]See, for example, R. D. Ruth, "Single-Particle Dynamics in Circular Accelerators," and L. Michelotti, "Introduction to the Nonlinear Dynamics Arising from Magneic Multipoles," *Physics of Particle Accelerators* (SLAC Summer School 1985, Fermilab Summer School 1984), AIP Conf. Proc. 153, 1987.

[2]H. Goldstein, *Classical Mechanics*, 2nd ed., Addison-Wesley Publishing Co., 1981.

we progress. In basic mechanics, we are all familiar with point transformations in configuration space. That is, we introduce new coordinates Q related to the old positions q by n equations of the form $Q = Q(q)$. In phase space, more general transformations among all $2n$ variables are possible and useful. All we require is that the form of Hamilton's equations be preserved.

Suppose we transform from variables p, q to variables P, Q, and that the new Hamiltonian is $K(P, Q, t)$. Hamilton's equations will be valid in both sets of coordinates, provided both satisfy the modified Hamilton's principle:

$$\delta \int \left(p_i \dot{q}_i - H \right) dt = 0, \tag{4.67}$$

$$\delta \int \left(P_i \dot{Q}_i - K \right) dt = 0, \tag{4.68}$$

where summation over the repeated indices is implied. The "modified" means that both positions and momenta are varied independently between the end points. The above will be satisfied if the integrands differ by only the total time derivative of some function F:

$$\left(p_i \dot{q}_i - H \right) = \left(P_i \dot{Q}_i - K \right) + \frac{dF}{dt}. \tag{4.69}$$

The transformations that maintain the validity of Hamilton's equations are called canonical transformations, and F is called the generating function. Note that the modified Hamilton's principle will remain valid also in the case that the integrands are in the ratio of some constant factor λ:

$$\lambda \left(p_i \dot{q}_i - H \right) = P_i \dot{Q}_i - K. \tag{4.70}$$

This is also a canonical transformation, corresponding to a scale change of the variables. We will encounter instances of this type of transformation in the next section.

The function F is in general a function of both the old and new variables as well as the time. We will restrict ourselves to functions that contain half of the old variables and half the new; these are useful for determining the explicit form of the transformation. The function F may then take on any of the following four forms:

$$F = F_1(q, Q, t), \tag{4.71}$$

$$F = F_2(q, P, t) - Q_i P_i, \tag{4.72}$$

$$F = F_3(Q, p, t) + q_i p_i, \tag{4.73}$$

$$F = F_4(p, P, t) + q_i p_i - Q_i P_i. \tag{4.74}$$

Now, if we insert each of these into

$$(p_i \dot{q}_i - H) = (P_i \dot{Q}_i - K) + \frac{dF}{dt}, \tag{4.75}$$

we obtain the relationships between old and new quantities listed below.

$$p = \frac{\partial F_1}{\partial q}, \qquad P = -\frac{\partial F_1}{\partial Q}, \tag{4.76}$$

$$p = \frac{\partial F_2}{\partial q}, \qquad Q = \frac{\partial F_2}{\partial P}, \tag{4.77}$$

$$q = -\frac{\partial F_3}{\partial p}, \qquad P = -\frac{\partial F_3}{\partial Q}, \tag{4.78}$$

$$q = -\frac{\partial F_4}{\partial p}, \qquad Q = \frac{\partial F_4}{\partial P}. \tag{4.79}$$

In all four cases,

$$K = H + \frac{\partial F_i}{\partial t}. \tag{4.80}$$

4.3.2 The Hamiltonian for Small Transverse Oscillations

The relativistic Hamiltonian for a particle of charge e moving under the influence of an electromagnetic field characterized by vector and scalar potentials $\vec{A}$ and V is

$$\mathscr{H} = \sqrt{\left(\vec{p} - e\vec{A}\right)^2 c^2 + m^2c^4} + eV, \tag{4.81}$$

where $\vec{p}$ is the momentum conjugate to the Cartesian position coordinates of the particle. The magnetic field $\vec{B}$ and the electric field $\vec{E}$ are given by

$$\vec{B} = \nabla \times \vec{A}, \tag{4.82}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}. \tag{4.83}$$

Recall that the canonical momentum $\vec{p}$ is related to the kinematic momentum $\gamma m\vec{v}$ by

$$\gamma m\vec{v} = \vec{p} - e\vec{A}. \tag{4.84}$$

In Chapter 3, we developed an equation of motion for small transverse deviations from the reference orbit. We want to follow the prescriptions of the Hamiltonian formalism and arrive at the same point. First, perform a canonical transformation with generating function

$$F = F_3(p, Q, t) = \vec{p} \cdot (\rho\hat{x} + x\hat{x} + y\hat{y}). \tag{4.85}$$

Then, following the rules of the preceding section,

$$p_s = \frac{\partial F_3}{\partial s} = \vec{p} \cdot \hat{s}\left(1 + \frac{x}{\rho}\right), \tag{4.86}$$

$$p_x = \frac{\partial F_3}{\partial x} = \vec{p} \cdot \hat{x}, \tag{4.87}$$

$$p_y = \frac{\partial F_3}{\partial y} = \vec{p} \cdot \hat{y}. \tag{4.88}$$

Again, as in Chapter 3, we assume that the curvature is locally a constant to simplify the discussion. Note that p_s is not the tangential component of the conjugate momentum. In order to preserve the relationship between components of the momentum and components of the vector potential, we define a canonical vector potential according to

$$A_s = \vec{A} \cdot \hat{s}\left(1 + \frac{x}{\rho}\right), \tag{4.89}$$

$$A_x = \vec{A} \cdot \hat{x}, \tag{4.90}$$

$$A_y = \vec{A} \cdot \hat{y}. \tag{4.91}$$

The generating function does not contain the time explicitly, so the new Hamiltonian, $\mathscr{H}'$, is just the old Hamiltonian expressed in the new coordinates:

$$\mathscr{H}' = c\left[\frac{1}{(1 + x/\rho)^2}(p_s - eA_s)^2 + (p_x - eA_x)^2 + (p_y - eA_y)^2 + m^2c^2\right]^{1/2} + eV \tag{4.92}$$

Now we want to change the independent variable from t to s. Consider $x' \equiv dx/ds$:

$$x' \equiv \frac{dx}{ds} = \frac{dx/dt}{ds/dt} = \frac{\partial\mathscr{H}/\partial p_x}{\partial\mathscr{H}/\partial p_s}. \tag{4.93}$$

The last form may be transformed into a partial derivative at constant $\mathscr{H}$ using

$$d\mathscr{H} = \left(\frac{\partial \mathscr{H}}{\partial p_x}\right)_{p_s} dp_x + \left(\frac{\partial \mathscr{H}}{\partial p_s}\right)_{p_x} dp_s = 0, \tag{4.94}$$

or

$$\left(\frac{\partial p_s}{\partial p_x}\right)_{\mathscr{H}} = -\frac{(\partial \mathscr{H}/\partial p_x)_{p_s}}{(\partial \mathscr{H}/\partial p_s)_{p_x}}. \tag{4.95}$$

Then for x' we have

$$x' = \left(\frac{\partial(-p_s)}{\partial p_x}\right)_{\mathscr{H}, p_y, x, y, s}. \tag{4.96}$$

This has the form of a Hamilton's equation for x' with $-p_s$ playing the role of the Hamiltonian. If the same procedure is carried out for the entire set of Hamilton's equations, we find

$$x' = \frac{\partial H}{\partial p_x}, \qquad y' = \frac{\partial H}{\partial p_y}, \qquad t' = -\frac{\partial H}{\partial \mathscr{H}}, \tag{4.97}$$

$$p_x' = -\frac{\partial H}{\partial x}, \qquad p_y' = -\frac{\partial H}{\partial y}, \qquad \mathscr{H}' = \frac{\partial H}{\partial t}. \tag{4.98}$$

Therefore, the new pairs of canonical variables are x, p_x; y, p_y, and $t, -\mathscr{H}$, with the new Hamiltonian $H = -p_s$. Solving for p_s, we obtain for our new Hamiltonian

$$\begin{aligned} H &= -p_s \\ &= -\sqrt{\left[\left(\frac{\mathscr{H} - eV}{c}\right)^2 - m^2c^2 - (p_x - eA_x)^2 - (p_y - eA_y)^2\right]\left(1 + \frac{x}{\rho}\right)^2} \\ &\quad - eA_s. \end{aligned} \tag{4.99}$$

To proceed we consider, as before, only the case where the electric field is zero and where the magnetic field may be described by

$$\vec{B} = B_x(x, y)\hat{x} + B_y(x, y)\hat{y} \tag{4.100}$$

for which we need only a single nonvanishing component of the vector potential, $\vec{A}\cdot\hat{s} \equiv A_z$. Therefore, we may write

$$H = -\sqrt{\left(\frac{\mathscr{H}^2}{c^2} - m^2c^2 - p_x^2 - p_y^2\right)\left(1 + \frac{x}{\rho}\right)^2} - eA_s. \tag{4.101}$$

For constant energy, $\mathscr{H} = E$ and thus

$$\frac{\mathscr{H}^2 - m^2c^4}{c^2} = p^2, \tag{4.102}$$

which gives us

$$\begin{aligned} H &= -\sqrt{(p^2 - p_x^2 - p_y^2)\left(1 + \frac{x}{\rho}\right)^2} - eA_s \\ &= -p\left(1 + \frac{x}{\rho}\right)\sqrt{1 - \left(\frac{p_x}{p}\right)^2 - \left(\frac{p_y}{p}\right)^2} - eA_s \\ &\approx -p\left[1 + \frac{x}{\rho} - \frac{1}{2}\left(\frac{p_x}{p}\right)^2 - \frac{1}{2}\left(\frac{p_y}{p}\right)^2\right] - eA_s. \end{aligned} \tag{4.103}$$

Finally, we must consider the form of A_s. From the definition of the vector potential and from Maxwell's equations, we find

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\partial A_z}{\partial y}\hat{x} - \frac{\partial A_z}{\partial x}\hat{y}, \tag{4.104}$$

$$\vec{\nabla}\times\vec{B} = \hat{s}\,\nabla^2 A_z = 0 \tag{4.105}$$

$$\Rightarrow \quad A_z = \mathrm{Re}\left\{\sum_n \frac{B^{(n)}}{n!}(x + iy)^n\right\}, \tag{4.106}$$

or

$$A_z = -B_0 - \tfrac{1}{2}B'(x^2 - y^2) - \tfrac{1}{6}B''(x^3 - 3xy^2) - \cdots. \tag{4.107}$$

Thus, A_s may be obtained from

$$A_s = \vec{A}\cdot\vec{s}\left(1 + \frac{x}{\rho}\right). \tag{4.108}$$

Scaling the Hamiltonian and the conjugate momentum variable by a constant, namely the design momentum p_0,

$$H \to H/p_0, \tag{4.109}$$

$$p_x \to p_x/p_0, \tag{4.110}$$

the final form of the Hamiltonian becomes

$$H = -\left(1 + \frac{x}{\rho} - \tfrac{1}{2}p_x^2 - \tfrac{1}{2}p_y^2\right) + \left(\frac{eB_0}{p_0}x + \frac{1}{2}\frac{eB'}{p_0}(x^2 - y^2)\right)\left(1 + \frac{x}{\rho}\right) + \cdots . \tag{4.111}$$

We may now apply Hamilton's equations to generate the equations of motion:

$$x' = \frac{\partial H}{\partial p_x} = p_x, \tag{4.112}$$

$$p_x' = -\frac{\partial H}{\partial x} = \left(\frac{1}{\rho} - \frac{eB_0}{p_0}\right) - \frac{eB_0}{p_0\rho}x - \frac{eB'}{p_0}x$$
$$= \left(\frac{1}{\rho} - \frac{1}{\rho_0}\right) - \left(\frac{1}{\rho\rho_0} + \frac{eB'}{p_0}\right)x, \tag{4.113}$$

which, for the ideal momentum particle ($\rho = \rho_0$), becomes

$$p_x' = x'' = -\left(\frac{1}{\rho_0^2} + \frac{eB'}{p_0}\right)x, \tag{4.114}$$

which is the same as the result of Chapter 3.

4.3.3 Transformations of the Hamiltonian

In one degree of freedom, our small oscillation Hamiltonian is of the form

$$H = \frac{x'^2}{2} + \frac{K(s)x^2}{2} + \cdots . \tag{4.115}$$

The first two terms resemble the Hamiltonian for a simple harmonic oscillator, though with the typical variation of the spring constant with s. In the last section, we found that a phase-amplitude description of the motion was useful. In the Hamiltonian formalism, the corresponding quantities are called action-angle variables. We wish to identify them and carry out the appropriate canonical transformation. The resulting Hamiltonian will still be a function of s, and a further transformation will be employed to remove that dependence. At that point, the "unperturbed" Hamiltonian will be a constant of the motion. One could, of course, invert the order of these transformations and arrive at the same point.

The form of the first transformation is suggested by the solutions that we have already developed. Recall, for the linear motion, that

$$x = \mathscr{A}\sqrt{\beta}\cos(\psi + \delta), \tag{4.116}$$

$$\beta x' + \alpha x = -\mathscr{A}\sqrt{\beta}\sin(\psi + \delta). \tag{4.117}$$

The Hamiltonian H and the solution above suggest that the new unperturbed Hamiltonian, H_0, should be of the form

$$H_0 = H_0(\mathscr{A}) = \text{constant}. \tag{4.118}$$

So we are motivated to select ψ as the coordinate and look for a conjugate variable, J, which is related to the amplitude. That is,

$$x = \mathscr{A}(J)\sqrt{\beta}\cos\psi, \tag{4.119}$$

$$\beta x' + \alpha x = -\mathscr{A}(J)\sqrt{\beta}\sin\psi, \tag{4.120}$$

where the arbitrary constant δ has been absorbed into the definition of ψ.

We can easily express x' in terms of x and ψ:

$$x' = -\frac{x\tan\psi + \alpha x}{\beta}. \tag{4.121}$$

Therefore, we look for a generating function of the type $F_1(x, \psi, s)$, for which

$$x' = \frac{\partial F_1}{\partial x}, \qquad J = -\frac{\partial F_1}{\partial \psi}. \tag{4.122}$$

From the expression for x' above, integration with respect to x gives

$$F_1 = -\frac{\alpha + \tan\psi}{2\beta}x^2 + f(\psi), \tag{4.123}$$

where $f(\psi)$ is an arbitrary function which we may set equal to zero. Then,

$$J = -\frac{\partial F_1}{\partial \psi} = \frac{\sec^2\psi}{2\beta}x^2, \tag{4.124}$$

and from the expression for x in terms of $\mathscr{A}$ we see that

$$J = \frac{\mathscr{A}^2}{2}, \tag{4.125}$$

or

$$\mathscr{A}(J) = \sqrt{2J}. \tag{4.126}$$

The new "unperturbed" Hamiltonian becomes just

$$H_0 = \frac{J}{\beta(s)}. \tag{4.127}$$

Now we wish to remove the s-dependence. Let us choose a new dependent variable which advances linearly with s in the unperturbed problem (which ψ does not). The quantity

$$\int \frac{ds}{\beta} - 2\pi\nu\frac{s}{C} \tag{4.128}$$

represents the "flutter" of the phase with respect to the average phase advance. Here, C is the circumference of the accelerator. So we wish to define a new coordinate, θ, such that

$$\psi = \theta + \text{"flutter"} = \theta + \int \frac{ds}{\beta} - \nu\frac{s}{R}. \tag{4.129}$$

This expression contains the old and new coordinate variables. Therefore, an F_1 transformation is not appropriate. Let's try an F_2. We want

$$\theta = \psi + \nu\frac{s}{R} - \int \frac{ds}{\beta}, \tag{4.130}$$

$$I = J, \tag{4.131}$$

and

$$J = \frac{\partial F_2}{\partial \psi}, \qquad \theta = \frac{\partial F_2}{\partial I}. \tag{4.132}$$

Following the same reasoning as above, the generating function is

$$F_2 = I\left(\psi + \nu\frac{s}{R} - \int \frac{ds}{\beta}\right), \tag{4.133}$$

and the new unperturbed Hamiltonian is then

$$H_0 = \frac{\nu}{R} I. \tag{4.134}$$

4.3.4 The Third-Integer Resonance Revisited

Having found an appropriate form for the unperturbed Hamiltonian H_0, we may now treat the remaining terms of H as perturbations and thus investigate the effects of nonlinearities on the particle motion. Let us consider the next term in the expansion of the Hamiltonian, namely, that due to a sextupole field. We have

$$\begin{aligned} H &= \frac{1}{2}p_x^2 + \left(\frac{1}{2}\frac{eB'}{p_0}\right)x^2 + \frac{1}{6}\frac{eB''}{p_0}x^3 \\ &= \tfrac{1}{2}x'^2 + \tfrac{1}{2}K(s)x^2 + \tfrac{1}{3}S(s)x^3 \\ &= H_0 + \tfrac{1}{3}S(s)x^3. \end{aligned} \tag{4.135}$$

The transformations thus far have yielded the following relations:

$$x = \sqrt{2\beta I}\cos\chi, \tag{4.136}$$

$$x' = -\sqrt{\frac{2I}{\beta}}\,(\sin\chi + \alpha\cos\chi), \tag{4.137}$$

where

$$\chi \equiv \theta - \nu\frac{s}{R} + \int \frac{ds}{\beta}. \tag{4.138}$$

So the new Hamiltonian is

$$\begin{aligned} H &= \frac{\nu}{R}I + \tfrac{1}{3}S(s)(2\beta I)^{3/2}\cos^3\left(\theta - \nu\frac{s}{R} + \int\frac{ds}{\beta}\right) \\ &= H_0 + \tfrac{1}{3}S(s)(2\beta I)^{3/2}\cos^3\left(\theta - \nu\frac{s}{R} + \int\frac{ds}{\beta}\right). \end{aligned} \tag{4.139}$$

Now we are in a position to proceed in much the way that we did in the last section. The amplitude function and sextupole strength are periodic functions of s, so we may expand the factor containing them in a Fourier series:

$$\beta^{3/2}S(s) = \sum_m W_m \cos[m\phi(s)], \tag{4.140}$$

$$W_m \equiv \frac{1}{\pi R}\int \beta^{3/2}S(s)\cos(m\phi)\,ds, \tag{4.141}$$

where the angular variable ϕ is s/R, and for simplicity we write only the cosine terms. Then, also expanding the cosine-cubed term, the Hamiltonian becomes

$$\begin{aligned} H &= \frac{\nu}{R}I + \tfrac{1}{12}(2I)^{3/2}\sum_m W_m \cos m\phi(\cos 3\chi + 3\cos\chi) \\ &= \frac{\nu}{R}I + \tfrac{1}{24}(2I)^{3/2}\sum_m W_m[\cos(3\chi + m\phi) + \cos(m\phi - 3\chi) \\ &\qquad + 3\cos(\chi + m\phi) + 3\cos(m\phi - \chi)]. \end{aligned} \quad (4.142)$$

From the equation for the rate of change of the action,

$$\begin{aligned} \frac{dI}{ds} &= -\frac{\partial H}{\partial\theta} \\ &= \tfrac{1}{24}(2I)^{3/2}\sum_m W_m[-3\sin(3\chi + m\phi) + 3\sin(m\phi - 3\chi) \\ &\qquad - 3\sin(\chi + m\phi) + 3\sin(m\phi - \chi)], \end{aligned} \quad (4.143)$$

we see that if there is an integer m such that $3\chi \approx m\phi$, then the condition for a resonance is satisfied. To examine the phase space near resonance, we go to rotating coordinates as we did in the more elementary treatment. Now, of course, this requires yet another canonical transformation. We take

$$F_2 = I_1\left(\theta - \frac{\nu_0 s}{R}\right), \quad (4.144)$$

where $\nu_0 = m/3$ is the resonant tune. This transformation leaves the initial action I and the final action I_1 the same, so we will suppress the subscript. The new coordinate, θ_1 is

$$\theta_1 = \theta - \nu_0\frac{s}{R} \quad (4.145)$$

with the new Hamiltonian given by

$$H = \frac{\delta}{R}I + \tfrac{1}{24}(2I)^{3/2}W_m\cos\left(m\phi - 3\theta_1 + 3\delta\,\phi - 3\int\frac{ds}{\beta}\right). \quad (4.146)$$

As before, $\delta \equiv \nu - \nu_0$.

In these coordinates, Hamilton's equations give

$$\frac{dI}{ds} = \tfrac{1}{8}(2I)^{3/2}W_m\sin\left(m\phi - 3\theta_1 + 3\delta\,\phi - 3\int\frac{ds}{\beta}\right), \quad (4.147)$$

$$\frac{d\theta_1}{ds} = \delta + \tfrac{1}{16}(2I)^{1/2}W_m\cos\left(m\phi - 3\theta_1 + 3\delta\,\phi - 3\int\frac{ds}{\beta}\right). \quad (4.148)$$

At the fixed points, the derivatives above are each equal to zero. The first tells us that the argument of the cosine function in the Hamiltonian must be equal to an integer times π. From the second, the distance to the fixed points is given by

$$I = \left(\frac{2\delta}{W_m}\right)^2. \tag{4.149}$$

To the degree that I can be related to the amplitude $\mathscr{A}$ of the unperturbed motion according to the relationship used above, the amplitude of the fixed point is given by

$$\mathscr{A} = \frac{8\delta}{W_m}. \tag{4.150}$$

This is the same as the result in the last section, taking into account the different definitions of amplitudes and Fourier coefficients.

PROBLEMS

1. Derive the inhomogeneous equation of motion after the Floquet transformation has been applied (Equation 4.5).

2. For a picture frame dipole magnet where the cores meet perfectly on one side but are separated by a small gap h on the other, show that the quadrupole term generated is given by

$$b_1 = \frac{h}{gw},$$

   where g is the nominal gap height and w is the pole width.

3. Consider a unit square in the tune diagram (i.e. ν_V vs. ν_H) with corners at (n, n), $(n + 1, n)$, $(n, n + 1)$, $(n + 1, n + 1)$. Draw the lines representing all sum resonances through fourth order.

4. Using Equations 4.48 and 4.49, find the first integral to the equation of motion as given in the text.

5. Integrate the equation of motion (Equation 4.52) along the vertical separatrix for the resonance considered in the text. Verify that the

number of turns, n, to progress from y_0 to y is

$$n = \frac{1}{2\sqrt{3}\,\pi\delta} \ln\left\{ \frac{\dfrac{A}{4\pi\delta} y_0 + \sqrt{3}}{\dfrac{A}{4\pi\delta} y_0 - \sqrt{3}} \cdot \frac{\dfrac{A}{4\pi\delta} y - \sqrt{3}}{\dfrac{A}{4\pi\delta} y + \sqrt{3}} \right\}.$$

6. Assume that a single thin sextupole is placed in a ring. The point of observation is chosen to be at the midpoint of the sextupole. Its strength is such that the harmonic driving term A is

$$A \approx \beta \frac{B''L}{2(B\rho)} = 0.05\ \text{mm}^{-1}.$$

For positive δ, the separatrices will be oriented as sketched in the text above. Take $\delta = 0.006$. Using the result of the preceding problem, find the position after 3 turns of a particle that starts from $y_0 = 10$ mm.

7. The single sextupole case would appear to be a long way from the spirit of the derivation upon which the analytical results are based. Using a computer, carry out a turn by turn calculation for the same particle as that in the example above. That is, start at the midpoint of the sextupole and give the particle a deflection appropriate to half of the sextupole. Propagate around the ring with a linear matrix, then deliver another half-sextupole kick. Compare with the result of the preceding problem. How do you know if you are even on the separatrix?

8. The length scale and sextupole strength in Problem 6 are useful in relating the dynamics to realistic values in accelerators. But for calculational purposes, it is easier to cast the mappings in dimensionless form. Note that the driving term A of Problem 6 has the dimensions of inverse length. Using $1/A$ as the unit of length, introduce new variables with x and p_x scaled accordingly. State the mapping of the preceding problem in these new variables. Note that the only remaining parameter is the tune, and so phase space plots developed in these variables are characteristic of the tune only. Modify the program that you wrote for the preceding problem accordingly, and repeat the calculations. Compare the behavior near resonance ($\delta = 0.006$) with that far from resonance ($\delta = 0.09$).

9. Investigate the deformation of phase space from a circle as a single sextupole is turned on by an extension of the turn by turn calculation carried out in Problem 8. As the phase space occupied by the beam exceeds the stable region provided by the ring, observe that particles depart along or near the separatrices in the near resonance situation of

the static case provided the rate of variation of the sextupole strength is sufficiently slow. This is a multiparticle problem, and you will have to decide how the initial phase space is to be populated.

10. The field of an octopole magnet varies as the cube of the horizontal or vertical displacement from its center, so one might expect an average octopole moment to produce a fourth-power term in the "potential" for betatron oscillations. The oscillation frequency would then depend on the oscillation amplitude. Carry out the same steps as used in the derivation of the amplitude and phase equations but with octopole fields rather than sextupole fields. Assume that the tune is far from resonance, so that any harmonic driving terms are unimportant. Show that the equations of motion are

$$\frac{da}{dn} = 0,$$

$$\frac{d\psi}{dn} = \tfrac{3}{8}a^2 D$$

with

$$D \equiv \frac{\beta_0}{(B\rho)} \oint \left(\frac{\beta}{\beta_0} \right)^2 \frac{B'''}{6}\, ds.$$

The tune change with amplitude is then

$$\Delta\nu(a) = \frac{1}{2\pi}\frac{3}{8}Da^2.$$

11. Although the stopband width arising from quadrupole errors is a problem connected with the linear motion, we didn't calculate the width in Chapter 3. It is relatively easy to do at this point, as another example of the method used for treating nonlinear motion. Repeat the steps of the sextupole case, but with quadrupole terms instead. For simplicity, assume that the only nonvanishing integral is

$$Q = \frac{\beta_0}{(B\rho)} \oint \frac{\beta}{\beta_0} B' \cos 2\nu_0\phi\, ds.$$

Show that the equations of motion in x and p_x are

$$\frac{dx}{dn} = -\tfrac{1}{2}Qp_x + 2\pi\delta \cdot p_x,$$

$$\frac{dp_x}{dn} = -\tfrac{1}{2}Qx - 2\pi\delta \cdot x,$$

where δ is now the difference in tune from the half integer. Show that the motion is unstable for all particles over the tune range of the unperturbed ring given by

$$\Delta\nu = \frac{1}{2\pi}|Q|.$$

This is the stopband width.

12. The results of Problem 10 and Problem 11 may be combined to illustrate another approach to slow extraction—the so-called half-integer method. Suppose the unperturbed tune is just below one-half of an integer. In the presence of an average octopole moment, some large amplitude particles may find themselves in the half-integer stopband if the appropriate quadrupole harmonic is present. Show that, in lowest order of perturbation theory, the separatrices in this case consist of two intersecting circles.