An Introduction to the Physics of High Energy Accelerators
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CHAPTER 3

Transverse Linear Motion

In the previous chapter we concentrated on acceleration and energy stability and found a stability principle that causes particles initially near each other in energy to remain so in the presence of oscillatory accelerating fields. We referred to the associated energy oscillations as motion in the longitudinal degree of freedom. Reasonably enough, the other two degrees of freedom are termed transverse, as indicated in Figure 3.1.

One of our tasks in this chapter is to investigate transverse stability. Do particles of the same energy, but with slightly different transverse coordinates, either in position or direction, remain near each other in the course of their motion in the accelerator? We will find a criterion for such stability and discuss the solution of the associated equations of motion. We will show that, in general, stability consists in bounded oscillatory motion about the design trajectory. This motion is termed a *betatron oscillation* for historical reasons. We will see that the transverse oscillation frequencies are much greater than the typical frequency of phase oscillations, thus allowing us to treat the longitudinal degree of freedom independently. We will thus have identified stability principles for all three degrees of freedom which are passive in the sense that, for our ideal accelerator, they do not rely on feedback mechanisms.

In a linear accelerator, particles of different momenta follow the same ideal trajectory. In a circular accelerator this is not the case, and we will exhibit the closed orbits of particles differing in momenta from that of the ideal particle. Thus, for this variety of accelerator, there is a transverse attribute to phase oscillations.

We can choose our ideal accelerator such that the transverse restoring forces are linear in the transverse coordinates. For high energy accelerators, the restoring forces and the bending forces in circular accelerators are

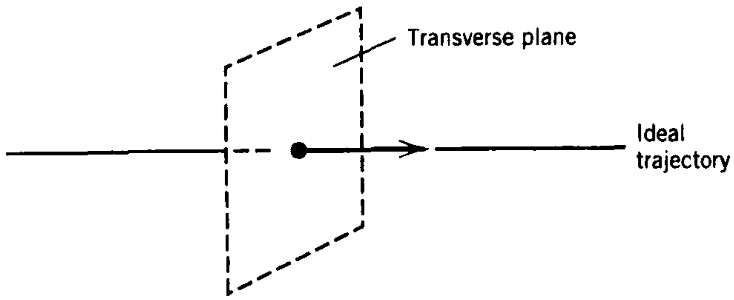

Figure 3.1. Characterization of transverse and longitudinal degrees-of-freedom of particle motion.

produced by magnetic rather than electric fields. For a nonrelativistic particle, in the low speed limit, electrostatic fields are more effective. But for a particle traveling near the speed of light, a magnetic field of 1 T and an electric field of 300 MV/m would each provide the same transverse deflecting force. The former field strength is typical of magnetic devices, while the latter is outside our present reach.

With our ideal linear restoring forces, the transverse degrees of freedom are also independent of one another; however, the two transverse frequencies are comparable and hence certain imperfections in the restoring fields can couple the motion. This topic of transverse coupled motion is left to a separate chapter. We conclude this chapter with an introduction to perturbations to the ideal linear magnetic fields which cause steering and focusing errors.

3.1 STABILITY OF TRANSVERSE OSCILLATIONS

The most basic magnetic field configuration—a uniform field— produces a form of focusing. Consider a particle traveling along a circular orbit in such a

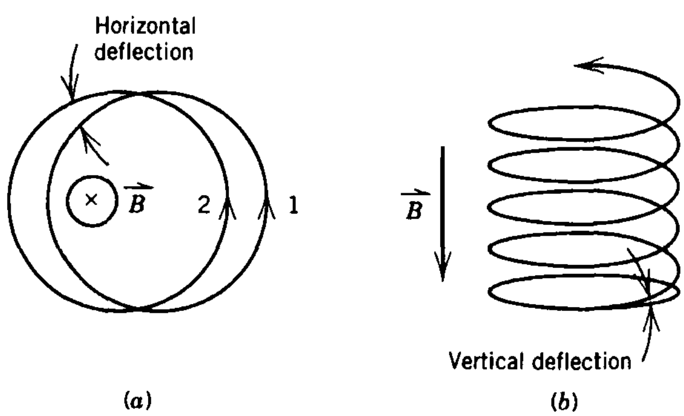

Figure 3.2. Charged particle motion in uniform magnetic field when perturbed by a deflection (a) perpendicular to the field lines, and (b) parallel to the field lines.

field. Suppose the particle receives an angular deflection in the plane perpendicular to the magnetic field. The resulting orbit will be just another circle of the same radius as the first but with a different center, as illustrated in Figure 3.2. One can say that the second orbit is performing a stable oscillation about the first. Unfortunately, if the deflection has a component along the magnetic field lines, the particle will subsequently spiral away without limit—there is no focusing in this degree of freedom.

3.1.1 Weak Focusing

The situation of Figure 3.2 can be rectified by designing the source of the magnetic field so that the field lines bend outward as shown in Figure 3.3. Particles above the midplane will experience a force downward; those below will be forced upward. However, along the horizontal plane the vertical component of the magnetic field decreases with increasing radius, since the field lines get farther apart. Thus vertical focusing is achieved at the expense of radial focusing, and so there is a limit to the effectiveness of the focusing that can be achieved in both transverse degrees of freedom simultaneously.

Suppose the vertical component of the field along the midplane is given by

$$B_y = \frac{B_0}{r^n}. \tag{3.1}$$

Then, if $n = 0$, we would have a uniform field and no vertical focusing: just the situation discussed earlier. If $n > 1$, the field could not provide the necessary centripetal force to keep the particles moving in a circular path of constant radius. Hence, for stability, the field index n is constrained to lie between the values of 0 and 1.

This form of focusing is called *weak focusing*. It has the disadvantage that as the design energy, and hence the circumference of the orbit, is increased, so also does the required aperture increase for a given angular deflection. Because the focusing is weak, the radial oscillations are essentially those depicted in Figure 3.2(a); the maximum radial displacement of a deflected particle is directly proportional to the radius of the machine. The scale of the magnetic components of a synchrotron, for example, would become unreasonably large and costly. This circumstance led to the invention of alternating gradient focusing (also known as *strong focusing*) in 1952. This method is the

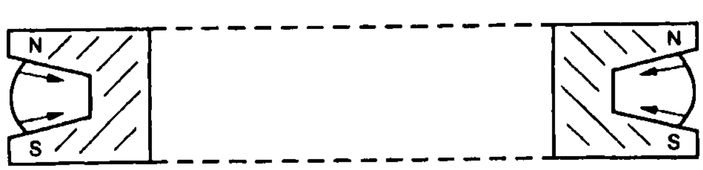

Figure 3.3. Cross section of weak focusing circular accelerator.

one most commonly used in accelerators today and we will consider its properties for the remainder of this chapter.

3.1.2 Strong Focusing

One would like the restoring force on a particle displaced from the design trajectory to be as strong as possible. The weak focusing scheme described above is limited. In the absence of current density, field gradients that provide restoring forces in both transverse degrees of freedom simultaneously are not possible. The condition $\vec{\nabla} \times \vec{B} = 0$ leads to

$$\frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y}, \tag{3.2}$$

where x and y are the two transverse coordinates. For small displacements, x and y, from the design trajectory, the field may be written as

$$\vec{B} = B_x\hat{x} + B_y\hat{y} \tag{3.3}$$

$$= \left(B_x(0,0) + \frac{\partial B_x}{\partial y}y + \frac{\partial B_x}{\partial x}x\right)\hat{x} + \left(B_y(0,0) + \frac{\partial B_y}{\partial x}x + \frac{\partial B_y}{\partial y}y\right)\hat{y}, \tag{3.4}$$

where $\hat{x}$ and $\hat{y}$ are unit vectors in the x and y directions, respectively. The last term in each component produces a force at right angles to the displacement and hence cannot represent a restoring force. The remaining coefficients of x and y are equal, according to the curl condition, Equation 3.2. Then the Lorentz force is focusing in one coordinate and defocusing in the other. The standard magnet that produces this focusing character is the quadrupole.

The focal length of a *thin lens* quadrupole magnet can be obtained easily. We imagine a charged particle moving through the quadrupole at a distance x from the magnet's axis of symmetry. The thin lens approximation implies that the length of the magnet, l, is short enough that the displacement x is unaltered as the particle passes through the magnet and hence the magnetic field experienced by the particle, $B_y = (\partial B_y/\partial x)x$, is constant along the particle trajectory. In this *paraxial* approximation, the angle is equal to the slope of the particle's trajectory, $x' \equiv dx/ds$ (where s is distance measured along the ideal trajectory). As depicted in Figure 3.4, the slope of the particle's transverse trajectory thus will be altered by an amount

$$\Delta x' = -\frac{l}{\rho} = -l \cdot \left(\frac{eB_y}{p}\right) = -\left(\frac{eB'l}{p}\right)x, \tag{3.5}$$

where ρ is the radius of curvature of the trajectory through the magnetic field

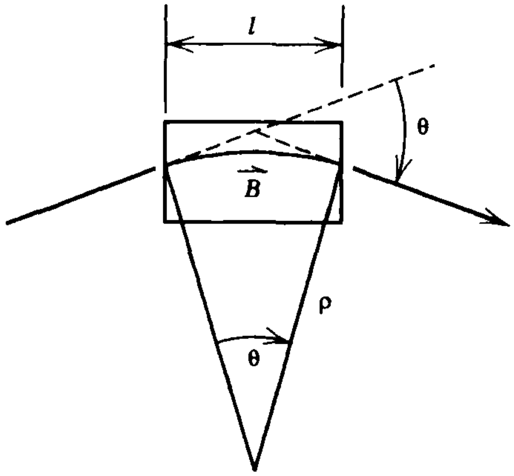

Figure 3.4. Deflection of particle by thin magnetic element.

and $B' \equiv \partial B_y/\partial x$ is the *gradient* of the quadrupole magnet, and where use has been made of Equation 2.1.

Since a ray parallel to the optic axis will be bent toward the focal point of the lens, as depicted in Figure 3.5, the change in slope is simply $\Delta x' = -x/f$, where f is the focal length of the quadrupole lens. The focal length is thus given by

$$\frac{1}{f} = \frac{eB'l}{p}. \tag{3.6}$$

The ratio of momentum to charge, p/e, is often called the *magnetic rigidity* and written $(B\rho)$; we will follow this latter convention in much of the text. Remember that $(B\rho)$ is just a single symbol. In the MKS system of units, $(B\rho)$ can be calculated from

$$(B\rho) = \frac{10}{2.9979} p_{(\mathrm{GeV}/c)} \qquad \text{tesla-meters.} \tag{3.7}$$

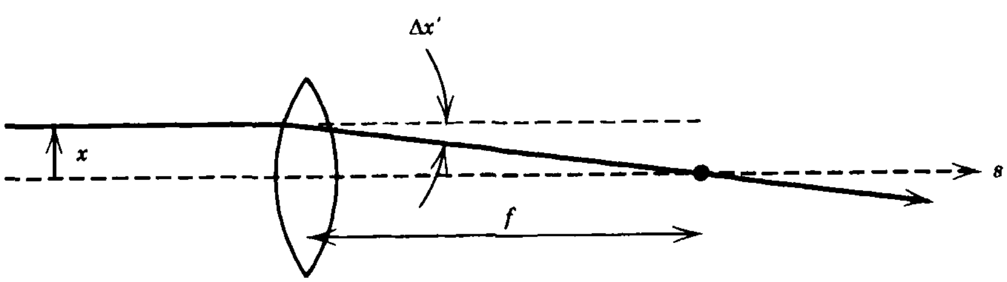

Figure 3.5. Ray initially parallel to the optical axis is bent by a convex lens causing it to pass through its focal point a distance f away.

We can therefore express our focal length relationship by

$$\frac{1}{f} = \frac{B'l}{(B\rho)}. \tag{3.8}$$

As mentioned earlier, quadrupole lenses focus in one plane and defocus in the other. Obviously, an accelerator cannot be made up of magnets that focus only in one plane. But we can recall from geometrical optics that a combination of equal strength convex and concave lenses will produce a net focusing. To see this, we recast Equation 3.5 in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}. \tag{3.9}$$

For a concave lens, the focal length is of opposite sign. In this language, the progress of a ray through the interlens space of length L is given by

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}. \tag{3.10}$$

Therefore, the matrix corresponding to transport of the ray through first a concave lens, then a drift, and then a convex lens may be written as

$$\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1 + \frac{L}{f} & L \\ -\frac{L}{f^2} & 1 - \frac{L}{f} \end{pmatrix}. \tag{3.11}$$

At least in the case where L is small compared to f, it is clear that there is net focusing; in this approximation the resulting matrix is that of a thin lens of net focal length $f^2/L > 0$. If the two lenses were interchanged, the net result would still be focusing. Hence a system of alternating gradient thin quadrupole magnets could, in principle, focus in both degrees of freedom at once. In fact, the focal length need not be large compared with the lens spacing for this to occur, which is the point of one of the problems at the end of the chapter.

The above discussion suggests that one can focus in two degrees of freedom simultaneously using a system of magnetic elements whose gradients alternate in sign. The discussion was based on tracing the trajectories of particles through a single pass of a system of two lenses of alternating focal lengths. In modern accelerators, where particles must be transported through great distances, the stability of particle motion through repetitive encounters with such structures must be studied. Below we develop a criterion for stable motion through an infinite number of passages through a focusing structure and apply the result to the thin lens alternating gradient system. The type of focusing achieved using alternating gradients is called *strong focusing*. As we shall see, strong focusing leads to beam sizes which are dependent upon the spacing of the lenses and their strengths, and independent of the scale of the accelerator.

3.1.3 Stability Criterion

In a synchrotron or long beam transport composed of alternately focusing and defocusing lenses, it is not obvious at the outset what relationships between lens strengths and spacing lead to stable oscillations as opposed to oscillations that grow in amplitude with time. The matrix language introduced in the preceding section can be used to establish a condition which distinguishes between these alternatives.

The detailed description of the way in which magnets and intervening spaces are placed to form the accelerator is conventionally called the *lattice*. After having developed the matrices appropriate to the elements of the accelerator, the motion of a particle can be followed through the lattice. If a particle traverses a series of elements having matrices $M_1, M_2, \ldots, M_n$, then the input and output conditions through these elements are related by the matrix

$$M = M_n \cdots M_2 M_1.$$

If the sequence of elementary matrices above is encountered repetitively, as is the case, for instance, if they represent the components all the way around a circular accelerator, then we can use this matrix to enquire into the stability of transverse oscillations.

For an oscillation to be stable, the quantity

$$M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}, \tag{3.12}$$

where M is the matrix for one turn or repetition period, must remain finite for arbitrarily large n. Let V_1, V_2 be the two eigenvectors of M, corresponding to the eigenvalues λ_1, λ_2. Any initial condition can be expressed in terms of V_1, V_2:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{in} = AV_1 + BV_2, \tag{3.13}$$

where A and B are constants. Propagation for n periods is then represented by

$$M^n \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}} = A\lambda_1^n V_1 + B\lambda_2^n V_2, \tag{3.14}$$

and so the requirement for stability is equivalent to the requirement that λ_1^n and λ_2^n not grow with n.

But note that M is the product of matrices each of which has determinant equal to unity, so M itself is unimodular. The eigenvalues of M are thus reciprocals of each other:

$$\lambda_2 = 1/\lambda_1, \tag{3.15}$$

and we can in general write

$$\lambda_1 = e^{i\mu},$$

$$\lambda_2 = e^{-i\mu},$$

where μ is a complex number. For stability, we see that μ must be real.

Now, solve the eigenvalue equation for M. Setting

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the eigenvalue equation

$$\det(M - \lambda I) = 0 \tag{3.16}$$

becomes

$$(ad - bc) - (a + d)\lambda + \lambda^2 = 0. \tag{3.17}$$

Noting that $ad - bc = \det M = 1$ and rearranging gives

$$\lambda^{-1} + \lambda = a + d \equiv \operatorname{Tr} M, \tag{3.18}$$

where $\operatorname{Tr} M$ stands for the trace of M. Finally, expressing λ in terms of μ gives

$$e^{i\mu} + e^{-i\mu} = 2\cos\mu = \operatorname{Tr} M, \tag{3.19}$$

and the stability condition is just

$$-1 \le \tfrac{1}{2}\operatorname{Tr} M \le 1. \tag{3.20}$$

Note that the stability condition is independent of the starting point, since the trace of a product of matrices is invariant under cyclic permutation of the matrices.

The significance of the angle μ will appear in the next section, where it will be identified as the phase advance of the transverse oscillation through the interval contained in M.

As an example, consider a lattice which consists only of equally spaced focusing and defocusing lenses, each of which we will assume to be thin. (This is referred to as a FODO lattice.) If the order is first the focusing lens, then a drift of length L, third a defocusing lens, and finally another drift of length L, the matrix is

$$M = \begin{pmatrix} 1 - \frac{L}{f} - \left(\frac{L}{f}\right)^2 & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{pmatrix}. \tag{3.21}$$

Application of the stability condition gives

$$-1 \le 1 - \frac{1}{2}\left(\frac{L}{f}\right)^2 \le 1,$$

or, simplifying,

$$\left|\frac{L}{2f}\right| \le 1. \tag{3.22}$$

So we obtain the remarkably simple result that the motion is stable provided the focal length is greater than half the lens spacing.

Now we are in a position to suggest one of the significant advantages of strong focusing. In Figure 3.6, we sketch an oscillation of a particle traversing a sequence of focusing and defocusing lenses for the case that the focal length is half the lens spacing. We see that the wavelength of an oscillation is just four times the lens spacing and therefore is unrelated to the size of the accelerator. The alternating gradient principle enables us to decouple the transverse aperture requirement from the size (hence, energy) of the accelerator. Even though for simplicity of the illustration we have made this

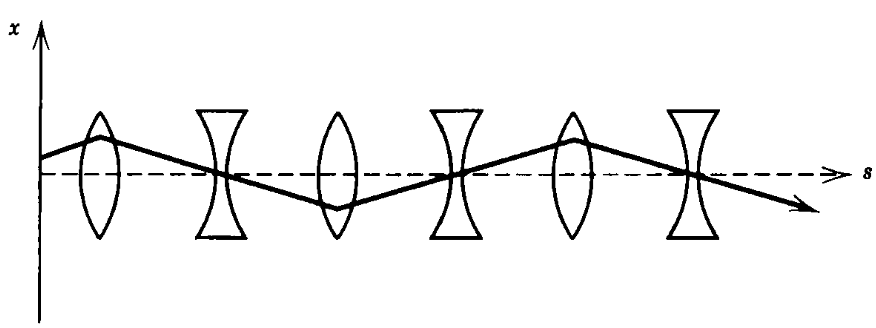

Figure 3.6. Example of a particle oscillation through a system of lenses where $f = L/2$. The maximum displacement is independent of the size of the accelerator.

argument at the limit of stability, it of course remains true for focal lengths within the stability bounds.

3.2 EQUATION OF MOTION

We are dangerously close to being able to write down the differential equations of motion for transverse oscillations. Consider a particle passing through a magnetic field with gradient $B' = \partial B_y/\partial x$ over a distance Δs. Recalling Equation 3.5, we see that the slope of a particle's trajectory, $x' = dx/ds$, changes by an amount $\Delta x' = -[B'\Delta s/(B\rho)]x$. Thus,

$$\frac{\Delta x'}{\Delta s} = -\frac{B'(s)}{(B\rho)}x. \tag{3.23}$$

Taking the limit as $\Delta s \to 0$, we obtain the second order differential equation

$$x'' + \frac{B'(s)}{(B\rho)}x = 0. \tag{3.24}$$

If there were a nonzero magnetic field on the design trajectory, as in a bending magnet, then Equation 3.24 would represent the difference between the slope changes of the particle in question and that of the ideal particle. While we have obtained the essentials of the equation of motion in just two steps, we have an obligation to the reader to provide a more rigorous derivation, which we now proceed to do.

Let us limit ourselves to the situations in which the design trajectory is a straight line or a single planar closed curve. By implication, we are developing the equation of motion for betatron oscillations in a linac or synchrotron.

Suppose that the geometry is as sketched in Figure 3.7. Locally, the design trajectory (reference orbit) has curvature ρ. The path length along this curve is s. Ultimately, s will be the independent variable. At any point along the reference orbit, we can define three unit vectors: $\hat{s}$, $\hat{x}$, $\hat{y}$. The position of a particle can then be expressed as a vector $\vec{R}$ in the form

$$\vec{R} = r\hat{x} + y\hat{y}, \tag{3.25}$$

where $r \equiv \rho + x$. We are interested in the behavior of the deviations x and y from the reference orbit.

The equation of motion is

$$\frac{d\vec{p}}{dt} = e\vec{v} \times \vec{B}, \tag{3.26}$$

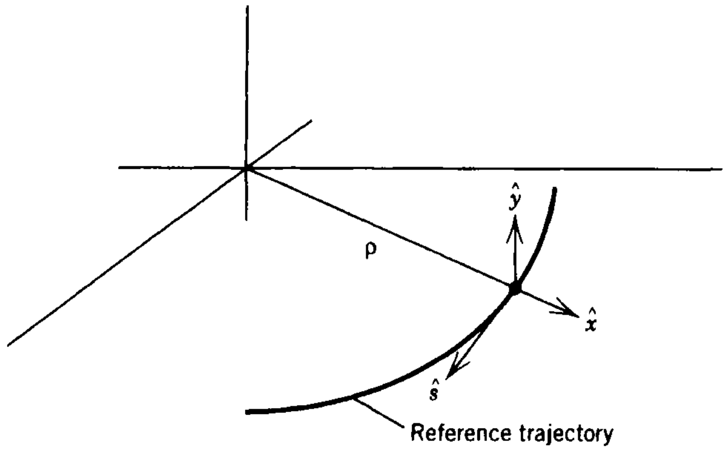

Figure 3.7. Coordinate system for development of equation of motion.

and we assume that there are radial and vertical components of $\vec{B}$—we will ignore the possible $\hat{s}$-component of $\vec{B}$ for now. So the cross product on the right hand side becomes

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{s} \\ v_x & v_y & v_s \\ B_x & B_y & 0 \end{vmatrix} = -v_s B_y \hat{x} + v_s B_x \hat{y} + (v_x B_y - v_y B_x)\hat{s}. \quad (3.27)$$

If we ignore the radiation created by an accelerating charge for now, then the energy and hence the Lorentz factor γ do not change in a static magnetic field. (Synchrotron radiation due to accelerating charges will be discussed in a later chapter.) The left hand side thus becomes

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\gamma m \dot{\vec{R}} = \gamma m \ddot{\vec{R}}, \quad (3.28)$$

and so

$$\ddot{\vec{R}} = \frac{e\vec{v} \times \vec{B}}{\gamma m}. \quad (3.29)$$

Now we must evaluate $\ddot{\vec{R}}$ in these coordinates:

$$\vec{R} = r\hat{x} + y\hat{y}, \quad (3.30)$$

$$\dot{\vec{R}} = \dot{r}\hat{x} + r\dot{\hat{x}} + \dot{y}\hat{y}. \quad (3.31)$$

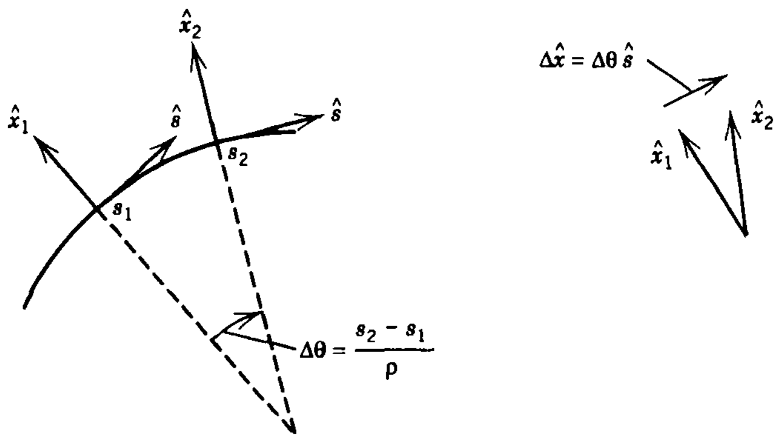

Figure 3.8. Time rate of change of unit vector $\hat{x}$.

We have to include the $\dot{\hat{x}}$-term in the above because if there is motion in the s-direction, the unit vector $\hat{x}$ will have a derivative. From Figure 3.8 we see that

$$\dot{\hat{x}} = \dot{\theta}\hat{s}, \tag{3.32}$$

where $\dot{\theta} \equiv v_s/r$. Therefore,

$$\dot{\vec{R}} = \dot{r}\hat{x} + r\dot{\theta}\hat{s} + \dot{y}\hat{y}, \tag{3.33}$$

and differentiating again,

$$\ddot{\vec{R}} = \ddot{r}\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + r\dot{\theta}\dot{\hat{s}} + \ddot{y}\hat{y}. \tag{3.34}$$

The new quantity in the above is $\dot{\hat{s}}$. By the same argument as used to obtain $\dot{\hat{x}}$, we have

$$\dot{\hat{s}} = -\dot{\theta}\hat{x}, \tag{3.35}$$

and so

$$\ddot{\vec{R}} = (\ddot{r} - r\dot{\theta}^2)\hat{x} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + \ddot{y}\hat{y}. \tag{3.36}$$

Thus, in the $\hat{x}$-direction the equation of motion is

$$\ddot{r} - r\dot{\theta}^2 = -\frac{ev_sB_y}{\gamma m} = -\frac{ev_s^2B_y}{\gamma m v_s}. \tag{3.37}$$

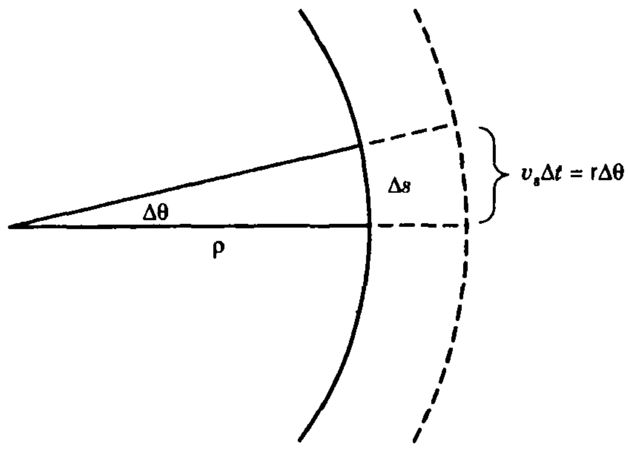

Figure 3.9. Comparison of reference orbit path length ds and particle path length $v_s\,dt$.

Since $v_x \ll v_s$ and $v_y \ll v_s$, to a very good approximation the total momentum p of the particle is $\gamma m v_s$. So

$$\ddot{r} - r\dot{\theta}^2 = -\frac{e v_s^2 B_y}{p}. \tag{3.38}$$

Now, change to s as the independent variable. Then

$$\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds}, \tag{3.39}$$

and from Figure 3.9 we see that

$$ds = \rho\, d\theta = v_s\, dt \frac{\rho}{r}. \tag{3.40}$$

Hence, assuming $d^2s/dt^2 = 0$,

$$\frac{d^2}{dt^2} = \left(\frac{ds}{dt}\right)^2 \frac{d^2}{ds^2} = \left(v_s \frac{\rho}{r}\right)^2 \frac{d^2}{ds^2}. \tag{3.41}$$

Replacing r with $\rho + x$, the equation of motion becomes

$$\frac{d^2x}{ds^2} - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2, \tag{3.42}$$

where $(B\rho) = p/e$. A similar treatment yields for the equation of motion in the y-direction

$$\frac{d^2y}{ds^2} = \frac{B_x}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2. \tag{3.43}$$

In general, these equations will be nonlinear. But let us restrict ourselves to fields that are linear functions of x and y, and keep only the lowest order terms in x and y. Later, we will treat the higher order terms as perturbations of the basic linear motion that we consider here.

We may use the field expansion introduced in Section 3.1.1, namely

$$B_x = B_x(0,0) + \frac{\partial B_x}{\partial y}y + \frac{\partial B_x}{\partial x}x, \tag{3.44}$$

$$B_y = B_y(0,0) + \frac{\partial B_y}{\partial x}x + \frac{\partial B_y}{\partial y}y. \tag{3.45}$$

Since we are considering a planar accelerator, $B_x(0,0) = 0$. We also do not wish the motion to be coupled in the basic design, and hence the coefficients $\partial B_y/\partial y$ and $\partial B_x/\partial x$ are assumed to be zero. Finally, the equations of motion become

$$\frac{d^2x}{ds^2} + \left[\frac{1}{\rho^2} + \frac{1}{(B\rho)}\frac{\partial B_y(s)}{\partial x}\right]x = 0, \tag{3.46}$$

$$\frac{d^2y}{ds^2} - \frac{1}{(B\rho)}\frac{\partial B_y(s)}{\partial x}y = 0, \tag{3.47}$$

where use of the curl condition has been made to eliminate B_x.

These equations resemble Equation 3.24 closely; the equation in x differs only in the addition of a "centripetal" term originating in our choice of curvilinear coordinates. [This term is the origin of the focusing in x as shown in Figure 3.2(a).] For large accelerators the centripetal term is usually small in comparison with the gradient term. The equations of motion above are both of the form

$$x'' + K(s)x = 0 \tag{3.48}$$

and so differ from a simple harmonic oscillator only in that the "spring constant" K is a function of position s.

We will discuss two methods of solution. First we note that within a single magnetic component of the accelerator, K is normally a constant by design, as depicted in Figure 3.10. So within each component, we can use harmonic oscillator solutions, piecing them together at the interfaces. Second, we will examine a closed form solution using the properties of Hill's equation.

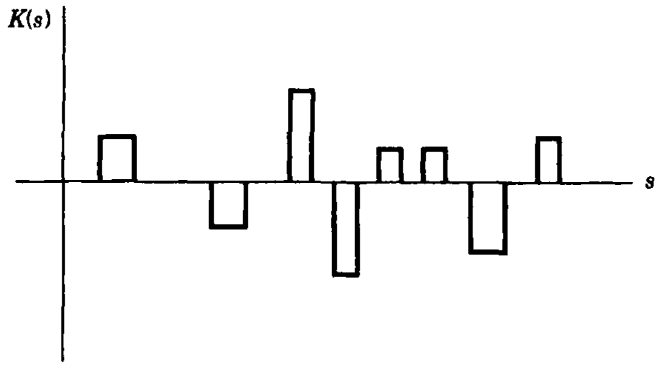

Figure 3.10. The spring constant K varies with position, but is normally constant within individual components of the accelerator.

3.2.1 Piecewise Method of Solution

In the same spirit as the geometrical optics argument of Section 3.1.2, we may describe the motion of a particle through an element of the accelerator by a 2×2 matrix. There are only three cases to consider: K vanishes, K is positive, and K is negative. The matrix for the first is the same as that for a drift space L between lenses in our earlier argument:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}. \tag{3.49}$$

In the y (vertical) plane, this corresponds either to a drift space between magnets or to propagation through a magnet with constant B_y. In the x (horizontal) plane, this corresponds either to a drift space between magnets or to a situation in which the centripetal term $(1/\rho^2)$ is exactly balanced by the field gradient. The latter is an unusual circumstance. Frequently, for other than exact calculation, the radius of curvature of a high energy accelerator is so large that the centripetal term may be neglected.

For $K > 0$ over a distance l, the equation of motion is just that of a simple harmonic oscillator, so in matrix form the solution is

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cos(\sqrt{K}\,l) & \dfrac{1}{\sqrt{K}}\sin(\sqrt{K}\,l) \\ -\sqrt{K}\sin(\sqrt{K}\,l) & \cos(\sqrt{K}\,l) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}, \tag{3.50}$$

while for $K < 0$, the corresponding result is

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{\text{out}} = \begin{pmatrix} \cosh\left(\sqrt{|K|}\,l\right) & \dfrac{1}{\sqrt{|K|}}\sinh\left(\sqrt{|K|}\,l\right) \\ \sqrt{|K|}\sinh\left(\sqrt{|K|}\,l\right) & \cosh\left(\sqrt{|K|}\,l\right) \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{\text{in}}. \tag{3.51}$$

Note that the thin lens limit emerges from the last two forms if one keeps Kl finite as $l \to 0$. In the limit, Kl tends to the reciprocal of the focal length f, as can be seen by comparison with the matrix for a thin lens in Equation 3.9.

Using the matrices above, the motion of a particle can be followed through an arrangement of accelerator elements. If a particle traverses a series of elements having matrices $M_1, M_2, \ldots, M_n$, then as stated earlier the input and output conditions through these elements are related by the matrix

$$M = M_n \cdots M_2 M_1.$$

3.2.2 Closed Form Solution

The second method of solution is based on the observation that our equation of motion is a form of Hill's equation—a differential equation studied extensively in the nineteenth century—and that general solutions can be written for it that closely resemble simple harmonic oscillations.

The equation of motion

$$x'' + K(s)x = 0 \tag{3.52}$$

has the property that though the "spring constant" K is a function of the independent variable s, for an important class of accelerators K is periodic. That is, there is a distance C such that

$$K(s + C) = K(s). \tag{3.53}$$

The repeat distance of the hardware, C, may be as large as the circumference of a synchrotron or it may be less; in any event, we will take K to be a periodic function of position. The result of nineteenth century mathematics that we will use is that the general solution of the equation of motion can be expressed in the form

$$x = Aw(s)\cos[\psi(s) + \delta], \tag{3.54}$$

where A and δ are the two constants of integration reflecting the initial conditions, and $w(s)$ can be required to be a periodic function with periodicity C. Note the similarity to the harmonic oscillator solution. For K everywhere a positive constant, we would immediately write

$$x = A\cos[\psi(s) + \delta] \tag{3.55}$$

with $\psi = \sqrt{K}s$, and A, δ the constants of integration. When K becomes a periodic function of position, the solution will differ from that for the simple harmonic oscillator by a factor representing a spatially varying amplitude and a phase which no longer develops linearly with s.

Now we must find how $w(s)$ and $\psi(s)$ are to be determined. Substitution of the general solution into the differential equation gives

$$x'' + Kx = A(2w'\psi' + w\psi'')\sin(\psi + \delta) + A(w'' - w\psi'^2 + Kw)\cos(\psi + \delta) = 0. \quad (3.56)$$

Since we want the functions w and ψ to be independent of δ (which depends upon the particular motion), we will require that the coefficients of the sine and cosine terms individually vanish. Multiplying the sine term by w we have

$$2ww'\psi' + w^2\psi'' = (w^2\psi')' = 0, \quad (3.57)$$

or

$$\psi' = \frac{k}{w(s)^2}, \quad (3.58)$$

where k is an arbitrary constant of integration.

Using this relationship between ψ and w, the coefficient of the cosine term yields the differential equation that w must satisfy:

$$w^3(w'' + Kw) = k^2. \quad (3.59)$$

Strictly speaking, $w(s)$ need not be periodic; it only has to be a solution of Equation 3.59. But if the motion we are trying to describe is that of a particle traveling through a periodic section of the accelerator, for instance through thousands of revolutions about a circular accelerator, it is much more useful to choose the unique periodic solution for $w(s)$. Hence we will restrict our attention for now to solutions of this equation with periodicity C.

In the previous section, the matrix propagating a transverse oscillation (betatron oscillation) from one place to another in a lattice was found as a product of matrices representing basic components of the accelerator. We can also express the same matrix in terms of the parameters introduced in this section. If we rewrite Equation 3.55 as

$$x = w(s)(A_1 \cos\psi + A_2 \sin\psi) \quad (3.60)$$

and

$$x' = \left(A_1 w' + \frac{A_2 k}{w}\right)\cos\psi + \left(A_2 w' - \frac{A_1 k}{w}\right)\sin\psi, \quad (3.61)$$

then for the initial conditions x_0, x'_0, at $s = s_0$, the constants A_1 and A_2 are

$$A_1 = \frac{x_0}{w}, \tag{3.62}$$

$$A_2 = \frac{x'_0 w - x_0 w'}{k}. \tag{3.63}$$

Now by requiring that the function w be periodic over the distance C, we may write down the matrix for propagation from s_0 to $s_0 + C$. The resulting matrix equation describing the motion is

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos\Delta\psi_C - \dfrac{ww'}{k}\sin\Delta\psi_C & \dfrac{w^2}{k}\sin\Delta\psi_C \\ -\dfrac{1+(ww'/k)^2}{w^2/k}\sin\Delta\psi_C & \cos\Delta\psi_C + \dfrac{ww'}{k}\sin\Delta\psi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}. \tag{3.64}$$

The phase of the particle's oscillation advances through the repeat period by an amount

$$\psi(s_0 \to s_0 + C) \equiv \Delta\psi_C = \int_{s_0}^{s_0+C} \frac{k\,ds}{w^2(s)}. \tag{3.65}$$

Because $w(s)$ is periodic, this integral is independent of the choice of s_0.

3.2.3 Courant-Snyder Parameters

Inspection of the matrix in Equation 3.64 reveals that the function $w^2(s)$ and its derivative both scale with the arbitrary constant k. Since the motion of the particle, and in particular the advance of the phase of its motion, is what's observed, choosing a different value of k must simply lead to a different value for the function $w^2(s)$, scaled by a factor of k. Since $w^2(s)$ and its derivative are the more fundamental quantities of the problem, it is customary to define new variables

$$\beta(s) \equiv \frac{w^2(s)}{k}, \tag{3.66}$$

$$\alpha(s) \equiv -\frac{1}{2}\frac{d\beta(s)}{ds} = -\frac{1}{2}\frac{d}{ds}\left(\frac{w^2(s)}{k}\right), \tag{3.67}$$

$$\gamma \equiv \frac{1+\alpha^2}{\beta}, \tag{3.68}$$

and rewrite the equation for one passage through the repeat period as

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0+C} = \begin{pmatrix} \cos \Delta\psi_C + \alpha \sin \Delta\psi_C & \beta \sin \Delta\psi_C \\ -\gamma \sin \Delta\psi_C & \cos \Delta\psi_C - \alpha \sin \Delta\psi_C \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}. \quad (3.69)$$

Here, the phase advance is

$$\Delta\psi_C = \int_{s_0}^{s_0+C} \frac{ds}{\beta(s)}, \quad (3.70)$$

and $\beta(s)$ may be interpreted as the local wavelength of the oscillation divided by 2π. The quantities α, β, and γ are usually referred to as Courant-Snyder parameters collectively; from now on, the function β will be referred to as the amplitude function.

So the general solution to the equation of motion can be written as

$$x(s) = A\sqrt{\beta(s)} \cos[\psi(s) + \delta], \quad (3.71)$$

where the constant k has been absorbed into the constant A. From Equation 3.59, the amplitude function $\beta(s)$ must satisfy the differential equation

$$2\beta\beta'' - \beta'^2 + 4\beta^2 K = 4. \quad (3.72)$$

It is often easier to remember this equation when written in terms of the Courant-Snyder parameters:

$$K\beta = \gamma + \alpha'. \quad (3.73)$$

The matrix of Equation 3.69 is often written in a compact way as

$$M = I \cos \Delta\psi_C + J \sin \Delta\psi_C, \quad (3.74)$$

where

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}. \quad (3.75)$$

Noting that $J^2 = -I$, where I is the identity matrix, one may also write Equation 3.74 in even more compact form:

$$M = e^{J\Delta\psi_C}. \quad (3.76)$$

The latter form often permits simplification of algebraic manipulations.

Computation of the Courant-Snyder parameters may be performed by comparing the two ways of expressing the matrix through a repeat period.

Suppose that multiplying all the individual matrices of the repeat period gives

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.77}$$

Equating the two versions of M,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\Delta\psi_C + \alpha\sin\Delta\psi_C & \beta\sin\Delta\psi_C \\ -\gamma\sin\Delta\psi_C & \cos\Delta\psi_C - \alpha\sin\Delta\psi_C \end{pmatrix}. \tag{3.78}$$

Then, first of all,

$$\cos\Delta\psi_C = \tfrac{1}{2}(a + d) = \tfrac{1}{2}\,\mathrm{Tr}\,M. \tag{3.79}$$

Comparison of this relation with the identical one satisfied by μ in the stability discussion of Section 3.1.3 enables us to identify μ as the phase advance through a repeat period.

Knowing $\cos\Delta\psi_C$ gives us the magnitude but not the sign of $\sin\Delta\psi_C$. But β must be a positive quantity, so the sign of $\sin\Delta\psi_C$ is whatever the sign of the matrix element b happens to be. Then

$$\beta = \frac{b}{\sin\Delta\psi_C}, \tag{3.80}$$

and by subtraction of the diagonal elements

$$\alpha = \frac{a - d}{2\sin\Delta\psi_C}. \tag{3.81}$$

Thus, we have the Courant-Snyder parameters at one point of the periodic lattice. But the same procedure works between any pair of corresponding points in the lattice, so one can find $\beta(s)$ for all s. With β determined at all points of the lattice, the particle motion from one point to another can be described by the matrix equation

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s_2} = M(s_1 \to s_2)\begin{pmatrix} x \\ x' \end{pmatrix}_{s_1}. \tag{3.82}$$

We can arrive at an explicit representation of the matrix $M(s_1 \to s_2)$ in terms of the amplitude function through the use of Equations 3.60 and 3.61. Suppose x_1 and x_1' are the initial conditions at $s = s_1$. Then the constants A_1 and A_2 are

$$A_1 = \frac{x_1}{w_1}, \tag{3.83}$$

$$A_2 = \frac{x_1' w_1 - x_1 w_1'}{k}. \tag{3.84}$$

Inserting these values for A_1 and A_2 into Equations 3.60 and 3.61 and rewriting in terms of the amplitude functions, the matrix $M(s_1 \to s_2)$ may be written as

$$\begin{pmatrix} \left(\dfrac{\beta_2}{\beta_1}\right)^{1/2}(\cos\Delta\psi + \alpha_1 \sin\Delta\psi) & (\beta_1\beta_2)^{1/2}\sin\Delta\psi \\ -\dfrac{1+\alpha_1\alpha_2}{(\beta_1\beta_2)^{1/2}}\sin\Delta\psi + \dfrac{\alpha_1-\alpha_2}{(\beta_1\beta_2)^{1/2}}\cos\Delta\psi & \left(\dfrac{\beta_1}{\beta_2}\right)^{1/2}(\cos\Delta\psi - \alpha_2\sin\Delta\psi) \end{pmatrix}. \tag{3.85}$$

Here, $\Delta\psi$ is the phase advance from s_1 to s_2.

The phase advance between any two points now can be uniquely determined via

$$\Delta\psi(s_1 \to s_2) = \int_{s_1}^{s_2} \frac{ds}{\beta(s)}. \tag{3.86}$$

In particular, for a circular machine, the number of oscillations per turn,

$$\nu \equiv \frac{1}{2\pi}\oint \frac{ds}{\beta(s)}, \tag{3.87}$$

is called the *tune* of the accelerator. Since β can be interpreted as an oscillation's local wavelength divided by 2π, a sense of scale for the values of β is obtained. While the actual particle oscillations might have small amplitudes (e.g., millimeters), the amplitude function β should be expected to take on numerical values of the scale of the repeat period. This also tells us that the numerical value of the constant A describing a particle's motion will be comparatively rather small. Note that A has dimensions of $(\text{length})^{1/2}$, and β has dimensions of length.

It should be pointed out that the solution to the equation of motion, Equation 3.71, explicitly implies stable motion. The solution must also be able to describe unstable motion. Demonstration of how the amplitude function and phase advance are altered for the case of an unstable lattice is left to the problems at the end of the chapter.

We have just gone through rather a lot of algebra to develop a way of representing a betatron oscillation that is at first sight a good deal more complicated than propagation using elementary matrices. In the thin lens approximation—not a bad approximation for large separated function synchrotrons—the betatron oscillation is after all just a sequence of straight line segments. We've managed to express these line segments in harmonic oscillator language; presumably there is some benefit. We'll try to illustrate the advantages as time goes on. Let us just point out that any oscillation can be easily constructed once one has a tabulation of the Courant-Snyder parameters and the phase advance as a function of position.

3.2.4 Emittance and Admittance

Now we are in a position to approach the important questions of the space demanded by the beam and the space provided by the accelerator. We are still working in the context of a perfect accelerator—no field imperfections.

In our solution for a betatron oscillation,

$$x(s) = A\sqrt{\beta(s)}\cos[\psi(s) + \delta], \tag{3.88}$$

the constant A can be expressed in terms of x and x' by eliminating the trigonometric functions. Forming the combination

$$\alpha(s)x(s) + \beta(s)x'(s) = -A\sqrt{\beta(s)}\sin[\psi(s) + \delta], \tag{3.89}$$

then squaring and summing Equations 3.88 and 3.89, we obtain

$$A^2 = \gamma(s)x(s)^2 + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2. \tag{3.90}$$

This Courant-Snyder invariant is analogous to the total energy of a harmonic oscillator. At any point in the accelerator, the invariant form describes an ellipse as depicted in Figure 3.11.

For the case of a circular accelerator, each time that the particle passes a particular position in the ring, its betatron oscillation coordinates will appear as a point on the ellipse given by the amplitude function and its slope at that point, as sketched in Figure 3.12.

For different locations through the lattice, the ellipses will have different shapes and orientations, but they will all have the same value of A. This

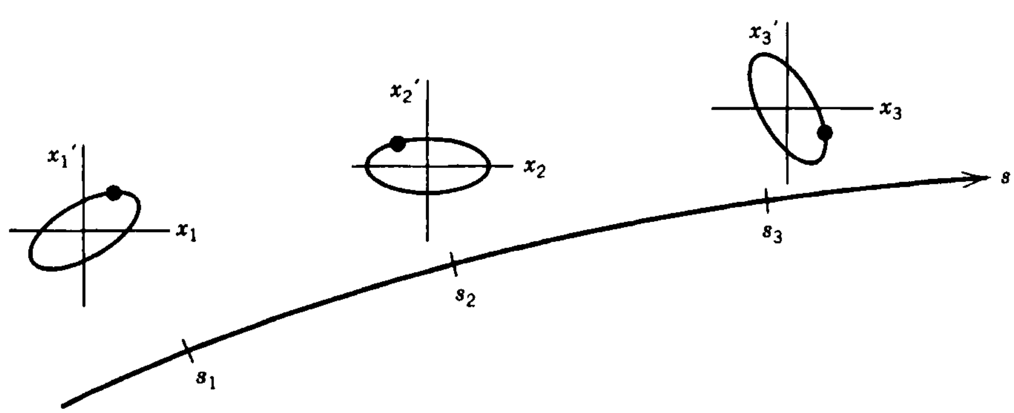

Figure 3.11. Phase space ellipses along the design trajectory.

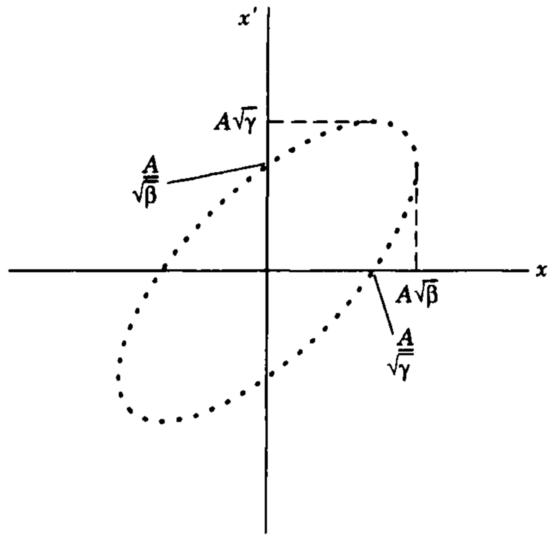

Figure 3.12. Phase space mapping from turn to turn in a circular accelerator.

means that they all have the same area. For, consider the general equation of an ellipse:

$$ax^2 + 2bxy + cy^2 = d. \tag{3.91}$$

According to analytic geometry, the area of this ellipse is

$$\frac{\pi d}{\sqrt{ac - b^2}}, \tag{3.92}$$

which in our case becomes

$$\frac{\pi A^2}{\sqrt{\beta\gamma - \alpha^2}} = \pi A^2. \tag{3.93}$$

The coordinates x, x' define a phase space for the motion that we are discussing, and we have shown that the area in this phase space enclosed by an unaccelerated particle is constant. Only a slight modification is necessary to find the invariant appropriate for accelerated motion and this is the subject of the next subsection.

The *admittance* is the phase space area associated with the largest ellipse that the accelerator will accept. From the preceding discussion, we would estimate the admittance as follows. At any point in the accelerator, the maximum value of x is $A\sqrt{\beta}$. If the half aperture available to the beam is $a(s)$, then somewhere there will be a minimum in $a(s)/\sqrt{\beta(s)}$. Then the

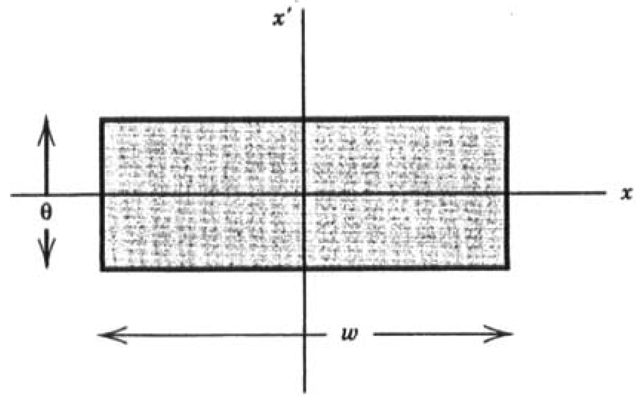

Figure 3.13. Phase space of particles emanating from a source of width w and angular spread θ.

admittance will be

$$\left(\pi \frac{a^2}{\beta}\right)_{\min}. \tag{3.94}$$

In the special case of a uniform half aperture a with no intruding septa, electrodes, and so on, then the minimum in $a/\sqrt{\beta}$ would occur at the maximum value of the amplitude function, $\beta_{\max}$; then

$$\text{admittance} = \frac{\pi a^2}{\beta_{\max}}. \tag{3.95}$$

The phase space area occupied by the beam is called the *emittance*, and is frequently denoted by ϵ. The ideal beam, of course, would have zero cross-sectional area and all the particles would be headed in exactly the same direction. All the particles would occupy the same point in the phase space of this transverse degree of freedom. But the most elementary model of a particle source leads to a nonzero emittance. Suppose that you have a source of width w from each point of which particles are produced within an angle θ. The phase space plot for the beam at the source will look like that depicted in Figure 3.13, enclosing a phase space area $w\theta$.

The phase space distribution of a beam is certainly not a uniformly populated rectangle, so a general definition of emittance will take this circumstance into account. For practical purposes, the phase space boundary of the beam may be considered to be an ellipse. Suppose that an irregularly shaped area is injected into an accelerator, with an initial state as shown in Figure 3.14. The subsequent motion of individual particles will lie on the elliptical invariant curves, as shown. As a result, the phase space demanded by the beam will be the area of the dashed curve. As time progresses, the

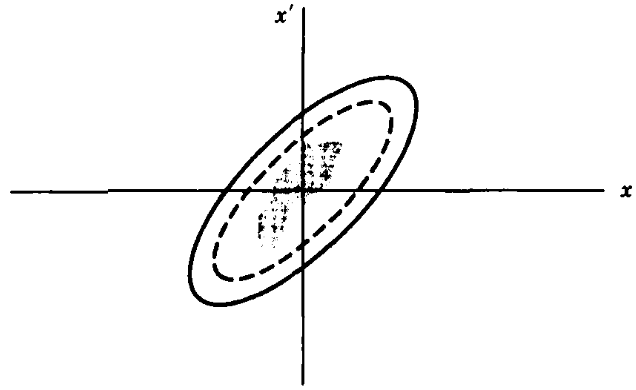

Figure 3.14. A beam with an irregularly shaped phase space distribution will conform to the elliptical phase space dictated by the optical properties of the accelerator, as a result of nonlinear forces inevitably present in a real machine.

initial phase space will tend to smear out and conform to the shape characteristic of the accelerator lattice as a result of field nonlinearities. In the static case that we are considering, the area in phase space remains constant (according to Liouville's theorem) but is so distorted that the area has increased in effect. This process of phase space dilution is called filamentation, and is avoided, insofar as is possible, by matching the injected beam shape to that of the invariant contours provided by the accelerator lattice. Emittance dilution will be examined in Chapter 7.

To summarize, then, if a beam in a synchrotron has emittance ϵ, then the phase space area is bounded by a curve

$$\frac{\epsilon}{\pi} = \gamma x^2 + 2\alpha x x' + \beta x'^2. \tag{3.96}$$

It is often convenient to speak of the emittance for a particular particle distribution in terms of the rms transverse beam size. As an example, we will consider a beam in a synchrotron in which the particles follow a Gaussian distribution in one transverse degree of freedom. This is the natural choice for electron storage rings, since synchrotron radiation ensures that the equilibrium distributions are Gaussian provided that particle losses are insignificant. The Gaussian distribution function is also a reasonable approximation for proton beams as well, in most instances.

Suppose that the distribution in the transverse coordinate x normalized to one particle is given by the density function

$$n(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}\,dx, \tag{3.97}$$

and that the distribution is stationary in time at a particular location s. That is, the beam is in an "equilibrium" situation where the distribution is indistinguishable from turn to turn. Since trajectories in x-$(\alpha x + \beta x')$ phase space are circles, as can be seen by examining Equations 3.88 and 3.89, then in this equilibrium situation the distribution in the coordinate $\alpha x + \beta x'$ will also be Gaussian with standard deviation σ. For, after all, the population on a given circle just rotates through an angle corresponding to the phase advance from turn to turn, and so the equilibrium distribution is independant of position along a circle, and depends only on the radius of the circle. The two-dimensional phase space distribution in these coordinates will be

$$n(x, \alpha x + \beta x')\,dx\,d(\alpha x + \beta x') = \frac{1}{2\pi\sigma^2} e^{-[x^2+(\alpha x+\beta x')^2]/(2\sigma)^2}\,dx\,d(\alpha x + \beta x'). \tag{3.98}$$

We switch to polar coordinates, where the radial coordinate is

$$r^2 = x^2 + (\alpha x + \beta x')^2, \tag{3.99}$$

and then the distribution is

$$n(r,\theta)\,r\,dr\,d\theta = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2} r\,dr\,d\theta. \tag{3.100}$$

If we define a radius a within which a fraction F of the particles are contained, then

$$F = \int_0^{2\pi}\int_0^a n r\,dr\,d\theta = \int_0^a e^{-r^2/2\sigma^2}\frac{r\,dr}{\sigma^2}, \tag{3.101}$$

and, solving for a,

$$a^2 = -2\sigma^2 \ln(1 - F). \tag{3.102}$$

Multiplying Equation 3.96 by β, we see that

$$\beta\epsilon/\pi = x^2 + (\alpha x + \beta x')^2. \tag{3.103}$$

If this emittance is the area in x-x' phase space that contains the fraction F of the particles, then $\beta\epsilon/\pi = a^2$. Thus,

$$\pi a^2 = \beta\epsilon = -2\pi\sigma^2 \ln(1 - F), \tag{3.104}$$

or

$$\epsilon = -\frac{2\pi\sigma^2}{\beta}\ln(1 - F). \tag{3.105}$$

Table 3.1 The fraction F of a Gaussian beam associated with various commonly used definitions of the emittance

ϵ	F (%)
σ^2/β	15
$\pi\sigma^2/\beta$	39
$4\pi\sigma^2/\beta$	87
$6\pi\sigma^2/\beta$	95

Equation 3.105 gives the area in x-x' phase space which contains the fraction F of a Gaussian beam with transverse rms beam size σ at a point in the lattice where the amplitude function is β. Various authors and institutions make different choices for the fraction F, and so there is not a standard significance to the numbers quoted for emittance. Table 3.1 lists a number of common definitions of emittance and their associated fractions F. The first entry is the near-standard choice in the electron accelerator community, where the quantity of primary interest is the rms beam size. The third and fourth entries tend to be used in proton accelerators in circumstances where one wishes to characterize the total beam size. For the remainder of the text we will primarily use the second entry, as it combines the significance of rms quantities with the emphasis on emittance as a phase space area.

The two most frequently used relations are those that give the maximum displacement and angle anywhere around the ring:

$$x_{\max} = \sqrt{\frac{\epsilon\beta_{\max}}{\pi}} \tag{3.106}$$

and

$$x'_{\max} = \sqrt{\frac{\epsilon\gamma_{\max}}{\pi}} \,. \tag{3.107}$$

Of course, the total number of particles contained within these maxima will depend on the choice of F in the relations above.

3.2.5 Adiabatic Damping of Betatron Oscillations

In the previous discussion we considered only the motion of particles with constant total momentum. We wish now to show how the amplitude of the motion varies as a function of the particle momentum, assuming that the momentum is a slowly changing function of time, or equivalently, of longitudinal position in the case of a linear accelerator. To illustrate the principle but avoid complexity, we begin with the equation of motion for a charged particle in the presence of a magnetic field of the form $\vec{B} = \hat{y}B'x$:

$$\frac{d}{dt}(p_x) = (e\vec{v} \times \vec{B})_x = -ev_sB_y = -evB'x. \tag{3.108}$$

Noting that $p_x = px'$,

$$v\frac{d}{ds}(px') = v(px'' + p'x') = -evB'x, \tag{3.109}$$

or

$$x'' + \frac{p'}{p}x' + \frac{eB'}{p}x = 0. \tag{3.110}$$

This is just Hill's equation with an added damping term.

To solve the above differential equation, we assume the solution to be of the form $x = uv$. Then

$$vu'' + \left(2v' + \frac{p'}{p}v\right)u' + \left(v'' + \frac{p'}{p}v' + \frac{eB'}{p}v\right)u = 0. \tag{3.111}$$

We now choose the function v such that the u' term is zero. That is,

$$2\frac{v'}{v} = -\frac{p'}{p}, \tag{3.112}$$

which says that v is of the form

$$v = v_0\left(\frac{p_0}{p}\right)^{1/2}. \tag{3.113}$$

Since the momentum is changing slowly, the v'' and $p'v'$ terms may be neglected. To see this, consider the coefficient of u in the differential equation above. With the form of v just obtained, this coefficient is

$$v_0\left(\frac{p_0}{p}\right)^{1/2}\left[\frac{eB'}{p} - \frac{1}{4}\left(\frac{p'}{p}\right)^2 - \frac{1}{2}\frac{p''}{p}\right]. \tag{3.114}$$

Since p is changing slowly, p'' is negligible with respect to the p'^2 term. Now the second term is on the order of

$$\left(\frac{p'}{p}\right)^2 = \left(\frac{\Delta p}{2\pi R}\frac{1}{p}\right)^2 = \frac{1}{4\pi^2R^2}\left(\frac{\Delta p}{p}\right)^2, \tag{3.115}$$

where Δp, the momentum increase per passage of the accelerating stations, is typically very much smaller than p. Since this term is much smaller than the centripetal term $\sim 1/\rho^2$ which we have already neglected from the outset of this discussion, the differential equation reduces to

$$\left(u'' + \frac{eB'}{p}u\right)v = 0, \tag{3.116}$$

or,

$$u'' + \frac{eB'}{p}u = 0, \tag{3.117}$$

which is Hill's equation. Therefore, the complete solution is of the form

$$x = uv = A_0\left(\frac{p_0}{p}\right)^{1/2}\beta^{1/2}(s)\cos[\psi(s) + \delta]. \tag{3.118}$$

The amplitude of the betatron oscillation is thus damped as the energy of the beam is adiabatically increased. Since the beam emittance is defined as a phase space area bounded by a Courant-Snyder invariant curve, and since this area is proportional to the square of the betatron amplitude, as shown earlier, we see that the beam emittance varies inversely with the beam momentum. The use of a *normalized* emittance,

$$\epsilon_N \equiv \epsilon \times \left(\gamma\frac{v}{c}\right), \tag{3.119}$$

permits comparisons of phase space areas independent of kinematic factors. This normalized emittance should, in the ideal world, remain constant throughout the entire acceleration process. The fact that it does not is exemplied in Chapters 7 and 8.

3.3 MOMENTUM DISPERSION

We have just examined the motion of particles having the same momentum as the ideal particle but differing transverse position and direction. Now we want to study the motion of particles differing in momentum from that of the ideal particle. The source of such differences is the bend fields that establish the ideal trajectory. This is a moot point for the case of a linear accelerator, but for circular accelerators it is a primary design consideration. We will find that these *off-momentum* particles in general undergo betatron oscillations about a new class of closed orbits in circular accelerators. The displacement of these closed orbits from that of the ideal particle will be described by a new lattice function—the momentum dispersion function.

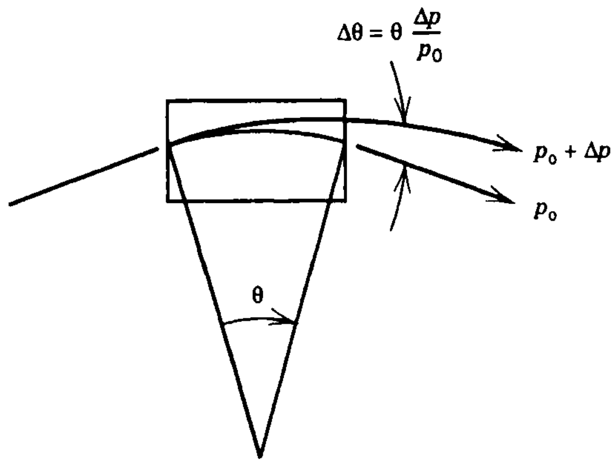

Figure 3.15. A bending magnet deflects particles of momentum higher than that of the ideal particle through a lesser angle, leading to a variety of closed orbits for particles of differing momenta.

The momentum dispersion function has its origin in the simple fact that a particle of higher momentum is deflected through a lesser angle in a bending magnet, as illustrated in Figure 3.15. It is as though a bending magnet which deflects the ideal particle by θ is the source of an angular perturbation $\theta\,\Delta p/p$ to the trajectory of an off-momentum particle entering on the design orbit, where $\Delta p/p$ is the fractional momentum difference relative to the ideal momentum.

In addition, higher momentum particles are bent less effectively in the focusing elements. That is, there is an effect completely analogous to chromatic aberration in conventional optics. The dependence of focusing on momentum will bring in its wake a dependence of betatron oscillation tune on momentum; the parameter quantying this relationship is called the chromaticity.

The compensation of chromaticity is accomplished by sextupole magnets, and we are led to the introduction of nonlinear elements into our heretofore linear focusing structure.

3.3.1 Equation of Motion for an Off-Momentum Particle

Since all of the treatment of the previous two sections was based on particles having the same momentum, we must begin again with the equation of motion to find the orbits for particles of differing momenta. We start from Equation 3.42, where we had

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2. \tag{3.120}$$

Recalling that the magnetic rigidity is the momentum per unit charge, we will let $(B\rho)$ represent the magnetic rigidity of the ideal particle with momentum p_0 and include a factor of p_0/p on the right hand side of the above equation of motion. That is,

$$x'' - \frac{\rho + x}{\rho^2} = -\frac{B_y}{(B\rho)}\left(1 + \frac{x}{\rho}\right)^2 \frac{p_0}{p}. \tag{3.121}$$

Furthermore, we still consider fields which vary linearly with transverse position, i.e.,

$$B_y = B_0 + B'x, \tag{3.122}$$

and expand the above equation of motion, neglecting terms quadratic in x/ρ and higher. We get

$$x'' + \left\{\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{(B\rho)}\frac{p_0}{p}\right\}x = \frac{1}{\rho}\frac{\Delta p}{p}, \tag{3.123}$$

where $\Delta p \equiv p - p_0$.

Let us write the closed orbit of an off-momentum particle in the form

$$x = D(p, s)\frac{\Delta p}{p_0} \tag{3.124}$$

and then look for the closed solution. That is, we look for the solution subject to the condition

$$D(p, s + C) = D(p, s), \tag{3.125}$$

where, as before, C is the repeat distance of the hardware. This function is referred to as the dispersion function. Since Equation 3.124 is a solution to the inhomogeneous Hill's equation, the general solution will differ from this particular (closed) solution by the addition of a solution of the homogeneous equation.

The equation to be solved for D is thus

$$D'' + \left\{\frac{1}{\rho^2}\frac{2p_0 - p}{p} + \frac{B'}{(B\rho)}\frac{p_0}{p}\right\}D = \frac{1}{\rho}\frac{p_0}{p}. \tag{3.126}$$

This is an inhomogeneous Hill's equation. The right hand side indicates that bending is the source of momentum dispersion. A bend center increments the slope of the dispersion function by the bending angle at the momentum in question, just as suggested by Figure 3.15. Since betatron oscillation wavelengths are apt to be long compared to the lengths of bending magnets, it is a good approximation to say that such a magnet changes the slope of the dispersion function by the magnet's bend angle at its bend center. (For a pure dipole magnet, this is an exact statement.)

3.3.2 Solution of Equation of Motion

We can proceed in the same fashion as in the "piecewise" method used for the betatron oscillations. Our equation is

$$D'' + K(s)D = \frac{1}{\rho}\frac{p_0}{p} = \frac{eB_0(s)}{p}, \tag{3.127}$$

where K is taken to be constant in each element of the lattice. Let us also assume B_0 is a constant within each element.

We can adapt the matrix approach used in the discussion of betatron oscillations to the present case. Since a particular solution to the dispersion function equation is a constant in each element, it may be readily shown that the general solution may be written as

$$\begin{pmatrix} D \\ D' \end{pmatrix}_{\text{out}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} D \\ D' \end{pmatrix}_{\text{in}} + \begin{pmatrix} e \\ f \end{pmatrix}, \tag{3.128}$$

where the two-by-two matrix is the same as that for the treatment of betatron oscillations (the homogeneous solution). By the addition of a third trivial equation, $1 = 1$, the equation above can be expressed in terms of a three-by-three matrix:

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{\text{out}} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}_{\text{in}}. \tag{3.129}$$

Table 3.2 shows the values of the matrix elements e and f for the cases where $K < 0$, $K = 0$, and $K > 0$.

By multiplying the various matrices for the pieces of the ring, we can find a matrix M for one turn (or, for one repeat period). The condition for the displaced equilibrium orbit to be closed is

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = M\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix}, \tag{3.130}$$

and its solution yields the dispersion function at the starting point.

Table 3.2 Values of the matrix elements M_{13} and M_{23} for various ranges of the spring constant K

K	e	f
< 0	$\frac{e}{p\|K\|} B_0(\cosh\sqrt{\|K\|}\,l - 1)$	$\frac{e}{p\sqrt{\|k\|}} B_0 \sinh\sqrt{\|K\|}\,l$
0	$\frac{1}{2}\frac{eB_0 l}{p} l$	$\frac{eB_0 l}{p}$
> 0	$\frac{e}{pK} B_0(1 - \cos\sqrt{K}\,l)$	$\frac{e}{p\sqrt{K}} B_0 \sin\sqrt{K}\,l$

Note that the same 3×3 matrices propagate either the dispersion function or the trajectory itself. That is, M operates on the vector

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} \tag{3.131}$$

or on the vector

$$\begin{pmatrix} x \\ x' \\ \dfrac{\Delta p}{p_0} \end{pmatrix}. \tag{3.132}$$

The same procedure can be carried out starting at any point in the lattice, and so the dispersion function can be computed everywhere. Alternatively, having found the dispersion function and its derivative at a particular starting point, this solution can be propagated forward using either the differential equation or the matrices describing individual elements.

In simple situations, the dispersion function will be everywhere positive; that is, orbits of higher momenta than the design orbit are at larger radius. The difference in perimeter between off-momentum orbits and the design orbit is characterized by the *compaction factor*, which is unfortunately designated by the overworked symbol α and is defined by the relation

$$\frac{\Delta C}{C} = \alpha \frac{\Delta p}{p_0}, \tag{3.133}$$

where C is the accelerator circumference. The form of this equation suggests that the name stems from the circumstance that this parameter is less than unity. That is indeed the case.

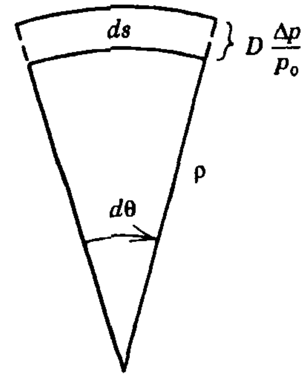

Figure 3.16. Illustration of increment in path length difference between off-momentum particle and ideal particle.

To illustrate the calculation of the compaction factor, let us suppose that the bending is provided by sector magnets. A *sector magnet* is one in which the ideal orbit enters and exits at right angles to the magnet ends. The procedure for calculating the difference in path between a particle of the central momentum and an off-momentum particle is sketched in Figure 3.16. The circumference change, ΔC, is given by

$$\Delta C = \oint \left(\rho + D \frac{\Delta p}{p_0} \right) d\theta - \oint (\rho \, d\theta). \tag{3.134}$$

For the fractional circumference change we therefore have

$$\frac{\Delta C}{C} = \frac{\oint D \, ds/\rho}{\oint ds} \frac{\Delta p}{p_0} = \left\langle \frac{D}{\rho} \right\rangle \frac{\Delta p}{p_0}, \tag{3.135}$$

or

$$\alpha \equiv \frac{1}{\gamma_t^2} = \left\langle \frac{D}{\rho} \right\rangle. \tag{3.136}$$

For a simple lattice, $\gamma_t \approx$ tune, as is the subject of a problem at the end of this chapter. The tune of an alternating gradient lattice scales with the number of cells, and so γ_t increases at the designer's discretion with the size of the accelerator. So again we have a circumstance where an aperture requirement—in this case, the aperture set aside to accommodate the momentum spread—is decoupled from the overall scale of the accelerator. This is another major advantage of the strong focusing principle.

We can build on our discussion of emittance to characterize the total beam size due to both betatron oscillations and momentum spread. For, we can write the displacement from the ideal trajectory of a particle as the sum of two terms:

$$x = D\frac{\Delta p}{p_0} + x_\beta, \tag{3.137}$$

where the first term represents the contribution of the closed orbit of the off-momentum particle and the second the free oscillation about that closed orbit. Averaging the square of this expression yields for the rms displacement

$$\sigma^2 = \frac{\epsilon\beta}{\pi} + D^2\left\langle\left(\frac{\Delta p}{p_0}\right)^2\right\rangle, \tag{3.138}$$

and we leave it to the reader to determine which of the various definitions of transverse emittance has been used to arrive at this last expression.

3.4 LINEAR DEVIATIONS FROM THE IDEAL LATTICE

We have now brought to an end our discussion of the linear transverse dynamics of the ideal accelerator. Thus far we have established the basic principles underlying single particle accelerator design. This is the framework upon which we will build in future chapters. We conclude this chapter with an introductory treatment of linear deviations from the ideal lattice, including steering and tune errors and adjustments.

3.4.1 Steering Errors and Corrections

Thus far we have described an accelerator where each magnetic or electric element performs exactly its prescribed task. In particular, for a circular accelerator, there is by design an ideal reference orbit which closes on itself and about which betatron oscillations would occur for particles with nonzero emittance. Suppose, however, that a particular bending magnet has a field somewhat different from its intended value. We will show that the field imperfection leads to a new closed orbit near, but differing from, the ideal design orbit.

Suppose that in our otherwise ideal circular accelerator a single steering error of magnitude $\theta = \Delta B l/(B\rho)$ is located at $s = 0$, where ΔB is an unintentional uniform field over the path length l. We want to find the new closed orbit, for x identically zero is no longer a solution of the equation of motion. Everywhere except at the location of the steering error, however, the equation of motion is still that of a betatron oscillation—only at the point of the angular impulse is the equation inhomogeneous.

Let the orbit immediately downstream of the deflection θ be specied by x_0, x_0'. To propagate this initial condition around the ring, we just multiply by the single-turn matrix M. Now we are immediately upstream of the deflection; to close the orbit we need only add the angle θ and demand that we be back to x_0, x_0'. In symbols,

$$M\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \theta \end{pmatrix} = \begin{pmatrix} x_0 \\ x_0' \end{pmatrix}. \tag{3.139}$$

Solving for x_0 and x_0',

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M)^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix}. \tag{3.140}$$

The matrix $(I - M)^{-1}$ can be recast using Equation 3.76. Then

$$(I - M)^{-1} = (I - e^{J2\pi\nu})^{-1} = \left[e^{J\pi\nu}(e^{-J\pi\nu} - e^{J\pi\nu})\right]^{-1} \tag{4.141}$$

$$= -(2J\sin\pi\nu)^{-1}(e^{J\pi\nu})^{-1} \tag{3.142}$$

$$= \frac{1}{2\sin\pi\nu}Je^{-J\pi\nu} \tag{3.143}$$

$$= \frac{1}{2\sin\pi\nu}(J\cos\pi\nu + I\sin\pi\nu), \tag{4.144}$$

and the closed orbit at $s = 0$ will be

$$\begin{pmatrix} x_0 \\ x_0' \end{pmatrix} = (I - M)^{-1}\begin{pmatrix} 0 \\ \theta \end{pmatrix} = \frac{\theta}{2\sin\pi\nu}\begin{pmatrix} \beta_0\cos\pi\nu \\ \sin\pi\nu - \alpha_0\cos\pi\nu \end{pmatrix}. \tag{3.145}$$

The closed orbit may now be expressed as a function of position or phase around the ring by, for instance, applying the matrix for propagation between one point and another (Equation 3.85). After a small bit of algebra, one finds

$$x(s) = \frac{\theta\beta^{1/2}(s)\beta_0^{1/2}}{2\sin\pi\nu}\cos[\psi(s) - \pi\nu] \tag{3.146}$$

for $0 < \psi < 2\pi\nu$.

The solution for the closed orbit is sketched in Figure 3.17. We see that the new closed orbit in the presence of a single steering error exhibits a *cusp* at the location of the error, where the trajectory experiences a *kink* through an angle θ. A local maximum in the oscillation occurs at the point in the accelerator where $\psi = \pi\nu$, i.e., half way around the accelerator for typical lattice designs. Particles whose initial conditions do not coincide with this closed orbit will still be influenced by the steering error each turn and will, in fact, perform betatron oscillations about the new closed orbit.

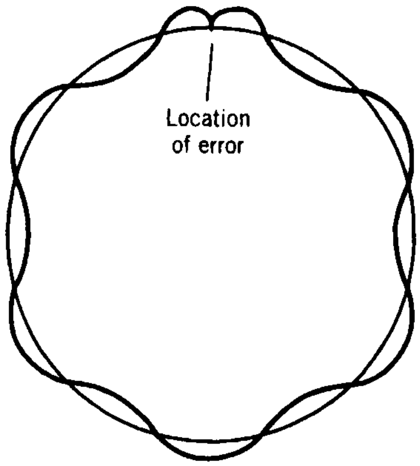

Figure 3.17. Sketch of closed orbit in presence of a single steering error.

In reality there will be as many steering errors as there are magnetic components in the accelerator. Let us continue by looking at the case of a simple FODO lattice in a synchrotron. Ideally, bending is provided by dipole magnets having a constant magnetic field in their transverse coordinates, while focusing is provided by quadrupole lenses. Most present day high energy synchrotrons employ this *separated function* lattice or a closely related variant. The dipole magnets will produce steering errors in both transverse degrees of freedom due to construction errors as well as installation and alignment errors. Variations in the total vertical component of the field as seen along the ideal trajectory will cause deflections in the horizontal plane as the particle passes through the magnet. Likewise, horizontal field components, such as those present when a dipole magnet is rotated slightly about its longitudinal axis, will cause vertical steering errors. A transverse alignment error of a quadrupole magnet will place a transverse field component on the ideal trajectory and so also contribute to the sources of steering errors.

As a numerical example, consider a quadrupole magnet which is displaced from the ideal trajectory by an amount δ in one of the transverse coordinates. Then, as a particle passes through this element, the slope of its trajectory will change by an amount $\Delta x' = \delta/F \equiv \theta$, where F is the focal length of the quadrupole. Take the case of the Tevatron, where the standard quadrupole focal length is 25 m. Then, an alignment error of 0.5 mm will generate an angular steering error of $\theta = 20$ μrad. If the error is at a focusing quadrupole where the amplitude function is 100 m, and one takes the tune to be about 19.4, then Equation 3.146 yields for a maximum closed orbit distortion at the focusing quadrupoles a value of

$$\Delta \hat{x} = \left| \frac{(20\ \mu\text{rad})(100\ \text{m})}{2 \sin \pi(19.4)} \right| = 1\ \text{mm}.$$

This is a linear problem, so we may use the superposition principle to combine the effects of many such errors. If there are N errors in the accelerator located at equivalent values of the amplitude function, we would expect the rms value of the closed orbit distortion to be larger than the preceding figure by a factor of order $N^{1/2}$ with the placement error δ also reinterpreted as an rms value. In the Tevatron, $N \approx 100$, so we come to an estimate of 10 mm for the rms orbit distortion due to quadrupole placement errors, and we would expect peaks larger than the rms by some factor on the order of 3 or 4, depending upon the distribution. As a consequence, we need some means of correcting steering errors as a basic design feature in synchrotrons of this scale.

Correction of closed orbit distortions can be carried out with the aid of a set of independently powered steering dipole magnets. The same set of steering dipoles can be used to make intentional closed orbit distortions to facilitate a variety of accelerator functions. More rigorous and quantitative calculations of the above are the subject of some of the problems at the end of the chapter.

As a final note, referring to Equation 3.146, we observe that there is no closed orbit if the tune is an integer. This is the most elementary example of a resonance. Of course, we didn't need to go through any algebra to find that out. If the tune were an integer, the steering errors would just reinforce from turn to turn until the oscillation amplitude became large enough to strike the walls of the vacuum chamber. The implication in the formula that the orbit goes to infinity is just an artifact of our approximations. But since infinity is only a few centimeters away, the approximations are pretty good.

3.4.2 Focusing Errors and Corrections

A gradient error would be expected to alter the tune of a circular accelerator. Let there be a single gradient error equivalent to a thin lens quadrupole with focal length f. The matrix M for a single turn is then

$$M = M_0 \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix}, \tag{3.147}$$

where M_0 is the matrix for the ideal ring. From the trace of M it follows that

$$\cos 2\pi\nu = \cos 2\pi\nu_0 - \frac{1}{2}\frac{\beta_0}{f}\sin 2\pi\nu_0, \tag{3.148}$$

where ν and ν_0 are the new and old tunes respectively, and β_0 is the original amplitude function at the point of the perturbation. For the ideal ring, presumably ν_0 is real by design. But depending on the sign and magnitude of the gradient error term, ν can become complex; that is, the motion can become unstable. Since, for small magnitudes of the gradient error term, the

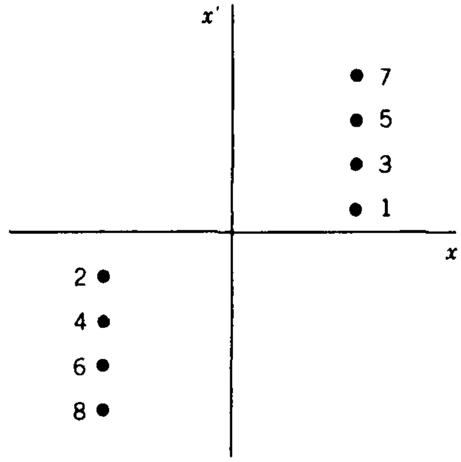

Figure 3.18. Phase space development of particle trajectory in presence of half-integer resonance.

instability will occur for ν near an integer or half integer, these instabilities are called half-integer resonances. There will be a range of values of ν_0 for which the motion is unstable; this range is called a *stopband*.

Just as in the case of dipole error resonances, we didn't need to use any algebra to demonstrate that quadrupole errors can produce resonance effects. Figure 3.18 represents the phase space history of a particle on successive turns as it passes the gradient error. The initial motion, in the absence of the error, was one in which the tune was an odd multiple of one-half. Successive passages of the gradient error just add constant vectors parallel to the vertical axis.

If the tune is not near a half-integer and the perturbation is sufficiently small, we can obtain a useful expression for the tune shift due to a gradient error by writing

$$\nu = \nu_0 + \delta\nu \tag{3.149}$$

and expanding the cosine term on the left hand side of the last equation. The result is

$$\delta\nu = \frac{1}{4\pi}\frac{\beta_0}{f}. \tag{3.150}$$

If there is a distribution of gradient errors, this last result generalizes to

$$\delta\nu = \frac{1}{4\pi}\sum_i \frac{\beta_i}{f_i} \rightarrow \frac{1}{4\pi}\oint \frac{\beta(s)B'(s)}{(B\rho)}\,ds \tag{3.151}$$

and is the lowest order (in gradient error) approximation to the tune shift.

In analogy with steering errors and corrections, one can make adjustments to the tunes of the accelerator by intentionally introducing perturbations on the gradients. The capability to adjust the tune is essential to modern high energy synchrotron operation. We will expand on this remark later in the book.

A gradient puts a kink in the amplitude function analogous to the deflection of the orbit produced by a dipole field. Let us compare the slope of the amplitude function on either side of the gradient error, or, equivalently, the parameter α, by calculating

$$\Delta M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} M_0 - M_0 \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}, \tag{3.152}$$

from which

$$\Delta\alpha = \frac{\beta}{f}, \tag{3.153}$$

where, in this case, β is the amplitude function of the perturbed lattice at the location of the error. Given the change in slope of the amplitude function produced by a gradient error, one can go on and calculate the deviation in the amplitude function throughout the ring. The principle of superposition is not valid for amplitude function perturbation due to combinations of gradient errors even in the linear lattice, in contrast to the situation for steering errors; but for sufficiently small gradient error, the treatments and results are quite similar. These matters are to be found in the problems at the end of the chapter.

3.4.3 Chromaticity

As mentioned during our treatment of the dispersion function, the equation of motion for an off-momentum particle includes a "spring constant" term which depends upon the particle momentum. One would thus suspect that the frequency of betatron oscillations of a particle, that is, the tune, would depend also upon its momentum. The change in tune due to momentum is called the chromaticity and is defined by

$$\delta\nu = \xi(p)\frac{\Delta p}{p_0}, \tag{3.154}$$

where $\Delta p/p_0$ is the momentum deviation relative to the ideal momentum.

The chromaticity can be calculated by Equation 3.150 relating tune shifts to gradient errors. Recall that we had

$$\delta\nu = \frac{1}{4\pi}\sum\frac{\beta_i}{f_i}, \tag{3.155}$$

where the f_i are to be identied as the focal lengths of the "error" quadrupoles that represent the difference between the off-momentum and central momentum states. That is, for a quadrupole of focal length F,

$$\frac{1}{f} = \frac{p_0}{p}\frac{1}{F} - \frac{1}{F} = -\frac{1}{F}\frac{\Delta p}{p}, \tag{3.156}$$

and so

$$\xi = -\frac{1}{4\pi}\sum \frac{\beta_i}{F_i}, \tag{3.157}$$

or, equivalently,

$$\xi = -\frac{1}{4\pi}\oint K\beta \, ds. \tag{3.158}$$

For a simple lattice, the chromaticity is about equal in magnitude and opposite in sign to the tune, as is to be demonstrated in one of the problems. In the more complex lattices of storage rings, the chromaticity is apt to be considerably larger in magnitude. This arises as follows. In order to focus the beams to a small spot at the collision point, one makes near-parallel beams on either side; that is, the amplitude function becomes large in quadrupoles which tend to be stronger than those elsewhere in the ring. The integrand in the last equation is therefore unusually large in such regions.

Another way of looking at the origin of the larger than normal chromaticity of storage rings is to make use of the differential equation satisfied by the amplitude function

$$K\beta = \alpha' + \gamma \tag{3.159}$$

to rewrite the expression for the chromaticity as

$$\xi = -\frac{1}{4\pi}\oint(\alpha' + \gamma)\, ds. \tag{3.160}$$

The first term vanishes when integrated around the ring. The term in γ will be large where the amplitude function is small, which is the situation at the collision point. Since γ is a constant in a drift space, the straight section in which collisions occur will contribute to the chromaticity a quantity equal to the straight section length divided by the amplitude function at the crossing for typical interaction region designs in which β is a minimum at the crossing point.

The source of chromaticity discussed here is the dependence of focusing strength on momentum for the ideal accelerator fields; the resulting chromaticity is called the *natural chromaticity*. There will be additional sources, as we shall see in Chapter 4.

Why worry about chromaticity? There are two reasons. If the beam has a large momentum spread, then a large chromaticity may place some portions of the beam on resonances. Secondly, the value of the chromaticity may determine whether or not certain intensity dependent motion is stable or unstable, as will be discussed in Chapter 6.

The capabilities for chromaticity adjustment provided by the linear lattice are limited; some other means must be used to modify this quantity. What is needed is a magnet that presents a gradient that is a function of momentum. A distribution of sextupole magnets is normally used for this purpose. In the horizontal plane, the sextupole field is of the form

$$B = kx^2, \tag{3.161}$$

and so the field gradient on a displaced equilibrium orbit is

$$B' = 2kx = 2kD\frac{\Delta p}{p_0}, \tag{3.162}$$

and the contribution to the chromaticity from sextupoles may be readily calculated.

A standard way of adjusting the chromaticity in both transverse degrees of freedom is to place a sextupole magnet at each main quadrupole location. The sextupoles are connected in two circuits; those at horizontally focusing quadrupoles are powered in one series circuit, and those at vertically focusing quadrupoles in the other. For the usual FODO lattice, the chromaticity changes due to the sextupoles would then be given by

$$\Delta\xi_H = \frac{N}{2\pi}(\beta_{\max}D_{\max}S_F + \beta_{\min}D_{\min}S_D), \tag{3.163}$$

$$\Delta\xi_V = -\frac{N}{2\pi}(\beta_{\min}D_{\max}S_F + \beta_{\max}D_{\min}S_D). \tag{3.164}$$

Here, N is the number of cells, and the sextupole strengths S_F and S_D are defined by $(\partial^2 B_y/\partial x^2)\cdot \text{length}/(2B\rho)$ evaluated at the focusing and defocusing quadrupole locations, respectively. Unfortunately, the sextupoles inevitably introduce intrinsically nonlinear aberrations, and we have taken our first step away from a design based upon purely linear dynamics.

PROBLEMS

1. In the "weak focusing" accelerators, the field index is defined as

$$n \equiv -\frac{dB/B}{dr/r}.$$

   (a) Show that the equation of motion of a particle in the vertical (axial) degree of freedom is

$$\frac{d^2z}{dt^2} + \omega^2 nz = 0,$$

   where ω is the angular rotation frequency. Thus, vertical oscillations are stable so long as $n > 0$.

   (b) If the design radius for the machine is R and a particle's radial coordinate is $r = R + x$, where $x \ll R$, show that the equation of motion in the radial (horizontal) degree of freedom for small oscillations is

$$\frac{d^2x}{dt^2} + \omega^2(1 - n)x = 0.$$

   Therefore, radial stability requires $n < 1$; stability in both transverse degrees of freedom simultaneously is assured only if $0 < n < 1$.

2. Consider a system made up of two thin lenses each of focal length f, one focusing and one defocusing, separated by a distance L. Show that the system is focusing if $|f| > L$.

3. Evaluate the matrix for a quadrupole of length 10 meters, with a gradient of 80 tesla/meter, and traversed by a particle with an energy of 20 TeV. Compare with a product of thin lens matrices occupying the same length.

4. Suppose that a particle traverses, first, a focusing lens with a focal length F; second, a drift of length L; third, a defocusing lens with focal length F; and, fourth, another drift of length L. Show that the matrix for this cell is given by

$$M = \begin{pmatrix} 1 - \dfrac{L}{F} - \left(\dfrac{L}{F}\right)^2 & 2L + \dfrac{L^2}{F} \\ -\dfrac{L}{F^2} & 1 + \dfrac{L}{F} \end{pmatrix}.$$

5. Consider a lattice made up entirely of thick gradient magnets, each of length L, alternating in "spring constant" between K and $-K$ (that is, assume that the $1/\rho^2$ term in the horizontal equation of motion is negligible). For what values of $\sqrt{K}L$ is the transverse motion in such a system stable?

6. Using a graphics terminal or home computer, and with $F = L$ in Problem 4 above, show that the motion is stable in transverse phase space. That is, start out particles with various values of x and x' and demonstrate that no divergence to large amplitude motion is indicated. Set $F = L/3$ and repeat; is the motion still stable?

7. Find the eigenvalues and eigenvectors for the FODO case. None is real—is that a problem?

8. Why is the transport matrix unimodular? Prove that it must be so under our assumptions. Suppose the particle energy changes—does the determinant still have to equal unity? If the matrix is

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

show (easily) that the inverse is

$$M^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

9. Suppose that one elects to write the matrix M for a periodic section as an exponential operator e^K. Using the properties of M as developed in Section 3.1.3, show that the trace of K is zero.

10. Going from point 1 to point 2, you traverse a sequence of elements that yield a matrix

$$M(1,2) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

From point 2 to point 3, you traverse the same elements but in reverse order. Show that the matrix from 2 to 3 is

$$M(2,3) = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

11. Show that the amplitude function is a solution of the linear differential equation

$$\beta''' + 4\beta' K + 2\beta K' = 0.$$

Within a lattice element, where K is constant, the solution must be one of the three forms

$$\beta = a + bs + cs^2$$
$$= a\cos 2\sqrt{K}s + b\sin 2\sqrt{K}s + c$$
$$= a\cosh 2\sqrt{|K|}s + b\sinh 2\sqrt{|K|}s + c.$$

In each case evaluate a, b, c in terms of α_0 and β_0, the parameters at the beginning of the element.

12. Show that the maximum and minimum values of the amplitude function for the simple FODO cell are given by

$$\beta_{\max} = 2F\left(\frac{1+\sin(\mu/2)}{1-\sin(\mu/2)}\right)^{1/2} = 2L\left(\frac{1+\sin(\mu/2)}{\sin\mu}\right),$$

$$\beta_{\min} = 2F\left(\frac{1-\sin(\mu/2)}{1+\sin(\mu/2)}\right)^{1/2} = 2L\left(\frac{1-\sin(\mu/2)}{\sin\mu}\right).$$

Evaluate these for a quadrupole spacing of 100 m and phase advance per cell of 80°.

13. Suppose that a particle traveling along the design orbit experiences an angular deflection θ. Show that thereafter its motion is given by

$$x = \theta\beta_0^{1/2}\beta^{1/2}(s)\sin\psi,$$

where β_0 is the amplitude function at the point of deflection, and phase ψ is measured relative to that point. Calculate the oscillation amplitude associated with a 100 μrad deflection at a maximum β point in the lattice of the preceding problem.

14. Given the Courant-Snyder parameters, or equivalently the J-matrix, at one point in the ring, they may readily be found at other points with the use of the appropriate transfer matrices. Suppose J_1 is the matrix representing a known set of parameters, and we want to find J_2. Let $M(1,2)$ be the matrix propagating the motion from point 1 to point 2. Show that the J-matrices at the two points are related by

$$J_2 = M(1,2)J_1M^{-1}(1,2).$$

Show that the parameter relations are

$$\beta_2 = m_{11}^2\beta_1 - 2m_{11}m_{12}\alpha_1 + m_{12}^2\gamma_1,$$
$$\alpha_2 = -m_{11}m_{21}\beta_1 + (m_{11}m_{22} + m_{12}m_{21})\alpha_1 - m_{12}m_{22}\gamma_1,$$
$$\gamma_2 = m_{21}^2\beta_1 - 2m_{21}m_{22}\alpha_1 + m_{22}^2\gamma_1,$$

where the m_{ij} are the matrix elements of $M(1,2)$. We have used the relation between the elements of $M(1,2)$ and the elements of the inverse in writing the above.

15. Show that the phase advance from point 1 to point 2 through a section described by the transport matrix $M(1,2)$ is given by

$$\Delta\psi = \tan^{-1}\left(\frac{m_{12}}{\beta_1 m_{11} - \alpha_1 m_{12}}\right)$$

where the m_{ij} are the matrix elements of $M(1,2)$ and β_1 and α_1 are the values of the amplitude functions at point 1.

16. For the simple thin lens FODO cell, verify that the phase advance as calculated by

$$\psi = \int \frac{ds}{\beta}$$

agrees with the result given by the trace of the matrix. Why is agreement assured in this case, while in general (for larger lattice segments) there may be ambiguities?

17. In order to achieve high luminosity in a colliding beam accelerator, the amplitude function is made small at the point where the beams are brought into collision. The length of the detector occupying this straight section will be large compared to the value of β at the interaction point. Show that the phase advance through this straight section will be approximately 180°.

18. Show that $J^2 = -I$. Show that n repetitions of

$$M = I\cos\mu + J\sin\mu$$

give the result akin to de Moivre's theorem,

$$M^n = I\cos n\mu + J\sin n\mu.$$

19. The discussion in this chapter has been explicitly carried out in the context of stable motion, where amplitude functions and phase advances are real numbers. Of course, the formalism works equally well for unstable motion. Consider the following case. Suppose one starts with a synchrotron with a one-turn matrix given by

$$M_0 = \begin{pmatrix} 0 & \beta_0 \\ -\dfrac{1}{\beta_0} & 0 \end{pmatrix}$$

and adds a thin lens quadrupole of focal length $f = 4\beta_0$. Evaluate the new amplitude function at this point and the new phase advance modulo 2π.

20. Suppose that a 10 GeV/c proton beam with normalized (39%) emittance of 2π mm mrad is injected into a synchrotron having a half cell length of 30 m and a cell phase advance of 68°. Estimate the boundaries of the beam excursions in displacement and angle.

21. The synchrotron into which the beam of the preceding problem is to be injected offers a half aperture of 50 mm in the horizontal plane. Calculate the admittance. Normally it is necessary that the admittance be much larger than the beam emittance for reasonable performance.

22. Suppose that there are many uncorrelated angular deflections distributed around a ring which average to zero and have an rms value θ_{rms}.

(a) Show that the rms (over an ensemble of synchrotrons) orbit distortion at some point of observation where the amplitude function is β_0 is

$$\langle x^2 \rangle^{1/2} = \frac{\beta_0^{1/2}}{2|\sin \pi\nu|}\left(\sum_i \theta_{\text{rms}}^2 \beta_i \sin^2(\pi\nu - \psi_i)\right)^{1/2}$$
$$= \frac{(\beta_0\bar{\beta})^{1/2}}{2\sqrt{2}\,|\sin \pi\nu|} N^{1/2}\theta_{\text{rms}},$$

where $\bar{\beta}$ is the average of the amplitude function at the N kick locations. In proceding from the first to the second form of the result, note that a term of order unity compared with N is neglected; for many purposes, this is an excellent approximation.

(b) Calculate the rms orbit distortion expected in a 20 TeV scale synchrotron, if the only source of the angular deviations is quadrupole alignment errors characterized by an rms value of 1 mm. For amplitude functions and focal lengths, use values characteristic of a thin lens FODO cell with a length of 180 m and a phase advance of 90°.

23. In Problem 22, only the rms orbit distortion was calculated. One would like to know the distribution of peak distortions, or, to put it more precisely, to know the odds of the maximum distortion exceeding the rms by a given factor. This is an easy job for a small computer. Generate an ensemble of synchrotrons, each populated with a set of steering errors having unit standard deviation and following a Gaussian distribution. Find the maximum orbit deviation for each synchrotron in units of the rms and plot the integral distribution. You should find, for example, that the probability of the peak distortion exceeding the rms value by a factor of two is 60%.

24. Orbits can be corrected and adjusted by using steering dipoles. One standard algorithm is based on so-called *three-bumps*. A local orbit distortion can be made by three steering dipoles. Let the three steering angles be θ_1, θ_2, and θ_3. Show that if these angles are related according to

$$\theta_2 = -\theta_1\left(\frac{\beta_1}{\beta_2}\right)^{1/2}\frac{\sin\psi_{13}}{\sin\psi_{23}},$$

$$\theta_3 = \theta_1\left(\frac{\beta_1}{\beta_3}\right)^{1/2}\frac{\sin\psi_{12}}{\sin\psi_{23}}$$

then the orbit distortion is localized between the first and third steering elements.

25. If a single quadrupole is hooked up backwards, so its focusing character is the reverse of what it should be, is it likely that the betatron oscillations will still be stable in both transverse planes? As usual, use a ring of simple FODO cells for this problem.

26. In the same spirit as Problem 22, derive an expression for the rms tune shift arising from uncorrelated gradient errors. Apply your expression to the case of a 20 TeV scale ring in which the main quadrupoles exhibit an rms fractional error in their focal lengths of 0.1%.

27. Suppose that a quadrupole of negligible length l and strength $q \equiv B'l/(B\rho)$ is placed in a ring at a point where the amplitude function has value β_1. Assume that upstream of this point the amplitude function is unperturbed. Show that downstream of the quadrupole the fractional deviation in β is given by

$$\frac{\Delta\beta}{\beta} \equiv \frac{\beta(s)-\beta_0(s)}{\beta_0(s)} = -q\beta_1\sin 2\psi_0 + \frac{1}{2}(q\beta_1)^2(1-\cos 2\psi_0)$$

where $\beta_0(s)$ is the original amplitude function, and the original phase ψ_0 is measured from the location of the quadrupole.

28. For sufficiently small quadrupole errors, the nonlinear term in the equation of the preceding problem can be neglected. Then the fractional deviation in the amplitude function obeys rules identical to the orbit distortions which arise from steering errors. Show that the rms fractional change in β associated with quadrupole errors is, in this approximation,

$$\left(\frac{\Delta\beta}{\beta}\right)_{\text{rms}} = \frac{1}{2\sqrt{2}\,|\sin 2\pi\nu|}\left\langle\sum_i q_i^2\beta_i^2\right\rangle^{1/2},$$

where the q's are defined as in the preceding problem.

29. Let the bending of a thin lens FODO cell be performed by pure dipole magnets providing a bend angle θ with the bend center at the midpoint of each half cell. Show that the maximum and minimum values of the dispersion function for a ring made entirely of such cells are

$$D_{\text{max, min}} = \frac{\theta L}{\sin^2 \frac{1}{2}\mu}\left(1 \pm \tfrac{1}{2}\sin\tfrac{1}{2}\mu\right),$$

where L is the half-cell length. For typical proton ring designs, the product θL tends to be about 1 meter. Since the phase advance per cell, μ, also is apt to be selected from a narrow range, the dispersion is about the same regardless of the peak energy of the synchrotron. It will be interesting to see if this scaling perseveres in the future.

30. Suppose that the basic repeat period of a synchrotron consists of $n - 1$ FODO cells of the sort used in the preceding problem, followed by a single FODO cell without bending. Show that in the bending cells the total dispersion is the sum of that appropriate to the FODO cell alone and a free oscillation having initial values (at the beginning of the superperiod)

$$\begin{pmatrix}\Delta D\\ \Delta D'\end{pmatrix} = \frac{(I - M^{-n})(M - I)}{2(1 - \cos n\mu)}\begin{pmatrix}D\\ D'\end{pmatrix}_{\text{FODO only}},$$

where M is the 2×2 matrix for propagation of a betatron oscillation through each cell. Note that the dispersion can now take on arbitrarily large positive or negative values. This is an example of a mismatched lattice function; the juxtaposition of cells with different intrinsic dispersion functions can lead to a possible unacceptable total dispersion.

31. Extend the phase-amplitude form of the transfer matrix to include momentum dispersion. The three additional elements needed are

$$\begin{aligned} m_{13} &= D_2 - m_{11}D_1 - m_{12}D'_1,\\ m_{23} &= -m_{21}D_1 - m_{22}D'_1 + D'_2,\\ m_{13} &= 1, \end{aligned}$$

where the m's on the right hand side are elements of the 2×2 matrix as written down in Section 3.2 for transport of a betatron oscillation from point 1 to point 2.

32. To illustrate the principle of matching of lattice functions, here is one method of proceeding from a bend (nonzero dispersion) region to a straight section without setting up a dispersion wave. Coming from the arc, the beam first encounters a FODO cell in which the bending is reduced to a fraction $1 - x$ of the standard cell deflection and then a cell in which the bending fraction is x. As before, all bend centers are at the midpoint of the half cell. Prove that the condition for the dispersion function and its derivative to be unchanged in the arc and vanish at the entry to the straight section is

$$x = \frac{1}{2(1 - \cos\mu)},$$

where μ is the phase advance of each of the FODO cells making up the ring. It may be useful to use the matrix of the preceding problem, and decide what happens to m_{13} and m_{23} as the bend strength is changed.

33. For the simple FODO lattice synchrotron, verify that the transition gamma, γ_t, is about equal in magnitude to the tune.

34. A steering error of strength θ produces a new closed orbit in a synchrotron which may have a path length different from the ideal path length by an amount ΔC. Show that $\Delta C = \theta D$, where D is the value of the dispersion function at the location of the steering error.

35. For the simple FODO lattice synchrotron, verify that the chromaticity is about equal in magnitude and opposite in sign to the tune if the only elements contributing to the chromaticity are the main quadrupoles.

36. Assume that the bending magnets exhibit a systematic sextupole moment. The field can be written in the form

$$B = B_0(1 + b_2 x^2),$$

where $b_2 = B''/2B_0$ is the sextupole coefficient. Show that the associated contribution to the chromaticity is

$$\Delta\xi = \pm\langle D\beta b_2\rangle$$

with the sign depending on the degree of freedom. Under some circumstances, $\Delta\xi$ can be large. At low fields, superconducting magnets can have signicant sextupole moments due to persistent currents. A possible value for b_2 is 3 m^{-2}. Estimate the resulting chromaticity for a 20 TeV scale ring.

37. Derive the expressions for chromaticity adjustment written down in the last section.

38. Consider a *dogleg* as shown in the diagram below.

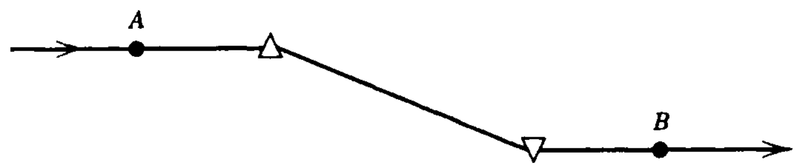

Figure for Problem 38.

(a) If the dispersion function at A is $D = D' = 0$, what is the value of the dispersion function at B? (Assume that $\theta \approx d/L \ll 1$.)

(b) At Fermilab, the 150 GeV Main Ring injector is located 25.5 in. above the Tevatron synchrotron. Beams are transferred between the two rings in a long straight section, free of quadrupoles. How much vertical dispersion is "generated" just by the transfer process?

(c) Since the design vertical dispersion function of the Tevatron is zero everywhere, there would be a mismatch of the dispersion function between the Main Ring and Tevatron if left uncorrected. Would you expect this mismatch to affect the emittance of the beam after injection? Why or why not?