CHAPTER 8

Synchrotron Radiation

Synchrotron radiation radiation is the dominant factor in the design of high energy electron synchrotrons and is the obstacle to exceeding 100 GeV or so in this type of accelerator. It has also brought about the spectacular success of synchrotron light sources. Only today is synchrotron radiation becoming a design consideration for proton synchrotrons. In the proton case, single particle motion, to a very good approximation, exemplifies a Hamiltonian system. Particle motion in electron synchrotrons, on the other hand, is inherently dissipative.

In this chapter, we look at some of the basic properties of the radiation process, the power, and the characteristic photon energy. The energy loss due to synchrotron radiation and its replacement by the RF acceleration system leads to a variation with time of the oscillation amplitudes in all three degrees of freedom, and we compute the related time constants. We will reproduce an elegant theorem—Robinson's theorem—which relates these three time constants and demonstrates that there is a net damping effect that can be apportioned among the degrees of freedom at the choice of the designer. An electron storage ring will be designed to damp in each degree of freedom.

The fact that the radiation process is quantized implies that there are statistical fluctuations in the radiation rate. These fluctuations cause excitation of synchrotron oscillations, and of betatron oscillations in at least one transverse degree of freedom. The interplay between the quantum fluctuations and damping will result for an ensemble of particles in an equilibrium beam distribution, which we will find to be Gaussian.

8.1 RADIATION FROM RELATIVISTIC PARTICLES

If a slowly moving particle of charge e undergoes an acceleration a, then the radiated power P is given by the Larmor formula:

$$P = \frac{1}{6\pi\epsilon_0}\frac{e^2a^2}{c^3}. \tag{8.1}$$

The angular distribution of the radiation varies as $\sin^2\theta$, where θ is the angle between the direction of the acceleration and the point of observation.

We can find the radiated power for relativistic charges by using the fact that radiated power is a Lorentz invariant. To arrive at the latter conclusion, we can argue as follows. Suppose a photon of angular frequency ω' is traveling at a direction θ with respect to the x'-axis in a "primed" frame that is moving parallel to the x-axis of the "unprimed" laboratory frame. Transformation to the unprimed frame gives

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}, \tag{8.2}$$

$$\omega = \gamma\omega'(1 + \beta\cos\theta'), \tag{8.3}$$

where γ is the Lorentz factor characterizing the relative motion of the two frames. If two photons are emitted at angles θ' and $\theta' + \pi$ with the same angular frequency ω', then in the laboratory frame the total energy will be proportional to

$$\omega_1 + \omega_2 = 2\gamma\omega'. \tag{8.4}$$

If the emission takes place in a short interval τ', then in terms of radiated power, the relation above can be written

$$P\tau = P'\tau'\gamma, \tag{8.5}$$

or $P = P'$, after recognition of the effect of time dilation. That is, the power that is lost to the Doppler shift in one direction is gained back in the other. So long as the angular distribution of radiation in the primed frame has the appropriate symmetry, we can conclude that the power is an invariant. Though not developed in all generality, this is enough for our purposes here.

Look at two cases: acceleration perpendicular to and parallel to the direction of motion of a relativistic charge. The first corresponds to a particle undergoing deflection in a bending magnet. In an inertial frame traveling at the speed of the particle and tangent to the orbit at the time of arrival of the particle, at that instant the particle will be at rest and undergoing acceleration in the $-y'$ direction. This situation is shown in Figure 8.1. In this

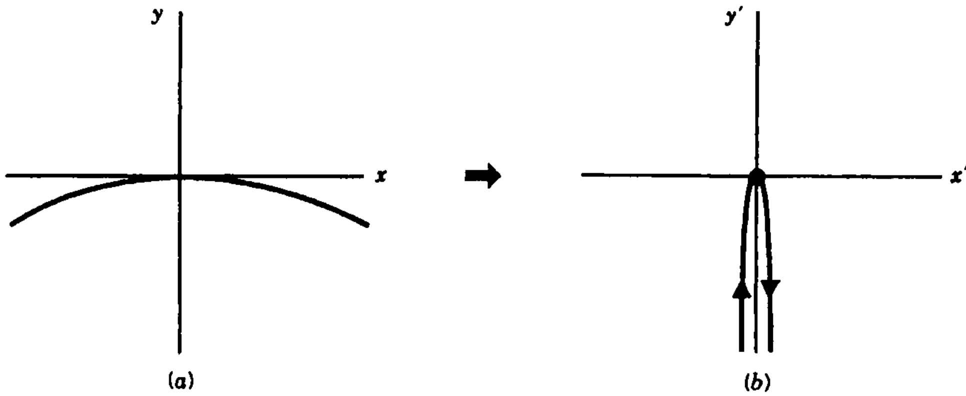

Figure 8.1. A particle undergoing circular motion in the laboratory frame (a) will come momentarily to rest when viewed in a frame moving tangent to its trajectory (b).

(primed) frame the power radiated is given by the Larmor expression, with a' inserted for the acceleration. For acceleration transverse to the relative direction of motion of the two frames, $a' = \gamma^2 a$. In the primed frame, the power distribution has the proper front-back symmetry needed for the argument of the preceding paragraph to be valid, so the power is an invariant. As a result, the power in the laboratory frame is

$$\begin{aligned} P &= \frac{1}{6\pi\epsilon_0}\frac{e^2a^2}{c^3}\gamma^4 \\ &= \frac{1}{6\pi\epsilon_0}\frac{e^2c}{\rho^2}\gamma^4 \\ &= \frac{1}{6\pi\epsilon_0}\frac{e^4}{m^4c^5}B^2E^2. \end{aligned} \tag{8.6}$$

In the second form, a has been replaced by the centripetal acceleration c^2/ρ of a relativistic electron. The third form will be useful when we discuss radiation damping; here, E is the total energy, $E = \gamma mc^2$, and B is the magnetic field producing the curvature of the particle's path.

In contrast, suppose that the acceleration is in the direction of motion of the charge. In the primed frame, the angular distribution just rotates by $\pi/2$, so our invariance-of-power argument is still all right. But now $a' = \gamma^3 a$, and so the power radiated in the laboratory frame is

$$P = \frac{1}{6\pi\epsilon_0}\frac{e^2a^2}{c^3}\gamma^6. \tag{8.7}$$

At first glance, this result looks even more ominous than the one for transverse acceleration. But such is not the case—acceleration in the direction of motion of a rapidly moving particle is not as easily produced as transverse acceleration. Equation 8.7 can be recast in the form

$$P = \frac{2}{3}\frac{r_0}{mc}\dot{p}^2, \tag{8.8}$$

where r_0 is the classical radius of the particle. Take one of the factors of $\dot{p}$, express it in terms of the rate of change of energy of the particle, $\dot{E}$, and compare P with $\dot{E}$. Write the other factor of $\dot{p}$ just as the force F. Then we have

$$\frac{P}{\dot{E}} = \frac{2}{3}\frac{r_0 F}{mc^2\beta}, \tag{8.9}$$

and for the ratio on the left to be significant, the particle must experience an energy gain or loss comparable with its rest energy within a distance equal to its classical radius. That is possible on the atomic or nuclear scale (e.g., bremsstrahlung), but not with laboratory sized accelerator components.

So we will be concerned only with radiation arising from transverse acceleration. The radiation loss per turn on the design orbit of a synchrotron will be

$$U_0 = \int_0^{2\pi R} P\,dz/c \tag{8.10}$$

$$= C_\gamma E^4 R \left\langle \frac{1}{\rho^2} \right\rangle, \tag{8.11}$$

where

$$C_\gamma = \frac{4\pi}{3}\frac{r_0}{(mc^2)^3} = 8.85 \times 10^{-5}\,\frac{\text{meters}}{\text{GeV}^3}, \tag{8.12}$$

and the square of the curvature $1/\rho$ is averaged over the circumference $2\pi R$ of the ring. The numerical coefficient has been evaluated for the electron, using $r_0 = 2.818 \times 10^{-15}$ m. The average power radiated is

$$\langle P \rangle = fU_0, \tag{8.13}$$

where f is the orbit frequency $c/2\pi R$.

8.2 DAMPING OF OSCILLATIONS

With the inclusion of synchrotron radiation, the transverse and longitudinal oscillations of a single particle no longer have invariant amplitudes, for the system is now dissipative. In this section we calculate the damping rates. Note that a characteristic time for synchrotron radiation effects is the time τ_0 in which an electron of energy E would radiate E, i.e.,

$$\tau_0 \equiv \frac{E}{\langle P \rangle}, \tag{8.14}$$

and the rates will be expressed in terms of τ_0.

Damping of vertical betatron oscillations is easy to understand. Synchrotron radiation reduces the momentum of a particle in the direction of its motion, while the acceleration system restores momentum parallel to the central orbit. Consider the case in which there is no net acceleration, as in beam storage. On the average, the two momentum increments are equal in magnitude. If, in an element of path ds, the particle radiates energy du and receives the same energy increment from the acceleration system, then the momenta before and after, $\vec{p}_1$ and $\vec{p}_2$, are related by

$$\vec{p}_2 = \vec{p}_1 - \frac{du}{c}\frac{\vec{p}_1}{|\vec{p}_1|} + \frac{du}{c}\hat{s}. \tag{8.15}$$

as seen in Figure 8.2. In terms of the transverse and longitudinal components,

$$p_{2y} = p_{1y} - \frac{du}{c}\frac{p_{1y}}{|\vec{p}_1|}, \tag{8.16}$$

$$p_{2s} = p_{1s} - \frac{du}{c}\frac{p_{1s}}{|\vec{p}_1|} + \frac{du}{c} \tag{8.17}$$

Division of the first by the second gives the relationship between $y' = p_y/p_s$ before and after traversing ds:

$$\begin{aligned} y_2' &= y_1' \frac{1 - du/E}{1 - du/E + du/(cp_s)} \\ &= y_1'(1 - du/E), \end{aligned} \tag{8.18}$$

where E is the total energy of the particle, and we have kept only the lowest order term in du/E in the second equation. Therefore, the equation of

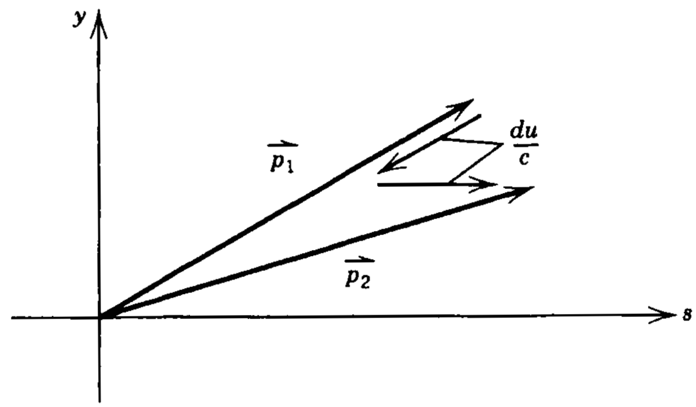

Figure 8.2. A particle undergoing a vertical betatron oscillation radiates in its direction of motion, and the momentum is restored in the direction of the design trajectory.

motion in y contains a term

$$y'' = -\frac{1}{E}\frac{du}{ds}y' \tag{8.19}$$

in addition to the focusing term proportional to y. For a damping rate slow compared to the betatron oscillation frequency, the free oscillation then is just modified by the multiplicative factor

$$\exp\left(-\frac{1}{2}\int\frac{1}{E}\frac{du}{ds}\,ds\right) = \exp\left(-\frac{t}{2\tau_0}\right), \tag{8.20}$$

and so the damping time constant is

$$\tau_y = 2\tau_0. \tag{8.21}$$

Identification of the damping time constant for synchrotron oscillations is almost as easy. Suppose the deviation from the synchronous energy of a particle is ΔE. In traversing an infinitesimal element of a turn, ΔE will change according to

$$\Delta E_2 = \Delta E_1 - du(\Delta E_1) + du(0), \tag{8.22}$$

where the second and third terms on the right are the energy loss due to synchrotron radiation at energy displaced from the synchronous energy ΔE_1 and the energy gain from the radiofrequency system at $\Delta E = 0$ respectively.

In terms of the radiated power,

$$du(\Delta E_1) = P(\Delta E_1)dt_1$$
$$= P(0)\left[1 + 2\frac{\Delta E_1}{E} + 2\frac{\Delta B}{B}\right]\left[1 + \frac{D}{\rho}\frac{\Delta E_1}{E}\right]dt_0, \quad (8.23)$$

$$du(0) = P(0)dt_0, \quad (8.24)$$

where dt_1 has been expressed in terms of the time element dt_0 on the synchronous orbit using Figure 3.16, and $P(\Delta E)$ has been written in terms of $P(0)$ using Equation 8.6. The ΔB in the first bracket of Equation 8.24 can be eliminated in favor of the fractional energy difference by the use of

$$\Delta B = B'x = B'D\frac{\Delta E}{E}. \quad (8.25)$$

Then to lowest order in the Δ's, the change in ΔE per turn due to synchrotron radiation becomes

$$\frac{d\Delta E}{dn} = -\Delta E\int_0^T \frac{P(0)}{E}\left[2 + \frac{D}{\rho} + 2D\frac{B'}{B}\right]dt_0. \quad (8.26)$$

Here, T is the period of the synchronous orbit, and as usual, we assume that changes in energy are small in that time scale. The first term in the integral is just $2U_0/E$, where, as before, U_0 is the energy radiated in one turn by the synchronous particle. If we change to time as the independent variable by multiplying both sides by the orbit frequency, f, we obtain

$$\frac{d\Delta E}{dt} = -\Delta E\left[2\frac{fU_0}{E} + \int_0^T \frac{fP(0)dt_0}{E}D\left(\frac{1}{\rho} + 2\frac{B'}{B}\right)\right]. \quad (8.27)$$

The quantity fU_0/E is $1/\tau_0$, where τ_0 is the characteristic time for radiation processes. If we take τ_0 outside of the brackets, then, after cancellation of various coefficients, the result is

$$\frac{d\Delta E}{dt} = -\frac{\Delta E}{\tau_0}(2 + \mathscr{D}), \quad (8.28)$$

$$\mathscr{D} \equiv \frac{\left\langle \frac{D}{\rho^2}\left(\frac{1}{\rho} + 2\frac{B'}{B}\right)\right\rangle}{\left\langle \frac{1}{\rho^2}\right\rangle}. \quad (8.29)$$

When this term is added to the equations of motion for a synchrotron oscillation, the solution for the motion will contain the factor

$$\exp\left[-\frac{1}{2\tau_0}(2+\mathscr{D})\right], \tag{8.30}$$

and so the time constant for damping in this degree of freedom is

$$\tau_s = \frac{2\tau_0}{2+\mathscr{D}}. \tag{8.31}$$

The analogous argument for the horizontal betatron oscillation is longer. It's more elegant to use *Robinson's theorem*, which deduces the sum of the damping rates for all three degrees of freedom.[1] Since we already know the results for two of the modes, the theorem gives us the third immediately.

The derivation goes as follows. Consider the transfer matrix of the six-vector $x, x', y, y', \phi, \Delta E$ through a path element ds. The diagonal elements for x' and y' will differ from unity by the quantity $-du/E$, as we have seen above. The diagonal element for ΔE will differ from unity by $-2\,du/E$. The only terms in the determinant of the matrix that are first order in ds come from the diagonal elements. So for this infinitesimal matrix

$$\det dM = 1 - 4\,du/E, \tag{8.32}$$

and since the determinant of a product of matrices is the product of their determinants, to lowest order for one revolution

$$\det M = 1 - 4\frac{U_0}{E}. \tag{8.33}$$

But the determinant is also the product of the eigenvalues. For oscillatory modes, the eigenvalues can be expressed as $\exp(\gamma_k)$. The six γ_k occur in conjugate pairs, so the imaginary parts do not contribute to the product. If we call the real parts α_x, α_y, and α_s, then

$$\alpha_x + \alpha_y + \alpha_s = -2\frac{U_0}{E}. \tag{8.34}$$

The α's are the decrements per turn; multiplication by the orbit frequency

[1] K. W. Robinson, "Radiation Effects in Circular Electron Accelerators," Phys. Rev. **111**, No. 2 (1958).

gives the result for the time constants:

$$\frac{1}{\tau_x} + \frac{1}{\tau_y} + \frac{1}{\tau_s} = \frac{2}{\tau_0}. \tag{8.35}$$

Therefore, using Robinson's theorem, we see that the damping time in the horizontal degree of freedom is

$$\tau_x = \frac{2}{1 - \mathscr{D}}\tau_0. \tag{8.36}$$

For a separated function synchrotron, where the focusing and bending are performed by separate elements, then $\mathscr{D}$ is small, and so for this case $\tau_x = 2\tau_0$ and all three degrees of freedom damp. This is, of course, the behavior one wishes for a storage ring.

8.3 QUANTUM FLUCTUATIONS AND EQUILIBRIUM BEAM SIZE

If the results of the preceding section were the end of the subject, we could design an electron storage ring in which all three degrees of freedom damped, and the emittances would shrink to zero. But such is not the case. The radiation process proceeds through the emission of discrete quanta, and the fluctuations in this random process produce an excitation of horizontal betatron oscillations and synchrotron oscillations.

To see how this excitation comes about, suppose that a particle is traveling along its synchronous orbit and emits a photon of energy w. The position of the particle doesn't change, so it suddenly finds itself starting a synchrotron oscillation with an initial energy offset $-w$ and a horizontal betatron oscillation with initial conditions $x = Dw/E$ and $x' = D'w/E$. Because of the random character of the photon emission, synchrotron radiation contributes a constant term to the growth of the horizontal and longitudinal emittances. This is just the situation we encountered in Chapter 7 when discussing emittance growth due to RF noise. Using that result, we expect the variance of the horizontal particle distribution to increase at the rate

$$\frac{d\sigma_x^2}{dt} = \tfrac{1}{2}Nf_0\langle \mathscr{H} \rangle \beta_x \frac{\langle w^2 \rangle}{E^2}, \tag{8.37}$$

where $\mathscr{H}$ is defined in Equation 7.106, β_x is the amplitude function, and N is the number of photon emissions per turn:

$$N = \frac{\langle P \rangle}{f_0 \langle w \rangle} = \frac{U_0}{\langle w \rangle}. \tag{8.38}$$

In an ideal planar synchrotron, where there is no vertical dispersion, $\mathscr{H}$ is zero in the vertical degree of freedom and so quantized photon emission does not stimulate vertical emittance growth.

The behavior of the rms energy spread due to quantum fluctuations follows from a similar argument, and so, with the inclusion of the damping terms, the equations of motion for the variances in the three degrees of freedom are:

$$\frac{d\sigma_x^2}{dt} = -\frac{2}{\tau_x}\sigma_x^2 + \tfrac{1}{2}Nf_0\langle\mathscr{H}\rangle\beta_x\frac{\langle w^2\rangle}{E^2}, \tag{8.39}$$

$$\frac{d\sigma_y^2}{dt} = -\frac{2}{\tau_y}\sigma_y^2, \tag{8.40}$$

$$\frac{d\sigma_E^2}{dt} = -\frac{2}{\tau_s}\sigma_E^2 + \tfrac{1}{2}Nf_0\langle w^2\rangle. \tag{8.41}$$

These equations are easily integrated to yield

$$\sigma_x^2(t) = \sigma_x^2(0)e^{-2t/\tau_x} + \tfrac{1}{4}Nf_0\langle\mathscr{H}\rangle\beta_x\tau_x\frac{\langle w^2\rangle}{E^2}(1 - e^{-2t/\tau_x}), \tag{8.42}$$

$$\sigma_y^2(t) = \sigma_y^2(0)e^{-2t/\tau_y}, \tag{8.43}$$

$$\left(\frac{\sigma_E}{E}\right)^2(t) = \left(\frac{\sigma_E}{E}\right)^2(0)e^{-2t/\tau_s} + \tfrac{1}{4}Nf_0\tau_s\frac{\langle w^2\rangle}{E^2}(1 - e^{-2t/\tau_s}). \tag{8.44}$$

We see that within a few radiation damping times (assuming that all three degrees of freedom are damped) equilibrium transverse emittances and energy spread are reached:

$$\epsilon_x \equiv \gamma\sigma_x^2/\beta_x \to \frac{1}{2}\left(\frac{\langle\mathscr{H}\rangle}{1-\mathscr{D}}\right)\frac{\langle w^2\rangle}{mc^2\langle w\rangle}, \tag{8.45}$$

$$\epsilon_y \equiv \gamma\sigma_y^2/\beta_y \to 0, \tag{8.46}$$

$$\sigma_E/E \to \left[\frac{1}{2}\left(\frac{1}{2+\mathscr{D}}\right)\frac{\langle w^2\rangle}{E\langle w\rangle}\right]^{1/2}. \tag{8.47}$$

Note it is common practice to quote emittances using $F = 15\%$ in Table 3.1 when discussing electron storage rings. The resulting equilibrium distributions will be Gaussian, as anticipated by our discussion of the central limit theorem in Chapter 7.

While $\langle\mathscr{H}\rangle$ and $\mathscr{D}$ are functions of the accelerator lattice, $\langle w\rangle$ and $\langle w^2\rangle$ are uniquely determined from the photon spectrum generated by synchrotron

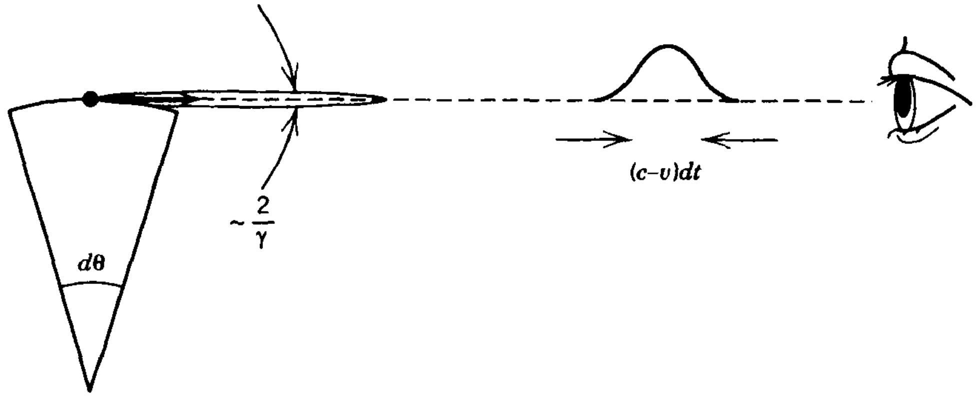

Figure 8.3. An observer sees a pulse of synchrotron radiation of time duration dt' as the cone of radiation sweeps out an angle $d\theta \approx 2/\gamma$.

radiation. A characteristic photon energy can be estimated as follows. Consider a highly relativistic charged particle traveling in a circular trajectory. We know that the radiated energy is concentrated in a cone of angular extent approximately given by $\pm 1/\gamma = mc^2/E$, as shown in Figure 8.3. An observer will see a pulse of radiation which lasts for a time on the order of

$$dt' = \frac{(c-v)dt}{c} = \left(1 - \frac{v}{c}\right)dt \approx \frac{dt}{2\gamma^2} = \frac{1}{2\gamma^2}\frac{d\theta}{\omega_0} = \frac{1}{\gamma^3\omega_0}, \tag{8.48}$$

where ω_0 is the instantaneous angular frequency of the circular motion. From Fourier analysis we know that the spectrum of such a pulse will contain frequencies up to about $f_{\max} \approx \pi\gamma^3\omega_0$. To see this, consider the Fourier coefficient

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(z)\cos nz\, dz \tag{8.49}$$

of the function $f(z)$. If $f(z)$ is a pulse of unit height and duration τ which repeats after a period τ_0, then

$$a_n = \frac{1}{\pi}\int_0^{2\pi\tau/\tau_0} \cos nz\, dz = \frac{1}{n\pi}\sin\frac{2\pi n\tau}{\tau_0}$$
$$= \frac{2\tau}{\tau_0}\frac{\sin(2\pi n\tau/\tau_0)}{2\pi n\tau/\tau_0}. \tag{8.50}$$

The coefficients which contribute to the Fourier series will cut off at about $n \approx \tau_0/2\tau$. For our case, $n \approx 2\pi/2\omega_0 dt' = \pi\gamma^3$. Thus, the maximum pho-

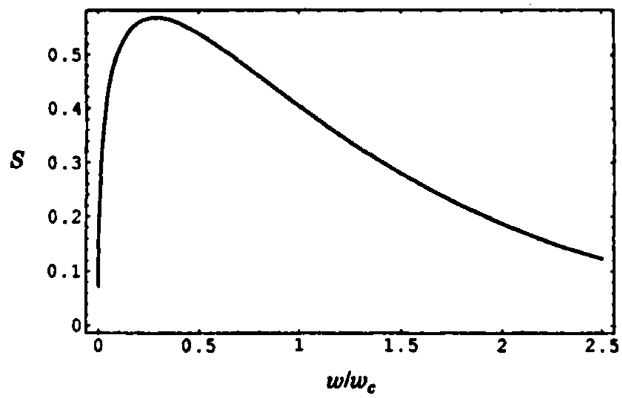

Figure 8.4. Synchrotron radiation power spectrum. The function S is defined in the text.

ton energy should be on the order of

$$w_t \equiv \hbar n \omega_0 = \pi \gamma^3 \hbar \omega_0. \qquad (8.51)$$

We will not derive the actual power spectrum. It is given by

$$\frac{dP(w)}{dw} = \frac{P}{w_c/\hbar} S(w/w_c) \qquad (8.52)$$

$$S(u) \equiv \frac{9\sqrt{3}}{8\pi} u \int_u^\infty K_{5/3}(v)\, dv \qquad (8.53)$$

$$w_c \equiv \tfrac{3}{2}\gamma^3 \hbar \omega_0, \qquad (8.54)$$

where K is a modified Bessel function, and w_c is termed the *critical energy*.[2] The function S is shown in Figure 8.4.

In terms of the critical energy, the mean and variance of the distribution are given by

$$\langle w \rangle = \frac{8}{15\sqrt{3}} w_c, \qquad (8.55)$$

$$\langle w^2 \rangle = \frac{11}{27} w_c^2. \qquad (8.56)$$

[2]See, for example, J. D. Jackson, *Classical Electrodynamics*, Wiley, New York, 1975, and M. Sands, "The Physics of Electron Storage Rings—An Introduction," in *Physics with Intersecting Storage Rings*, ed. B. Touschek, Academic Press, New York, 1971.

We may therefore rewrite our equilibrium conditions as

$$\epsilon_x = \frac{55\sqrt{3}}{2^4 3^2}\left(\frac{\langle \mathscr{H} \rangle}{1 - \mathscr{D}}\right)\frac{w_c}{mc^2}, \tag{8.57}$$

$$\epsilon_y = 0, \tag{8.58}$$

$$\frac{\sigma_E}{E} = \left[\frac{55\sqrt{3}}{2^4 3^2}\left(\frac{1}{2 + \mathscr{D}}\right)\frac{w_c}{\gamma mc^2}\right]^{1/2}. \tag{8.59}$$

For a synchrotron with constant bending radius, the critical energy may be written as

$$w_c = \frac{9}{8\pi}\frac{\hbar c}{r_0}\left(\frac{\rho}{R}\right)\frac{U_0}{E}. \tag{8.60}$$

For typical designs in the $E = 10$ GeV range, U_0 is in the neighborhood of 10 MeV. To allow for straight sections, let's put $\rho/R \approx \frac{3}{4}$. Then $w_c = 19$ keV, that is, in the hard x-ray part of the spectrum. Let $\mathscr{D} \approx 0$ (separated function lattice). If the dispersion function is on the order of 2 m and the amplitude function is on the order of 40 m in the bending regions, then $\langle \mathscr{H} \rangle \approx (2\text{ m})^2/40\text{ m} = 0.10$ m. So the equilibrium emittance and energy spread for our example would be $\epsilon_x = 2500$ mm mrad and $\sigma_E/E = 0.8 \times 10^{-3}$. The contribution to the horizontal beam size from the transverse emittance would be $\sigma_x = (\beta_x \epsilon_x/\gamma)^{1/2} = 2.2$ mm. The total horizontal beam size is $\sigma = (\beta_x \epsilon_x/\gamma + D^2 \sigma_E^2/E^2)^{1/2} = 2.7$ mm.

Our idealized results imply that the vertical beam size is zero. In reality, some portion of the transverse emittance will be coupled into the vertical degree of freedom, as will some small part of the horizontal dispersion. Though the vertical beam size will not be zero, for a corrected lattice it will be an order of magnitude or more smaller than the horizontal beam size. As a result, the words "ribbon beam" are often applied to describe the transverse bunch cross section.

PROBLEMS

1. Calculate the radiation per turn lost in synchrotron radiation by
   (a) the Cornell 10 GeV electron synchrotron, whose radius of curvature is 0.1 km, and
   (b) a 5 TeV electron synchrotron built around the earth's equator.
   In each also estimate the bend field.

2. Suppose that a 20 TeV proton storage ring has a circumference of 87 km, a bending radius of 10 km, and a stored current of 70 mA. Calculate the

power going into synchrotron radiation. If that power must be removed from the superconducting magnets, at a temperature of 4 K by refrigerators operating at 20% of ideal Carnot efficiency, estimate the refrigeration power demand.

3. For a separated function synchrotron, oscillations in all three degrees of freedom are damped. Express their time constants in terms of τ_0. Suppose the Fermilab Main Ring were to be used as an electron accelerator. Evaluate the time constants at 20 GeV. (For the Main Ring, $R = 1$ km and $\rho = 0.75$ km.)

4. In a combined function alternating gradient ring, all three degrees of freedom do not damp; in particular, radial betatron oscillations are generally antidamped. Show that this is so for the simple combined function ring of Problem 5 of Chapter 3.

5. In a separated function electron storage ring, show that the (un-normalized) horizontal emittance and the variance of the fractional momentum spread are both proportional to the square of the energy. Evaluate the constants for the Fermilab Main Ring under conditions of Problem 3 above.

6. In an *undulator* electrons traverse a series of magnets, producing alternating up and down fields. The integrated field through this device is zero, so the orbit suffers no net deflection. The angular deviation within a given magnet is within the $1/\gamma$ cone of the synchrotron radiation, so coherence is maintained in the radiation from one magnet to the next. If the pattern of up and down fields has a period length L, show that the synchrotron radiation will have a characteristic wavelength $L/(2\gamma^2)$. This estimate can be made by a variant of the argument used to obtain w_t in the text.

7. In this chapter, the power radiated by a bunch containing n particles is n times the power radiated by a single particle, whereas the factor would be n^2 if the radiation were coherent. Justify the choice of a factor of n.

8. The synchrotron radiation power spectrum, Equation 8.52, is calculated on the basis of classical electrodynamics. We could expect this result to be valid provided the critical energy, w_c, is small compared with the energy of the particle. Some of the challenging parameter sets that have been put forward for linear electron colliders imply operation in a quite different regime. For order-of-magnitude purposes, suppose n particles of (total) energy E are uniformly distributed in a cylinder of radius r and length L. A particle traveling in the opposite direction with the same energy intercepts the bunch at radius r; it will emit synchrotron radiation (called "beamstrahlung") due to the electromagnetic fields of the bunch. Estimate the ratio of the critical energy to the total energy based on the classical picture for $\gamma = 10^7, n = 10^9, r = 10^{-9}$ m, and $\mathrm{L} = 10^{-6}$ m.