An Introduction to the Physics of High Energy Accelerators
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CHAPTER 7

Emittance Preservation

In our main examples of accelerator applications in Chapter 1—the luminosity of a collider and the brightness of a synchrotron light source—we commented on the importance of producing and maintaining small beam size. The beam quality aspect of small beam size is small emittance. While it was demonstrated in Chapter 3 that the properly normalized emittances are adiabatic invariants, there are, unfortunately, a variety of processes which will lead to emittance growth.

Examples of this sort include the various scattering and diffusion processes afflicting a beam. The scattering of beam particles by interactions with the residual gas in the vacuum chamber will lead to emittance growth and beam loss. Scattering among the particles of a single beam can lead to growth of the beam dimensions in all three degrees of freedom; this *intrabeam scattering* can limit the luminosity lifetime of a hadron-hadron collider. Random noise in the radiofrequency acceleration system or in the magnet power system can lead to emittance dilution in the various degrees of freedom. Quantum fluctuations in the synchrotron radiation process excite transverse and longitudinal oscillations in electron rings. Another important source of emittance growth is errors in the transfer of a beam from one accelerator to another.

There is a distinct difference between electron accelerators and proton accelerators insofar as emittance preservation is concerned. As will be seen in Chapter 8, the radiation produced by an accelerated charge and the replenishment of the energy by the RF accelerating system causes the emittances of the beam in a synchrotron to vary with time. With appropriate choices of parameters, the system will damp oscillations in all three degrees of freedom. In electron synchrotrons, where radiation plays a dominant role, the emittances of the beam are virtually predetermined; while mechanisms for emittance growth are present, with quantized emission of energy being the dominant source, the damping of the oscillations will force the beam size to an equilibrium value. Proton synchrotrons are less forgiving, however. Since the radiation effects are many orders of magnitude smaller than in electron synchrotrons, other emittance growth mechanisms in proton synchrotrons can be of more serious concern for these devices. In this chapter we will hence concentrate on emittance preservation within hadron circular accelerators and beam transfers between such rings. The emittances of electron beams in circular accelerators will be discussed in Chapter 8.

We will discuss various emittance growth processes as applied to the transverse degrees of freedom. The longitudinal emittance of a bunched beam can also grow via many of the same processes; in fact, the longitudinal emittances of many modern proton synchrotrons are often increased intentionally, to provide Landau damping for instance. For cases where preservation of the longitudinal emittance is of interest, many of the same types of arguments as we will go through for transverse processes can also be applied. Some of these situations are presented in the problems at the end of the chapter.

Many of the sources of emittance dilution can be grouped into two important categories. The first contains mechanisms which cause single abrupt changes in the particle phase space distribution, resulting in a larger than desired phase space area demanded by the beam. The most common examples of such processes include steering and gradient errors encountered during the transfer of beams into a synchrotron. The second category contains mechanisms which "continuously" afflict the particles' oscillation amplitudes. In the following two sections we discuss two examples for each category. Other examples are left to the problems.

Though the effects mentioned thus far lead to emittance growth, methods have been developed to reduce beam emittances in all three degrees of freedom. Emittance reduction techniques were instrumental in the development of accumulator rings for antiprotons. The last section of this chapter is devoted to basic descriptions of stochastic beam cooling methods as they have been applied to hadron storage rings.

7.1 INJECTION MISMATCH

Modern high energy accelerator facilities employ a series of accelerators of various intermediate energies. In the design of beam transport systems between accelerators, the primary concern is to match the amplitude functions, dispersion functions, and of course the ideal beam trajectory coming from the first synchrotron to those of the second synchrotron. If a proper match is not provided, an increase in the transverse emittance will result.

As an example, consider a distribution of particles all of the ideal energy entering a synchrotron with the centroid of the distribution offset from the

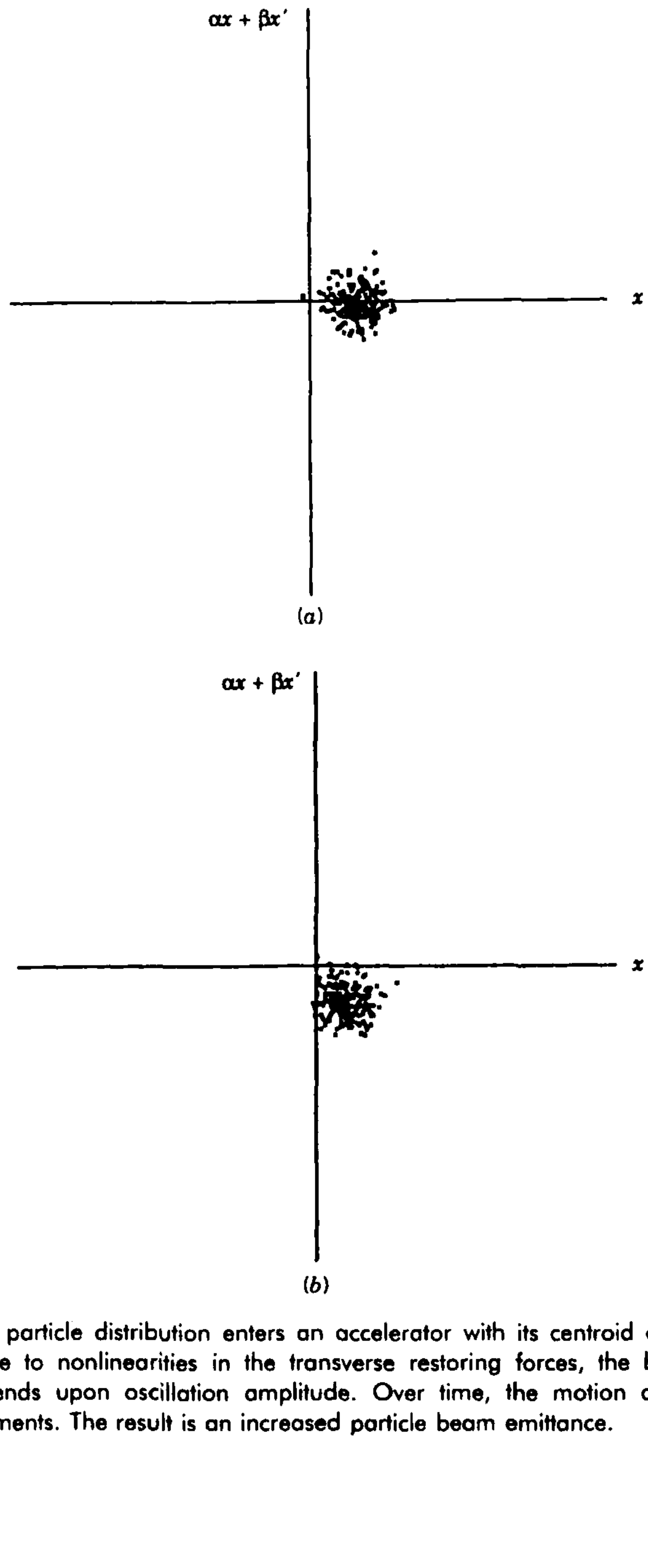


Figure 7.1. A particle distribution enters an accelerator with its centroid displaced from the ideal orbit. Due to nonlinearities in the transverse restoring forces, the betatron oscillation frequency depends upon oscillation amplitude. Over time, the motion decoheres and the distribution filaments. The result is an increased particle beam emittance.

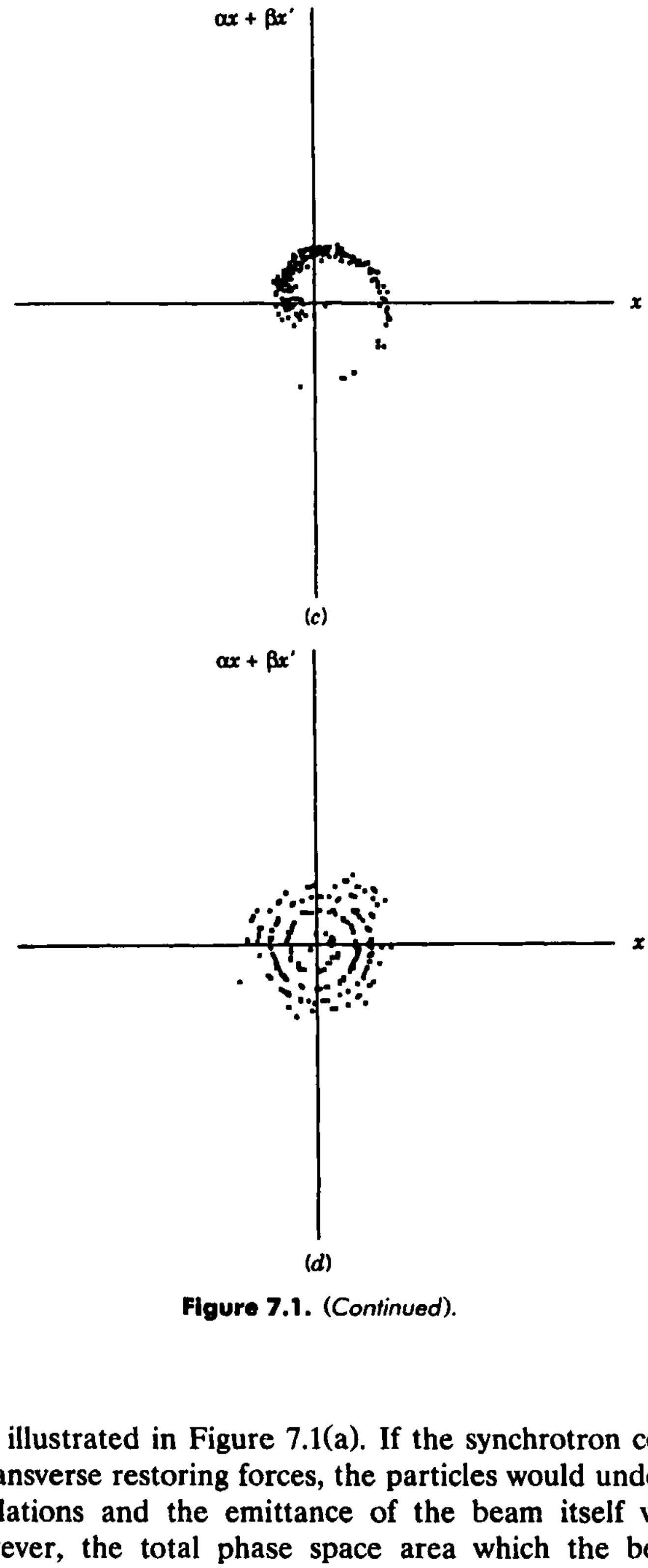
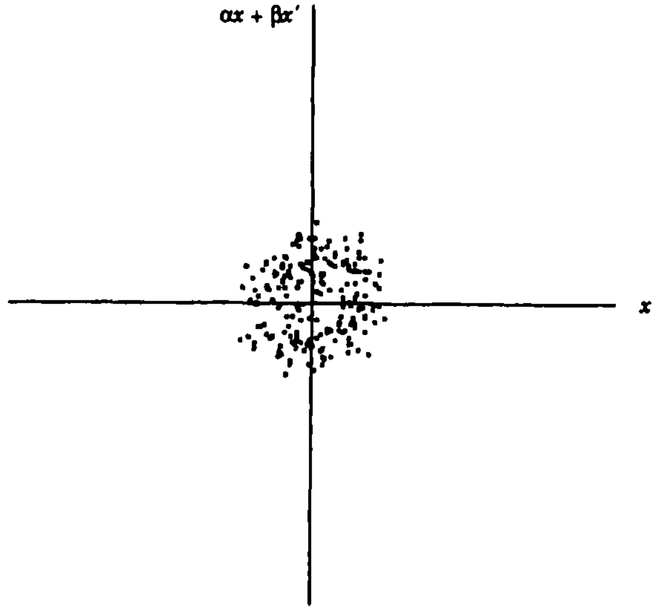


Figure 7.1. *(Continued).*

ideal orbit, as illustrated in Figure 7.1(a). If the synchrotron contained only ideal linear transverse restoring forces, the particles would undergo coherent betatron oscillations and the emittance of the beam itself would remain constant; however, the total phase space area which the beam explored would have effectively increased [Figure 7.1(b)]. In a more realistic accelerator, the magnetic fields will in general have nonlinear components, and thus



(e)

Figure 7.1. (*Continued*).

the oscillation frequency will depend upon the oscillation amplitude; the particle motion will eventually decohere and the beam distribution will filament, as shown in Figure 7.1(c) and (d). Finally, after enough time has elapsed, the phase space distribution might look like that in Figure 7.1(e), where cylindrical symmetry has been reestablished, but the emittance of the beam has increased.

For ease of computation we will assume in what follows that a particle's oscillation amplitude will remain invariant after injection into the synchrotron. Its tune may depend upon the amplitude, but whether the injected distribution continues to oscillate coherently or whether nonlinear fields cause the motion to decohere, the time average distribution of the particles will be the same provided the average is taken over a sufficiently long period.

Given a particle with an initial coordinate in phase space, the resulting time average distribution in the transverse coordinate may be obtained for that particle. Using this result, an expression for the final distribution of many particles, given their initial distribution, may be found. With this distribution function, the area in phase space which contains a certain fraction of the particles may be computed, as well as the rms beam size. Initial distributions generated by various forms of mismatch may then be inserted into these expressions to yield resulting time average distributions and emittance dilution factors.

We consider first the time average distribution of a single particle. If the transverse motion in one degree of freedom is observed at a particular longitudinal location s in the synchrotron, then the trajectory in x, p_x phase space, where $p_x \equiv \alpha x + \beta x'$, is

$$x^2 + p_x^2 = a^2, \tag{7.1}$$

where a is the amplitude of the particle motion at point s. The Courant-Snyder parameters α and β are evaluated at s.

Suppose a particle enters the accelerator and upon its first passage through point s the particle has phase space coordinates (x_0, p_{x0}). Upon subsequent revolutions about the machine, the particle will reappear at point s with phase space coordinates (x, p_x) which lie on a circle of radius $a = (x_0^2 + p_{x0}^2)^{1/2}$. The exact location on the circle after each revolution will depend upon the phase advance of the betatron oscillation for one complete revolution, which may in fact be dependent upon amplitude. Over a long period of time, the probability of finding the particle at a specific transverse displacement x may be computed. If x and p_x are parametrized by

$$x = a \cos \omega t, \tag{7.2}$$

$$p_x = -a \sin \omega t, \tag{7.3}$$

then the phase space distribution of the particle will be given by

$$g_1(x, p_x, t)\, dx\, dp_x = \delta(x - a \cos\omega t)\delta(p_x + a \sin \omega t)\, dx\, dp_x \tag{7.4}$$

where $\delta(u)$ is the Dirac δ-function. Integrating over p_x yields

$$g_2(x, t)\, dx = \delta(x - a \cos \omega t)\, dx. \tag{7.5}$$

To find the time average distribution in x, we may integrate $g_2(x, t)$ over a cycle of period $\tau = 2\pi/\omega$. In fact, due to the symmetry of the problem, integration over half a period is sufficient, which yields

$$n_a(x)\, dx = dx \frac{2}{\tau} \int_0^{\tau/2} \delta(x - a \cos \omega t)\, dt \tag{7.6}$$

$$= dx \frac{2}{\tau} \int_{-a}^{a} \delta(x - u) \frac{du}{a\omega\left[1 - (u/a)^2\right]^{1/2}}, \tag{7.7}$$

or

$$n_a(x)\, dx = \frac{1}{\pi a} \frac{dx}{\sqrt{1 - (x/a)^2}}. \tag{7.8}$$

Given the initial condition in transverse phase space (x_0, p_{x0}), over a long period of time the probability of finding the particle between x and $x + dx$ is $n_a(x)\, dx$.

Now given an initial distribution of particles $n_0(x, p_x)\,dx\,dp_x$ within the synchrotron at location s, then the resulting time average distribution of the particles may be found. Switching to polar coordinates, the number of particles which are located within a circle of radius a is given by

$$f(a) = \int_0^{2\pi}\int_0^a n_0(r,\theta)\,r\,dr\,d\theta, \tag{7.9}$$

and the number of particles between two circles of radii a and $a + da$ is

$$\frac{\partial f(a)}{\partial a}\,da = da\int_0^{2\pi} n_0(a,\theta)\,a\,d\theta. \tag{7.10}$$

Thus, the contribution of a particular ring of radius a and thickness da to the resulting time average distribution in x is

$$n_a(x)\,dx = \frac{1}{\pi}\frac{dx\,da}{\sqrt{1-(x/a)^2}}\int_0^{2\pi} n_0(a,\theta)\,d\theta. \tag{7.11}$$

Upon adding up all contributions due to all pertinent rings (i.e., $a \geq |x|$), the resulting time average distribution in x will be

$$n(x)\,dx = \frac{dx}{\pi}\int_{|x|}^{\infty}\int_0^{2\pi}\frac{n_0(a,\theta)}{\sqrt{1-(x/a)^2}}\,d\theta\,da. \tag{7.12}$$

Using this equation, the resulting time average distribution of particles in one degree of freedom may be computed given the initial distribution of particles delivered by the beamline. A perfect match of the beamline to the synchrotron would produce a resulting time average distribution of $n(x)\,dx = \int n_0(x, p_x)\,dp_x$.

The variance of the time average distribution can be obtained by integrating $\int x^2 n(x)\,dx$. An easier way of arriving at the average value of x^2 is to consider the symmetry of the distribution. The particle's trajectory in phase space is a circle, given by $x^2 + p_x^2 = a^2$. Averaging over time, we see that $\langle x^2\rangle + \langle p_x^2\rangle = a^2$, where the angle brackets denote time averages. But because the phase space trajectory is circular, $\langle x^2\rangle = \langle p_x^2\rangle$ and thus

$$\langle x^2\rangle = a^2/2. \tag{7.13}$$

Hence, while the average position of the particle is $x = 0$, its rms position will be $\sigma \equiv x_{\text{rms}} = a/\sqrt{2}$.

With these general results for the time average distribution of a single particle we can now address questions about mismatches of beams of particles upon entering a synchrotron.

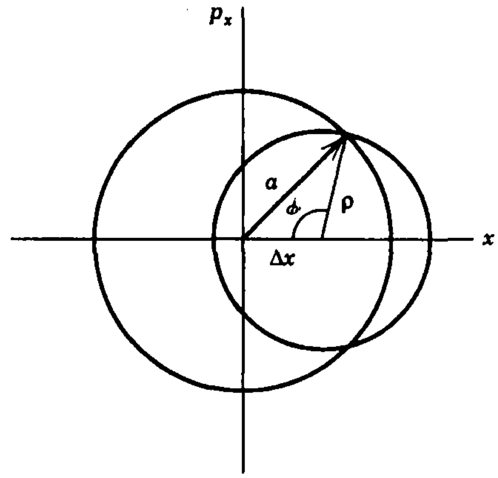


Figure 7.2. A particle distribution enters an accelerator with its centroid displaced by an amount Δx from the ideal orbit. A particle whose amplitude would have been ρ, had the central trajectory been matched, now oscillates with an amplitude a.

7.1.1 Steering Errors

Consider the effect of a steering error such as might occur at injection. The situation is illustrated in Figure 7.2. The origin of coordinates is on the design trajectory of the accelerator, but the beam enters with its centroid offset by an amount Δx in position and an amount $\Delta x'$ in slope. With respect to the centroid, the position of a particle may be characterized by the polar coordinates ρ, ϕ. However, upon entrance into the accelerator, the particle will undergo a betatron oscillation with amplitude a about the design orbit. The x, p_x axes may be rotated through an angle Θ given by

$$\tan \Theta = \frac{\Delta p_x}{\Delta x} = \frac{\beta\,\Delta x' + \alpha\,\Delta x}{\Delta x}, \tag{7.14}$$

so that the problem is equivalent to one in which the incoming distribution is displaced only in position by an amount

$$\Delta x_{\text{eq}} \equiv \sqrt{\Delta x^2 + (\beta\,\Delta x' + \alpha\,\Delta x)^2}\,. \tag{7.15}$$

From now on, Δx will be used to represent Δx_{eq}.

Let us assume that the incoming distribution is Gaussian with the form

$$n_0(x, p_x)\,dx\,dp_x = \frac{1}{2\pi\sigma_0^2} e^{-[(x-\Delta x)^2 + p_x^2]/2\sigma_0^2}\,dx\,dp_x \tag{7.16}$$

and compute the final time average distribution due to a position mismatch, $n_x(x)\,dx$. Switching to polar coordinates,

$$n_x(x)\,dx = \frac{dx}{2\pi^2\sigma_0^2}\int_{|x|}^{+\infty}\int_0^{2\pi} \frac{e^{-[a^2+\Delta x^2 - 2a\,\Delta x\cos\theta]/2\sigma_0^2}}{\sqrt{1-(x/a)^2}}\,d\theta\,da, \tag{7.17}$$

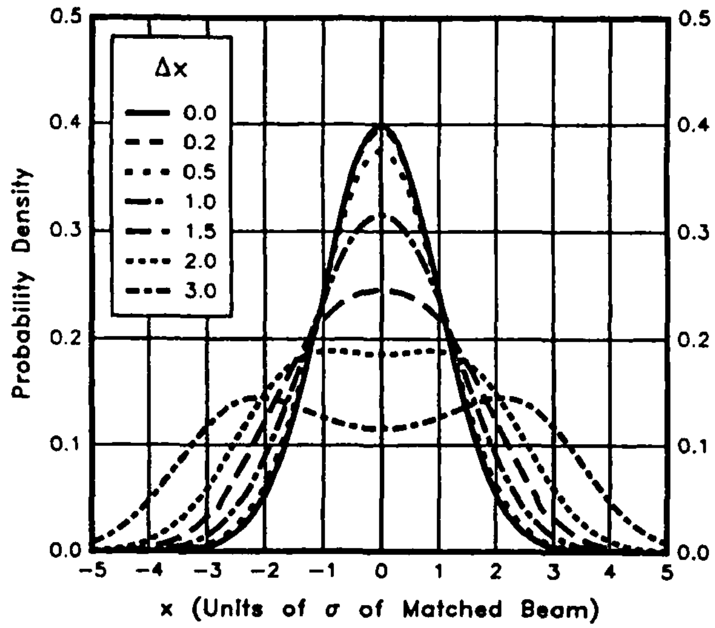


Figure 7.3. Particle distribution resulting from steering error at injection.

or, reducing the expression to a single integral,

$$n_x(x)\,dx = \frac{dx}{\pi\sigma_0^2}\int_{|x|}^{+\infty} e^{-(a-\Delta x)^2/2\sigma_0^2}\frac{e^{-|a\,\Delta x|/\sigma_0^2}I_0\left(a\,\Delta x/\sigma_0^2\right)}{\sqrt{1-(x/a)^2}}\,da, \quad (7.18)$$

where $I_0(z)$ is the modified Bessel function of order zero.[1] Numerical integration of this expression yields the curves shown in Figure 7.3. If the injected beam is displaced by more than about twice the standard deviation of the initial particle distribution, the resulting time average distribution exhibits a double hump.

The variance of the resulting particle distribution can be found using Equation 7.13. A single particle with initial coordinates corresponding to an amplitude a will provide a contribution of $a^2/2$ to the variance of the resulting time average distribution in x. From Figure 7.2 we see that

$$a^2 = \rho^2 + \Delta x^2 - 2\rho\,\Delta x\cos\phi. \quad (7.19)$$

An initial distribution having a variance σ_0^2 and which has rotational symmetry about the point $(\Delta x, 0)$ can be written in the form

$$n_0(\rho,\phi) = f(\rho)/2\pi. \quad (7.20)$$

[1]See, for example, M. Abramowitz, and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1970, p. 374.

The variance of the time average distribution can thus be computed as

$$\sigma^2 \equiv \langle x^2 \rangle = \int \frac{a^2}{2} n_0 \, d\Sigma \tag{7.21}$$

$$= \int \frac{\rho^2}{2} n_0 \, d\Sigma + \int \frac{\Delta x^2}{2} n_0 \, d\Sigma - \int \rho \Delta x \cos\phi n_0 \, d\Sigma$$

$$= \int \frac{\rho^2}{2} n_0 \, d\Sigma + \frac{\Delta x^2}{2} - \frac{\Delta x}{2\pi} \int \rho^2 f(\rho) \, d\rho \int \cos\phi \, d\phi \tag{7.22}$$

where $d\Sigma$ is a differential element of phase space area. While the third term is zero, the first term is just

$$\int \frac{\rho^2}{2} n_0 \, d\Sigma = \sigma_0^2, \tag{7.23}$$

which is the variance of the incoming distribution—that is, the variance the final distribution would have had if there were no mismatch. Thus, the variance of the resulting distribution is simply

$$\sigma^2 = \sigma_0^2 + \tfrac{1}{2} \Delta x^2. \tag{7.24}$$

While the distribution functions plotted in Figure 7.3 assume an initial Gaussian distribution, the expression for the variance in Equation 7.24 is completely general for any initial distribution with cylindrical symmetry in x, p_x phase space.

In Chapter 3 we derived an expression for the emittance of a beam distribution which is Gaussian in the transverse coordinate. If the incoming distribution were Gaussian, the resulting time average distribution due to an injection position mismatch would not be. However, provided that the mismatch is not too large, the area in phase space which will contain the beam is larger than the incoming emittance by the ratio

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{1}{2}\left(\frac{\Delta x}{\sigma_0}\right)^2. \tag{7.25}$$

Choosing one of the many definitions of the normalized emittance, such as $\epsilon_n = \pi\sigma^2(\gamma v/c)/\beta$, we see that a steering error at injection will generate an increase in the beam emittance by an amount

$$\Delta\epsilon_n = \frac{\pi(\gamma v/c)\,\Delta\sigma^2}{\beta} = \frac{\pi(\gamma v/c)}{2} \frac{\Delta x^2 + (\beta\,\Delta x' + \alpha\,\Delta x)^2}{\beta} \tag{7.26}$$

where β and α are the Courant-Snyder parameters at the location of the observed trajectory errors and here the expression includes both errors in position and slope. It is interesting to note that this result is independent of the incoming beam size.

A mismatch of the dispersion function which is delivered to a synchrotron from a beamline can be handled analogously, since this is simply a mismatch of the trajectories of off-momentum particles. For a particle of momentum $p + \Delta p$, where p is the ideal momentum, the equilibrium orbit lies on the phase space point $(x, p_x) = (D\,\Delta p/p, (\beta D' + \alpha D)\,\Delta p/p)$, where D is the dispersion function. A mismatch of the incoming dispersion function to that of the accelerator will result in a steering error $\Delta x = \Delta D\,\Delta p/p$ and $\Delta x' = \Delta D'\,\Delta p/p$, which will result in an increased transverse emittance.

Without going through the same steps as before, we state the results for the time average distribution function due to a mismatch of the dispersion function. Let D be the dispersion function of the accelerator at the observation point, and ΔD_e be the deviation of that value delivered by the beamline:

$$\Delta D_e \equiv \left[\Delta D^2 + (\beta\,\Delta D' + \alpha\,\Delta D)^2\right]^{1/2}. \tag{7.27}$$

Then

$$n_D(x)\,dx = \frac{dx}{\sqrt{2\pi^3}\,\sigma_0^2}\int_{-\infty}^{+\infty}\int_{|x-D\sigma_p\delta|}^{+\infty} e^{-\delta^2/2} e^{-(a-|\Delta D_e\sigma_p\delta|)^2/2\sigma_0^2} \times \frac{e^{-|a\,\Delta D_e\sigma_p\delta|/\sigma_0^2} I_0\left(a\,\Delta D_e\,\sigma_p\delta/\sigma_0^2\right)}{\sqrt{1-\left(\dfrac{x-D\sigma_p\delta}{a}\right)^2}}\,da\,d\delta. \tag{7.28}$$

Here $\delta \equiv (\Delta p/p)/\sigma_p$, where σ_p is the rms value of the relative momentum deviation; and we are assuming once again Gaussian distributions both in the initial transverse beam dimension and in momentum.

Note that for $\Delta D_e = 0$, the distribution function n_D becomes

$$n_D(x)\,dx = \frac{1}{\sqrt{2\pi\left(\sigma_0^2 + D^2\sigma_p^2\right)}} e^{-x^2/2(\sigma_0^2+D^2\sigma_p^2)}\,dx, \tag{7.29}$$

which is a Gaussian distribution with variance $\sigma_0^2 + D^2\sigma_p^2$. From Equation 3.138 we know that the variance of the beam distribution at a location in a beamline or synchrotron where the dispersion function is nonzero is given by $\sigma^2 = \sigma_t^2 + D^2\sigma_p^2$, where σ_t contains the transverse emittance. The result of a dispersion function mismatch is to increase the transverse emittance and hence increase σ_t. To see this effect, we examine the resulting time average distribution with D set to zero; the total variance of the distribution which one would actually observe would be $D^2\sigma_p^2$ plus the variance of $n_D(D = 0)$.

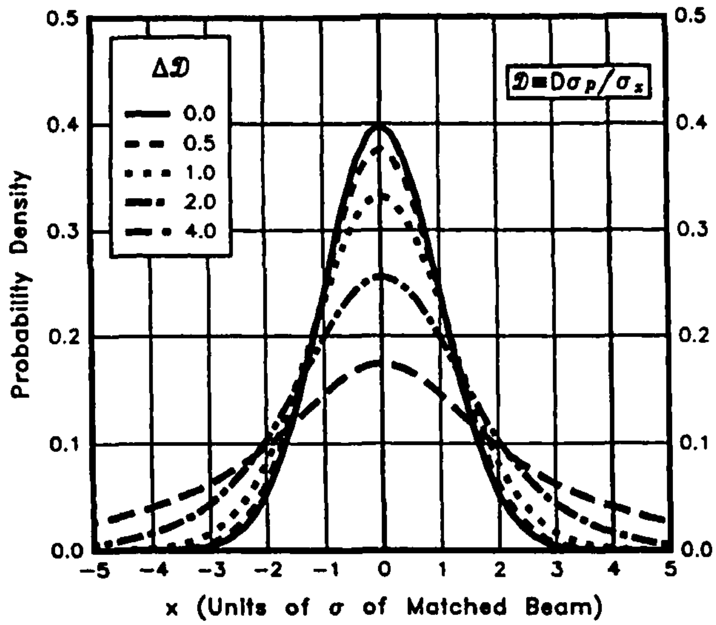


Figure 7.4. Particle distribution resulting from mismatch of dispersion function at injection. The curves are drawn for a point where the dispersion function of the synchrotron is zero.

Figure 7.4 shows the distribution $n_D(D = 0)$ for several values of the dispersion mismatch. The severity of the emittance dilution depends upon both ΔD_e and σ_p, as it must. If all of the particles were of the exact same momentum, the beam size would not increase no matter how large a value for ΔD_e was obtained. Likewise, any small deviation from the ideal dispersion function significantly affects the emittance of a beam which has a large enough momentum spread.

The variance of the distribution $n_D(D = 0)$ is given by

$$\sigma^2 = \sigma_0^2 + \tfrac{1}{2}\,\Delta D^2\,\sigma_p^2, \tag{7.30}$$

and likewise the change in the normalized emittance, under the same assumptions of sufficiently small mismatch, is

$$\Delta\epsilon_n = \frac{\pi(\gamma v/c)}{2}\,\frac{\Delta D^2 + (\beta\,\Delta D' + \alpha\,\Delta D)^2}{\beta}\sigma_p^2. \tag{7.31}$$

7.1.2 Focusing Errors

The treatment for an amplitude function mismatch can be carried out in a similar fashion. To begin with, we must differentiate between the amplitude function which is being delivered by the beamline and the periodic amplitude

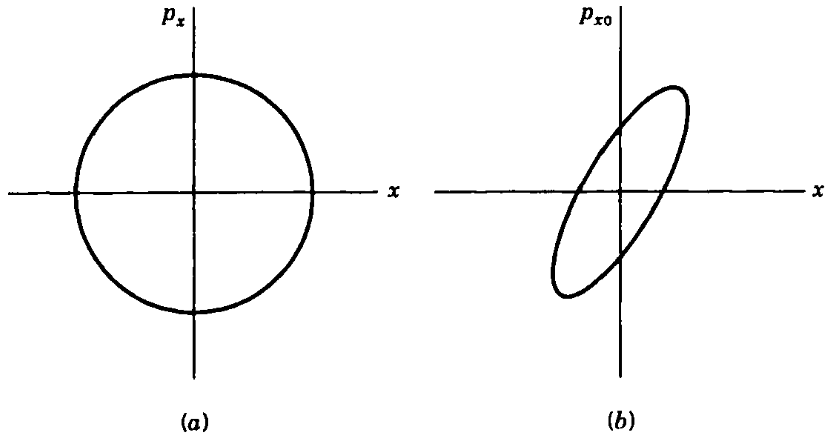


Figure 7.5. A phase space trajectory which is circular as viewed in terms of the beamline lattice functions (a) will in general appear elliptical when viewed in terms of the ring lattice functions (b).

function of the synchrotron. Suppose β and α are the Courant-Snyder parameters as delivered by the beamline to a particular point in an accelerator, and β_0, α_0 are the periodic lattice functions of the ring at that point. A particle with trajectory (x, x') can be viewed in the $(x, \beta x' + \alpha x) \equiv (x, p_x)$ phase space corresponding to the beamline functions, or in the $(x, \beta_0 x' + \alpha_0 x) \equiv (x, p_{x0})$ phase space corresponding to the lattice functions of the ring. If the phase space motion lies on a circle in the beamline view, then it will lie on an ellipse in the ring view, as indicated in Figure 7.5.

The equation of the ellipse can be obtained by noting that

$$x' = \frac{p_x - \alpha x}{\beta} = \frac{p_{x0} - \alpha_0 x}{\beta_0}, \tag{7.32}$$

or

$$p_x = \frac{\beta}{\beta_0} p_{x0} + \left(\alpha - \frac{\beta}{\beta_0}\alpha_0\right) x \tag{7.33}$$

$$\equiv \beta_r p_{x0} + \Delta\alpha_r x. \tag{7.34}$$

If the equation of the circle in the beamline view is

$$x^2 + p_x^2 = \beta A^2, \tag{7.35}$$

where A^2 is the Courant-Snyder invariant, then the equation of the ellipse in the ring system will be

$$(1 + \Delta\alpha_r^2)x^2 + 2\beta_r \Delta\alpha_r x p_{x0} + \beta_r^2 p_{x0}^2 = \beta A^2. \tag{7.36}$$

It will be useful to rotate the coordinate axes so that they correspond to the major and minor axes of the ellipse. This amounts to rotating through an angle Θ so that the cross term in the equation of the ellipse is eliminated. The angle is given by

$$\tan 2\Theta = \frac{2\beta_r \Delta\alpha_r}{1 + \Delta\alpha_r^2 - \beta_r^2}, \tag{7.37}$$

and the resulting equation in the rotated coordinates $x_e \equiv x \cos\Theta + p_x \sin\Theta$, $p_{xe} \equiv -x \sin\Theta + p_x \cos\Theta$ will be

$$\begin{aligned}\tfrac{1}{2}\left[(1 + \Delta\alpha_r^2 + \beta_r^2) + \sqrt{(1 + \Delta\alpha_r^2 - \beta_r^2)^2 + 4(\beta_r \Delta\alpha_r)^2}\right] x_e^2 \\ + \tfrac{1}{2}\left[(1 + \Delta\alpha_r^2 + \beta_r^2) - \sqrt{(1 + \Delta\alpha_r^2 - \beta_r^2)^2 + 4(\beta_r \Delta\alpha_r)^2}\right] p_{xe}^2 \\ = \beta A^2.\end{aligned} \tag{7.38}$$

But the expression under the radical can be simplified:

$$(1 + \Delta\alpha_r^2 - \beta_r^2)^2 + 4(\beta_r \Delta\alpha_r)^2 = (1 + \Delta\alpha_r^2 + \beta_r^2)^2 - 4\beta_r^2. \tag{7.39}$$

We also note that

$$\begin{aligned}1 + \Delta\alpha_r^2 + \beta_r^2 &= 1 + (\alpha - \beta_r \alpha_0)^2 + \beta_r^2 \\ &= \beta_r\left[\frac{1 + \alpha^2}{\beta_r} + \beta_r(1 + \alpha_0^2) - 2\alpha\alpha_0\right] \\ &= \beta_r\left[\beta_0 \frac{1 + \alpha^2}{\beta} + \beta \frac{1 + \alpha_0^2}{\beta_0} - 2\alpha\alpha_0\right] \\ &= \beta_r[\beta_0\gamma + \beta\gamma_0 - 2\alpha\alpha_0] \\ &= 2\beta_r F,\end{aligned} \tag{7.40}$$

where

$$F \equiv \tfrac{1}{2}(\beta_0\gamma + \beta\gamma_0 - 2\alpha\alpha_0). \tag{7.41}$$

Using Equations 7.39 and 7.40, Equation 7.38 becomes

$$\frac{F + \sqrt{F^2 - 1}}{\beta_0 A^2} x_e^2 + \frac{F - \sqrt{F^2 - 1}}{\beta_0 A^2} p_{xe}^2 = 1, \tag{7.42}$$

which is the standard form of the equation of an ellipse in terms of its major and minor axes.

Finally, we recognize that

$$\frac{1}{F - \sqrt{F^2 - 1}} = F + \sqrt{F^2 - 1} \tag{7.43}$$

and so the equation of the ellipse is of the form

$$b_r x_e^2 + \frac{1}{b_r} p_{xe}^2 = \beta_0 A^2, \tag{7.44}$$

where

$$b_r \equiv F + \sqrt{F^2 - 1}. \tag{7.45}$$

Note that for the special case where $\alpha = \alpha_0 = 0$ and $\beta \neq \beta_0$, then $\Theta = 0$, $F = [(\beta/\beta_0) + (\beta_0/\beta)]/2$, and $b_r = \beta/\beta_0 = (\beta_0 + \Delta\beta)/\beta_0 = 1 + \Delta\beta/\beta_0$. That is, $b_r - 1 = \Delta\beta/\beta_0$ represents the "amplitude" of a "beta function" mismatch.

We can now proceed to look at the time average distribution which results from an initial distribution created by an amplitude function mismatch. If the incoming distribution is a Gaussian with cylindrical symmetry when viewed in terms of the beamline lattice functions, then in the phase space corresponding to the ring lattice functions, the distribution will have the form

$$n_0(x, p_x)\, dx\, dp_x = \left(\frac{e^{-x^2/2b_r\sigma_0^2}}{\sqrt{2\pi b_r \sigma_0^2}} \right) \left(\frac{e^{-b_r p_x^2/2\sigma_0^2}}{\sqrt{2\pi\sigma_0^2/b_r}} \right) \tag{7.46}$$

where the subscripts e have been suppressed to simplify notation. Here, σ_0 corresponds to the rms displacement the particle distribution would have if the lattice functions were perfectly matched.

Upon transforming to polar coordinates, the integral for the resulting time average distribution due to an amplitude function mismatch becomes

$$n_\beta(x)\, dx = \frac{dx}{2\pi^2\sigma_0^2} \int_{|x|}^{+\infty} \int_0^{2\pi} \frac{\exp\left[-\dfrac{a^2}{2b_r\sigma_0^2} (\cos^2\theta + b_r^2 \sin^2\theta) \right]}{\sqrt{1 - (x/a)^2}}\, d\theta\, da, \tag{7.47}$$

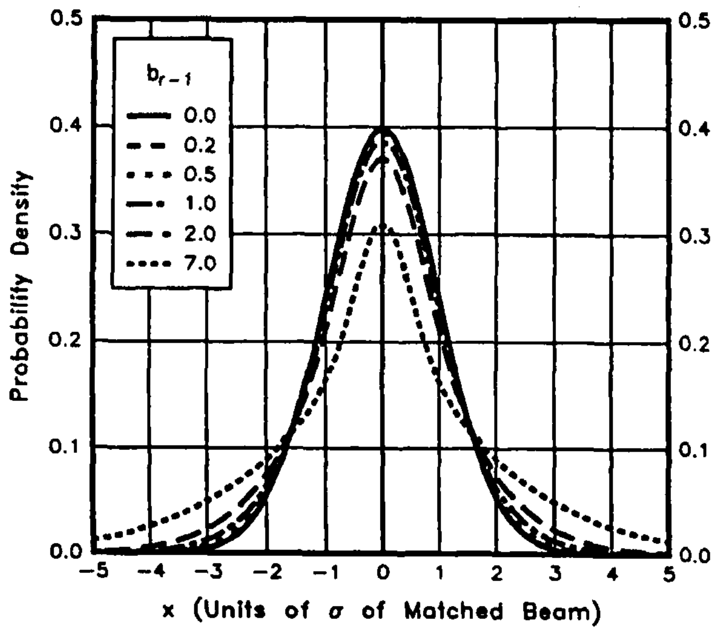


Figure 7.6. Particle distribution resulting from amplitude function mismatch at injection.

or

$$n_\beta(x)\,dx = \frac{dx}{\pi\sigma_0^2}\int_{|x|}^{+\infty} e^{-a^2/2b_r\sigma_0^2} \frac{\exp\left[-\left(\dfrac{a^2}{4}\dfrac{b_r^2-1}{b_r}\right)\right] I_0\left(\dfrac{a^2}{4}\dfrac{b_r^2-1}{b_r}\right)}{\sqrt{1-(x/a)^2}}\,da. \tag{7.48}$$

Figure 7.6 shows the resulting distribution. In contrast to the distributions resulting from steering errors, the centroid of the beam is hardly disturbed, and hence, as can be seen, the amplitude function must be greatly mismatched to produce a significant increase in the variance of the distribution.

Once again the variance of the resulting particle distribution can be found using Equation 7.13. If the initial coordinates of a particular particle are given by (x_0, p_{x0}) where

$$b_r x_0^2 + \frac{1}{b_r} p_{x0}^2 = \beta_0 A^2, \tag{7.49}$$

then this particle will commence describing a circular trajectory in phase space on subsequent passages through the synchrotron, where the radius of the phase space trajectory is given by $a = \sqrt{x_0^2 + p_{x0}^2}$. Upon averaging over the entire distribution, we have

$$\langle x_0^2 \rangle = b_r \sigma_0^2, \tag{7.50}$$

$$\langle p_{x0}^2 \rangle = \frac{1}{b_r}\sigma_0^2, \tag{7.51}$$

and hence

$$\langle a^2 \rangle = \langle x_0^2 \rangle + \langle p_{x0}^2 \rangle \tag{7.52}$$

$$= b_r \sigma_0^2 + \frac{1}{b_r}\sigma_0^2. \tag{7.53}$$

It follows that the resulting distribution will have variance

$$\sigma^2 = \frac{\langle a^2 \rangle}{2} = \frac{b_r^2 + 1}{2b_r}\sigma_0^2. \tag{7.54}$$

But $b_r = F + \sqrt{F^2 - 1}$ and hence $b_r^2 + 1 = 2Fb_r$. Therefore, upon averaging over time, the variance of the distribution will be increased by a factor

$$\sigma^2/\sigma_0^2 = F = \tfrac{1}{2}(\beta\gamma_0 + \beta_0\gamma - 2\alpha\alpha_0). \tag{7.55}$$

This expression can be made to look more like the expressions obtained for steering errors by rewriting it as (see the Problems)

$$F = 1 + \tfrac{1}{2}|\det \Delta J| \tag{7.56}$$

where J is the 2×2 matrix containing the Courant-Snyder parameters:

$$J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}. \tag{7.57}$$

For the case where the slope of the amplitude function is matched and equal to zero, we have

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2}\left(\frac{\Delta\beta/\beta_0}{\sqrt{1 + \Delta\beta/\beta_0}}\right)^2. \tag{7.58}$$

It is interesting to note that the change in emittance (or σ^2) generated by an amplitude function mismatch is proportional to the incoming emittance, in contrast to the effect of a steering error, where the emittance increase is independent of the initial emittance.

Table 7.1 summarizes the emittance dilution factors to injection amplitude function, dispersion function, and steering errors.

Table 7.1. Transverse emittance dilution factors; see text for explanations.

Amplitude function mismatch:

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \tfrac{1}{2}|\det \Delta J|$$

Dispersion function mismatch:

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2}\left(\frac{\Delta D^2 + (\beta\,\Delta D' + \alpha\,\Delta D)^2}{\sigma_0^2}\right)\sigma_p^2$$

Injection steering error:

$$\frac{\sigma^2}{\sigma_0^2} = 1 + \frac{1}{2}\left(\frac{\Delta x^2 + (\beta\,\Delta x' + \alpha\,\Delta x)^2}{\sigma_0^2}\right)$$

7.2 DIFFUSION PROCESSES

In this section we wish to discuss the general behavior of particle distributions in the presence of mechanisms which continuously stimulate emittance growth. Let us suppose we have an initial distribution of particles in transverse phase space, $f_0(x, x')$, where x is the transverse coordinate and x' is the slope of a particle's trajectory, $x' = dx/ds$, as observed at some particular point in the accelerator. In the absence of such mechanisms, we assume that this distribution will not change with time, and so

$$f(x, x', t) = f_0(x, x'). \tag{7.59}$$

However, if some process is randomly altering the betatron amplitudes of the particles in the beam, the extent of the distribution will grow with time and $f(x, x', t)$ will satisfy the diffusion equation.

To understand the diffusion equation, imagine a system of particles, constrained to move with one degree of freedom, with density function $f(x)$. Let J be the average number of particles per unit area crossing a plane perpendicular to the x-direction per unit time. We now look at a region around x, as sketched in Figure 7.7, and consider the number of particles flowing into and out of this region.

The number flowing into the region bounded by x and $x + \Delta x$ and with cross-sectional area A in the small time interval Δt is $AJ(x)\,\Delta t$. The

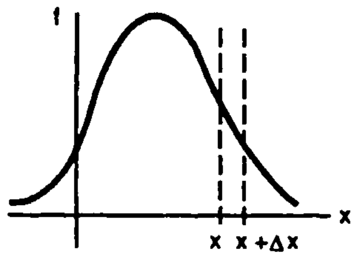


Figure 7.7. Diffusion of particles across a boundary at x.

number flowing out is given by $AJ(x + \Delta x)\,\Delta t$. Thus,

$$\frac{\partial}{\partial t}(fA\,\Delta x) = AJ(x) - AJ(x + \Delta x), \tag{7.60}$$

or

$$\frac{\partial f}{\partial t} = -\frac{\partial J}{\partial x}. \tag{7.61}$$

Now if f is uniform, the migration of particles due to some random process into the region between x and $x + \Delta x$ will equal the average flow of particles out of this region, and so $J(x)$ would be zero. If the density function had a greater value at x than at $x + \Delta x$ (i.e., if f has a nonzero gradient), then more particles are apt to wander into the region from the left than from the right. That is, one would expect J to be proportional (to good approximation) to the rate of change of f with respect to x:

$$J = -C\frac{\partial f}{\partial x}, \tag{7.62}$$

where C is a constant of proportionality. Substituting Equation 7.62 into Equation 7.61, we obtain the diffusion equation in one degree of freedom:

$$\frac{\partial f}{\partial t} = C\frac{\partial^2 f}{\partial x^2}. \tag{7.63}$$

The more general three-dimensional form of the diffusion equation is

$$\frac{\partial f}{\partial t} = C\,\nabla^2 f. \tag{7.64}$$

For the case of diffusion in one of the transverse degrees of freedom of a particle distribution in an accelerator, it is again useful to consider the distribution in x, p_x phase space. We consider only initial distributions with cylindrical symmetry in this phase space. Let r be the amplitude of a particle's transverse oscillation: $r^2 = x^2 + p_x^2 = x^2 + (\beta x' + \alpha x)^2$. Then the density function, in polar coordinates, satisfies

$$\frac{\partial f}{\partial t} = C\,\nabla^2 f = C\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right). \tag{7.65}$$

We can see the significance of the diffusion constant C by multiplying Equation 7.65 by r^2 and integrating over all of phase space. We find

$$\int r^2 \frac{\partial f}{\partial t}\, r\,dr = C\int r\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) r\,dr, \tag{7.66}$$

$$\frac{\partial}{\partial t}\int r^2 f r\,dr = C\int r^2\, d\left(r\frac{\partial f}{\partial r}\right),$$

$$\frac{\partial}{\partial t}\langle r^2\rangle = C\left[\left(r^3\frac{df}{dr}\right)_0^{\infty} - \int\left(r\frac{df}{dr}\right)(2r\,dr)\right]$$

$$= -2C\int r^2\,df$$

$$= -2C\left[(r^2 f)\big|_0^{\infty} - \int f\cdot 2r\,dr\right]$$

$$= 4C\int fr\,dr$$

$$= 4C, \tag{7.67}$$

or

$$C = \frac{1}{4}\frac{\partial}{\partial t}\langle r^2\rangle. \tag{7.68}$$

We thus see that C is related to the time rate of change of the emittance of the beam. We therefore transform coordinates to involve the Courant-Snyder invariant $W \equiv [x^2 + (\beta x' + \alpha x)^2]/\beta = r^2/\beta$. So we may write $f = f(W, t)$ and note that the variable W is independent of longitudinal location within the accelerator. The diffusion equation then becomes

$$\frac{\partial f}{\partial t} = \frac{4C}{\beta}\frac{\partial}{\partial W}\left(W\frac{\partial f}{\partial W}\right). \tag{7.69}$$

To allow for processes which may occur at various points along the circumference, we can define a new diffusion constant

$$R = \frac{d}{dt}\left\langle\frac{r^2}{\beta}\right\rangle = \frac{d}{dt}\langle W\rangle \tag{7.70}$$

and write our diffusion equation as

$$\frac{\partial f}{\partial t} = R \frac{\partial}{\partial W}\left(W \frac{\partial f}{\partial W} \right). \tag{7.71}$$

To proceed, we define two quantities:

$$Z = \frac{W}{W_a}, \tag{7.72}$$

$$\tau = \left(\frac{R}{W_a} \right) t, \tag{7.73}$$

where W_a is the Courant-Snyder invariant corresponding to the limiting aperture of the accelerator (i.e., the admittance). If a is the half aperture at a location where the amplitude function has the value β, then $W_a = a^2/\beta$. Notice that Z and τ are both dimensionless quantities.

In terms of Z and τ, the problem reduces to

$$\frac{\partial f}{\partial \tau} = \frac{\partial}{\partial Z}\left(Z \frac{\partial f}{\partial Z} \right) \tag{7.74}$$

subject to the boundary conditions

$$f(Z,0) = f_0(Z), \tag{7.75}$$

$$f(1,\tau) = 0. \tag{7.76}$$

The solution of the above differential equation is

$$f(Z,\tau) = \sum_n c_n J_0\left(\lambda_n \sqrt{Z}\right) e^{-\lambda_n^2 \tau/4} \tag{7.77}$$

with

$$c_n = \frac{1}{J_1(\lambda_n)^2} \int_0^1 f_0(Z) J_0\left(\lambda_n \sqrt{Z}\right) dZ, \tag{7.78}$$

where λ_n is the nth zero of the Bessel function $J_0(z)$ ($\lambda_1 = 2.405$, $\lambda_2 = 5.520, \ldots$).

We now consider a particular form of the initial distribution, namely a bi-Gaussian in x, x' phase space. For this situation, the function f_0 will be

$$f_0(x, p_x)\, dx\, dp_x = f_0(r) r\, dr \tag{7.79}$$

$$= \frac{1}{\sigma^2} e^{-r^2/2\sigma^2} r\, dr \tag{7.80}$$

$$= \tfrac{1}{2} e^{-r^2/2\sigma^2} d(r^2/\sigma^2), \tag{7.81}$$

or

$$f_0(Z)\, dZ = \frac{a^2}{2\sigma^2} e^{-(a^2/2\sigma^2)Z}\, dZ. \tag{7.82}$$

So the coefficients c_n become

$$c_n = \frac{\alpha}{J_1(\lambda_n)^2} \int_0^1 e^{-\alpha Z} J_0(\lambda_n \sqrt{Z})\, dZ, \tag{7.83}$$

where $\alpha = a^2/2\sigma^2$. If the entire initial beam distribution lies well within the aperture so that the integrand is sufficiently near zero before Z approaches 1, i.e., if α is greater than about 5, then the c_n's may be approximated by

$$c_n = \frac{1}{J_1(\lambda_n)^2} e^{-z_n}(\cosh z_n - \sinh z_n) \tag{7.84}$$

$$= \frac{1}{J_1(\lambda_n)^2} e^{-2z_n}, \tag{7.85}$$

where

$$z_n = \frac{\lambda_n^2}{4} \left(\frac{\sigma}{a}\right)^2. \tag{7.86}$$

If the initial distribution does not satisfy the above condition, then the c_n integrals may be performed numerically.

The development of the particle distribution with time, as well as the total beam intensity as a function of time, may now be computed. By integrating

$f(Z, \tau)$ over the range of Z, the number of particles $N(\tau)$ may be obtained, namely,

$$N(\tau) = \int_0^1 f(Z, \tau)\, dZ \tag{7.87}$$

$$= \int_0^1 \sum_n c_n J_0(\lambda_n \sqrt{Z}) e^{-\lambda_n^2 \tau/4}\, dZ, \tag{7.88}$$

$$= \sum_n c_n \int_0^1 J_0(\lambda_n \sqrt{Z}) e^{-\lambda_n^2 \tau/4}\, dZ \tag{7.89}$$

or

$$N(\tau) = 2 \sum_n \frac{c_n}{\lambda_n} J_1(\lambda_n) e^{-\lambda_n^2 \tau/4}. \tag{7.90}$$

For $\sigma \ll a$, this becomes

$$N(\tau) = 2 \sum_n \frac{1}{\lambda_n J_1(\lambda_n)} \exp\left\{ -\frac{\lambda_n^2}{4}\left[\tau + 2\left(\frac{\sigma}{a}\right)^2\right]\right\}. \tag{7.91}$$

Figure 7.8(a) shows how $f(Z, \tau)$ varies with time for the case $\sigma/a = 0.20$. The particle distribution grows in transverse size until it reaches the aperture ($Z = 1$), at which time the area under the curve quickly begins to decrease. The intensity $N(\tau)$ for this same case is displayed in Figure 7.8(b). From $\tau = 0$ to $\tau \approx 0.1$ the intensity is nearly constant. Upon reaching the aperture limit, the intensity rapidly falls off until $\tau \approx 0.5$, where the final lifetime $4/\lambda_1^2$ is reached.

When a substantial fraction of the initial Gaussian distribution lies outside the aperture limit (all particles with $Z > 1$ being lost immediately), the rather flat region of $N(Z)$ for small Z disappears. Figure 7.9(a) shows the intensity vs. time for the case $\sigma/a = 0.2$. The lifetime, determined by

$$\tau_L(\tau) = -\frac{N(\tau)}{dN/d\tau}, \tag{7.92}$$

is shown in Figure 7.9(b) for the same two cases. The one case begins with a very long lifetime which then decreases to the value $4/\lambda_1^2$, while the other case begins with a very short lifetime which rapidly approaches its asymptotic value.

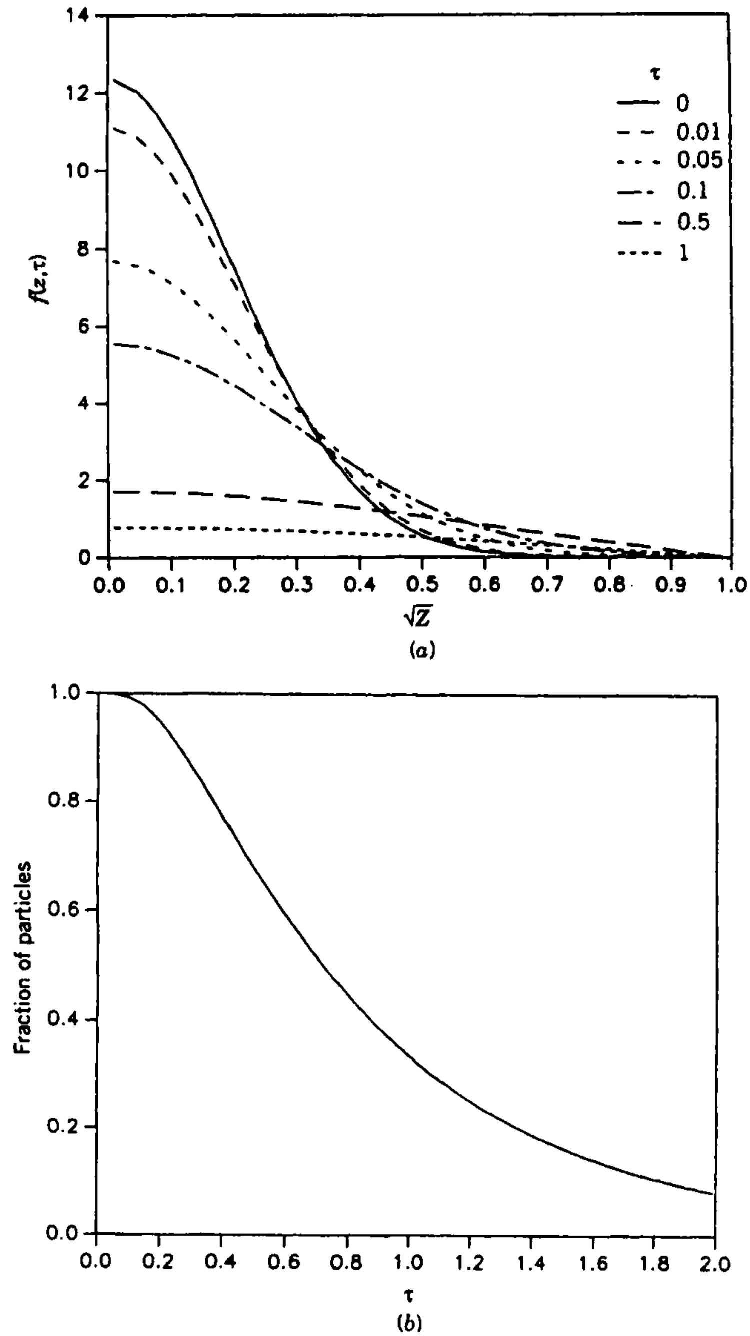


Figure 7.8. (a) Variation of density function f and (b) variation of particle beam intensity with time for an aperture at 5σ.

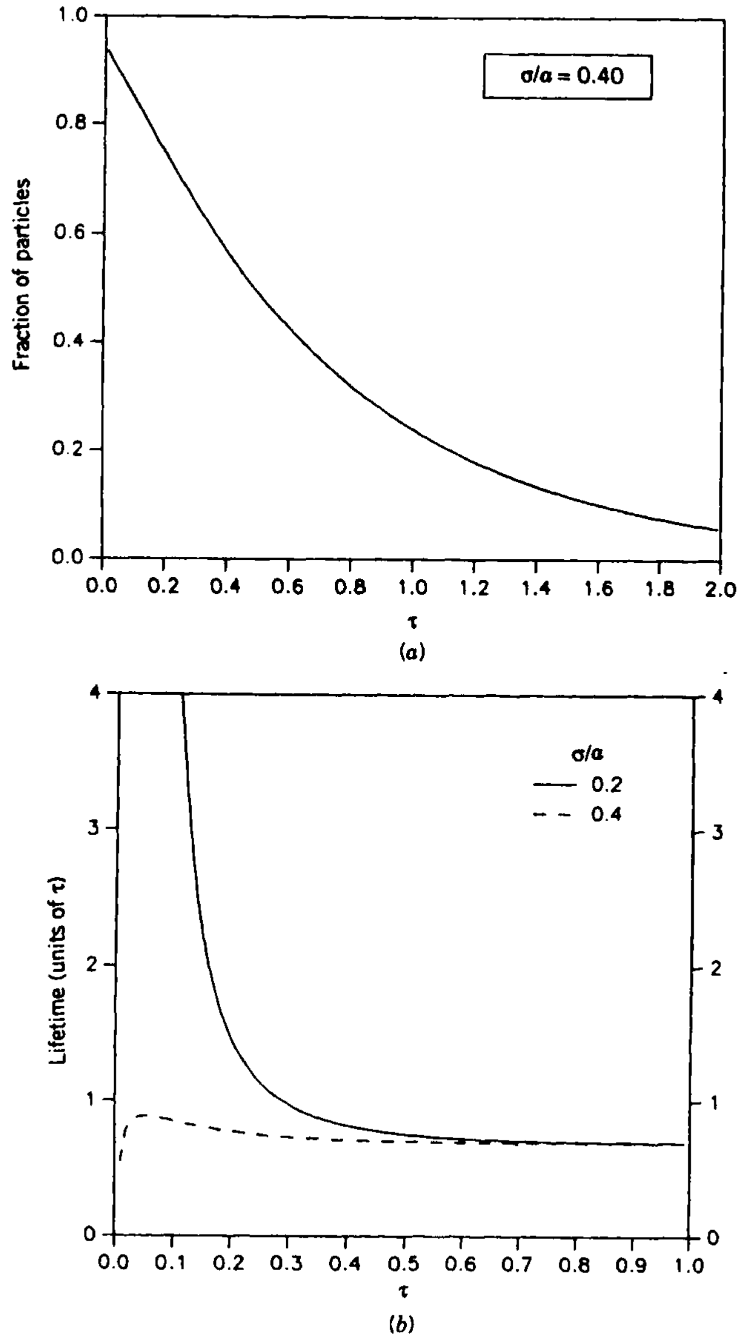


Figure 7.9. (a) Intensity and (b) lifetime vs. time for $\sigma / a = 0.2$ and 0.4.

The dimensionless quantity τ is related to time t by

$$\tau = \left(\frac{R}{W_a}\right)t, \tag{7.93}$$

and hence the asymptotic lifetime is given by

$$t_L = \frac{4}{\lambda_1^2}\frac{W_a}{R}. \tag{7.94}$$

If the emittance growth is caused by the changing of particle direction due to fluctuations in magnetic fields, elastic scattering off residual gas particles, etc., then the next step is to evaluate the diffusion constant R under these various circumstances.

7.2.1 RF Noise and Excitation of Oscillations

In this section, we wish to look at the development of betatron oscillations, and hence growth in emittance, if there is a sequence of random energy changes such as might be produced by noise in a radiofrequency accelerating system. We assume for this argument that each energy increment upon each passage through the accelerating station is uncorrelated with all other such occurrences. Now suppose the momentum of a particle changes abruptly by an amount Δp. If the particle were not already undergoing a synchrotron oscillation, one would now begin with initial conditions $\Delta E = \beta pcu$ and $\Delta\phi = 0$, where $u \equiv \Delta p/p$. If the dispersion D or its derivative D' is different from zero, a betatron oscillation will start with respect to the new off-momentum orbit with initial conditions $x = -Du$ and $x' = -D'u$, as indicated in Figure 7.10.

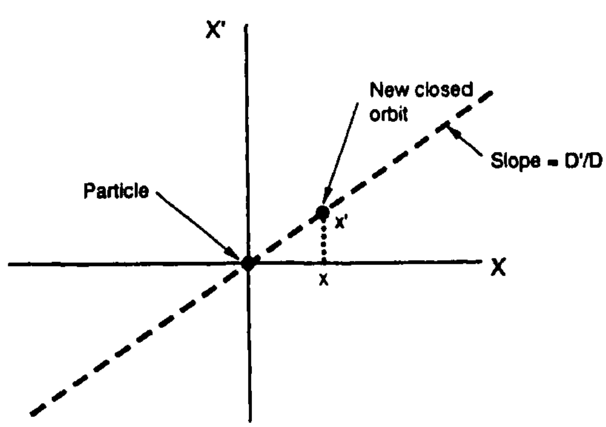


Figure 7.10. The dashed line represents the locus of closed orbits for various particle momenta.

Let's consider a sequence of fractional momentum changes $\{u_i\}$. In our familiar vector notation, after the first kick,

$$\begin{pmatrix} x_1 \\ x_1' \end{pmatrix} = -u_1 \begin{pmatrix} D \\ D' \end{pmatrix}. \tag{7.95}$$

After transformation by the single-turn matrix M, a second kick is delivered, so

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_1' \end{pmatrix} - u_2 \begin{pmatrix} D \\ D' \end{pmatrix} \tag{7.96}$$

$$= -(u_1 M + u_2) \begin{pmatrix} D \\ D' \end{pmatrix}, \tag{7.97}$$

and after n turns

$$\begin{pmatrix} x_n \\ x_n' \end{pmatrix} = -(u_1 M^{n-1} + u_2 M^{n-2} + \cdots + u_{n-1} M + u_n) \begin{pmatrix} D \\ D' \end{pmatrix} \tag{7.98}$$

$$= -\sum_{m=1}^{n} u_m M^{n-m} \begin{pmatrix} D \\ D' \end{pmatrix}. \tag{7.99}$$

We're interested in emittance growth, so we should look at the Courant-Snyder invariant, which is given by

$$W = \frac{r^2}{\beta} = \gamma x^2 + 2\alpha x x' + \beta x'^2 = -\vec{x}^T S J \vec{x}, \tag{7.100}$$

where

$$J \equiv \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}, \tag{7.101}$$

$$S \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7.102}$$

and A^T refers to the transpose of A.

Writing M in the form

$$M = I \cos \mu + J \sin \mu , \tag{7.103}$$

where μ is the phase advance for one revolution, each term in $W(n)$ will be of the form

$$-u_m u_k \vec{D}^T (I \cos \mu_1 + J^T \sin \mu_1) S J (I \cos \mu_2 + J \sin \mu_2) \vec{D}, \tag{7.104}$$

where $\mu_1 = (n-m)\mu$ and $\mu_2 = (n-k)\mu$. Noting that $J^T S = -SJ$ and $J^2 = -I$, this expression reduces to

$$\mathscr{H} u_m u_k \cos(m-k)\mu \tag{7.105}$$

with $\mathscr{H}$ defined as

$$\mathscr{H} = \gamma D^2 + 2\alpha DD' + \beta D'^2. \tag{7.106}$$

Then we have

$$W = \mathscr{H} \sum_{m,k}^{n} u_m u_k \cos(m - k)\mu. \tag{7.107}$$

Since we do not know the particular sequence of errors $\{u_i\}$, the best we can do is to average over a large ensemble of such sequences. Then

$$\langle W \rangle = \mathscr{H} \sum_{m,k}^{n} \cos(m - k)\mu \cdot \frac{1}{P} \sum_{p=1}^{P} u_{m,p} u_{k,p} \tag{7.108}$$

where P is the total number of sets in the ensemble. For uncorrelated kicks such as we would have from a truly random noise source,

$$\frac{1}{P} \sum_{p=1}^{P} u_{m,p} u_{k,p} = \delta_{mk} u_m^2 \tag{7.109}$$

and so

$$\langle W \rangle = \mathscr{H} \sum_{m=1}^{n} u_m^2 = n\mathscr{H}\langle u^2 \rangle, \tag{7.110}$$

which gives for the diffusion constant

$$R = \frac{d}{dt}\langle W \rangle = f_0 \mathscr{H}\langle u^2 \rangle = f_0 \mathscr{H} \frac{e^2 \langle \mathrm{v}^2 \rangle}{\left(\frac{v}{c}\right)^4 E^2}, \tag{7.111}$$

where $\sqrt{\langle \mathrm{v}^2 \rangle}$ is the rms voltage due to noise in the accelerating system and f_0 is the revolution frequency. In terms of a normalized emittance $\epsilon_N \equiv \pi\gamma(v/c)\sigma^2/\beta$,

$$\frac{d\epsilon_N}{dt} = \frac{\pi\gamma}{2} f_0 \mathscr{H} \frac{e^2 \langle \mathrm{v}^2 \rangle}{\left(\frac{v}{c}\right)^3 E^2}, \tag{7.112}$$

where we note that $\sigma^2 = \langle x^2 \rangle = \langle r^2 \rangle / 2$. Note that if the dispersion function and its derivative are both zero at the source of the noise, emittance growth can be avoided.

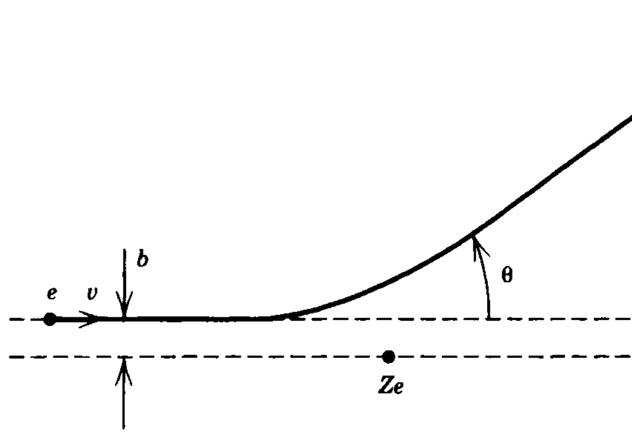


Figure 7.11. Coulomb scattering of electron from target nucleus.

7.2.2 Beam-Gas Scattering

As another example of a diffusion process we consider the small angle scattering of particles in the beam off of residual gas molecules in the vacuum chamber. We first consider the deflection of a single particle from a target nucleus and then the effect of multiple scattering off many such targets. From this we can obtain an expression for the time rate of change of the rms particle amplitude due to beam-gas scattering, and hence an emittance growth rate.

Coulomb Scattering Consider an incident particle of charge e approaching a target nucleus of charge Ze as shown in Figure 7.11. The particle approaches with a speed v and impact parameter b. The Coulomb repulsion (or attraction) changes the direction of motion of the particle by the angle θ. If the scattering angle is small, the transverse momentum $p_\perp$ that the particle acquires is given by

$$p_\perp = \int F_\perp\, dt = e\int E_\perp\, dt = \frac{e}{v}\int E_\perp\, dz \tag{7.113}$$

$$= \frac{e}{v}\frac{1}{2\pi b}\int E_\perp\, dA = \frac{e}{v}\frac{1}{2\pi b}\frac{Ze}{\epsilon_0}, \tag{7.114}$$

and

$$\theta = \frac{p_\perp}{p} = \frac{Ze^2}{2\pi\epsilon_0 pvb}. \tag{7.115}$$

The differential cross section $d\sigma/d\Omega$ is the area $d\sigma$ presented by a target particle for scattering an incident particle into the solid angle $d\Omega$. For a particle approaching the target with impact parameter b, $d\sigma = 2\pi b\,db$. For small angles, $d\Omega = 2\pi\theta\,d\theta$. Hence, the differential cross section is

$$\frac{d\sigma}{d\Omega} = \left|\frac{2\pi b\,db}{2\pi\theta\,d\theta}\right|. \tag{7.116}$$

The absolute value is taken because both the area and the solid angle are positive quantities.

Using

$$b = \frac{Ze^2}{2\pi\epsilon_0 pv\theta}, \tag{7.117}$$

we have

$$\frac{db}{d\theta} = -\frac{Ze^2}{2\pi\epsilon_0 pv\theta^2}, \tag{7.118}$$

from which

$$\frac{d\sigma}{d\Omega} = \frac{b}{\theta}\frac{db}{d\theta} = 4\left(\frac{Ze^2}{4\pi\epsilon_0 pv}\right)^2\frac{1}{\theta^4}. \tag{7.119}$$

This is the small angle limit of the famous Rutherford scattering cross section formula.

Multiple Coulomb Scattering We next imagine a thin layer of material through which a particle passes, interacting with many atoms along its way. Upon each interaction, the particle's transverse coordinate is changed very little, but its direction of motion is altered according to the results of the section above. For the interaction with one scattering center, the variance of the particle's scattering angle is given by

$$\langle\theta^2\rangle_1 = \frac{\int\theta^2(d\sigma/d\Omega)\,d\Omega}{\int(d\sigma/d\Omega)\,d\Omega}. \tag{7.120}$$

The limits of integration are

$$\theta_{\text{min}} \approx \frac{Ze^2}{2\pi\epsilon_0 pva}, \tag{7.121}$$

$$\theta_{\text{max}} \approx \frac{Ze^2}{2\pi\epsilon_0 pvR}, \tag{7.122}$$

where a is the "radius" of the target atom, and R is the "radius" of the target nucleus.

Using the Rutherford cross section above, we find that

$$\begin{aligned}\langle\theta^2\rangle_1 &= 2\theta_{\min}^2 \ln(\theta_{\max}/\theta_{\min}) \\ &= 8Z^2 r_e^2 \left(\frac{m_e c^2}{pv}\right)^2 \frac{\ln(a/R)}{a^2}\end{aligned} \tag{7.123}$$

where $r_e = e^2/(4\pi\epsilon_0 m_e c^2)$ is the classical radius of the electron. By adding up the contributions due to the scattering off the scattering centers within radius a of the particle's trajectory and through a thickness l of a material of density ρ and atomic weight A, we get

$$\langle\theta^2\rangle = \frac{N_A}{A}\rho(l\pi a^2)\langle\theta^2\rangle_1 \tag{7.124}$$

for the variance of the scattering angle distribution. Here N_A is Avagradro's number.

When dealing with radiation processes one often expresses lengths in units of the radiation length L_{rad}, given by

$$\frac{1}{L_{\mathrm{rad}}} \equiv 2\alpha\frac{N_A}{A}\rho Z^2 r_e^2 \ln\frac{a}{R}, \tag{7.125}$$

where $\alpha \approx 1/137$ is the fine structure constant. In tables, the material density is often suppressed and "lengths" are expressed in units of grams per square centimeter.

The variance of the scattering angle distribution may now be written as

$$\begin{aligned}\langle\theta^2\rangle &= \frac{4\pi}{\alpha}\left(\frac{m_e c^2}{pv}\right)^2 \frac{l}{L_{\mathrm{rad}}} \\ &= \left(\frac{E_s}{pv}\right)^2 \frac{l}{L_{\mathrm{rad}}},\end{aligned} \tag{7.126}$$

where $E_s = m_e c^2\sqrt{4\pi/\alpha} = 21$ MeV. If we now consider the projection onto one particular transverse plane, then

$$\langle\theta^2\rangle = \langle\theta_x^2\rangle + \langle\theta_y^2\rangle = 2\langle\theta_x^2\rangle. \tag{7.127}$$

Therefore, the variance of the scattering angle in one transverse degree of freedom is given by

$$\langle\theta_x^2\rangle = \left(\frac{15\ \mathrm{MeV}}{pv}\right)^2 \frac{l}{L_{\mathrm{rad}}}. \tag{7.128}$$

The Diffusion Constant Finally, we use the result of the previous section to discuss the scattering of the beam particles off of the residual particles in the vacuum chamber. The average rate of change of the transverse scattering angle in one degree of freedom is given by

$$\frac{d}{dt}\langle\theta^2\rangle = \left(\frac{15\ \mathrm{MeV}}{pv}\right)^2 \frac{c}{L_{\mathrm{rad}}} \tag{7.129}$$

$$= \left(\frac{15\ \mathrm{MeV}}{\gamma m v^2}\right)^2 \frac{c}{L_{\mathrm{rad}}} \tag{7.130}$$

$$= (3.3 \times 10^{-7}/\mathrm{sec})\frac{P[\mu\mathrm{Torr}]}{\gamma^2}, \tag{7.131}$$

where the last expression is given for a proton beam, assuming air for the residual gas in the vacuum chamber, and assuming $v \approx c$. Here, $P[\mu\mathrm{Torr}]$ is the average vacuum chamber pressure expressed in microtorrs.

The diffusion constant is $R = d\langle W\rangle/dt$ and $\Delta W = \Delta\{(x^2 + [\beta x' + \alpha x]^2)/\beta\} = (2\alpha x\,\Delta x' + \beta^2\Delta x'^2)/\beta$. So, when averaging over many scatterings,

$$R = \left\langle \beta \frac{d}{dt}\,\Delta x'^2 \right\rangle = \langle\beta\rangle\langle\dot{\theta}^2\rangle, \tag{7.132}$$

and the emittance growth rate, using $\epsilon_N \equiv \pi\gamma(v/c)\sigma^2/\beta$, is

$$\frac{d\epsilon_N}{dt} = \frac{\pi(v/c)}{2\gamma}\langle\beta\rangle\left(\frac{15\ \mathrm{MeV}}{mv^2}\right)^2 \frac{c}{L_{\mathrm{rad}}} \tag{7.133}$$

$$= \pi\langle\beta\rangle(1.6 \times 10^{-7}/\mathrm{sec})\frac{P[\mu\mathrm{Torr}]}{\gamma} \tag{7.134}$$

for the assumptions above.

7.3 EMITTANCE REDUCTION

The processes described in the last two sections are indicative of common and often unavoidable sources of emittance dilution in hadron synchrotrons. Potential major sources of emittance growth can often be identified, and modifications can be made during the design of the accelerator to improve the situation. One may still have residual effects, which may be difficult to identify, and one may still desire smaller emittance beams than can be readily produced by normal means. In particular, beams of exotic particles, such as antiprotons, which are produced through the targeting of primary beams will have inherently large transverse emittances as well as large energy spread. It was the desire to build proton-antiproton ($p\bar{p}$) colliding beam accelerators which led to the development of electron cooling and stochastic cooling. These emittance reduction techniques, in particular the latter, allowed antiprotons to be accumulated in reasonable quantities and with reasonable emittances so that $p\bar{p}$ collisions in the 0.5–2 TeV center of mass energy range could be achieved. The invention of stochastic beam cooling and the subsequent discovery of the vector bosons at CERN in the $Sp\bar{p}S$ collider led to the award of the Nobel Prize in physics to C. Rubbia and S. van der Meer. Over the past several years beam cooling techniques have been utilized in several accelerators in high energy and nuclear physics experimental facilities.

Electron cooling[2] involves the thermal interplay of a proton beam and an electron beam toward equilibrium transverse and longitudinal temperatures; the initial electron temperature is quite low (parallel beam with low momentum spread), and the initial proton beam temperature is significantly higher. While this technique is qualitatively simple to visualize, a complete quantitative treatment is beyond the scope of this section. Rather, we turn to a description of stochastic cooling. The basis of this technique is a shade less intuitive, yet its analysis is more straightforward.

7.3.1 Transverse Stochastic Cooling

The concept of a stochastic cooling system is remarkably simple. It is also remarkable that it is technically feasible.[3] Suppose we want to reduce the transverse emittance of a beam. A beam bunch contains a finite number of particles; thus the beam centroid will deviate from the central orbit of the bunch by a finite amount. If we detect and correct this deviation, the effective emittance of the bunch will be reduced. If we were indeed talking about a single bunch, we would have succeeded in making a minuscule reduction in the emittance and the process would be at an end. But if, on the other hand, our beam sample rapidly interchanges particles with other samples, the fluctuation in the centroid position is regenerated and the process can be repeated. In concept, at least, the system might consist of the arrangement in Figure 7.12. The centroid position is sensed at the pickup, and the signal is conveyed across the ring, amplified, and delivered to a kicker which provides an angular deflection proportional to the displacement sensed at the pickup. The kicker is located an odd number of quarter wavelengths in betatron phase downstream of the pickup. The signal path must be shorter than the

[2] G. I. Budker, *Proc. Intl. Symp. on Electron and Positron Storage Rings*, Saclay, 1966, p. II-1-1. For a general overview, see W. Kells, "Electron Cooling," in *Physics of High Energy Particle Accelerators*, AIP Conf. Proc. 87, New York, 1982.

[3] D. Möhl, G. Petrucci, L. Thorndahl, and S. van der Meer, "Physics and Technique of Stochastic Cooling," Physics Reports 58, No. 2 (1980). The following discussion has been adapted from A. V. Tollestrup and G. Dugan, "Elementary Stochastic Cooling," in *Physics of High Energy Particle Accelerators*, AIP Conf. Proc. 105, New York, 1983.

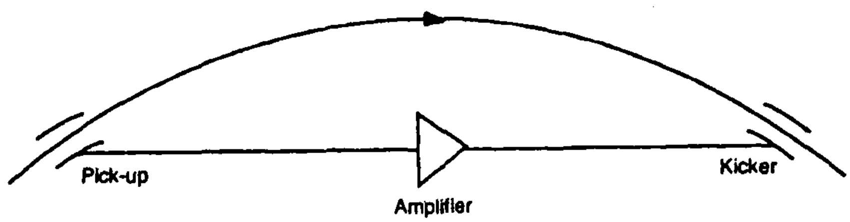


Figure 7.12. Stochastic cooling system consisting of pickup electrodes, amplifier, and beam deflector.

orbital path length between the two devices to ensure that the signal reaches the kicker at the same time as the beam.

It is desirable to reduce the transverse emittance of a beam containing a very large number of particles. If the system had infinitely fine time resolution, each particle could be sensed and corrected and the process could be successfully concluded in a few revolutions. However, a real system does not have this capability, and so there will be only finitely many particles in the sample that are corrected. Clearly, the smaller the number of particles in the sample, the closer one approaches the ideal.

First, we relate the sample size to the system bandwidth. Suppose there are N particles uniformly distributed around the ring. If the sample size is such that the beam is divided into k samples, each containing N_s particles, then the minimum wavelength that can be resolved in the analysis of the data is

$$\lambda_{\min} = \frac{2C}{k}, \tag{7.135}$$

where C is the ring circumference. Therefore, the frequency content of the information extends to

$$f_{\max} = \frac{v}{\lambda_{\min}} = \frac{kv}{2C} = \frac{k}{2T}, \tag{7.136}$$

where T is the revolution period. For a system with a flat frequency response from $f = 0$ to $f = W$, W determines $f_{\max}$. So the number of particles in a sample, in terms of the bandwidth W, is given by

$$N_s = \frac{N}{k} = \frac{N}{2TW}. \tag{7.137}$$

We now consider a measurement of a particular sample. Each particle in the sample receives a correction proportional to the sample's mean displacement $\langle x \rangle$. So an individual particle's displacement after the kick is $x - g\langle x \rangle$. To get at the emittance reduction, we need to consider the change of the rms of the distribution. For the kth particle,

$$x_k^2 \to (x_k - g\langle x\rangle)^2 = x_k^2 - 2gx_k\langle x\rangle + g^2\langle x\rangle^2. \tag{7.138}$$

We write

$$\langle x\rangle = \frac{1}{N_s}\sum_i x_i = \frac{1}{N_s}x_k + \frac{1}{N_s}\sum_{i\neq k} x_i. \tag{7.139}$$

Then

$$\begin{aligned}(x_k - g\langle x\rangle)^2 &= x_k^2 - \frac{2g}{N_s}x_k^2 - \frac{2g}{N_s}x_k\sum_{i\neq k}x_i + \frac{g^2}{N_s^2}\left(x_k + \sum_{i\neq k}x_i\right)^2\\ &= x_k^2 - \left(\frac{2g}{N_s} - \frac{g^2}{N_s^2}\right)x_k^2\\ &\quad - \left(\frac{2g}{N_s} - \frac{2g^2}{N_s^2}\right)\sum_{i\neq k}x_i x_k + \frac{g^2}{N_s^2}\left(\sum_{i\neq k}x_i\right)^2.\end{aligned} \tag{7.140}$$

Averaging over all the particles, we get

$$\begin{aligned}\frac{1}{N_s}\sum_k (x_k - g\langle x\rangle)^2 &= \frac{1}{N_s}\sum_k x_k^2 - \left(\frac{2g}{N_s} - \frac{g^2}{N_s^2}\right)\frac{1}{N_s}\sum_k x_k^2\\ &\quad - \left(\frac{2g}{N_s} - \frac{2g^2}{N_s^2}\right)\sum_{k,i\neq k}x_i x_k\\ &\quad + \frac{g^2}{N_s}\sum_k\left(\frac{1}{N_s}\sum_{i\neq k}x_i\right)^2.\end{aligned} \tag{7.141}$$

The second term on the right hand side is the sum of the contribution of each particle acting back upon itself. This coherent term is offset in part by the fourth term, representing the incoherent contribution of the other particles in the sample. Since the individual particle displacements are uncorrelated, the sum present in the third term is zero.

We can analyze the last term as follows:

$$\begin{aligned}
\sum_k \left(\frac{1}{N_s} \sum_{i \neq k} x_i \right)^2 &\approx \sum_k \left(\frac{1}{N_s} \sum_i x_i \right)^2 \\
&= \sum_k \frac{1}{N_s^2} \sum_i \sum_j x_i x_j \\
&= \frac{1}{N_s} \sum_k \frac{1}{N_s} \sum_i x_i^2 \\
&= \frac{1}{N_s} \sum_k \langle x^2 \rangle \\
&= \frac{1}{N_s} \cdot N_s \langle x^2 \rangle \\
&= \langle x^2 \rangle,
\end{aligned} \tag{7.142}$$

where in the second step we have assumed that the various x_i are uncorrelated. Thus, keeping terms up to first order in $1/N_s$, we have for the rate of change of $\langle x^2 \rangle$

$$\frac{d\langle x^2 \rangle}{dn} = -\frac{2g}{N_s} \langle x^2 \rangle + \frac{g^2}{N_s} \langle x^2 \rangle. \tag{7.143}$$

The cooling rate is then

$$\frac{1}{\epsilon} \frac{d\epsilon}{dn} = -\left(\frac{2g - g^2}{N_s} \right), \tag{7.144}$$

or, in terms of time,

$$\frac{1}{\tau} \equiv -\frac{1}{\epsilon} \frac{d\epsilon}{dt} = -\frac{1}{\epsilon} \frac{d\epsilon}{dn} \frac{1}{T} = \frac{2g - g^2}{N_s T} = \frac{2W}{N}(2g - g^2). \tag{7.145}$$

Let's add two refinements to this relationship. System noise is an important consideration in the design of a cooling ring. Suppose that the noise introduced at the kicker is equivalent to a position error x_n at the pickup. Then the correction applied to each particle becomes

$$x - g(\langle x \rangle + x_n). \tag{7.146}$$

Proceeding as before,

$$[x - g(\langle x\rangle + x_n)]^2 = x^2 - 2gx(\langle x\rangle + x_n) + g^2(\langle x\rangle^2 + 2x_n\langle x\rangle + x_n^2) \tag{7.147}$$

for a single particle, and averaging over the sample gives

$$\langle[x - g(\langle x\rangle + x_n)]^2\rangle = \langle x^2\rangle - 2g\langle x\rangle^2 - 2g\langle x\rangle\langle x_n\rangle + g^2\langle x\rangle^2 + g^2\langle x_n\rangle\langle x\rangle + g^2\langle x_n^2\rangle. \tag{7.148}$$

Averaging over many samples, $\langle x_n\rangle = 0$ and so

$$\frac{1}{\langle x^2\rangle}\frac{d\langle x^2\rangle}{dn} = [-2g + g^2(1 + U)]\frac{1}{N_s}, \tag{7.149}$$

where $U \equiv \langle x_n^2\rangle/\langle x\rangle^2$ is the ratio of the expected noise to the expected signal power.

Our second refinement is to take into account the fact that the fluctuation in the centroid position may not be regenerated independently from one turn to the next. In other words, if particles move rapidly from one sample to another, each sample will rerandomize during the course of one turn and we will have the ideal situation. But the "mixing" may not be perfect, and we have to allow for this possibility.

The movement from sample to sample is due to the spread in orbital frequencies arising from the spread in particle momentum. The number of revolutions required for a particle of momentum $\Delta p/p$ to pass from one sample to another is

$$M = \frac{T_s}{\Delta T}, \tag{7.150}$$

where $T_s = (N_s/N)T = 1/(2W)$ is the sample time, and ΔT is the change in the revolution period due to the momentum deviation $\Delta p/p$. Then

$$M = \frac{1}{2WT|\eta|(\Delta p/p)}. \tag{7.151}$$

For ideal mixing, $M = 1$. Intuitively, one would expect the cooling rate to degrade by a factor of M as we depart from perfect mixing. Actually, this factor of M appears only in the incoherent term, and so the emittance decreases according to

$$\epsilon = \epsilon_0 e^{-t/\tau}, \tag{7.152}$$

where we have for the cooling rate

$$\frac{1}{\tau} = \frac{2W}{N}\left[2g - g^2(M + U)\right]. \tag{7.153}$$

7.3.2 Longitudinal Stochastic Cooling

The transverse cooling sketched in the last section is able to reduce the transverse emittances of antiprotons to the level appropriate for p$\bar{\text{p}}$ collider operation. We have not thus far addressed the question of how one accumulates large numbers of antiprotons. It is inherent in the production process that antiprotons are produced over a broad range of momenta, and this spread needs to be reduced. Longitudinal cooling is able to achieve both goals, as we shall see.

Suppose we detect momentum differences by their related orbital frequency differences. We need a way of applying no correction if the frequency is correct; this can be accomplished by adding a filter to the layout shown in the preceding section. If a correction is required, it will be applied by a longitudinal kick, rather than a transverse kick as was done in the previous section. Because the cooling systems have a wide bandwidth, this implies that the filter remove not only the fundamental of the derived frequency but its harmonics as well.

In a system devised for accumulation of particles it is natural to speak in terms of particle flux and density functions. The time evolution of the density function $\psi(E)$ will represent a trade off between the diffusive effects of the incoherent interactions and the collective flow arising from the coherent forces. The equation that describes the time evolution of a density function subject to these processes is called the Fokker–Planck equation.

As in the discussion of beam-gas scattering, the flux arising from diffusion can be written in the form

$$\vec{J} = -D\,\nabla\psi, \tag{7.154}$$

where $\vec{J}$ is the particle flux. In the case under consideration here, since energy is the only degree of freedom,

$$J = -D(E)\frac{\partial\psi}{\partial E}, \tag{7.155}$$

where the diffusion "constant" may be a function of energy. To this, we must add coherent forces. If the rate of energy gain is $C(E)$, then we must add $\psi C(E)$ to the flux, obtaining

$$J = C(E)\psi - D(E)\frac{\partial\psi}{\partial E}. \tag{7.156}$$

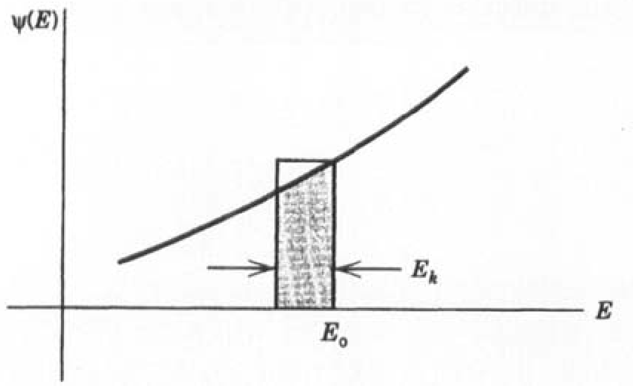


Figure 7.13. Particle density function $\psi(E)$.

We can now obtain the time rate of change of ψ from the continuity equation:

$$\frac{\partial \psi}{\partial t} = -\nabla \cdot \vec{J} = -\frac{\partial}{\partial E}\left[C(E)\psi - D(E)\frac{\partial \psi}{\partial E}\right]. \tag{7.157}$$

This is called the Fokker–Planck equation.

Let's arrive at the expression for the flux by an alternative route so that we may identify the coefficients $C(E)$ and $D(E)$ in terms of the kicker voltage. Suppose $\psi(E)$ appears as in Figure 7.13. We are interested in the flux at E_0 due to an energy increment E_k generated by the kicker. The number of particles present in the shaded area is

$$n = \psi(E_0)E_k - \tfrac{1}{2}E_k\left(\frac{\partial \psi}{\partial E}E_k\right). \tag{7.158}$$

On the time scale associated with the kicker frequency, the particle distribution changes very little. Therefore, we may average over a time interval sufficiently short that ψ does not change, but sufficiently long to get meaningful averages of the kicker voltage factors. We get

$$\langle n \rangle = \psi(E_0)\langle E_k \rangle - \tfrac{1}{2}\langle E_k^2 \rangle \left(\frac{\partial \psi}{\partial E}\right)_{E_0}. \tag{7.159}$$

Therefore, the average flux of particles passing through this region will be

$$J = \frac{d\langle n \rangle}{dt} = \psi \frac{d}{dt}\langle E_k \rangle - \frac{1}{2}\frac{d}{dt}\langle E_k^2 \rangle \frac{\partial \psi}{\partial E}. \tag{7.160}$$

By comparison with our earlier expression for the flux, the coherent force coefficient and the diffusion (noise) coefficient are given by

$$C(E) = \frac{d}{dt}\langle E_k \rangle \tag{7.161}$$

$$D(E) = \frac{1}{2}\frac{d}{dt}\langle E_k^2 \rangle. \tag{7.162}$$

We may now apply the above relationships to momentum stacking and cooling. We will examine two cases for which the particle density does not depend upon time. In such equilibrium circumstances the Fokker–Planck equation tells us that the flux is constant. We first consider a simple example in which the flux is zero. Suppose there is a coherent force driving particles toward some central energy E_0, where the force is proportional to the energy deviation $E - E_0$, and suppose that the diffusion force is a constant. Then $C(E) = -\alpha(E - E_0)$ and $D(E) = D_0$. So this static situation is described by

$$J = -\alpha(E - E_0)\psi - D_0\frac{\partial\psi}{\partial E} = 0. \tag{7.163}$$

The solution to the above equation is the Gaussian

$$\psi = \psi_0 e^{-\alpha(E-E_0)2/2D_0}. \tag{7.164}$$

A large particle density results if the noise (D_0) is small and if the restoring force (α) is large.

Now suppose we were to introduce a small group of particles with central energy E_i, where $E_i - E_0$ is large compared to the rms of the distribution above. Then the coherent force would dominate the force due to diffusion, and this small group would be driven toward the larger distribution over some time interval, as depicted in Figure 7.14. Such a scenario is referred to as momentum stacking; small pulses of particles are continuously injected into the synchrotron at an energy E_i and then are collected into an equilibrium distribution with a central core at E_0.

The remarks concerning momentum stacking in the preceding paragraph suggest the basis for the method of antiproton accumulation used at CERN and Fermilab. In this method—the Van der Meer method—the flux is constant with time, with particles continuously being injected into the accumulator storage ring. We note that the coherent force, in terms of the voltage $V(E)$ applied by the kicker each turn, would be

$$C(E) = \frac{eV(E)}{T} \tag{7.165}$$

where T is the revolution period. To arrive at an approximate expression for

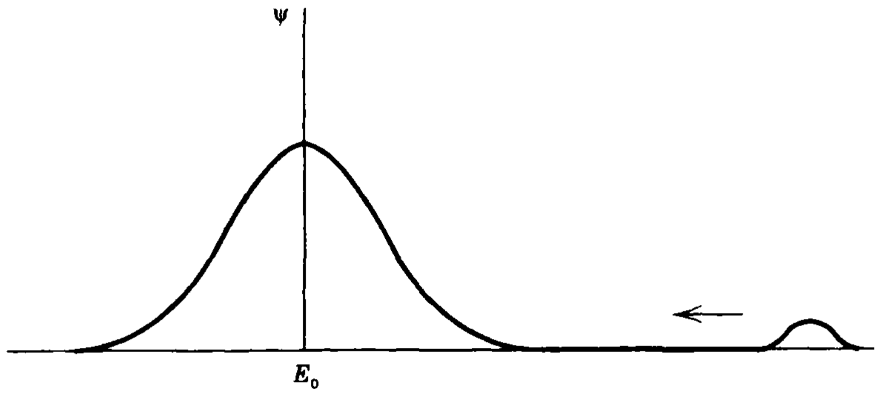


Figure 7.14. Momentum stacking.

the diffusion coefficient, assume that the expectation value of E_k^2 arises solely from the incoherent noise in the sample. Then, recalling the argument in the preceding section relating $\langle x^2 \rangle$ to $\langle x \rangle$,

$$\langle E_k^2 \rangle = \langle E_k \rangle^2 \times N_s. \tag{7.166}$$

In this case, since we are sampling frequencies (and therefore energies), the number of particles in the sample is proportional to ψ. Hence, we expect the diffusion coefficient to be of the form

$$D(E) = AV^2\psi, \tag{7.167}$$

where A is a constant determined by the design of the cooling system.

So, putting this all together, the constant flux is given by

$$J = \frac{eV}{T}\psi - AV^2\psi\,\frac{\partial\psi}{\partial E} = J_0. \tag{7.168}$$

Solving for $\partial\psi/\partial E$, we have

$$\frac{\partial\psi}{\partial E} = -\frac{J_0}{AV^2\psi} + \frac{e}{AVT}. \tag{7.169}$$

We now choose a kicker voltage which will make $\partial\psi/\partial E$ as large as possible:

$$V = \frac{2TJ_0}{e\psi}. \tag{7.170}$$

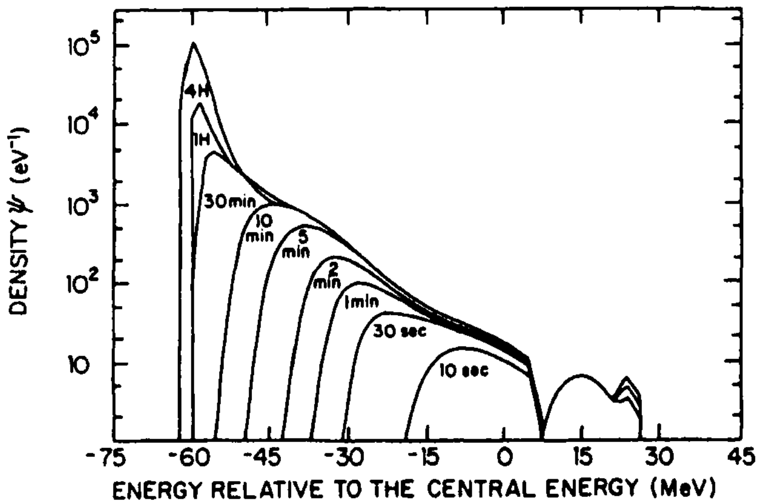


Figure 7.15. Design curves for antiproton energy density at FNAL Accumulator Ring, showing development of the "$\bar{p}$ stack" over time. From Tollestrup and Dugan, with permission.

So we have

$$\frac{\partial\psi}{\partial E} = -\frac{e^2\psi}{4J_0T^2A} + \frac{e^2\psi}{2J_0T^2A} = \frac{e^2\psi}{4AT^2J_0} \equiv \frac{\psi}{E_d}, \tag{7.171}$$

or

$$\psi = \psi_0 e^{(E-E_i)/E_d}. \tag{7.172}$$

Therefore, in the equilibrium state, there is a constant flux of particles being injected at energy E_i; over time, the particle density increases exponentially with the energy difference $E - E_i$ as shown in Figure 7.15. [In this figure, the particle flux is negative (from the right) and hence the density increases to the left as shown.]

To generate the density profile described above, the kicker voltage must be given by

$$V = \frac{2TJ_0}{e\psi} = \frac{2TJ_0}{e\psi_0}e^{-(E-E_i)/E_d}, \tag{7.173}$$

which tells us that the particles in the higher density region need to receive less kick in an exponential fashion. As can be seen in Figure 7.15, the particle density in the core can be increased many orders of magnitude in just a few hours. For a sense of scale, the central energy of this accumulator ring is 8 GeV.

7.4 SOME REMARKS ON BEAM DISTRIBUTIONS

We have often, throughout the text, assumed that particle distributions are Gaussian in order to perform calculations. The question always arises whether or not this is a reasonable assumption. It is easy to conceive of ways of producing beam distributions which are not Gaussian, such as those produced in a hadron storage ring by deflecting the beam with a pulsed kicker magnet and allowing filamentation to occur; indeed, this experiment has been

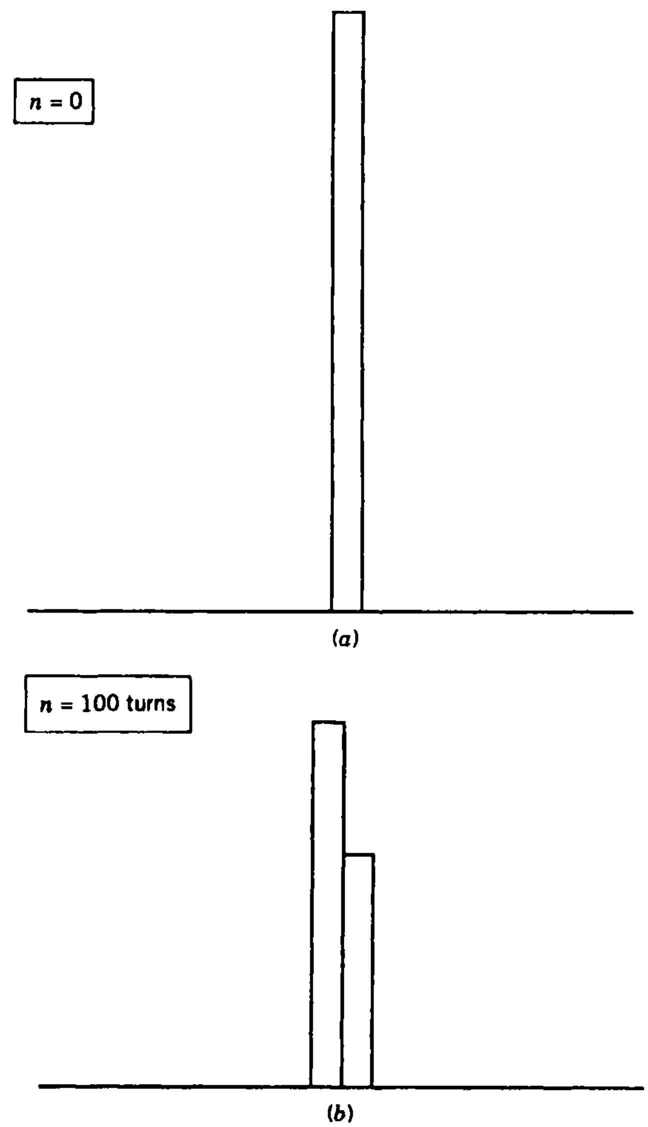


Figure 7.16. Histograms showing the development of a particle distribution due to random changes in particle trajectories. The initial particle distribution function is given by $n(x) = N\delta(0)$, where N is the total number of particles (200 for the case shown) and $\delta(x)$ is the Dirac δ-function. Independent of the choice of the random process, the resulting distribution will appear Gaussian after enough time has elapsed.

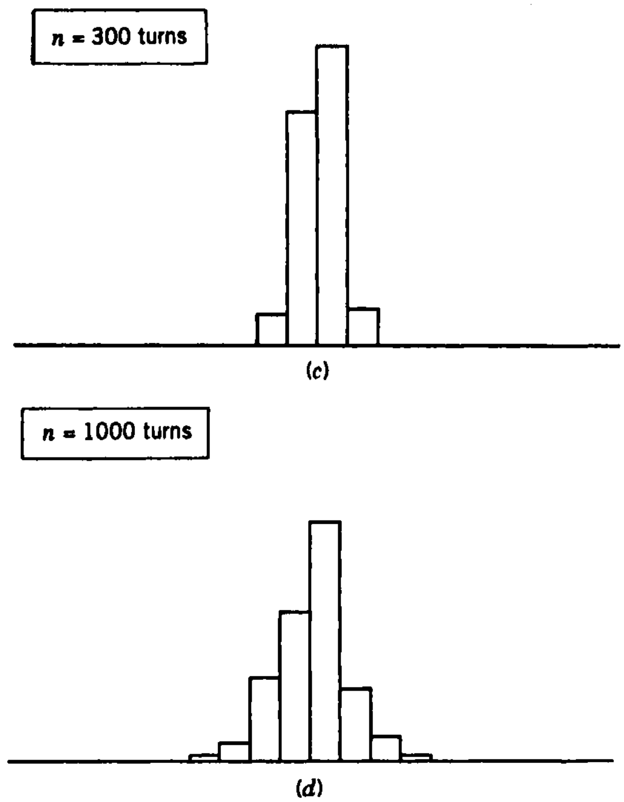


Figure 7.16. *(Continued).*

performed many times. But if care is taken to match trajectories and optical properties at injection and to minimize other coherent disturbances to the beam, then abrupt changes to individual particle trajectories are apt to be dominated by purely random processes.

Consider, for example, a zero emittance initial beam distribution—that is, all the particles start out at the origin of phase space. Now suppose that on each turn about the accelerator each particle receives a random change in the slope of its trajectory—either a positive increment of one unit, a negative increment of one unit, or no change at all. Each particle will undergo its own independent random walk. The results of such a scenario are shown in Figure 7.16. Over a large number of such increments and for a large number of particles, the distribution of transverse positions will tend to have the shape of the normal (Gaussian) distribution function. This is just the manifestation of the central limit theorem of probability theory:[4] the distribution function for a sum of random variables approaches the normal distribution function as the number of variables in the sum increases. The most powerful aspect of the theorem is that the result is independent of how the fluctuations are distributed, just so long as they are random.

[4]See, for example, H. J. Larson, *Introduction to Probability Theory and Statistical Inference*, John Wiley & Sons, New York, 1974.

Hence, one can imagine a beam of protons, each particle having suffered various random deflections due to scattering from gas molecules, intrabeam scattering, perturbations in magnetic fields, power supply noise, mechanical vibrations, and so on. So long as the various events are random and uncorrelated among the various particles, the distribution function describing the position of a particle which has been subjected to all of these random processes will be Gaussian, to good approximation.

At the beginning of this chapter it was mentioned that discussion of electron beam emittance would be left to Chapter 8. There we will see that the dominant source of emittance fluctuation is the emission of photons due to synchrotron radiation, an inherently random process. Since the ensuing damping times are, in general, short, one would expect that even correlated processes such as injection errors might be swamped by the synchrotron radiation effects. Thus, electron beams in circular accelerators are certainly expected to have Gaussian distributions, and that is what is observed. The fact that proton beams in large synchrotrons also typically appear Gaussian suggests that the central limit theorem is at work here as well.

PROBLEMS

1. Compute the increase in the normalized (39%) emittance due to a 1 mm amplitude steering error observed at a point where $\beta = 100$ m for injection energies of (a) 8 GeV, (b) 150 GeV, and (c) 2 TeV.

2. Suppose an injection line leading into a planar synchrotron leads to a nonzero value for the vertical dispersion function. Show that the variance of the resulting vertical distribution after dilution takes place is related to the variance of the incoming distribution by

$$\sigma^2 = \sigma_0^2 + \frac{1}{2}\left(\Delta D \frac{\sigma_p}{p}\right)^2,$$

where σ_p/p is the rms of the distribution in $\Delta p/p$, and ΔD is the value of the dispersion function delivered by the beamline to the observation point. The slope of the dispersion function is assumed to be well matched.

3. Show that the quantity F used in the discussion of amplitude function mismatch may be written as

$$F = 1 - \tfrac{1}{2}\det(J - J_0),$$

where the J's are the matrices of Courant-Snyder parameters used in the text. Here, J_0 reflects the values of the synchrotron lattice, while J contains the parameters delivered by the beamline.

4. Using the result of the previous problem, show that a quadrupole located in a transfer line between two accelerators which has a field error of $\Delta B'/B'$ will produce an emittance dilution given by

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{1}{2}\left(\frac{\beta_0}{f}\right)^2\left(\frac{\Delta B'}{B'}\right)^2,$$

where f is the nominal focal length of the quadrupole and β_0 is the design value of the amplitude function at the location of the quadrupole.

5. Show that $\det(\Delta J)$ is an invariant within an unperturbed lattice, independent of longitudinal coordinate s. Thus, the choice of an injection point is arbitrary when discussing the mismatch of Courant-Snyder parameters.

6. Show that the emittance dilution factor due to an amplitude function mismatch can be written as

$$F = 1 + \frac{1}{2}\left[\frac{(\Delta\beta/\beta_0)^2 + (\alpha_0(\Delta\beta/\beta_0) - \Delta\alpha)^2}{1 + (\Delta\beta/\beta_0)}\right]$$

7. Downstream of a gradient perturbation, show that the amplitude function mismatch propagates through the unperturbed lattice of a synchrotron obeying the differential equation

$$\frac{d^2}{d\phi^2}\left(\frac{\Delta\beta}{\beta_0}\right) + 4\nu_0^2\left(\frac{\Delta\beta}{\beta_0}\right) = -2\nu_0^2\det(\Delta J)$$

where $\phi \equiv \psi/\nu_0$ is the reduced phase and ν_0 is the unperturbed tune.

8. Suppose a synchronous transfer occurs between two accelerators which have their RF frequencies and bucket areas properly matched in order to preserve longitudinal emittance in the transfer. If the bunch area in phase space is much smaller than the bucket area, compute the change in longitudinal emittance due to a small (a) energy mismatch, (b) phase mismatch.

9. In the text, we discuss how injection errors lead to an increase in emittance. Liouville's theorem states that phase space density is conserved for Hamiltonian systems. Explain this apparent contradiction.

10. Consider the $x, (\beta x' + \alpha x)$ phase space distribution generated by a steering error of amplitude Δx inflicted upon a Gaussian beam of initial rms size σ_0. Compute the radius a_0 of the phase space circle which contains 39% of the injected particles. Define a dilution factor a_0^2/σ_0^2, and compare this with the expression $\sigma^2/\sigma_0^2 = 1 + \frac{1}{2}\Delta x^2$ found in the text.

11. Verify that

$$f(Z,\tau) = \sum_n c_n J_0(\lambda_n \sqrt{Z}) e^{-\lambda_n^2 \tau/4}$$

is indeed the appropriate solution of the diffusion equation, as was stated in Equation 7.77.

12. Suppose the normalized emittance of the beam injected into the Fermilab Main Ring is 15π mm mrad (95%). Assume that the average vacuum pressure is 5×10^{-7} Torr. If the limiting half aperture of the accelerator is 10 mm at a location of $\langle\beta\rangle = 50$ m, estimate the fractional beam loss due to scattering with the residual gas after 4 seconds for an injection energy of (a) 8 GeV and (b) 20 GeV. It may be helpful to make use of the graphs in the text.

13. Consider a beam which uniformly populates $x, (\beta x' + \alpha x)$ phase space out to a radius a_0. Show that the solution to the diffusion equation is

$$f(Z,\tau) = \sum_n \frac{2J_1(\lambda_n a_0/a)}{\lambda_n J_1^2(\lambda_n)} \frac{a}{a_0} J_0(\lambda_n \sqrt{Z}) e^{-\lambda_n^2 \tau/4},$$

$$N(\tau) = 4\frac{a}{a_0} \sum_n \frac{J_1(\lambda_n a_0/a)}{\lambda_n^2 J_1(\lambda_n)} e^{-\lambda_n^2 \tau/4},$$

where a is the limiting aperture.

14. It is now common to inject negative hydrogen ions delivered by a linear accelerator into the first synchrotron of a large proton accelerator facility. The electrons are stripped by passing the incoming beam through a carbon foil. Estimate the emittance increase if the particles pass through a foil of 25 μm thickness every 1.6 μsec for a total of 16 μsec.

15. Suppose the luminosity lifetime

$$\frac{1}{\tau} \equiv -\frac{1}{\mathscr{L}} \frac{d\mathscr{L}}{dt}$$

in the Tevatron collider is 10 hours, and suppose it is due to transverse emittance growth of both particle species. If the emittance is attributed to RF noise, estimate the rms noise voltage. Assume $\epsilon_N(95\%) \approx 20\pi$ mm mrad. The horizontal lattice functions at the accelerating stations are $\beta = 72$ m, $\alpha = -0.47$, $D = 2.4$ m, and $D' = 0.02$. Comment on the role of gas scattering in the luminosity lifetime.

16. Transverse betatron cooling in the Fermilab Debuncher ring is carried out with a 2–4 GHz system. The orbit period is 1.6 μsec, the slip factor η is 0.006, the momentum spread may be characterized as 0.3%, and the initial noise-to-signal ratio is ~ 2. Calculate the optimum cooling rate for a beam of 10^7 antiprotons.

17. The appeal of electron cooling is easy to illustrate. In conventional kinetic theory, the gas temperature is related to the mean energy of the molecules by

$$\frac{\langle p^2 \rangle}{2m} = \frac{3}{2} kT.$$

So for an ion beam, one can define a "temperature" for each degree of freedom by

$$\frac{\langle p_{x0}^2 \rangle}{m}, \qquad \frac{\langle p_{y0}^2 \rangle}{m}, \qquad \frac{\langle p_{s0}^2 \rangle}{m},$$

where the Boltzmann constant has been supressed. The subscript "0" implies that the momenta are measured with respect to the rest frame of the beam centroid.

(a) Show that for one transverse degree of freedom

$$T_x = mc^2 \left(\frac{v}{c} \right)^2 \gamma^2 (\sigma')^2 = mc^2 \left(\frac{v}{c} \right) \gamma \frac{\epsilon_N}{\pi \beta},$$

where all the quantities are now measured in the laboratory frame. Here, $(\sigma')^2 = \langle (x')^2 \rangle$. Note that, because of the presence of the amplitude function β, the temperature is a function of position.

(b) Evaluate T_x for typical injection parameters from a proton linac into a synchrotron. Take $(v/c)\gamma \approx 0.7$, $\epsilon_N = \pi/2$ mm mrad, and $\beta = 10$ m.

(c) Repeat the calculation for the longitudinal degree of freedom. Show that

$$T_s = mc^2 \left(\frac{v}{c} \right)^2 \sigma_p^2, \qquad \sigma_p^2 \equiv \left\langle \left(\frac{\Delta p}{p} \right)^2 \right\rangle.$$

Take $\sigma_p = 10^{-3}$ to obtain a numerical estimate.

(d) In electron cooling, an electron beam traveling at the same speed as the ion beam centroid interchanges energy with the ion beam. Estimate the temperature of an electron beam emitted from a hot cathode, in the same units as that used for the ion temperatures above.

18. In this chapter, we have concentrated on transverse emittance growth, in large part because of the importance of transverse emittance to the luminosity of a collider. But longitudinal emittance cannot be ignored, for eventually dilution processes may lead to loss of particles from stable buckets. Derive an expression for longitudinal emittance growth analogous to Equation 7.112.