Wakefield Notes

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1. Longitudinal Wakes

We want to obtain the monopole, longitudinal, single particle (or "delta function") wake. From Palumbo, Eq. 25:

$$W_z(\mathbf{r},\tau) = \frac{1}{q_1}\int_{-\infty}^{\infty} i_b(\tau')w_z(\mathbf{r},\tau-\tau')d\tau' \tag{1}$$

Here W_z is the wake produced by a bunch (obtained from T3P), sampled at transverse position $\mathbf{r}$ and time τ. q_1 is the total bunch charge, i_b is the instantaneous beam current, and w_z is the single particle wake. Note that both W_z and w_z have units of V/C. We want to solve for w_z.

First, let's rewrite everything in terms of discrete quantities. Eq. 1 becomes:

$$W_z(\tau) = \frac{1}{q_1}\sum_{\tau'=0}^{\tau} i_b(\tau')w_z(\tau-\tau')\Delta\tau \tag{2}$$

Here we have broken up the bunch into discrete chunks of length $\Delta\tau$, and charge $i_b(\tau')\Delta\tau$. We define $\tau' = 0$ as the "head" of the bunch (typically 4σ before the center), and enforce causality by truncating the sum at $\tau' = \tau$ are defined as where the bunch "starts" and "ends" . We have also dropped $\mathbf{r}$, since the monopole wake has no transverse dependence.

Using the convolution theorem, we obtain:

$$W_z(\tau) = \frac{\Delta\tau}{q_1}\mathbf{F^{-1}}[\hat{i_b}(\omega)\hat{w_z}(\omega)] \tag{3}$$

where $\mathbf{F^{-1}}$ is the (discrete) inverse Fourier transform, and a $\hat{i_b}$ and $\hat{w_z}$ are transformed quantities. Taking the Fourier transform of both sides and solving for $\hat{w_z}$ gives us

$$\hat{w_z}(\tau) = \frac{q_1}{\Delta\tau}\frac{\hat{W}_z(\omega)}{\hat{i_b}} \tag{4}$$

Taking the inverse transform, we end up with

$$w_z(\tau) = \frac{q_1}{\Delta\tau}\mathbf{F^{-1}}\left[\frac{\hat{W}_z(\tau)}{\hat{i_b}}\right] \tag{5}$$

2. Transverse Wakes

The Panofsky-Wenzel theorem states that

$$\frac{1}{c}\frac{\partial}{\partial\tau}\vec{w_\perp}(\vec{r},\vec{r_1};\tau) = \nabla_{\perp,r}w_z(\vec{r},\vec{r_1};\tau) \tag{6}$$

If we expand to first order in $\vec{r_1}$ (assuming a small displacement for the leading particle), this becomes (P92):

$$\frac{1}{c}\frac{\partial}{\partial\tau}\vec{w_\perp}(\vec{r},\vec{r_1};\tau) = \nabla_{\perp,r}\left(w_z(\vec{r},0,\tau) + \left[\nabla_{\perp,r_1}w_z(\vec{r},\vec{r_1};\tau)\right]|_{r_1=0}\cdot\vec{r_1}\right) \tag{7}$$

In an axially symmetric structure, w_z expanded to 2nd order is:

$$w_z(\vec{r},\vec{r1};\tau) \approx w_{z,0}(\tau) + rr_1cos(\phi)\overline{w_{z,1}}(\tau) \tag{8}$$

So Equation 7 simplifies to:

$$\vec{w_\perp}(\vec{r},\vec{r_1};\tau) = -cr_1\int_{-\infty}^{\tau}\overline{w_{z,1}}(\tau')d\tau' \tag{9}$$

Note that the transverse dipole wake depends only on the displacement of the leading particle.

2.1. Without symmetry

Unfortunately, in CESR we don't have axial symmetry. Instead, let's write out w_z to second order in cartesian coordinates:

$$
\begin{aligned}
w_z(x,y,x_1,y_1,\tau) = w_{z,0} &+ \frac{\partial w_z}{\partial x}x + \frac{\partial w_z}{\partial y}y + \frac{\partial w_z}{\partial x_1}x_1 + \frac{\partial w_z}{\partial y_1}y_1 \\
&+ \frac{\partial^2 w_z}{\partial x^2}x^2 + \frac{\partial^2 w_z}{\partial x \partial y}xy + \frac{\partial^2 w_z}{\partial x \partial x_1}xx_1 + \frac{\partial^2 w_z}{\partial x \partial y_1}xy_1 + \frac{\partial^2 w_z}{\partial y^2}y^2 \\
&+ \frac{\partial^2 w_z}{\partial y \partial x_1}yx_1 + \frac{\partial^2 w_z}{\partial y \partial y_1}yy_1 + \frac{\partial^2 w_z}{\partial x_1^2}x_1^2 + \frac{\partial^2 w_z}{\partial x_1 \partial y_1}x_1 y_1 + \frac{\partial^2 w_z}{\partial y_1^2}y_1^2 \quad (10)
\end{aligned}
$$

Going through these terms we observe:

- The longitudinal monopole wake ($w_{z,0}$), which does not depend on displacement.
- Terms that are linear in the displacement of the leading particle (x_1, y_1), or trailing particle (x, y). These will be nonzero only if we don't have top/down or left/right symmetry.
- Higher order terms, which we can neglect for now (though they are important for calculating the transverse wake).

Plugging Eq. 10 into Eq. 7 gives us:

$$
\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial \tau}\vec{w_\perp}(\vec{r},\vec{r_1};\tau) = &\left(\frac{\partial w_z}{\partial x} + 2\frac{\partial^2 w_z}{\partial x^2}x + \frac{\partial^2 w_z}{\partial x \partial y}y + \frac{\partial^2 w_z}{\partial x \partial x_1}x_1 + \frac{\partial^2 w_z}{\partial x \partial y_1}y_1\right)\hat{x} \\
&+ \left(\frac{\partial w_z}{\partial y} + 2\frac{\partial^2 w_z}{\partial y^2}y + \frac{\partial^2 w_z}{\partial x \partial y}x + \frac{\partial^2 w_z}{\partial y \partial y_1}y_1 + \frac{\partial^2 w_z}{\partial y \partial x_1}x_1\right)\hat{y} \quad (11)
\end{aligned}
$$

Note that there are actually four general types of transverse wake:

- A "transverse monopole wake" (e.g. $\frac{\partial w_z}{\partial y}$), which is not dependent on position. This is caused by a lack of top/down or left/right symmetry.
- A "quadrupolar" or "detuning" wake (e.g. $2\frac{\partial^2 w_z}{\partial y^2}y$), proportional to the displacement of the trailing particle. This comes from a lack of axial symmetry.
- The familiar "dipole" transverse wake (e.g. $\frac{\partial^2 w_z}{\partial y \partial y_1}y_1$), proportional to the displacement of the leading particle.

- Scary looking coupling terms (e.g. $\frac{\partial^2 w_z}{\partial y \partial x_1} x_1$), which appear when we have *neither* top/down nor left/right symmetry.

Fortunately, in CESR we do have approximate top-down and left-right symmetry for most elements. Therefore $w_z(x, y, x_1, y_1, \tau) = w_z(-x, y, -x_1, y_1, \tau)$ and $w_z(x, y, x_1, y_1, \tau) = w_z(x, -y, x_1, -y_1, \tau)$. So Eq. 10 simplifies to:

$$\begin{aligned} w_z(x, y, x_1, y_1, \tau) &= w_{z,0} + \frac{\partial^2 w_z}{\partial x^2} x^2 + \frac{\partial^2 w_z}{\partial x \partial x_1} x x_1 + \frac{\partial^2 w_z}{\partial y^2} y^2 \\ &\quad + \frac{\partial^2 w_z}{\partial y \partial y_1} y y_1 + \frac{\partial^2 w_z}{\partial x_1^2} x_1^2 + \frac{\partial^2 w_z}{\partial y_1^2} y_1^2 \end{aligned} \tag{12}$$

Plugging this expression into Eq. 7 gives us:

$$\frac{1}{c} \frac{\partial}{\partial \tau} \vec{w_\perp}(\vec{r}, \vec{r_1}; \tau) = \left(2 \frac{\partial^2 w_z}{\partial x^2} x + \frac{\partial^2 w_z}{\partial x \partial x_1} \vec{x_1} \right) \hat{x} + \left(2 \frac{\partial^2 w_z}{\partial y^2} y + \frac{\partial^2 w_z}{\partial y \partial y_1} y_1 \right) \hat{y} \tag{13}$$

The only terms that survive are the "dipole" and "quadrupolar" terms.

If we assume left/right but not top/down symmetry (e.g. for the lump pumps), we get:

$$\begin{aligned} \frac{1}{c} \frac{\partial}{\partial \tau} \vec{w_\perp}(\vec{r}, \vec{r_1}; \tau) = &\left(2 \frac{\partial^2 w_z}{\partial x^2} x + \frac{\partial^2 w_z}{\partial x \partial x_1} x_1 \right) \hat{x} \\ &+ \left(\frac{\partial w_z}{\partial y} + 2 \frac{\partial^2 w_z}{\partial y^2} y + \frac{\partial^2 w_z}{\partial y \partial y_1} y_1 \right) \hat{y} \end{aligned} \tag{14}$$

So the first three types of transverse wake listed above are significant.

2.2. Calculating Transverse Wakes

So how do we determine $\vec{w_\perp}$ from T3P? The first two terms in Eq. 13 can be determined simply by varying the witness position for a given on-axis wake. The last two terms are more difficult. If we displace both the leading and trailing charge by an amount Δy and subtract the on-axis wake, we get:

$$w_z(0, \Delta y, 0, \Delta y) - w_{z,0} = \left(\frac{\partial^2 w_z}{\partial y^2} + \frac{\partial^2 w_z}{\partial y \partial y_1} + \frac{\partial^2 w_z}{\partial y_1^2} \right) \Delta y^2 \tag{15}$$

If the leading and trailing are displaced in the opposite direction, we get:

$$w_z(0, \Delta y, 0, -\Delta y) - w_{z,0} = \left(\frac{\partial^2 w_z}{\partial y^2} - \frac{\partial^2 w_z}{\partial y \partial y_1} + \frac{\partial^2 w_z}{\partial y_1^2} \right) \Delta y^2 \tag{16}$$

The T3P method of using an electric boundary in the center of the chamber is equivalent to having a positive charge at Δy and a negative charge at $-\Delta y$:

$$\begin{aligned} w_z(0, \Delta y, 0, \Delta y) - w_z(0, \Delta y, 0, -\Delta y) &= \left(\frac{\partial^2 w_z}{\partial y^2} + \frac{\partial^2 w_z}{\partial y \partial y_1} + \frac{\partial^2 w_z}{\partial y_1^2} \right) \Delta y^2 \\ &\quad - \left(\frac{\partial^2 w_z}{\partial y^2} - \frac{\partial^2 w_z}{\partial y \partial y_1} + \frac{\partial^2 w_z}{\partial y_1^2} \right) \Delta y^2 \\ &= 2 \frac{\partial^2 w_z}{\partial y \partial y_1} \Delta y^2 \end{aligned} \tag{17}$$

Note that, if we had cylindrical symmetry, $\frac{\partial^2 w_z}{\partial y^2} = \frac{\partial^2 w_z}{\partial y_1^2} = 0$, and all of these methods would be equivalent. But, since we don't, the electric boundary method is preferred for determining the transverse wake.