Inflector Acceptance

D.Rubin

September 25, 2013

Twiss Parameters

Consider linear transport in one dimension through a beam line, The phase space 2-vector

$$\mathbf{x} = (x, x').$$

x and x' are offset and angle with respect to some reference orbit. As long as the forces are linear, (quadrupole, dipole, drift) or nearly so, the vector is propagated through those elements with a 2X2 unity determinant matrix M.

$$\mathbf{x}^f = M_{fi}\mathbf{x}^i$$

In general the forces may be nonlinear. Then the matrix M is the Jacobian of the mapping from $\mathbf{x}^i$ to $\mathbf{x}^f$. If $\mathbf{x}^f(\mathbf{x}^i)$ then

$$M = \begin{pmatrix} \frac{\partial x_1^f}{\partial x_1^i} & \frac{\partial x_1^f}{\partial x_2^i} \\ \\ \frac{\partial x_2^f}{\partial x_1^i} & \frac{\partial x_2^f}{\partial x_2^i} \end{pmatrix}$$

The Jacobean, M, has unit determinant.

Scalar invariant

Define the scalar

$$s = \mathbf{x}^T A\mathbf{x}$$

Then define A, so that

$$s = \begin{pmatrix} x & x' \end{pmatrix} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

A can be any 4 parameters that we like. No loss of generality by setting $A = A^T$ since we only need three numbers to define the most general scalar combination of x and x'. Now suppose we propagate $\mathbf{x}_b \to \mathbf{x}_e$ with the help of M. Then $\mathbf{x}_e = M\mathbf{x}_b$ and

$$s = \mathbf{x}_b^T M^T (M^T)^{-1} A_b M^{-1} M\mathbf{x}_b = \mathbf{x}_e^T (M^T)^{-1} A_b M^{-1}\mathbf{x}_e = \mathbf{x}_e^T A_e\mathbf{x}_e$$

Evidently s is invariant as long as

$$A_e = (M^T)^{-1} A_b M^{-1} \tag{1}$$

Or

$$\begin{pmatrix} \gamma_e & \alpha_e \\ \alpha_e & \beta_e \end{pmatrix} = (M^T)^{-1} \begin{pmatrix} \gamma_b & \alpha_b \\ \alpha_b & \beta_b \end{pmatrix} M^{-1} \tag{2}$$

We use the transfer matrix to propagate the twiss parameters. Another thing, from Equation 1 we see that $|A_e| = |A_b|$. The determinant of the twiss matrix is invariant. We set it to unity for convenience. Then $\gamma\beta - \alpha^2 = 1$. It should be clear that except for the unit determinant requirement, the twiss parameters (α, β, γ) are totally unconstrained. We assign them whatever values we like at one location along the beam line and they are determined everywhere else. But so far not much there.

In a ring, we typically choose the twiss parameters so that they are single valued. That is $A(s) = A(C + s)$ where C is the circumference. That way there is a unique set of α and β at each point around the ring. We have

$$A = (M^T)^{-1} A M^{-1}$$

where M is the full turn matrix. Turns out that

$$M = \begin{pmatrix} \cos\mu + \alpha \sin\mu & \beta \sin\mu \\ -\gamma \sin\mu & \cos\mu - \alpha \sin\mu \end{pmatrix}$$

where $\mu = 2\pi\nu$ and ν is the tune.

From the full turn matrix we can determine the stability of the system by extracting the eigenvalues.

$$M \vec{v}_i = \lambda_i \vec{v}_i$$

The behavior after n turns is

$$M^n \vec{v}_i = \lambda_i^n \vec{v}_i$$

The system is stable if the eigenvalues are unimodular. (Remember that $\det M = 1 \rightarrow \lambda_1 \lambda_2 = 1$.) Then $\lambda_i = e^{i\mu}$ where μ is real. For a ring, we can define twiss parameters by requiring that they be single valued. Twiss parameters in a transfer line are determined by the distribution of particles. It is clear from the above, that the twiss parameters will establish how the phase space coordinates are correlated, how x and x' are related. Consider the matrix of second moments. (The average of the first moments is zero).

$$\Sigma = \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}$$

The matrix is constructed as

$$\mathbf{x}\mathbf{x}^T = \begin{pmatrix} x \\ x' \end{pmatrix} \begin{pmatrix} x & x' \end{pmatrix}$$

and

$$\mathbf{x}_e \mathbf{x}_e^t = M \mathbf{x}_b \mathbf{x}_b^T M^T$$

Then

$$\langle \mathbf{x}_e \mathbf{x}_e^t \rangle = \langle M \mathbf{x}_b \mathbf{x}_b^T M^T \rangle = M \langle \mathbf{x}_b \mathbf{x}_b^T \rangle M^T$$

or

$$\begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}_e = M \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}_b M^T$$

which looks almost like the rule for propagating the twiss matrix A. In fact we had that

$$A_e = (M^T)^{-1} A_b M^{-1}.$$

Then

$$A_e^{-1} = M A_b^{-1} M^T$$

and the matrix A_e^{-1} transforms the same as the Σ matrix. The elements of the two matrices are evidently related. In particular

$$A^{-1} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}$$

The ϵ is some unknown scale factor.

$$|A^{-1}| = |\Sigma| \rightarrow \epsilon^2 = \langle xx \rangle \langle x'x' \rangle - \langle xx' \rangle^2$$

Since

$$\sigma^2 = \langle xx \rangle, \quad (\sigma')^2 = \langle x'x' \rangle$$

we have that

$$\beta = \frac{\sigma^2}{\epsilon}, \quad \gamma = \frac{(\sigma')^2}{\epsilon}$$

The twiss parameters are determined by the distribution of the phase space coordinates of the trajectories.

Computing transfer matrix with tracking

Sometimes it is difficult to construct the transfer matrix from first principles. The matrix conveys the focusing effect of the element but to build the matrix we essentially need to know all the gradients etc. Alternatively we can do tracking. Remember that the transfer matrix is the Jacobian of the map

$$M = \begin{pmatrix} \frac{\partial x_1^f}{\partial x_1^i} & \frac{\partial x_1^f}{\partial x_2^i} \\ \\ \frac{\partial x_2^f}{\partial x_1^i} & \frac{\partial x_2^f}{\partial x_2^i} \end{pmatrix}$$

The strategy is essentially to compute the derivatives numerically. If we know the reference trajectory (uniquely defined in a circular machine, but not so straightforward in a transfer line like the entrance through the backlog iron and into the inflector), we can calculate trajectories displaced by Δx and $\Delta x'$ from the reference and build M. In principle we need only three non degenerate trajectories to determine the 2X2 matrix for horizontal or vertical motion as well as the reference. Write

$$\begin{aligned} M_{i\rightarrow f}(\mathbf{x}_{in} - \mathbf{x}_{ref}) &= \mathbf{x}_f - \mathbf{x}_{ref} \\ M_{i\rightarrow f}\mathbf{x}_{in} - (M_{i\rightarrow f} - I)\mathbf{x}_{ref} &= \mathbf{x}_f \\ M_{i\rightarrow f}\mathbf{x}_{in} - \mathbf{x}_0 &= \mathbf{x}_f \end{aligned} \tag{3}$$

Next construct

$$N = \begin{pmatrix} M_{i\rightarrow f} & \mathbf{x_0} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & x_0 \\ m_{21} & m_{22} & x_0' \\ 0 & 0 & 1 \end{pmatrix}$$

and Equation 3 becomes

$$N \begin{pmatrix} \mathbf{x}_{in} \\ -1 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_f \\ -1 \end{pmatrix}.$$

The goal remember is to compute $M_{i \to f}$ and $\mathbf{x}_{ref}$. Choose three distinct values for $\mathbf{x}_{in}$, namely $\mathbf{x}_{in}^i$, $i = 1, 2, 3$, track each to $\mathbf{x}_f^i$ and we get

$$N \begin{pmatrix} \mathbf{x}_{in}^1 & \mathbf{x}_{in}^2 & \mathbf{x}_{in}^3 \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_f^1 & \mathbf{x}_f^2 & \mathbf{x}_f^3 \\ -1 & -1 & -1 \end{pmatrix}$$

Finally

$$N = \begin{pmatrix} \mathbf{x}_f^1 & \mathbf{x}_f^2 & \mathbf{x}_f^3 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{x}_{in}^1 & \mathbf{x}_{in}^2 & \mathbf{x}_{in}^3 \\ -1 & -1 & -1 \end{pmatrix}^{-1}$$

Extract M and $\mathbf{x}_{ref}$ from N as per above. The strategy is readily extended to the full 6 dimensional phase space where

$$\mathbf{x} \to \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix}$$

where $\delta = \Delta E/E$. So to determine the evolution of the phase space (that is the twiss parameters) through the iron and inflector into the ring we simply compute 7 trajectories. We can in principle use the same 7 trajectories to determine the transfer matrix between any two points along the reference orbit.

Description of transfer matrix

We can write the most general 2X2 determinant 1 matrix as

$$M = e^{\mu J}, \qquad J = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}$$

and $|J| = 1$. Then we see that $J^n = (-1)^n J$ and

$$e^{\mu J} = \cos \mu + J \sin \mu.$$

All well and good. Let's return to our expression for the invariant

$$s = \mathbf{x}^T A \mathbf{x}$$

Suppose we normalize the phase space vector so that

$$s = \mathbf{u}^T \mathbf{u} = \mathbf{x}^T G^T G \mathbf{x}$$

with $\mathbf{x} = G^{-1}\mathbf{u}$, which requires that $G^T G = A$. Then

$$G = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & -\sqrt{\beta} \end{pmatrix}$$

and

$$G^TG = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

Recall that

$$\begin{aligned} \mathbf{x}_1 &= M\mathbf{x}_0 \\ G_1^{-1}\mathbf{u_1} &= MG_0^{-1}\mathbf{u_0} \\ \rightarrow \mathbf{u_1} &= G_1MG_0^{-1}\mathbf{u_0} \end{aligned}$$

But the transformation that preserves the length of $\mathbf{u}$ is orthogonal and therefore

$$G_1MG_0^{-1} = R(\theta)$$

where R is a two by two rotation. Finally we can write

$$\begin{aligned} M &= G_1^{-1}R(\theta)G_0 \\ &= \begin{pmatrix} \sqrt{\beta_1} & 0 \\ -\frac{\alpha}{\sqrt{\beta_1}} & -\frac{1}{\sqrt{\beta_1}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_0}} & 0 \\ -\frac{\alpha_0}{\sqrt{\beta_0}} & -\sqrt{\beta_0} \end{pmatrix} \end{aligned}$$

The map from one point to another can always be written in terms of the twiss parameters at each point and the phase advance between them.

Next include dispersion. Write the 6x6 transfer matrix, assuming zero horizontal vertical coupling but finite horizontal and/or vertical dispersion

$$T = \begin{pmatrix} M_x & 0 & m_x \\ 0 & M_y & m_y \\ n_x & n_y & M_z \end{pmatrix}.$$

The dispersion vector with energy offset $\delta = 1$ is $(\eta_x, \eta_x', eta_y, \eta_y', l, 1)$. The dispersion vector is propagated the the transfer matrix T. If there is no RF element(something that changes the energy), it must be true that

$$m_{01} = \begin{pmatrix} 0 & \eta_1 \\ 0 & \eta_1' \end{pmatrix} - M_{01}\begin{pmatrix} 0 & \eta_0 \\ 0 & \eta_0' \end{pmatrix}$$

since then

$$T\begin{pmatrix} \eta_0(x) \\ \eta_0'(x) \\ \eta_0(y) \\ \eta_0'(y) \\ \sim \\ 1 \end{pmatrix} = \begin{pmatrix} M(x)_{01}\begin{pmatrix} \eta_0(x) \\ \eta_0'(x) \end{pmatrix} + m(x)_{01}\begin{pmatrix} \sim \\ \delta \end{pmatrix} \\ M(y)_{01}\begin{pmatrix} \eta_0(y) \\ \eta_0'(y) \end{pmatrix} + m(y)_{01}\begin{pmatrix} \sim \\ \delta \end{pmatrix} \\ \sim \end{pmatrix} = \begin{pmatrix} \eta_1(x) \\ \eta_1'(x) \\ \eta_1(y) \\ \eta_1'(y) \\ \sim \\ \delta \end{pmatrix}$$

which gives us all of the pieces we need to construct the transfer matrix from a point with twiss parameters $\beta_x^1, \alpha_x^1, \beta_y^1, \alpha_y^1, \eta_x^1, {\eta_x'}^1, \eta_y^1, {\eta_y'}^1$ to $\beta_x^2, \alpha_x^2, \beta_y^2, \alpha_y^2, \eta_x^2, {\eta_x'}^2, \eta_y^2, {\eta_y'}^2$ with horizontal and vertical phase advances $\phi_x^2 - \phi_x^1$ and $\phi_y^2 - \phi_y^1$.

Injection through inflector

Consider the mapping of the phase space as the muon beam passes through the hole in the backleg iron and cryostat, across the gap and through the inflector. The Jacobian of the map captures the field gradients that correspond to focusing. While the details of the field along the trajectory of the injected beam are complex, the outstanding feature is that the vertical field is close to zero at the boundary of the backlog iron and that it increases approximately linearly to 1.45T just inside the pole gap. The distance from the iron to the gap is about 40 cm. The corresponding gradient

$$\frac{\partial B_y}{\partial x} = \frac{1.45}{0.4}[\mathrm{T/m}].$$

The effect on the horizontal phase space is to defocus. Since $\nabla \times \mathbf{B} = 0$, the vertical gradient is related to the horizontal according to $\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x} = 0$. We know the gradient so we can compute the transfer matrix and propagate twiss parameters through the region. Another effect of the field non uniformity is to steer the beam along a trajectory with increasing curvature towards the center of the ring until it enters the inflector. In the inflector, we have a superposition of the magnet fringe and the inflector field. At the upstream end the inflector overcompensates and the beam curvature is outward. About 1/3 of the way into the inflector, the net field is very nearly zero and the trajectory straightens out. But the inflector field is uniform and has no effect on the gradient. We refer to the field map that Nathan extracted from Hugh Brown's notes. The vertical field along the path of the injected muons is shown in Figure 1

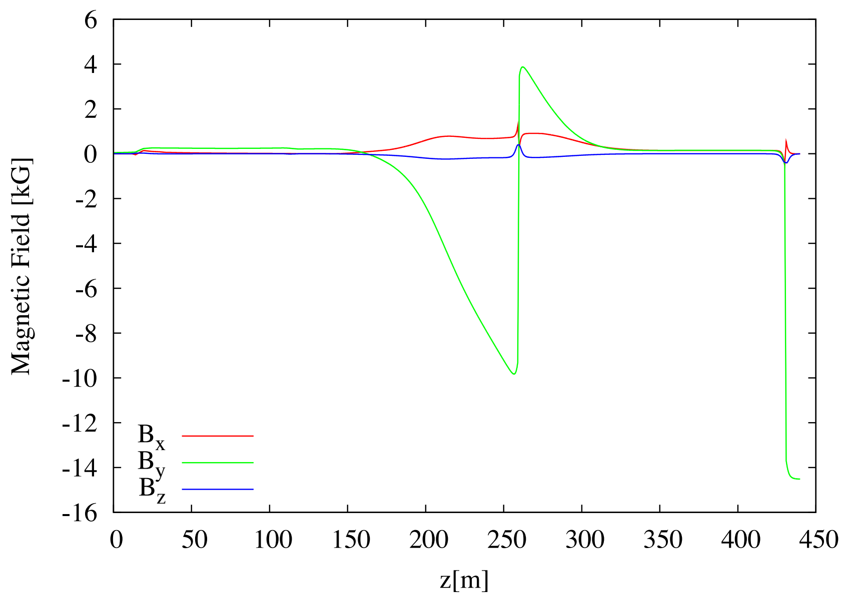

Figure 1: Magnetic field along path of injected muons

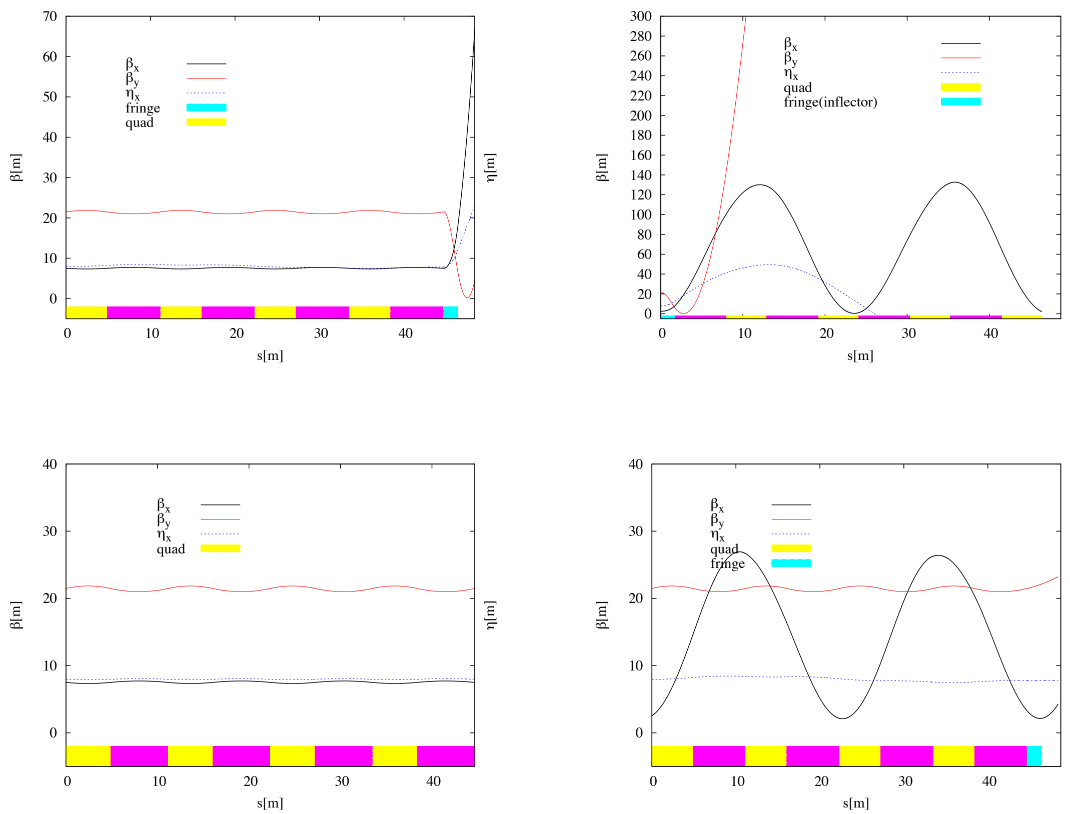

Figure 2: Twiss parameters