EXPANSION OF MAGNETIC FIELD IN CYLINDRICAL COORDINATES.

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In order to incorporate measurements of the azimuthal dependence of vertical and radial fields into a tracking program, we

- Write the general solution to LaPlace's equation in cylindrical coordinates, $\nabla^2\psi(\rho,\theta,z) = 0$
- Compute $\nabla\psi = \mathbf{B}$
- Fit $B_\rho(\rho=\rho_0,\theta,z=0)$ to the measured radial field data to determine coefficients
- Fit $B_z(\rho=\rho_0,\theta,z=0)$ to the measured vertical field data the rest of the coefficients

The general solution is

$$\psi = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)\left(\sinh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi) + \cosh(k_{mn}z)(C_{mn}\sin m\phi + D_{mn}\cos m\phi)\right)$$

Then

$$\begin{aligned} B_z &= -\frac{\partial\psi}{\partial z} \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho)k_{mn}\left(\cosh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi)\right. \\ &\qquad \left. - \sinh(k_{mn}z)(C_{mn}\sin m\phi + D_{mn}\cos m\phi)\right) \end{aligned}$$

At $\rho=\rho_0, z=0$,

$$B_z(\rho_0,\phi,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(k_{mn}\rho_0)k_{mn}(A_{mn}\sin m\phi + B_{mn}\cos m\phi)$$

$$\begin{aligned} B_\rho &= -\frac{\partial\psi}{\partial\rho} \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\partial J_m}{\partial\rho}(k_{mn}\rho)\left(\sinh(k_{mn}z)(A_{mn}\sin m\phi + B_{mn}\cos m\phi)\right. \\ &\qquad \left. + \cosh(k_{mn}z)(C_{mn}\sin m\phi + D_{mn}\cos m\phi)\right) \end{aligned}$$

$$B_\rho(\rho_0,\phi,0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{\partial J_m}{\partial\rho}(k_{mn}\rho_0)(C_{mn}\sin m\phi + D_{mn}\cos m\phi)$$

Write the measured fields along the magic radius

$$\begin{aligned} B_z^m &= \sum_m (a_m \cos m\phi + b_m \cos m\phi) \\ B_\rho^m &= \sum_m (c_m \cos m\phi + d_m \cos m\phi) \end{aligned}$$

Then

$$\begin{aligned} a_m &= \sum_{n=1}^{\infty} J_m(k_{mn}\rho_0) k_{mn} A_{mn} \\ b_m &= \sum_{n=1}^{\infty} J_m(k_{mn}\rho_0) k_{mn} B_{mn} \\ c_m &= \sum_{n=1}^{\infty} \frac{\partial J_m}{\partial \rho}(k_{mn}\rho_0) k_{mn} C_{mn} \\ d_m &= \sum_{n=1}^{\infty} \frac{\partial J_m}{\partial \rho}(k_{mn}\rho_0) k_{mn} D_{mn} \end{aligned}$$

Define

$$\begin{aligned} A'_{mn} &= J_m(k_{mn}\rho_0) k_{mn} A_{mn} \\ B'_{mn} &= J_m(k_{mn}\rho_0) k_{mn} B_{mn} \\ C'_{mn} &= \frac{\partial J_m}{\partial \rho}(k_{mn}\rho_0) k_{mn} C_{mn} \\ D'_{mn} &= \frac{\partial J_m}{\partial \rho}(k_{mn}\rho_0) k_{mn} D_{mn} \end{aligned}$$

And

$$\sum_n A'_{mn} = a_m$$

etc. It appears that since I have no information at nonzero z or $\rho \neq \rho_0$ that all of the k_{mn} are degenerate. So I might as well only include $n = 1$ in the sum. The result is a perfectly good solution although perhaps not the most general. At least it will obey Maxwell and be consistent with the boundary conditions. Then

$$\begin{aligned} a_m &= J_m(k_{m1}\rho_0)k_{m1}A_{m1} \\ b_m &= J_m(k_{m1}\rho_0)k_{m1}B_{m1} \end{aligned}$$

$$\begin{aligned} c_m &= \frac{\partial J_m}{\partial \rho}(k_{m1}\rho_0)k_{m1}C_{m1} \\ d_m &= \frac{\partial J_m}{\partial \rho}(k_{m1}\rho_0)k_{m1}D_{m1} \end{aligned}$$

Let's drop the subscript n

Assume that $|\mathbf{B}| = \mathbf{B_z}$. a_m, b_m and c_m, d_m are the coefficients of the fits to the measured B_z and B_r respectively. Define

$$\begin{aligned} A_m &= a_m/(J_m(k_{m1}\rho_0)k_{m1}) \\ B_m &= b_m/(J_m(k_{m1}\rho_0)k_{m1}) \end{aligned}$$

$$\begin{aligned} C_m &= c_m/(\frac{\partial J_m}{\partial \rho}(k_{m1}\rho_0)k_{m1}) \\ D_m &= d_m/(\frac{\partial J_m}{\partial \rho}(k_{m1}\rho_0)k_{m1}) \end{aligned}$$

Then the fields

$$\begin{aligned} B_z &= \sum_{m=0}^{\infty} J_m(k_m\rho)k_m \left(\cosh(k_m z)(A_m \sin m\phi + B_m \cos m\phi)\right. \\ &\quad \left. - \sinh(k_m z)(C_m \sin m\phi + D_m \cos m\phi)\right) \\ B_\rho &= \sum_{m=0}^{\infty} \frac{\partial J_m}{\partial \rho}(k_m\rho) \left(\sinh(k_m z)(A_m \sin m\phi + B_m \cos m\phi)\right. \\ &\quad \left. + \cosh(k_m z)(C_m \sin m\phi + D_m \cos m\phi)\right) \\ B_\phi &= \sum_{m=0}^{\infty} \frac{J_m(k_m\rho)}{\rho} m \left(\sinh(k_m z)(A_m \cos m\phi - B_m \sin m\phi)\right. \\ &\quad \left. + \cosh(k_m z)(C_m \cos m\phi - D_m \sin m\phi)\right) \end{aligned}$$