national accelerator laboratory

TM-375
0402

HALF INTEGRAL RESONANT EXTRACTION FROM THE MAIN RING

L. C. Teng

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Because of the non-zero stopband width of the half-integral resonance, this resonance is more advantageous for extraction than the third-integral resonance which has zero stopband width, especially when the linear betatron-oscillation wave number has a sizable wobble due to current ripples in the main quadrupoles. The proper non-linear field to use in the half-integral extraction is the octupole.

Neglecting the y motion (vertical) and the curvature of the closed orbit the resonant x motion (horizontal) is given by

$$\left\{\begin{aligned} \frac{d\phi}{d\theta} &= \varepsilon-2B_o-B_1\cos(2\phi-\psi_1) \\ &\quad -2R^2\left[6D_o+4D_1\cos(2\phi-\chi_1)+D_2\cos(4\phi-\chi_2)\right] \\ \frac{dR}{d\theta} &= -B_1R\sin(2\phi-\psi_1) \\ &\quad -2R^3\left[2D_1\sin(2\phi-\chi_1)+D_2\sin(4\phi-\chi_2)\right] \end{aligned}\right. \tag{1}$$

with the first integral

$$R^2\left[\varepsilon-2B_o-B_1\cos(2\phi-\psi_1)\right]$$
$$-R^4\left[6D_o+4D_1\cos(2\phi-\chi_1)+D_2\cos(4\phi-\chi_2)\right] = K = \text{const.} \qquad (2)$$

where

$$\theta = \int \frac{dz}{\nu\beta} = \text{linear oscillation phase advance},$$

R and ϕ are related to x and p by

$$\begin{cases} x = R\sqrt{\beta}\cos\left(\phi - \frac{41}{2}\theta\right) \\ p = \frac{R}{\sqrt{\beta}}\frac{\sin\left(\phi-\phi_o-\frac{41}{2}\theta\right)}{\cos\phi_o} \qquad \tan\phi_o \equiv \alpha \end{cases} \qquad (3)$$

or, conversely

$$\begin{cases} R = \left(\gamma x^2 + 2\alpha xp + \beta p^2\right)^{1/2} \\ \phi = \frac{41}{2}\theta + \tan^{-1}\left(\alpha + \beta\frac{p}{x}\right) \end{cases} \qquad (4)$$

and

$$\begin{cases} \varepsilon = \frac{41}{2} - \nu = \text{deviation of } \nu \text{ from resonant value } 20\frac{1}{2} \\ B_o = \text{0th harmonic (average) of } \left[\frac{\nu}{4}\beta^2\frac{\Delta B'}{(B\rho)}\right] \\ B_1\cos(41\theta-\psi_1) = \text{41st harmonic of } \left[\frac{\nu}{4}\beta^2\frac{\Delta B'}{(B\rho)}\right] \\ D_o = \text{0th harmonic (average) of } \left[\frac{\nu}{192}\beta^3\frac{B'''}{(B\rho)}\right] \end{cases}$$

$$\left\{\begin{array}{l}
D_1 \cos(41\theta-\chi_1) = \text{41st harmonic of } \left[\frac{\nu}{192}\,\beta^3 \frac{B'''}{(B\rho)}\right] \\[2ex]
D_2 \cos(82\theta-\chi_2) = \text{82nd harmonic of } \left[\frac{\nu}{192}\,\beta^3 \frac{B'''}{(B\rho)}\right] \\[2ex]
\Delta B'(\theta) = \Delta\frac{\partial B_y}{\partial x} = \text{additional quadrupole field} \\[2ex]
B'''(\theta) = \frac{\partial^3 B_y}{\partial x^3} = \text{octupole field} \\[2ex]
(B\rho) = \text{rigidity of particle} \\[2ex]
\alpha,\ \beta,\ \gamma = \text{conventional linear-oscillation amplitude functions.}
\end{array}\right.$$

In these equations we have also dropped the small kinematic term--an R^4 term in K.

We, now, make the following simplifying assumptions:

1. The effect of the 0th harmonic of $\Delta B'$, namely B_o is simply a modification of ε. We shall, therefore, set $B_o = 0$ and consider it incorporated in ε.
2. We shall assume that the normally present error quadrupole field is either properly compensated by trim quadrupoles or incorporated in B_1.
3. We assume, also, that the normally present error octupole field (especially D_o and D_2) is properly compensated by trim octupole magnets, and that the octupole magnets for manipulating the resonance are placed in pairs at opposite ends of ring diameters and oppositely excited so that no even harmonic of B''' is present and we can put $D_o = D_2 = 0$.

There is, now, no need for subscripts and they shall be dropped. Eqs. (1) and (2) are, then, simplified to

$$\begin{cases} \dfrac{d\phi}{d\theta} = \varepsilon - B\cos(2\phi-\psi) - 8DR^2\cos(2\phi-\chi) \\ \\ \dfrac{dR}{d\theta} = -BR\sin(2\phi-\psi) - 4DR^3\sin(2\phi-\chi) \end{cases} \tag{5}$$

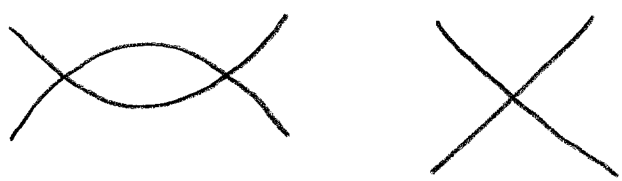
$$K = R^2\left[\varepsilon - B\cos(2\phi-\psi)\right] - 4DR^4\cos(2\phi-\chi). \tag{6}$$

A. <u>(R,ϕ) Phase-Plane Topology</u>

The central fixed point $R = 0$ (controlled by the R^2 term in K) is stable for $\varepsilon > B$ and becomes unstable for $\varepsilon < B$. (We shall discuss only cases for which $\varepsilon \geq 0$. The extension to negative values of ε is obvious. Note also that B and D are, by definition, positive.) The overall phase-plane topology depends on the relative phase $\psi-\chi$ between the quadrupole and the octupole fields.

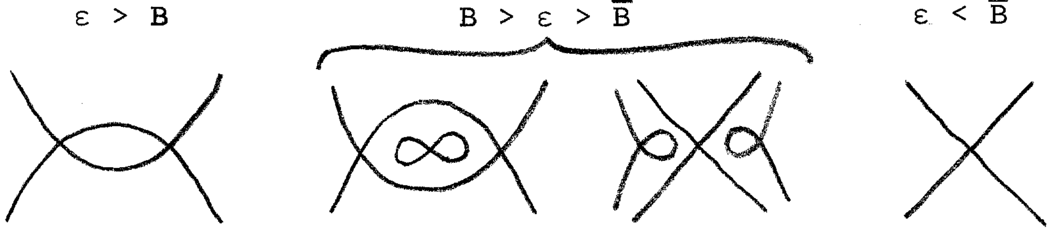
For $0 < |\psi-\chi| < \frac{\pi}{2}$ the entire phase plane opens up for $\varepsilon < B$, and the topology is as follows:

$\varepsilon > B$ $\qquad\qquad$ $\varepsilon < B$

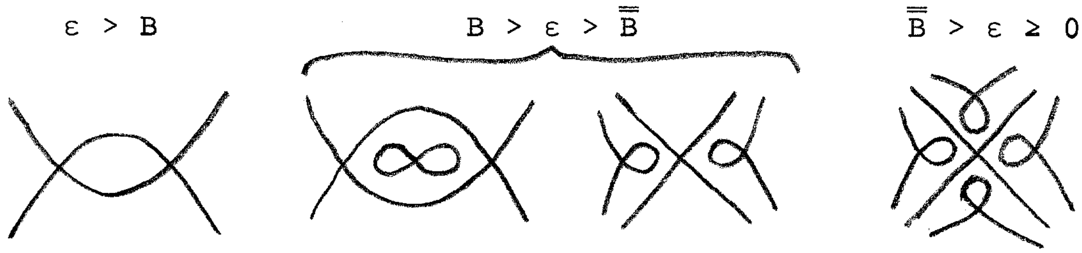


For $\frac{\pi}{2} < |\psi-\chi| < \sin^{-1}\frac{1}{3}$ the phase plane opens up only for $\varepsilon < \overline{B}$ where $\overline{B}$ is some value smaller than B. The phase-plane topology is as follows:

$\varepsilon > B$ $\qquad\qquad$ $B > \varepsilon > \overline{B}$ $\qquad\qquad$ $\varepsilon < \overline{B}$



For $\sin^{-1}\frac{1}{3} < |\psi-\chi| < \pi$ the phase plane never opens up completely. The topology is shown below



For extraction, therefore, we should have $0 < |\psi-\chi| < \frac{\pi}{2}$. Specifically, we will arrange things so that $\psi = \chi$, namely the quadrupole and the octupole fields are exactly in phase. For this case the central closed area shrinks as ε approaches B from a larger value and becomes zero at $\varepsilon = B$. Extraction will commence when the central closed area shrinks to a value equal to the horizontal emittance of the beam and all beam will be extracted when $\varepsilon = B$.

B. Fixed Points and Separatrices

Setting $\psi = \chi$ and performing some scaling we can rewrite Eq. (5) and (6) as

$$\begin{cases} \dfrac{d\phi}{d\lambda} = \delta - \cos(2\phi-\psi) - 2r^2\cos(2\phi-\psi) \\[2ex] \dfrac{dr}{d\lambda} = -r\sin(2\phi-\psi) - r^3\sin(2\phi-\psi) \end{cases} \tag{7}$$

$$r^2[\delta - \cos(2\phi-\psi)] - r^4\cos(2\phi-\psi) = k = \text{constant} \tag{8}$$

where

$$\lambda = B\theta, \qquad r^2 = \frac{4D}{B} R^2, \qquad \text{and} \qquad \delta = \frac{\varepsilon}{B}.$$

Hence, $\frac{4D}{B}$ controls the "magnification" and B controls the "speed" in the phase plane.

For $\delta > 1$ ($\varepsilon > \beta$) there are three fixed points given by $\frac{d\phi}{d\lambda} = \frac{dr}{d\lambda} = 0$. These are

$$\begin{cases} r = 0 \quad \text{(central stable)} \\ r = r_u = \sqrt{\frac{\delta-1}{2}} \qquad \phi = \frac{\psi}{2},\ \pi + \frac{\psi}{2}. \quad \text{(unstable)} \end{cases}$$

All three fixed points coalesce at $r = 0$ and become one unstable fixed point for $\delta < 1$. The separatrices are given by

$$\begin{cases} r^4 \cos(2\phi-\psi) - r^2[\delta-\cos(2\phi-\psi)] + \frac{(\delta-1)^2}{4} = 0 & \delta \geq 1 \\ r^2 \cos(2\phi-\psi) - [\delta-\cos(2\phi-\psi)] = 0. & \delta \leq 1 \end{cases} \tag{9}$$

At the end of extraction $\delta = 1$ and the separatrices become

$$(1+r^2) \cos(2\phi-\psi) = 1. \tag{10}$$

C. Locations of Quadrupole and Octupole

At the electrostatic septum $\theta = 0$ (by definition) $\beta = \beta_s = 98\text{m}$, $\alpha = \alpha_s = \tan \phi_{os} = 0.46$. We would like the beam to stream out parallel to the septum, i.e., $p_s = 0$. Eq. (3), then, gives $\phi = \phi_{os}$ and

$$\begin{cases} x_s = R\sqrt{\beta_s}\cos\phi_{os}, & R = \dfrac{x_s}{\sqrt{\beta_s}\cos\phi_{os}}. \\ p_s = 0 & \end{cases} \tag{11}$$

During extraction the streaming "direction" along an outgoing separatrix varies slightly. We shall adjust the parameters for the end of extraction when the separatrix is given by Eq. (10). We want to adjust ψ so that the outgoing branch of this separatrix passes through $\phi = \phi_{os}$ at $R = \dfrac{\bar{x}_s}{\sqrt{\beta_s}\cos\phi_{os}}$ where $\bar{x}_s = 3.5$ cm $= 0.035$ m corresponds to the middle of the septum aperture which extends from $x_{s1} = 3.0$ cm $= 0.03$ m to $x_{s2} = 4.0$ cm $= 0.04$ m. Thus, the proper phase ψ is

$$\psi = 2\phi_{os} + \sec^{-1}\left(1 + \frac{4D}{B}\,\frac{\bar{x}_s^{\,2}}{\beta_s\cos^2\phi_{os}}\right). \tag{12}$$

The proper locations for the quadrupole(s) and octupole(s) (all assumed to be short) are, therefore

$$\begin{cases} \theta_{quad} = \dfrac{1}{41}(\psi + m\pi) \\ \theta_{oct} = \dfrac{1}{41}(\psi + n\pi) \end{cases} \qquad m, n = \text{integers} \tag{13}$$

D. Strengths of Quadrupole and Octupole

The streaming "speed" also varies slightly during extraction. Again, we shall adjust parameters for the end of extraction. Eliminating ϕ between Eq. (10) and the second of Eq. (7) we get

$$\frac{dr}{d\lambda} = r\sqrt{(1+r^2) - 1}.$$

The solution of this equation giving the variation of r from r_1 to r_2 when λ varies from λ_1 to λ_2 is

$$2(\lambda_2-\lambda_1) = \sqrt{1 + \frac{2}{r_1^2}} - \sqrt{1 + \frac{2}{r_2^2}} \; . \tag{14}$$

At the septum r^2 should increase from

$$r_1^2 = \frac{4D}{B} R_1^2 = \frac{4D}{B} \frac{x_{s1}^2}{\beta_s \cos^2\phi_{os}} \qquad \text{to} \qquad r_2^2 = \frac{4D}{B} R_1^2 = \frac{4D}{B} \frac{x_{s2}^2}{\beta_s \cos^2\phi_{os}}$$

in two revolutions, namely in $\theta_2 - \theta_1 = 4\pi$ or $\lambda_2 - \lambda_1 = 4\pi B$. Eq. (14), then gives

$$8\pi B = \left(1 + \frac{2\beta_s \cos^2\phi_{os}}{x_{s1}^2} \frac{B}{4D}\right)^{1/2} - \left(1 + \frac{2\beta_s \cos^2\phi_{os}}{x_{s2}^2} \frac{B}{4D}\right)^{1/2} . \tag{15}$$

Eq. (15) gives sets of proper values of quadrupole strength B and octupole strength D which will yield the desired streaming "speed." We note also that the stopband width of the half-integral resonance is $\Delta\nu = \pm B$. Therefore, we would like to have a large B value.

For short quadrupoles and octupoles with strengths $\Delta B'\ell$ and $B'''\ell$ and placed at locations having β values β_B and β_D respectively, we have

$$\begin{cases} B = \dfrac{\beta_B}{4\pi} \dfrac{\Delta B'\ell}{(B\rho)} \\[2ex] D = \dfrac{\beta_D^2}{192\pi} \dfrac{B'''\ell}{(B\rho)} \end{cases} \tag{16}$$

E. Onset of Extraction

For $\delta > 1$ the area of the central stable region (taken to be approximately elliptical) is given approximately by the first of Eq. (9) as

$$\begin{aligned} A &\cong \pi(r \text{ at } 2\phi-\psi = 0) \quad (r \text{ at } 2\phi-\psi = \pi) \\ &= \pi \sqrt{\frac{\delta-1}{2}} \left(\sqrt{\frac{\delta^2+1}{2}} - \frac{\delta+1}{2} \right)^{1/2} \\ &= \pi \, r_u \left[\sqrt{\left(1+r_u^2\right)^2 + r_u^4} - \left(1+r_u^2\right) \right]^{1/2} . \end{aligned} \tag{17}$$

Extraction starts when A equals the horizontal emittance of the beam which is estimated to be $E = \frac{\pi}{4} \times 10^{-6}$ m-rad at 200 GeV. This gives the starting values of r_u and δ through

$$\begin{cases} r_u \left[\sqrt{\left(1+r_u^2\right)^2 + r_u^4} - \left(1+r_u^2\right) \right]^{1/2} = \dfrac{4D}{B} \dfrac{E}{\pi} \\ \delta-1 = 2r_u^2 \end{cases} \tag{18}$$

Correspondingly, the beam half-width at the start of extraction is

$$x_{su} = r_u \sqrt{\frac{B}{4D}} \sqrt{\beta_s} \cos\phi_{os} . \tag{19}$$

Since we do not want the streaming "direction" and "speed" to vary too much during extraction x_{su} should not be too close to the septum position at x_{s1}. The upper limit of x_{su} is about $\frac{1}{2}x_{s1}$. This gives an upper limit on the quadrupole strength B.

F. Numerical Results

The parameters assumed are

at septum:

$\beta_s = 98$ m $\alpha_s = \tan\phi_{os} = 0.46$ $\phi_{os} = 0.43$

$x_{s1} = 0.03$ m $x_{s2} = 0.04$ m $\bar{x}_s = 0.035$ m

at quadrupoles and octupoles:

$\beta_B = \beta_D = 90$ m $(B\rho) = 6700$ kGm for 200 GeV

and various equations become:

$$\psi = 0.86 + \sec^{-1}\left[1+(0.151 \times 10^{-4}\mathrm{m})\,\frac{4D}{B}\right]$$

$$\begin{cases} \theta_{quad} = \dfrac{1}{41}(\psi + m\pi) \\ \theta_{oct} = \dfrac{1}{41}(\psi + n\pi) \end{cases}$$

$$8\pi B = \left[1 + \left(18.0 \times 10^{4}\mathrm{m}^{-1}\right)\frac{B}{4D}\right]^{1/2} - \left[1 + \left(10.1 \times 10^{4}\mathrm{m}^{-1}\right)\frac{B}{4D}\right]^{1/2}$$

$$\begin{cases} \Delta B'\ell = (936\ \mathrm{kG})B \\ B'''\ell = (499\ \mathrm{kG/m})D \end{cases}$$

$$r_u\left[\sqrt{\left(1+r_u^2\right)^2 + r_u^4} - \left(1+r_u^2\right)\right]^{1/2} = \left(\frac{1}{4} \times 10^{-6}\mathrm{m\text{-}rad}\right)\frac{4D}{B}$$

$$\begin{cases} \varepsilon - B = B(\delta - 1) = 2Br_u^2 \\ x_{su} = \sqrt{\dfrac{B}{4D}}\left(9.0\mathrm{m}^{1/2}\right) r_u \end{cases}$$

The numerical results are summarized in the following table:

	Quadrupole and Octupole					Onset of Extraction		
	Location	Strength						
B/4D (10^{-4}m)	ψ (rad)	B	D (m^{-1})	$\Delta B'\ell$ (kG)	$B'''\ell$ $\left(10^3 \frac{kG}{m^2}\right)$	r_u	x_{su} (cm)	ε-B
0	2.433	0	391.7	0	195.5	∞	0.56	.00122
0.01	2.371	.00147	366.8	1.37	183.1	.768	0.69	.00173
0.1	2.024	.01013	253.3	9.48	126.4	.334	0.95	.00226
0.25	1.761	.0185	185.4	17.35	92.6	.244	1.10	.00221
0.5	1.558	.0279	139.3	26.1	69.5	.193	1.23	.00208
1.0	1.381	.0407	101.8	38.1	50.8	.153	1.38	.00190
1.5	1.294	.0505	84.1	47.2	42.0	.133	1.47	.00180
2.0	1.240	.0586	73.3	54.8	36.6	.121	1.54	.00172
2.5	1.202	.0658	65.8	61.5	32.8	.112	1.60	.00166
3.0	1.174	.0722	60.2	67.6	30.0	.106	1.65	.00162
3.5	1.151	.0781	55.8	73.1	27.9	.101	1.69	.00158
4.0	1.133	.0836	52.3	78.3	26.1	.096	1.73	.00155
4.5	1.118	.0888	49.3	83.1	24.6	.092	1.76	.00152
5.0	1.105	.0937	46.8	87.7	23.4	.089	1.80	.00149

The boxed row gives the best compromise. For this case:

1. Quadrupole strength = $\Delta B'\ell$ = 47.2 kG

   (Two quadrupoles each having ℓ = 0.3 m, say, and B' = 78.7 kG/m, diametrically located, and oppositely excited to give only odd harmonics.)

2. Octupole strength = $B'''\ell$ = 42,000 kG/m^2

   (Two octupoles each having ℓ = 0.5 m, say, and B''' = 42,000 kG/m^3 corresponding to a pole-tip field of 0.875 kG at an aperture radius of 0.05 m. These octupoles should be diametrically located and oppositely excited to give only odd harmonics.)

3. Quadrupole locations--The two quadrupoles should be located at

$$\theta = \begin{cases} \frac{1}{41}(\psi + n\pi) & \psi = 1.294 = 74.1^\circ \\ \frac{1}{41}(\psi + n\pi) + \pi & n = 0,1,2,\ldots 80,81 \end{cases}$$

   where $\theta = \frac{1}{\nu}$ (betatron-oscillation phase) when $\nu \to \frac{41}{2}$. We have also assumed β_B = 90 m at the quadrupoles. If β_B is different from 90 m at the quadrupoles the quadrupole strength should be adjusted accordingly.

4. Octupole locations--Same as those for the quadrupoles but can have a different n value. Here, also, if β_D is different from 90 m at the octupoles the octupole strength should be adjusted accordingly.

5. Stopband width $\Delta\nu = \pm 0.0505$.

6. Betatron-oscillation tune at start of extraction

$$\nu = 20.4477.$$

7. Beam half-width at start of extraction $x_{su} = 1.47$ cm.

It remains to check that the streaming "direction" and "speed" do not vary too much during extraction. The values of p_s and $\frac{dr}{d\lambda}$ at the start of extraction ($\delta = 1.0356$) and the end of extraction ($\delta = 1$) given by Eqs. (3), (7) and (9) are

	Start ($\delta = 1.0356$)	End ($\delta = 1$)
"direction" p_s	0.0167 mrad	0 mrad
"speed" $\frac{dr}{d\lambda}$	0.121 ($\Delta x_s = 0.83$ cm in 2 turns	0.146 ($\Delta x_s = 1$ cm in 2 turns)

These variations are certainly tolerable.

This calculation serves only as a first-order design guide. The effects of the momentum spread in the beam, the tune ripple, and the vertical motion must be studied more in detail using a computer.

The study of the phase-plane topology given in Section A was made by W. Lee.

TM-375A
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HALF INTEGRAL RESONANT EXTRACTION FROM THE MAIN RING--ADDENDUM

L. C. Teng

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With δ-function quadrupole and octupole

$$2B_o = B_1 \equiv B, \qquad 2D_o = D_1 = D_2 \equiv D$$

and when they are all in phase Eqs. (1) and (2) in TM-375 become

$$\left\{\begin{aligned} \frac{d\phi}{d\theta} &= \varepsilon - B - B\cos(2\phi-\psi) \\ &\quad - 8DR^2\left[\frac{3}{4} + \cos(2\phi-\psi) + \frac{1}{4}\cos 2(2\phi-\psi)\right] \\ &= \varepsilon - B - B\cos(2\phi-\psi) - 4DR^2\left[1+\cos(2\phi-\psi)\right]^2 \\ \frac{dR}{d\theta} &= -BR\sin(2\phi-\psi) \\ &\quad - 4DR^3\left[\sin(2\phi-\psi) + \frac{1}{2}\sin 2(2\phi-\psi)\right] \\ &= -BR\sin(2\phi-\psi) - 4DR^3\sin(2\phi-\psi)\left[1+\cos(2\phi-\psi)\right] \end{aligned}\right. \tag{1A}$$

At the end of extraction $\delta = 1$ and the separatrices become

$$r^2\left[1 + \cos(2\phi-\psi)\right]^2 + 2\left[1 + \cos(2\phi-\psi)\right] - 4 = 0 \qquad (6A)$$

or

$$\cos(2\phi-\psi) = \frac{1}{r^2}\left[\sqrt{1 + 4r^2} - 1\right] - 1. \qquad (7A)$$

B. Locations of Quadrupoles and Octupoles

With $\phi = \phi_{os} = \tan^{-1}\alpha_s$ we have at the septum

$$\begin{cases} x_s = \dfrac{R}{\sqrt{\gamma_s}} & \gamma_s = \dfrac{1+\alpha_s^2}{\beta_s} \\ p_s = 0 & \end{cases}$$

In order that the outgoing separatrix (7A) passes through $\phi = \phi_{os}$ and $r = \bar{r} = \sqrt{\frac{4D}{B}}\,\bar{R} = \sqrt{\frac{4D}{B}}\,\sqrt{\gamma_s}\,\bar{x}_s$ where $\bar{x}_s = 0.035\text{m}$ we must have

$$\psi = 2\phi_{os} + \cos^{-1}\frac{1}{\bar{r}^2}\left[\sqrt{1+4\bar{r}^2} - \left(1+\bar{r}^2\right)\right]. \qquad (8A)$$

C. Strengths of Quadrupoles and Octupoles

For $\Delta\lambda = 4\pi B$ (2 turns) the second of Eq. (3A) gives

$$\frac{1}{4\pi B}\frac{\Delta r}{r} = \frac{1}{4\pi B}\frac{\Delta x_s}{\bar{x}_s} = -\sin(2\phi-\psi)\left\{1 + \bar{r}^2\left[1+\cos(2\phi-\psi)\right]\right\} \qquad (9A)$$

E. Numerical Results

With

$$\beta_s = 98 \text{ m} \qquad \alpha_s = \tan \phi_{os} = 0.46 \qquad (\phi_{os} = 0.43)$$

$$\Delta x_s = 0.01 \text{ m} \qquad \bar{x}_s = 0.035 \text{ m}$$

and at the quadrupole and octupole

$$\beta_B = \beta_D = 90 \text{ m} \qquad (B\rho) = 6700 \text{ kGm at } 200 \text{ GeV}$$

we get the table following (p. 6). The values of ψ, $B'\ell$ and $B'''\ell$ are plotted against B/4D in Fig. 1 for easy interpolation. The boxed row is a good compromise. For this case the variations of streaming "direction" and "speed" from the start to the end of extraction are given below.

	Start (ν = 20.44228)	End (ν = 20.44435)
"direction" p_s	0.00685 mrad	0 mrad
"speed" $\frac{1}{r}\frac{dr}{d\lambda}$	0.731 (Δx_s = 0.895 cm in 2 turns)	0.817 (Δx_s = 1 cm in 2 turns)

These variations are tolerable.

Comparison of these parameters with those given in TM-375 with all even harmonics of the octupole field eliminated shows that this present case has a slight advantage.

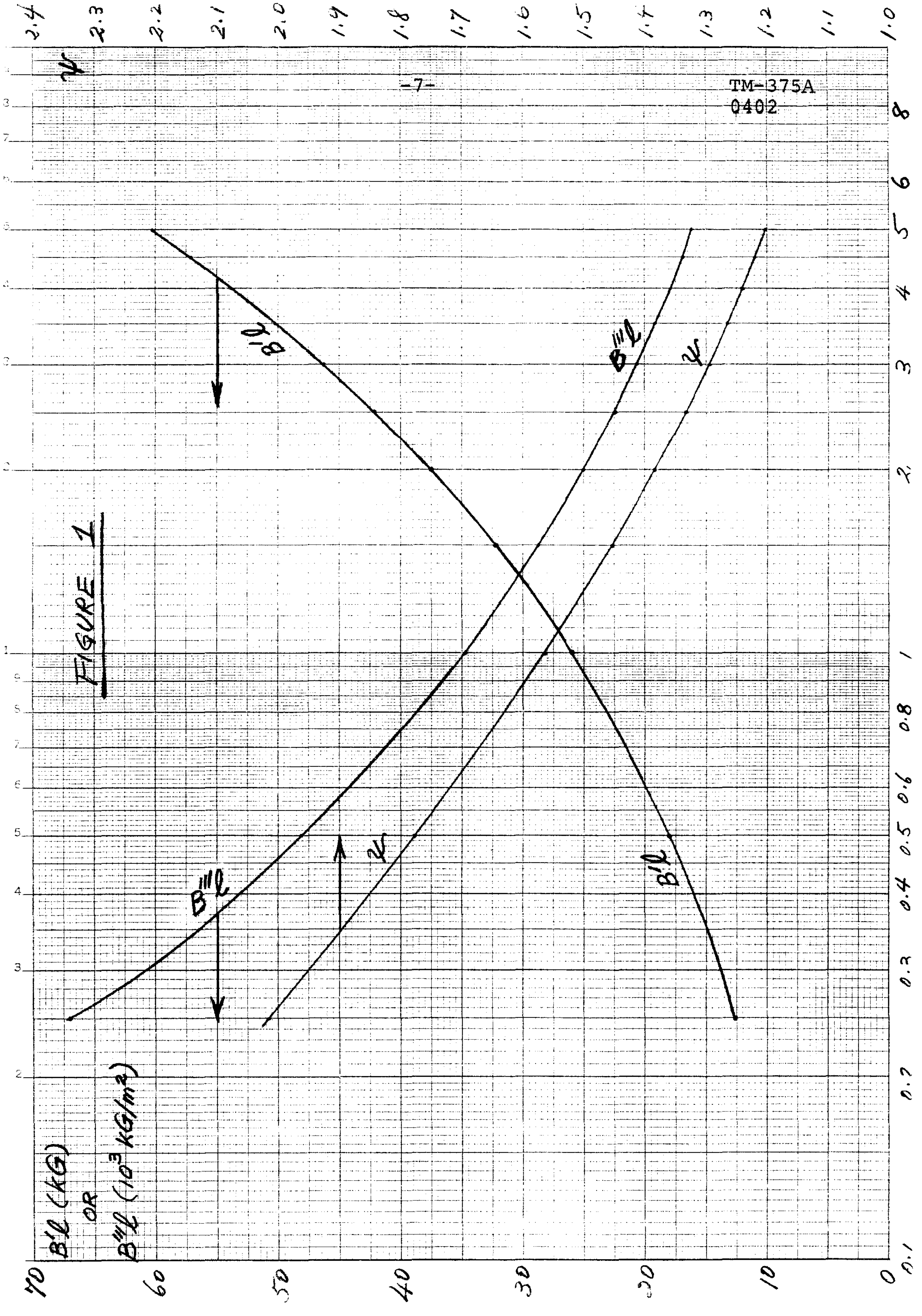
ψ
2.4
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
B'l (KG)
OR
B''l (10³ KG/m²)
70
60
50
40
30
20
10
0
0.1
0.2
0.3
0.4
0.5
0.6
0.8
1
2
3
4
5
6
8
B'l
B''l
ψ

FIGURE 1

TM-375B
0402

HALF INTEGRAL RESONANT EXTRACTION FROM THE MAIN RING--ADDENDUM 2

L. C. Teng

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In the case of one δ-function quadrupole and one δ-function octupole the phase-plane topology also becomes rather complex when the quadrupole and the octupole are not in phase. Instead of an exhaustive study of the general case we shall investigate only the special case when they are out of phase by $\Delta\psi = \pi$ (or $\Delta\theta = \frac{\pi}{41}$). This corresponds to locating the quadrupole and the octupole diametrically opposite and is, hence, of special interest for application.

We shall retain ψ to specify the location of the octupole and change ψ to $\psi-\pi$ for the quadrupole. We observe that aside from a redefinition of δ, this is equivalent to a change in sign of B. (Changing the sign of B also changes the sign of the zeroth harmonic or the average of the quadrupole field, hence the sign of the tune shift.) Denoting a special case of signs and phase difference by where

$$\lambda = |B|\theta, \qquad r^2 = \frac{4D}{|B|} R^2, \text{ and}$$

$$\delta = \begin{cases} \frac{\varepsilon}{B} - 1 & \text{for} \quad (+,+)_o \\ \frac{\varepsilon}{|B|} + 1 & \text{for} \quad (-,+)_\pi \end{cases}$$

Case B

$$\begin{cases} \frac{d\phi}{d\lambda} = \delta + \cos(2\phi-\psi) - r^2\left[1 + \cos(2\phi-\psi)\right]^2 \\ \frac{dr}{d\lambda} = r\,\sin(2\phi-\psi) - r^3\,\sin(2\phi-\psi)\left[1 + \cos(2\phi-\psi)\right] \end{cases}$$

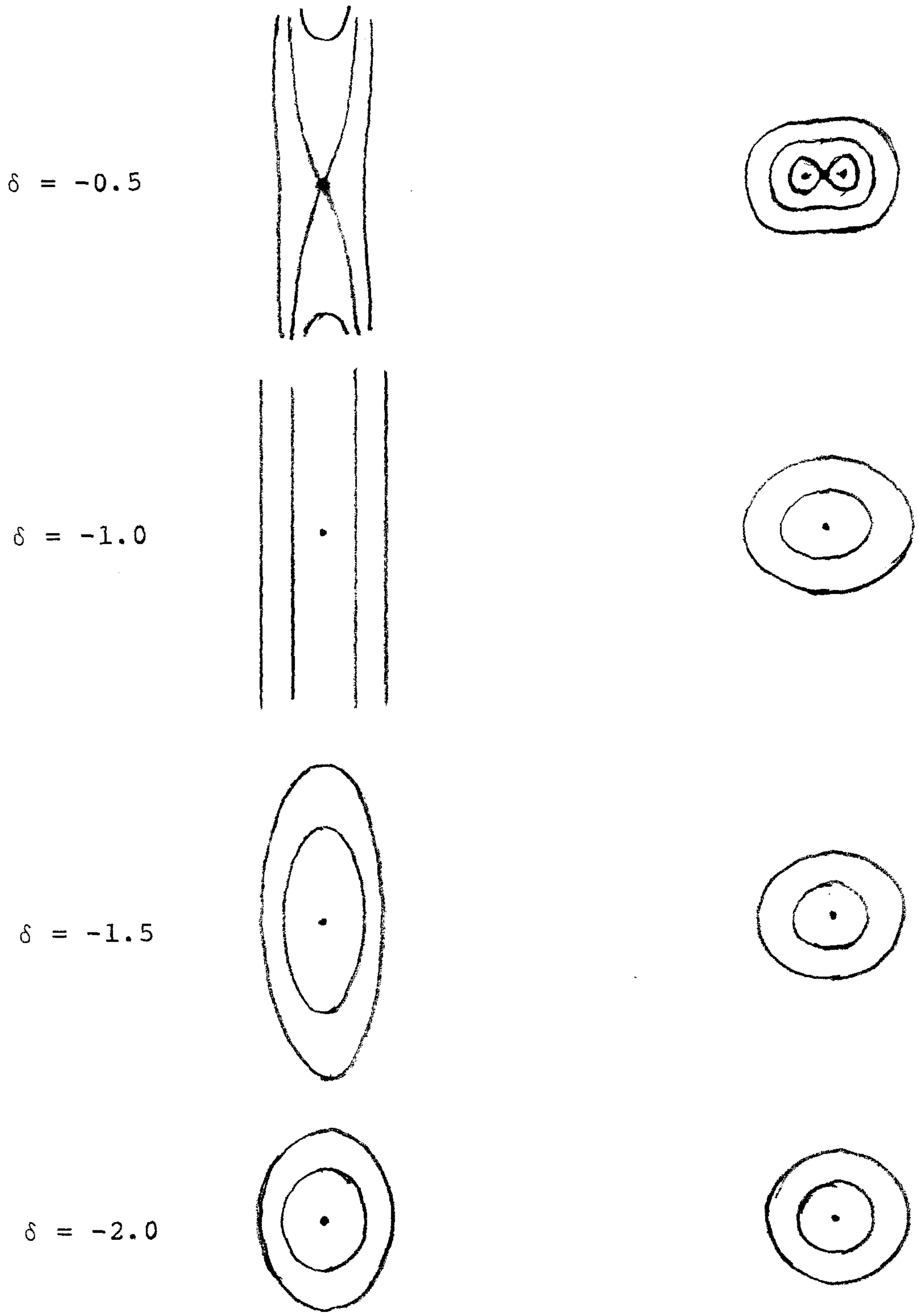
$$r^2\left[\delta + \cos(2\phi-\psi)\right] - \frac{1}{2}\,r^4\left[1 + \cos(2\phi-\psi)\right]^2 = k$$

where

$$\lambda = |B|\theta, \qquad r^2 = \frac{4D}{|B|} R^2, \text{ and}$$

$$\delta = \begin{cases} \frac{\varepsilon}{|B|} + 1 & \text{for} \quad (-,+)_o \\ \frac{\varepsilon}{B} - 1 & \text{for} \quad (+,+)_\pi \end{cases}$$

The phase-plane topology for these cases is shown in the following sketches:


δ = -0.5
δ = -1.0
δ = -1.5
δ = -2.0

TM-375
0402

HALF INTEGRAL RESONANT EXTRACTION FROM THE MAIN RING--Addendum 3

L. C. Teng

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S. Ohnuma discovered that the extraction proceeds quite well with only the zeroth harmonic octupole field. With δ-function quadrupole and only zeroth harmonic octupole, Eqs. (1) and (2) in TM-375 become

$$\begin{cases} \frac{d\phi}{d\theta} = \varepsilon - B - B\cos(2\phi-\psi) - 6DR^2 \\ \frac{dR}{d\theta} = -BR\sin(2\phi-\psi) \end{cases} \qquad (1B)$$

$$R^2\left[\varepsilon - B - B\cos(2\phi-\psi)\right] - 3DR^4 = K. \qquad (2B)$$

After scaling with

$$\lambda = B\theta, \qquad r^2 = \frac{6D}{B}R^2, \qquad \delta = \frac{\varepsilon}{B} - 1$$

we have

$$\begin{cases} \frac{d\phi}{d\lambda} = \delta - \cos(2\phi-\psi) - r^2 \\ \frac{dr}{d\lambda} = -r\sin(2\phi-\psi) \end{cases} \qquad (3B)$$

$$r^2\left[\delta - \cos(2\phi-\psi)\right] - \frac{1}{2}r^4 = k. \qquad (4B)$$

A. Fixed Points and Separatrices

Outside the stopband $\delta>1$ ($\varepsilon>2B$), the fixed points are

$$\begin{cases} r = 0 \text{ (central stable)} & \\ r = r_s = \sqrt{\delta+1} & \phi = \frac{\psi}{2}+\frac{\pi}{2},\ \frac{\psi}{2}+\frac{3\pi}{2} \\ & \text{(outboard stable)} \\ r = r_u = \sqrt{\delta-1} & \phi = \frac{\psi}{2},\ \frac{\psi}{2}+\pi \text{ (unstable)} \end{cases}$$

Now we have two additional stable fixed points on the outside which means that even inside the stopband ($\delta<1$) the phase space is closed on the outside. But since r_s is very much larger than r_u the closing of the separatrices at very large values of R does not impair the effectiveness for extraction.

The separatrices are constant k curves passing through the unstable fixed points and are given by

$$\begin{cases} r^4 - 2r^2\left[\delta - \cos(2\phi-\psi)\right] + (\delta-1)^2 = 0 & \delta\geq 1 \\ r^2 - 2\left[\delta - \cos(2\phi-\psi)\right] = 0. & \delta\leq 1 \end{cases} \tag{5B}$$

At the end of extraction $\delta = 1$ and the direction of the outward streaming branch of the separatrix is given by

$$\begin{cases} \cos(2\phi-\psi) = 1 - \frac{r^2}{2}\ . \\ \sin(2\phi-\psi) = -\frac{r}{2}\sqrt{4-r^2}. \end{cases} \tag{6B}$$

B. Locations of Quadrupole

With $\phi = \phi_{os} = \tan^{-1}\alpha_s$, we have at the septum

$$\begin{cases} x_s = \dfrac{R}{\sqrt{\gamma_s}} = \sqrt{\dfrac{B}{6D}}\,\dfrac{r}{\sqrt{\gamma_s}} \\ p_s = 0 \end{cases} \qquad \gamma_s \equiv \frac{1+\alpha_s^2}{\beta_s}$$

In order that the outgoing separatrix (6B) passes through $\phi = \phi_{os}$ and $r = \bar{r} \equiv \sqrt{\frac{6D}{B}}\sqrt{\gamma_s}\;\bar{x}_s$ we must have

$$\psi = 2\phi_{os} + \cos^{-1}\left(1 - \frac{\bar{r}^2}{2}\right). \tag{7B}$$

C. Strengths of Quadrupole and Octupole

The quantity x is related to r and ϕ by

$$x = \sqrt{\frac{B}{6D}}\sqrt{\beta}\; r \cos\left(\phi - \frac{41}{2}\theta\right)$$

which, together with Eq. (3B), gives, at the septum ($\theta = 0$) and at the end of extraction ($\delta = 1$)

$$\frac{1}{\sqrt{\beta_s}}\sqrt{\frac{6D}{B}}\frac{dx_s}{d\lambda} = \frac{dr}{d\lambda}\cos\phi - r\frac{d\phi}{d\lambda}\sin\phi$$

$$= r(r^2-1)\sin\phi_{os} - r\sin(\phi_{os}-\psi).$$

In 2 turns ($\Delta\lambda = 4\pi B$) and at $r = \bar{r}$ we get Δx_s given by

$$\frac{1}{\sqrt{\beta_s}}\sqrt{\frac{6D}{B}}\frac{\Delta x_s}{4\pi B} = \bar{r}\left[(\bar{r}^2-1)\sin\phi_{os} - \sin(\phi_{os}-\psi)\right]. \tag{8B}$$

D. Onset of Extraction

For $\delta > 1$ the area of the central stable region (taken to be approximately elliptical) is given by the first of Eq. (5B)

$$A \cong \pi (r \text{ at } 2\phi - \psi = 0) \quad (r \text{ at } 2\phi - \psi = \pi)$$

$$= \pi\sqrt{\delta - 1}\ (\sqrt{\delta} - 1) = \pi\ r_u \left(\sqrt{1 + r_u^2} - 1\right) .$$

Extraction starts when

$$A = \pi\ r_u \left(\sqrt{1 + r_u^2} - 1\right) = \frac{6D}{B}\,E, \qquad r_u^2 = \delta - 1. \tag{9B}$$

The beam half-width at the septum at the start of extraction is

$$x_{su} = \sqrt{\frac{B}{6D}}\ \frac{r_u}{\sqrt{\gamma_s}} . \tag{10B}$$

E. Numerical Results

Since the septum position $x_s \equiv \bar{x}_s - \frac{\Delta x_s}{2}$ can be easily adjusted by a local orbit bump, the computation procedure is changed. The parameters assumed are

$$\beta_s = 96.8481 \text{ m} \qquad \gamma_s = 0.0122255 \text{ m}^{-1}$$

$$\alpha_s = \tan \phi_{os} = 0.428976 \qquad \phi_{os} = 0.405234$$

$$\Delta x_s = 0.01 \text{ m} \qquad E = \frac{\pi}{4} \times 10^{-6} \text{ m-rad}$$

and at the quadrupole

$$\beta_B = 92.13 \text{ m}, \qquad (B\rho) = 6702.5 \text{ kGm at } 200 \text{ GeV}.$$

For given values of D = [D], the values of B = [B]

$B'\ell \equiv \frac{4\pi}{\beta_B}(B\rho)B$ = [B'L], $\frac{41}{2} - 2B$ = [TUNE], $x_s \equiv \bar{x}_s - \frac{\Delta x_s}{2}$ = [XS],

x_{su} = [XU], and $\varepsilon = B(\delta+1)$ = [EPSLN] are calculated as functions of ψ = [PSI]; and given in the following table. (The symbols in brackets are headings in the table.) These values will give the proper streaming speed ($\Delta x_s = 0.01$ m) and streaming direction ($p_s = 0$). Of course, one should check that $x_{xu} < x_s$. The value of D is related to the zeroth harmonic (average) octupole strength $B'''\ell$ by $D = \frac{\beta_D^2}{192\pi} \frac{B'''\ell}{(B\rho)}$. The proper value of β_D will depend on the source of $B'''\ell$ (either specially inserted octupole magnets or error octupole field in the main ring quadrupoles). The values of $B'''\ell$ = [B'''L] given in the table corresponds to $\beta_D = 92.13$ m.

The computation was made by W. W. Lee. I am grateful to S. Ohnuma for pointing out an error in the original calculation.

D= 125.9694 B'''L= 60000.0

PSI	B	B'L	TUNE	XS	XU	EPSLN
1.00	0.161972	148.0763	20.1761	0.020056	0.017586	0.002858
1.05	0.118076	107.9460	20.2638	0.022012	0.016689	0.002574
1.10	0.091411	83.5684	20.3172	0.023697	0.015997	0.002365
1.15	0.073736	67.4100	20.3525	0.025185	0.015440	0.002203
1.20	0.061287	56.0293	20.3774	0.026524	0.014976	0.002072
1.25	0.052118	47.6471	20.3958	0.027745	0.014581	0.001965
1.30	0.045130	41.2581	20.4097	0.028871	0.014240	0.001874
1.35	0.039657	36.2545	20.4207	0.029918	0.013941	0.001796
1.40	0.035275	32.2491	20.4294	0.030900	0.013675	0.001728
1.45	0.031704	28.9840	20.4366	0.031826	0.013438	0.001669
1.50	0.028748	26.2818	20.4425	0.032703	0.013225	0.001616
1.55	0.026270	24.0165	20.4475	0.033539	0.013031	0.001569
1.60	0.024170	22.0964	20.4517	0.034339	0.012855	0.001527
1.65	0.022373	20.4535	20.4553	0.035108	0.012694	0.001489
1.70	0.020822	19.0360	20.4584	0.035848	0.012546	0.001454
1.75	0.019475	17.8043	20.4610	0.036564	0.012410	0.001423
1.80	0.018297	16.7272	20.4634	0.037258	0.012285	0.001394
1.85	0.017261	15.7802	20.4655	0.037933	0.012169	0.001368
1.90	0.016346	14.9435	20.4673	0.038592	0.012062	0.001344
1.95	0.015534	14.2012	20.4689	0.039235	0.011962	0.001322
2.00	0.014811	13.5403	20.4704	0.039865	0.011870	0.001302

D= 188.9541 B'''L= 90000.0

PSI	B	B'L	TUNE	XS	XU	EPSLN
1.00	0.185412	169.5051	20.1292	0.016888	0.016816	0.003919
1.05	0.135163	123.5673	20.2297	0.018598	0.015959	0.003530
1.10	0.104639	95.6619	20.2907	0.020069	0.015299	0.003244
1.15	0.084406	77.1652	20.3312	0.021369	0.014767	0.003022
1.20	0.070156	64.1375	20.3597	0.022538	0.014324	0.002844
1.25	0.059661	54.5423	20.3807	0.023605	0.013947	0.002696
1.30	0.051661	47.2287	20.3967	0.024589	0.013622	0.002572
1.35	0.045396	41.5011	20.4092	0.025504	0.013336	0.002465
1.40	0.040380	36.9160	20.4192	0.026362	0.013083	0.002372
1.45	0.036292	33.1784	20.4274	0.027170	0.012857	0.002291
1.50	0.032908	30.0851	20.4342	0.027937	0.012653	0.002219
1.55	0.030072	27.4920	20.4399	0.028667	0.012469	0.002155
1.60	0.027668	25.2941	20.4447	0.029366	0.012301	0.002097
1.65	0.025611	23.4134	20.4488	0.030037	0.012147	0.002045
1.70	0.023836	21.7908	20.4523	0.030684	0.012007	0.001998
1.75	0.022293	20.3808	20.4554	0.031309	0.011877	0.001955
1.80	0.020945	19.1479	20.4581	0.031916	0.011758	0.001916
1.85	0.019759	18.0638	20.4605	0.032506	0.011647	0.001880
1.90	0.018711	17.1060	20.4626	0.033081	0.011545	0.001847
1.95	0.017782	16.2563	20.4644	0.033643	0.011451	0.001817
2.00	0.016954	15.4998	20.4661	0.034194	0.011363	0.001790

D= 251.9388 B'''L= 120000.0

PSI	B	B'L	TUNE	XS	XU	EPSLN
1.00	0.204072	186.5645	20.0919	0.014887	0.016291	0.004905
1.05	0.148766	136.0034	20.2025	0.016440	0.015462	0.004418
1.10	0.115170	105.2896	20.2697	0.017777	0.014823	0.004060
1.15	0.092901	84.9313	20.3142	0.018958	0.014308	0.003783
1.20	0.077217	70.5924	20.3456	0.020020	0.013880	0.003560
1.25	0.065665	60.0316	20.3687	0.020990	0.013515	0.003376
1.30	0.056860	51.9819	20.3863	0.021883	0.013200	0.003220
1.35	0.049964	45.6779	20.4001	0.022715	0.012924	0.003087
1.40	0.044444	40.6313	20.4111	0.023494	0.012680	0.002971
1.45	0.039944	36.5175	20.4201	0.024229	0.012461	0.002870
1.50	0.036220	33.1130	20.4276	0.024925	0.012264	0.002780
1.55	0.033098	30.2589	20.4338	0.025589	0.012086	0.002699
1.60	0.030452	27.8398	20.4391	0.026224	0.011924	0.002627
1.65	0.028188	25.7698	20.4436	0.026833	0.011775	0.002563
1.70	0.026235	23.9839	20.4475	0.027421	0.011639	0.002504
1.75	0.024537	22.4320	20.4509	0.027989	0.011514	0.002450
1.80	0.023053	21.0750	20.4539	0.028540	0.011399	0.002401
1.85	0.021747	19.8818	20.4565	0.029076	0.011292	0.002357
1.90	0.020594	18.8276	20.4588	0.029599	0.011194	0.002316
1.95	0.019571	17.8924	20.4609	0.030109	0.011103	0.002278
2.00	0.018661	17.0598	20.4627	0.030610	0.011018	0.002243

D= 314.9236 B'''L= 150000.0

PSI	B	B'L	TUNE	XS	XU	EPSLN
1.00	0.219830	200.9705	20.0603	0.013461	0.015895	0.005836
1.05	0.160253	146.5053	20.1795	0.014903	0.015087	0.005258
1.10	0.124063	113.4198	20.2519	0.016144	0.014464	0.004833
1.15	0.100075	91.4894	20.2999	0.017240	0.013962	0.004503
1.20	0.083179	76.0434	20.3336	0.018227	0.013545	0.004238
1.25	0.070735	64.6670	20.3585	0.019127	0.013190	0.004019
1.30	0.061251	55.9958	20.3775	0.019956	0.012883	0.003834
1.35	0.053822	49.2050	20.3924	0.020728	0.012614	0.003676
1.40	0.047876	43.7687	20.4042	0.021451	0.012376	0.003538
1.45	0.043029	39.3373	20.4139	0.022133	0.012163	0.003418
1.50	0.039017	35.6699	20.4220	0.022780	0.011972	0.003311
1.55	0.035654	32.5954	20.4287	0.023396	0.011798	0.003215
1.60	0.032804	29.9895	20.4344	0.023985	0.011640	0.003130
1.65	0.030365	27.7597	20.4393	0.024552	0.011496	0.003053
1.70	0.028260	25.8359	20.4435	0.025097	0.011363	0.002983
1.75	0.026432	24.1642	20.4471	0.025625	0.011242	0.002919
1.80	0.024833	22.7023	20.4503	0.026136	0.011129	0.002861
1.85	0.023427	21.4170	20.4531	0.026634	0.011026	0.002808
1.90	0.022185	20.2814	20.4556	0.027119	0.010930	0.002760
1.95	0.021083	19.2740	20.4578	0.027593	0.010841	0.002715
2.00	0.020102	18.3771	20.4598	0.028057	0.010759	0.002674

D= 377.9083 B'''L= 180000.0

PSI	B	B'L	TUNE	XS	XU	EPSLN
1.00	0.233604	213.5630	20.0328	0.012373	0.015579	0.006728
1.05	0.170295	155.6851	20.1594	0.013729	0.014788	0.006062
1.10	0.131837	120.5265	20.2363	0.014897	0.014178	0.005572
1.15	0.106345	97.2220	20.2873	0.015929	0.013686	0.005193
1.20	0.088391	80.8082	20.3232	0.016857	0.013278	0.004887
1.25	0.075168	68.7190	20.3497	0.017704	0.012931	0.004635
1.30	0.065088	59.5045	20.3698	0.018485	0.012630	0.004422
1.35	0.057195	52.2881	20.3856	0.019211	0.012367	0.004240
1.40	0.050876	46.5112	20.3982	0.019892	0.012134	0.004081
1.45	0.045725	41.8021	20.4086	0.020533	0.011926	0.003942
1.50	0.041462	37.9049	20.4171	0.021142	0.011738	0.003819
1.55	0.037888	34.6378	20.4242	0.021722	0.011568	0.003710
1.60	0.034859	31.8686	20.4303	0.022276	0.011414	0.003611
1.65	0.032267	29.4991	20.4355	0.022809	0.011273	0.003523
1.70	0.030031	27.4547	20.4399	0.023322	0.011143	0.003442
1.75	0.028088	25.6783	20.4438	0.023819	0.011024	0.003369
1.80	0.026389	24.1248	20.4472	0.024300	0.010914	0.003302
1.85	0.024895	22.7589	20.4502	0.024768	0.010813	0.003241
1.90	0.023575	21.5522	20.4529	0.025225	0.010719	0.003185
1.95	0.022404	20.4817	20.4552	0.025671	0.010633	0.003134
2.00	0.021361	19.5285	20.4573	0.026108	0.010552	0.003087

D= 440.8930 B'''L= 210000.0

PSI	B	B'L	TUNE	XS	XU	EPSLN
1.00	0.245921	224.8235	20.0082	0.011503	0.015317	0.007588
1.05	0.179274	163.8938	20.1415	0.012791	0.014540	0.006837
1.10	0.138788	126.8814	20.2224	0.013901	0.013941	0.006285
1.15	0.111953	102.3482	20.2761	0.014881	0.013458	0.005857
1.20	0.093052	85.0689	20.3139	0.015763	0.013057	0.005513
1.25	0.079131	72.3423	20.3417	0.016567	0.012716	0.005229
1.30	0.068520	62.6419	20.3630	0.017308	0.012421	0.004989
1.35	0.060211	55.0451	20.3796	0.017998	0.012162	0.004784
1.40	0.053558	48.9636	20.3929	0.018645	0.011933	0.004606
1.45	0.048136	44.0062	20.4037	0.019255	0.011729	0.004449
1.50	0.043648	39.9035	20.4127	0.019833	0.011545	0.004310
1.55	0.039886	36.4641	20.4202	0.020383	0.011378	0.004187
1.60	0.036697	33.5489	20.4266	0.020910	0.011226	0.004076
1.65	0.033969	31.0544	20.4321	0.021416	0.011088	0.003976
1.70	0.031615	28.9023	20.4368	0.021904	0.010961	0.003885
1.75	0.029569	27.0322	20.4409	0.022375	0.010844	0.003803
1.80	0.027780	25.3968	20.4444	0.022833	0.010736	0.003728
1.85	0.026207	23.9589	20.4476	0.023277	0.010637	0.003659
1.90	0.024818	22.6886	20.4504	0.023711	0.010545	0.003596
1.95	0.023585	21.5616	20.4528	0.024135	0.010460	0.003538
2.00	0.022487	20.5582	20.4550	0.024550	0.010381	0.003485