Electron Dynamics with Synchrotron Radiation

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❑ Introduction to electron dynamics

Coordinates and Hamiltonian for a storage ring.

Stochastic differential equations including radiation.

Robinson's Theorem

❑ Modes of oscillation

Betatron and synchrotron motion

Radiation damping and quantum fluctuations

Fokker Planck equation and distribution functions

❑ Advanced topics by illustration

Dynamical effects from radiation

Non-linear resonances

Limit cycles

Not too much detail or rigour in these talks!

See references for further details and justifications, formalism for more general cases, etc.

Suggested Reading

- M. Sands, The Physics of Electron Storage Rings, SLAC-121 (1970).

  Very clear for basic single particle dynamics and effects of synchrotron radiation.

- J.D. Jackson, Classical Electrodynamics, Wiley, New York 1975.

  Classic text on electrodynamics, includes relativistic dynamics and treatments of radiation from accelerated charges.

- A.W. Chao, J. Appl. Phys., **50 (2),** p. 595 (1979).

  Simple but general approach to linear theory.

- A.W. Chao, “Equations for Multiparticle Dynamics”, Joint US/CERN School, South Padre Island 1986, Springer Lecture Notes in Physics No. 296 (1986).

  Treatment of coupled motion, including quantu-lifetime calculations.

- D.P. Barber, K. Heinemann, H. Mais, G. Ripken, “Fokker-Planck treatment of stochastic Particle motion ….”, DESY 91-146 (1991)

  General linear theory including spin. See also references therein.

- J.M. Jowett, “Introductory Statistical Mechanics for Electron Storage Rings”, US Particle Accelerator School, AIP Conference Proceedings, No. 153 (1985).

  Base for present talk. Contains further details and many references.

- J.M. Jowett, “Electron Dynamics with Radiation and Non-linear Wigglers”, CERN Accelerator School, Oxford 1985, CERN Report 87-3 (1987).

  Similar treatment, contains some other material.

Motivation

❑ Proton (hadron) synchrotrons & storage rings

For single-particle dynamics, an adequate mathematical model is that of a classical particle moving in an applied external electromagnetic field.

State of system is $(\mathbf{q},\mathbf{p}) \in \mathrm{R}^6$, canonical variables.

Rich field for application of *Hamiltonian* dynamics.

N.B. Hamiltonian dynamics is a *very special* case.

❑ Electron synchrotrons and storage rings

"electron" ≡ "positron or electron" in this talk.

Hamiltonian model, and concepts and techniques derived from it remain useful but do not strictly apply because electrons generate significant electromagnetic fields of their own: **radiation reaction.**

Electron continually loses energy as synchrotron radiation.

Global system of particle + EM field remains Hamiltonian but, for purposes of describing particle dynamics in the ring, we are not interested in the dynamical variables of the EM field.

Employ *reduced description* in terms of canonical variables of classical particle with addditional dissipative terms in the equations of motion. These terms are stochastic and are *second order* in the EM coupling

$$\alpha = e^2 / \hbar c$$

Synchrotron Radiation

❑ Photon emission (incoherent only)

Intrinsically quantum-mechanical phenomenon
$\Rightarrow$ emission times and energies of quanta of energy are random quantities.

Provided the energies and magnetic fields are not too high, certain average quantities, such as the mean emission rate and the mean radiated power, may be calculated to good accuracy within classical electrodynamics.

Only when the magneticfields and energies become so high that the mean of the classical synchrotron radiation spectrum becomes comparable to the electron energy does it become necessary to include quantum corrections to the total radiated power and the frequency spectrum.

❑ Semi-classical picture

Orbital quantum numbers of electron very large.

Change during photon emission also large.

Approximate as instantaneous jump in energy since

$$\tau_\gamma \approx \frac{\rho}{\gamma c} << \frac{1}{\text{(characteristic frequencies of motion)}}$$

$$\frac{1}{\tau_\gamma} << \omega_c \equiv \frac{3c\gamma^3}{2\rho} \text{(critical frequency)}$$

means that freqeuncy spectrum is locally well - defined.

Results from electrodynamics

Radiation from accelerated charge, quoted without derivation (see, e.g., Jackson)

Power radiated by accelerated particle

$$P_{\text{rad}} = \left\{ -\frac{dE}{dt} \text{ in any frame} \right\} = \frac{2}{3} \frac{\left[e^2 / 4\pi\varepsilon_0 \right]}{m^2 c^3} \frac{dp^\mu}{d\tau} \frac{dp_\mu}{d\tau}$$

4 - vector form of the Lorentz force in purely transverse magnetic field $\mathbf{E} = 0,\ \mathbf{p} \cdot \mathbf{B} = \mathbf{0}$

$$\frac{dp^\mu}{d\tau} = \frac{e}{m} F^{\mu\nu} p_\mu = \frac{e}{mc} (0, \mathbf{p} \times \mathbf{B})$$

$$\boxed{b(\mathbf{x}) \equiv \frac{e}{p_0 c} \left| \mathbf{B}(x, y, s) \right|}$$

$$P_{\text{rad}} = \frac{2}{3} \frac{e^2 \left[e^2 / 4\pi\varepsilon_0 \right]}{m^4 c^3} (\mathbf{p} \times \mathbf{B})^2 = \frac{2}{3} \frac{e^2 r_e p_0^2}{m^3 c} p^2 b(\mathbf{x})^2$$

where p_0 is a reference value of the particle momentum and $b(\mathbf{x})$ is a normalised magnetic field (units of m^{-1}).

$$P_{\text{rad}} = -\frac{dE}{dt} = \frac{2}{3} \frac{\left[e^2 / 4\pi\varepsilon_0 \right]}{(mc)^3} |\mathbf{p}|^2 |c\mathbf{B}|^2$$

Energy lost per turn in a ring of constant bend radius ρ, where $|\mathbf{p}| \approx E / c = e|\mathbf{B}|\rho$

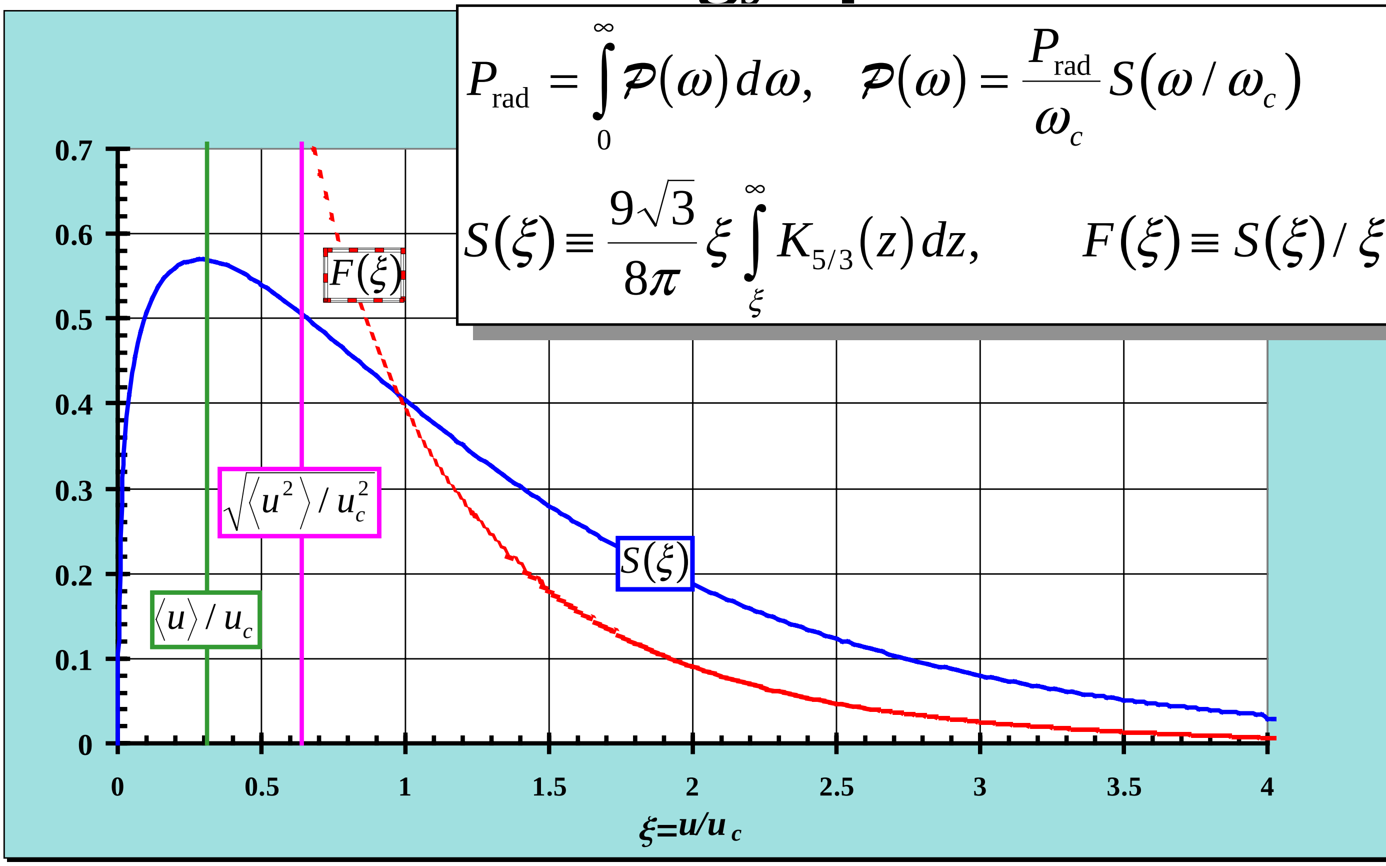
$$U_0 = \frac{2\pi\rho}{c} P_{\text{rad}} = \frac{4\pi}{3} \frac{r_e}{\left(mc^2\right)^3} \frac{E^4}{\rho} = C_\gamma \frac{E^4}{\rho}$$

where $C_\gamma = \dfrac{4\pi}{3} \dfrac{r_e}{\left(mc^2\right)^3} = 8.85 \times 10^{-5}\ \text{m.GeV}^{-3}$ (electrons)

Photon energy spectrum

$$P_{\text{rad}} = \int_0^\infty \mathcal{P}(\omega)\,d\omega, \quad \mathcal{P}(\omega) = \frac{P_{\text{rad}}}{\omega_c} S(\omega/\omega_c)$$

$$S(\xi) \equiv \frac{9\sqrt{3}}{8\pi}\xi \int_\xi^\infty K_{5/3}(z)\,dz, \qquad F(\xi) \equiv S(\xi)/\xi$$



❑ Classical frequency spectrum

re-interpret as energy spectrum of photons

$$P_{\text{rad}} = \int_0^\infty n_X(u)\,u\,du, \quad n_X(u) = \frac{\mathcal{P}(u/\hbar)}{\hbar u}$$

$N_X(s)$ = photon emission rate

$$= \int_0^\infty n_X(u,s)\,du = \frac{5\sqrt{3}}{6}\frac{c r_e p_0}{\hbar} b_X(s)$$

independent of p.

$$\langle u \rangle_X = \text{mean photon energy}$$

$$= \frac{8}{15\sqrt{3}} u_c$$

$$= \frac{4}{5\sqrt{3}} \frac{\hbar c p_0}{(mc)^3} p^2 b_X(s)$$

$$\langle u^2 \rangle_X = \text{mean-square photon energy}$$

$$= \frac{11}{12} \frac{(\hbar c p_0)^2}{(mc)^6} p^4 b_X(s)^2$$

Stochastic Radiation Power

❑ Random emission time and photon energy

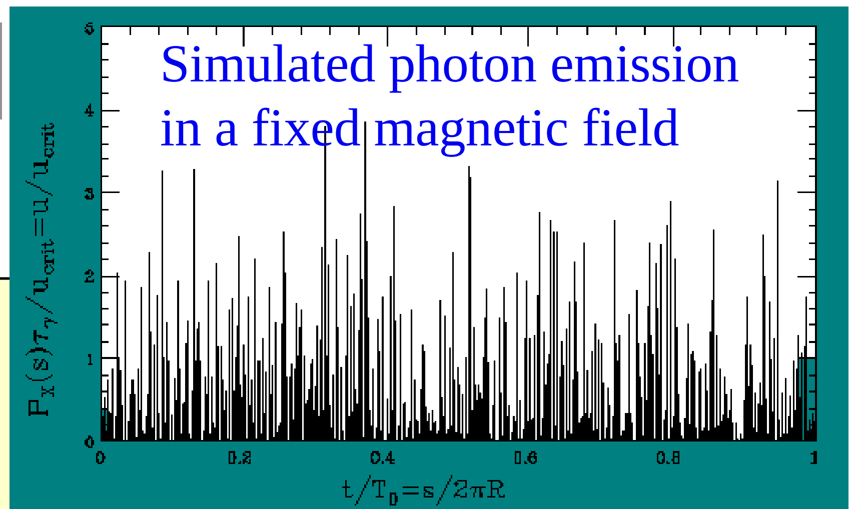
Density of a given realisation (fixed X, s):

$$\Omega_X(u,s) = \sum_j \delta(s - s_j)\delta(u - u_j), \quad \langle \Omega_X(u,s) \rangle = N_X(s) f_X(u;s)/c$$

$f_X(s) =$ energy spectrum of photons

❑ Instantaneous radiation power

$$P_X(s) = c\sum_j u_j \delta(s - s_j),$$



Stochastic average:

$$\langle P_X(s) \rangle = N_X(s) \langle u \rangle_X$$
$$= c_1 c^2 p^2 b_X(s)^2$$

$$c_1 = \frac{2 r_e p_0^2}{3(mc)^3}, \quad b_X(s) = \frac{e}{p_0 c} |\mathbf{B}(x,y,s)|$$

$$\langle \xi(s)\xi(s') \rangle = \delta(s - s_j)$$

$$c_2 = \frac{55}{24\sqrt{3}} \frac{r_e \hbar p_0^3}{(mc)^6}$$

Stochastic representation of fluctuating power:

$$P_X(s) = c^2 p^2 \left(c_1 b_X(s)^2 + \sqrt{c_2 b_X(s)^3}\, \xi(s) \right)$$

See SLAC Summer School for further details.
Representation of power can be *approximated or simplified* in various ways of which this is one.

Correlation function

Stochastic representation of fluctuating power:

$$P_X(s) = c^2 p^2 \left(c_1 b_X(s)^2 + \sqrt{c_2 b_X(s)^3}\, \xi(s) \right)$$

$$P_X(s) = \underbrace{\langle P_X(s) \rangle}_{\text{mean}} + \underbrace{\hat{P}_X(s)}_{\text{fluctuating part}}$$

Two - point correlation function of fluctuations:

$$\Rightarrow \left\langle \hat{P}_X(s) \hat{P}_X(s') \right\rangle = c N_X(s) \left\langle u^2 \right\rangle_{Xs} \delta(s - s')$$

Application of "Campbell's Theorem"

Classical deterministic radiation power has been supplemented with a term of order $\sqrt{\hbar}$.

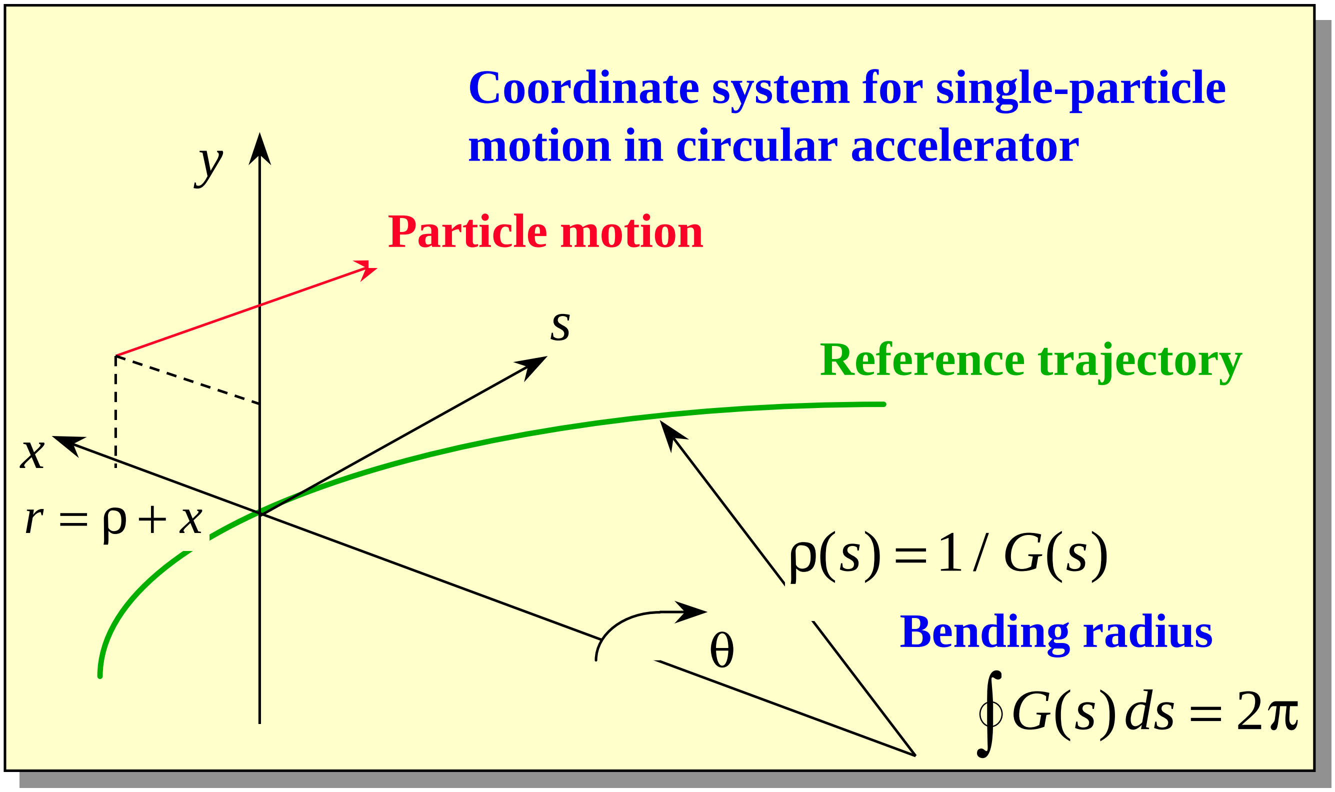
Average radiation power and its quantum fluctuations depend nonlinearly on the particle's coordinates through the instantaneous momentum and spatial dependences of the magnetic field.

Simulation

Simulation of photon emission involves emitting photons at random times along the particle's path with probability depending on the local magnetic field and *p*.

Each photon's energy must be generated randomly in accordance with a distribution depending on the same physical quantities.

Storage Ring Coordinates



- Azimuthal coordinate s (usually) plays role of time (independent variable) in accelerator dynamics.

  Time t becomes the coordinate for the third degree of freedom (different particles pass s at different times). Usually use time-delay w.r.t. reference particle.

- Particles move in a neighbourhood of a reference trajectory (ideally a curve passing through centres of all magnets)
- Each of the coordinates *(x,y,ct)* has a conjugate momentum variable

  Usually measure in units of reference longitudinal momentum so these momenta are dimensionless variables.

  In these units, p_x, p_y are equal to the (small) *angles* of particle trajectory with respect to reference trajectory.

Applied fields

❑ Vector potential in storage ring

For illustration describe only flat ring with dipoles, upright quadrupoles, sextupoles and RF cavities. Other terms can be added to describe other elements.

One relevant component of vector potential in Coulomb gauge:

$$A_s(x,y,t,s) = \mathbf{A}.\mathbf{e}_s(1+G(s)x)$$

$$= -\frac{p_0 c}{e}\left[xG(s)\left(1+G(s)\frac{x}{2}\right)+\tfrac{1}{2}K_1(s)\left(x^2-y^2\right)+\tfrac{1}{6}K_2(s)\left(x^3-3xy^2\right)+\ldots\right]$$

$$+\sum_k \frac{e\hat{V}_k}{\omega_{\mathrm{RF}}}\delta_C(s-s_k)\cos(\omega_{\mathrm{RF}}t+\phi_k)$$

$G(s) =$ curvature of reference orbit

❑ Hamiltonian

$$H(x,y,t,p_x,p_x,-E;s) = (\mathbf{p}+e\mathbf{A}).\mathbf{e}_s(1+G(s)x)$$

$$= -eA_s(x,y,t,s) - (1+G(s)x)\sqrt{\frac{E^2}{c^2}-m^2c^2-p_x^2-p_y^2}$$

Canonical transformation to simplify momentum-dependence

$$(t,-E) \mapsto \left(z_t = -ct\sqrt{1-m^2c^4/E^2},\, p\right)$$

$$H(x,y,z_t,p_x,p_x,p;s) = -eA_s(x,y,t(z_t,p),s)$$

$$-(1+G(s)x)\sqrt{p^2-p_x^2-p_y^2}$$

Equations of motion

$$x' = (1+Gx)\frac{p_x}{\sqrt{p^2 - p_x^2 - p_y^2}} \approx (1+Gx)\frac{p_x}{p}$$

$$y' = (1+Gx)\frac{p_y}{\sqrt{p^2 - p_x^2 - p_y^2}} \approx (1+Gx)\frac{p_y}{p}$$

$$z_t' = -(1+Gx)\frac{p}{\sqrt{p^2 - p_x^2 - p_y^2}} \approx (1+Gx) \approx ct'$$

$$p_x' = -G(p - p_0) - p_0\left(G^2 + K_1\right)x - \tfrac{1}{2} p_0 K_2\left(x^2 - y^2\right) + \ldots$$

$$p_y' = p_0 K_1 y + \tfrac{1}{2} p_0 K_2 xy + \ldots$$

$$p' \approx -\sum_k \frac{e\hat{V}_k}{c}\delta_C(s - s_k)\cos(\omega_{\mathrm{RF}} z_t / c + \phi_k)$$

Note dependences on canonical momenta.

Radiation has not yet been included.

How can we add terms describing the effect of radiation reaction?

Radiation Reaction Forces

Consider single photon emission

Sweeping many electrodynamic subtleties under the carpet (see e.g., Jackson for further discussion).

Photon emission is essentially collinear (opening angle of radiation from beam is $1/2\gamma$) with particle motion and is modelled as instantaneous momentum change:

$$\mathbf{p} \mapsto \mathbf{p} - \mathbf{u}/c, \quad \text{where } \mathbf{p}.\mathbf{u} > \mathbf{0}, \quad \mathbf{p} \times \mathbf{u} = 0$$

$$p \mapsto p - u/c$$

$$p_x \mapsto p_x - \frac{u}{c}\frac{p_x}{p} = p_x - \frac{u}{c}\frac{x'}{t'}$$

$$p_y \mapsto p_y - \frac{u}{c}\frac{p_y}{p} = p_y - \frac{u}{c}\frac{y'}{t'}$$

Consider short time interval around emission azimuth s

$$u = \int_{s-\sigma}^{s+\sigma} P_X(s')\,ds' \quad \text{as } \sigma \to 0$$

to construct stochastic differential equations:

$$dp = -P_X(s)dt/c \approx -P_X(s)z_t' ds/c^2$$

$$dp_x = -P_X(s)(x'/t')dt/c = -P_X(s)x' ds/c^2$$

$$dp_y = -P_X(s)(y'/t')dt/c = -P_X(s)y' ds/c^2$$

Must also include the Hamiltonian part of the equations of motion.

General equations of motion

Combine Hamiltonian and radiation reaction

Special form applies in *these coordinates* only.

$$x' = \frac{\partial H}{\partial p_x} \qquad p_x' = -\frac{\partial H}{\partial x} - \frac{P_X(s)}{c^2}\frac{\partial H}{\partial p_x},$$

$$y' = \frac{\partial H}{\partial p_y} \qquad p_y' = -\frac{\partial H}{\partial y} - \frac{P_X(s)}{c^2}\frac{\partial H}{\partial p_y},$$

$$z_t' = \frac{\partial H}{\partial p} \qquad p' = -\frac{\partial H}{\partial z_t} + \frac{P_X(s)}{c^2}\frac{\partial H}{\partial p}$$

Define radiation coupling functions, note dependences on momenta (root of Robinson Theorem):

$$\begin{aligned} p_x' &= -\frac{\partial H}{\partial x} - (1+Gx)pp_x\left(c_1 b_X(s)^2 + \sqrt{c_2 b_X(s)^3}\,\xi(s)\right) \\ &\equiv -\frac{\partial H}{\partial x} + \frac{p_0}{c}\Pi_x \\ p_y' &= -\frac{\partial H}{\partial y} - (1+Gx)pp_y\left(c_1 b_X(s)^2 + \sqrt{c_2 b_X(s)^3}\,\xi(s)\right) \\ &\equiv -\frac{\partial H}{\partial y} + \frac{p_0}{c}\Pi_y \\ p' &= -\frac{\partial H}{\partial z_t} + (1+Gx)p^2\left(c_1 b_X(s)^2 + \sqrt{c_2 b_X(s)^3}\,\xi(s)\right) \\ &\equiv -\frac{\partial H}{\partial z_t} - \frac{p_0}{c}\Pi_t \end{aligned}$$

Stochastic differential equations of motion

Robinson's Theorem

Non-Liouvillian flow

The equations of motion describe a flow which does *not* conserve phase space measure.

$$\mathbf{X} = \left(x, y, z_t, p_x, p_y, p,\right), \quad \mathbf{X}' = \mathbf{V}(\mathbf{X})$$

with **V** given by the RHS of the equations of motion.

$$\mathbf{V} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \frac{\partial H}{\partial \mathbf{X}} - \frac{P_X(s)}{c^2} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} \end{pmatrix} \frac{\partial H}{\partial \mathbf{X}}$$

where $\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

The divergence of this flow is

$$\nabla_{\mathbf{X}} \cdot \mathbf{V} = -\frac{\partial}{\partial p_x}\left[\frac{P_X(s)}{c^2}\frac{\partial H}{\partial c}\right] - \frac{\partial}{\partial p_y}\left[\frac{P_X(s)}{c^2}\frac{\partial H}{\partial p_y}\right] + \frac{\partial}{\partial p}\left[\frac{P_X(s)}{c^2}\frac{\partial H}{\partial p}\right]$$

$$= \frac{P_X(s)}{c^2}\left[-\frac{\partial^2 H}{\partial p_x^2} - \frac{\partial^2 H}{\partial p_y^2} + \frac{\partial^2 H}{\partial p^2}\right] + \frac{2 P_X(s)}{c^2}\frac{\partial H}{\partial p}$$

where we used the fact that $P_X(s)$ does not depend on p_x, p_y.

❑ Evaluate derivatives of *H* (exactly!)

Neglect synchrotron radiation in places (RF cavities) where vector potential depends on p (through original time-dependence), so we can evaluate:

$$\frac{\partial^2 H}{\partial p_x^2} = (1+Gx)\frac{p^2 - p_y^2}{\left(p^2 - p_x^2 - p_y^2\right)^{3/2}}$$

❑ Combine terms

$$\nabla_{\mathbf{x}} \cdot \mathbf{V} = -(1+Gx)\frac{4P_X(s)}{c^2\sqrt{p^2 - p_x^2 - p_y^2}}$$

From original Hamiltonian, we have

$$t' = (1+Gx)\frac{E}{c^2\sqrt{p^2 - p_x^2 - p_y^2}}$$

❑ Total "damping" is half of

$$\boxed{\left\{\begin{matrix}\text{Phase-space compression} \\ \text{per unit time}\end{matrix}\right\} = -\frac{4P_X(s)}{E}}$$

This is a *local* version of Robinson's Theorem and will hold for all canonical transformations we make.

It also includes fluctuations of the radiation power.

❑ Robinson's original version (1-turn average)

$$\alpha_{\text{tot}} \approx \frac{2U_0 f_0}{E} \quad \text{where} \quad U_0 = \oint P_X(s)t'\,ds, \quad \text{valid for } \frac{U_0}{E} << 1$$

Closed orbit with radiation

"Sawtooth" closed orbit satisfies:

$$x_0' = \left.\frac{\partial H}{\partial p_x}\right|_0 = \left.H_{p_x}\right|_0 \quad \text{and similar for } y$$

$$p_{x0}' = -\left.\frac{\partial H}{\partial x}\right|_0 - \frac{\left.\langle P_X(s)\rangle\right|_0}{p_0 c^2}\left.\frac{\partial H}{\partial p_x}\right|_0 = \left.H_x\right|_0 - \frac{\left.\langle P\rangle\right|_0}{p_0 c^2}\left.H_{p_x}\right|_0,$$

$$z_{t0}' = \left.\frac{\partial H}{\partial \delta}\right|_0 = \left.H_\delta\right|_0$$

$$p' = -\left.\frac{\partial H}{\partial z_t}\right|_0 + \frac{\left.\langle P_X(s)\rangle\right|_0}{p_0 c^2}\left.\frac{\partial H}{\partial \delta}\right|_0 = \left.H_{z_t}\right|_0 - \frac{\left.\langle P\rangle\right|_0}{p_0 c^2}\left.H_\delta\right|_0$$

Canonical transformation:

Take out closed orbit with **energy sawtooth**:

$$\mathbf{X} = (x_0(s) + \tilde{x}, \ldots, z_{t0}(s) + \tilde{z}, p_{x0}(s) + \tilde{p}_x, \ldots, \delta_0(s) + \varepsilon)$$

from generating function

$$F\left(x, y, z_t, \tilde{p}_x, \tilde{p}_y, \varepsilon\right) = \left[x - x_0(s)\right]\left[\tilde{p}_x + p_{x0}(s)\right] + \ldots$$
$$+ \left[z_t - z_{t0}(s)\right]\left[\varepsilon + \delta_0(s)\right]$$

$$\tilde{x} = \frac{\partial F}{\partial \tilde{p}_x}, \quad \cdots\cdots\cdots, \delta = -\frac{\partial F}{\partial \varepsilon}$$

❑ New Hamiltonian:

$$
\begin{aligned}
\tilde{H} &= H\left(X_0(s)+\tilde{X},s\right)+\frac{\partial F}{\partial s} \\
&= H\left(x_0(s),\ldots,p_{x0}(s),\ldots\right)+H_x\big|_0\tilde{x}+H_{p_x}\big|_0\tilde{p}_x \\
&\quad +\tfrac{1}{2}H_{xx}\big|_0\tilde{x}^2+H_{xp_x}\big|_0\tilde{x}p_x+\tfrac{1}{2}H_{p_xp_x}\big|\tilde{p}_x^{\,2}+\mathrm{O}\left(X^3\right) \\
&\quad +\cdots-x_0'(s)\left[\tilde{p}_x+p_{x0}(s)\right]+\left[x-x_0(s)\right]p_{x0}'-\cdots \\
&= \tfrac{1}{2}H_{xx}\big|_0\tilde{x}^2+H_{xp_x}\big|_0\tilde{x}p_x \\
&\quad +\tfrac{1}{2}H_{p_xp_x}\big|\tilde{p}_x^{\,2}+\cdots+\tfrac{1}{2}H_{xx}\big|_0\tilde{x}^2+H_{xp_x}\big|_0\tilde{x}p_x+\tfrac{1}{2}H_{p_xp_x}\big|_0+\mathrm{O}\left(X^3\right) \\
&\quad -\frac{\langle P\rangle|_0}{p_0c^2}H_{p_x}\big|_0\tilde{x}-\cdots+\frac{\langle P\rangle|_0}{p_0c^2}H_\delta\big|_0\tilde{z}
\end{aligned}
$$

But $H_{p_x}\big|_0 \approx \left(1+G(s)x_0(s)\right)p_{x0}(s),\ldots,$

$H_\delta\big|_0 \approx -\left(1+G(s)x_0(s)\right)$

so $\tilde{H}$ contains ***radiation terms*** of the form

$$
-\frac{\langle P\rangle|_0}{p_0c^2}\left(1+G(s)x_0(s)\right)\left(p_{x0}(s)\tilde{x}+p_{y0}(s)\tilde{y}-\tilde{z}\right)
$$

⇒ *next slide*

Stable phase angle

$$
\tilde{H} = \left\{ \begin{array}{l} \text{Usual terms, including} \\ \displaystyle\sum_k \frac{e\hat{V}_{\mathrm{RF}}}{p_0 c\,\omega_{\mathrm{RF}}}\,\delta_C(s-s_k)\cos\!\left(\frac{\omega_{\mathrm{RF}}}{c}\big(\tilde{z}+z_0(s)\big)-\cdots\right) \end{array} \right\}
$$

$$
-\frac{\langle P\rangle\big|_0}{p_0c^2}\big(1+G(s)x_0(s)\big)\Big(p_{x0}(s)\tilde{x}+p_{y0}(s)\tilde{y}-\tilde{z}\Big)
$$

$$
\Rightarrow \quad \varepsilon' = -\frac{\partial\tilde{H}}{\partial\tilde{z}} = \sum_k \frac{e\hat{V}_{\mathrm{RF}}}{p_0c^2}\,\delta_C(s-s_k)\sin\!\left(\frac{\omega_{\mathrm{RF}}}{c}\big(\tilde{z}+z_0(s)\big)-\cdots\right)
$$

$$
-\frac{\langle P\rangle\big|_0}{p_0c^2}\big(1+G(s)x_0(s)\big)
$$

Integrating over one turn (in limit of smallish Q_s, etc.):

$$
\Delta\varepsilon = \oint \varepsilon'\,ds = \sum_k \frac{e\hat{V}_{\mathrm{RF}}}{p_0c^2}\sin\!\left(\frac{\omega_{\mathrm{RF}}}{c}\big(\tilde{z}+z_0(s_k)\big)-\cdots\right)-\frac{U_0}{p_0c}
$$

Equilibrium gives stable phase angle as a component of the 6-dimensional closed orbit.

This Hamiltonian system was constructed explicity via canonical transformations.

Not the same as the “classical” part of the radiation.

In **optics and tracking programs**: if radiation terms are not included in the Hamiltonian, the quadratic terms (which determine the betatron and synchrotron oscillations, tunes, etc.) will be wrong. Focussing functions (which include the RF) must be defined on the true closed orbit even if damping terms are dropped.

MAD command:

```
BEAM,particle=positron,-radiate
TRACK
```

Energy sawtooth

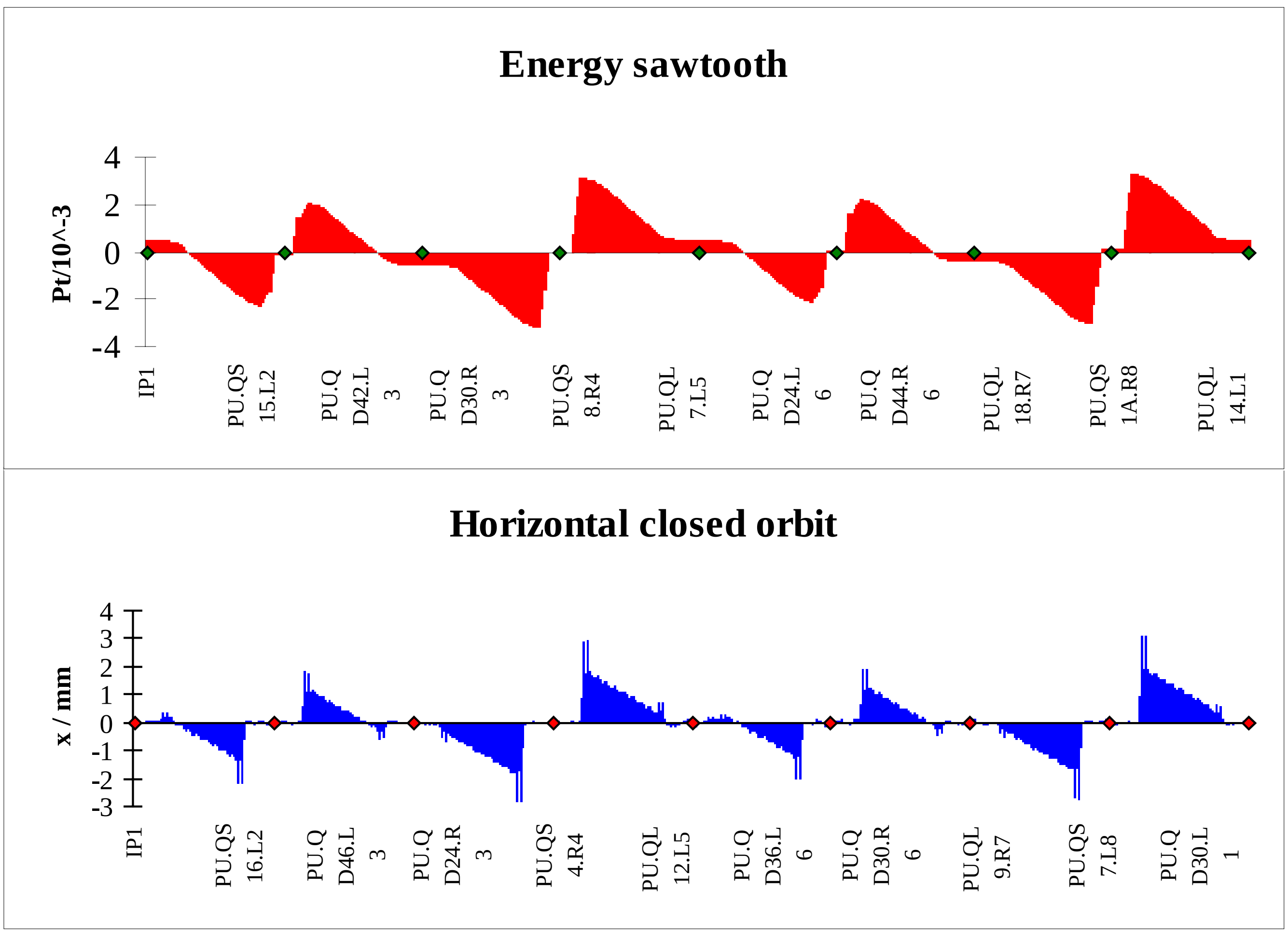
❑ Example from LEP

LEP2: w05v6, 90 GeV, e+ orbit, ideal RF

Ideal LEP2 optical and RF configuration with 192 SC cavities, 120 Cu cavities.

Example of energy sawtooth and closed orbit, which contains additional terms beyond that given directly by energy sawtooth (Bassetti effect):

$$x_c(s) = 0 + D_x \delta_s(s) + x_B(s)$$



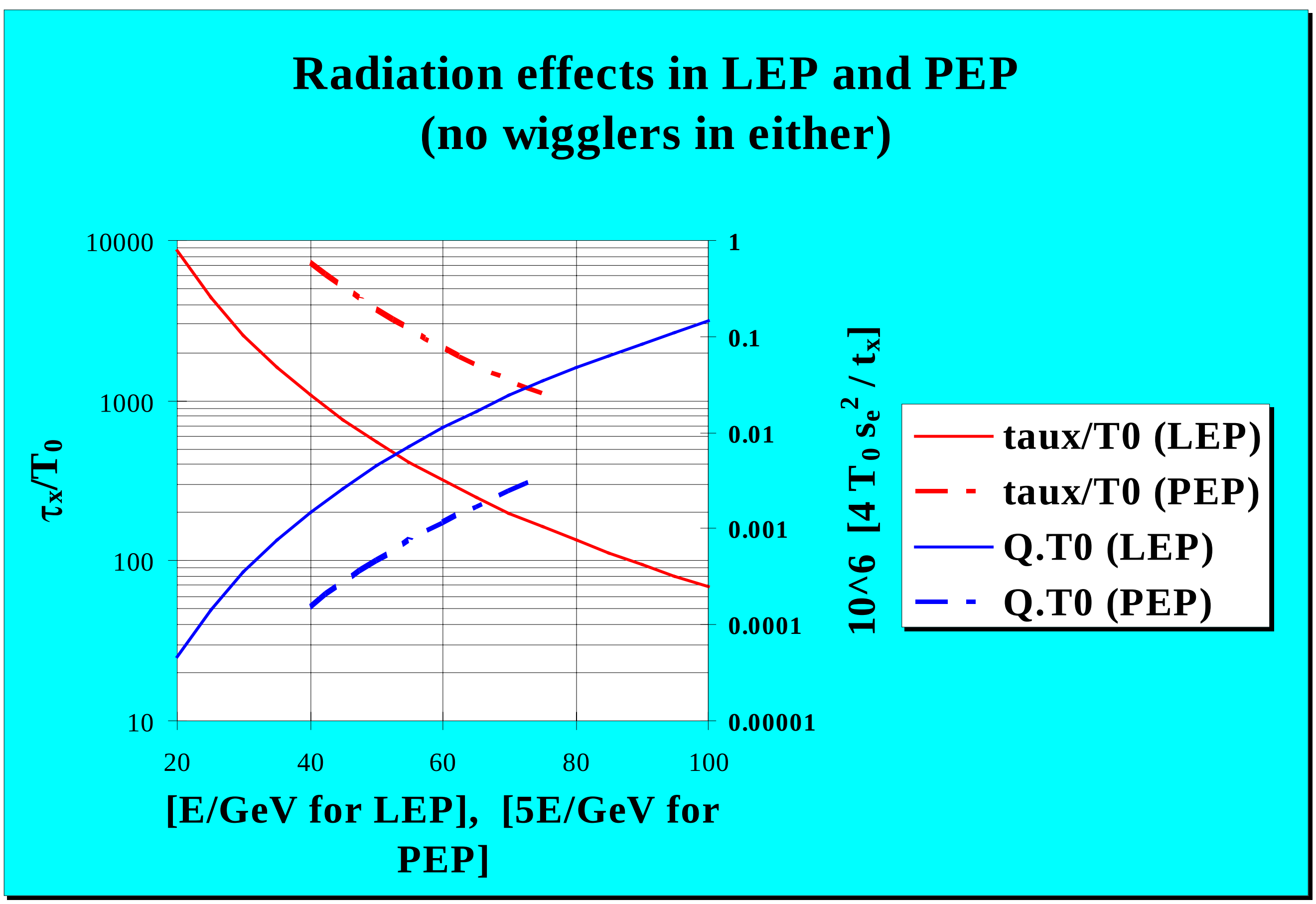
Consequences of LEP2 energy

❑Synchrotron radiation ⇒

- ♦Radiation damping
- ♦Quantum fluctuations
- ♦Local variation of energy (down in magnets, up in RF cavities)
  ⇒ "energy sawtooth", different for e^+ and e^-
  ⇒ different bending, focussing, nonlinearities, locally
  ⇒ different orbits, Twiss functions, dynamic aperture

Radiation damping per turn: $$\frac{T_0}{\tau_x} = \frac{\langle P_X \rangle}{E} \propto \frac{E^3}{\rho},$$

Quantum excitation per turn: $$\frac{N_X \langle u^2 \rangle T_0}{E^2} = \frac{4\sigma_\varepsilon^2}{\tau_x / T_0} \propto \frac{E^5}{\rho^2}$$



Tracking Radiating Particles

Radiation reaction forces (which change physical momenta only) are added to Hamilton's equations

$$x' = \frac{\partial H}{\partial p_x} \qquad p_x' = -\frac{\partial H}{\partial x} - \frac{P_X(s)}{p_0 c^2}\frac{\partial H}{\partial p_x},$$

$$y' = \frac{\partial H}{\partial p_y} \qquad p_y' = -\frac{\partial H}{\partial y} - \frac{P_X(s)}{p_0 c^2}\frac{\partial H}{\partial p_y},$$

$$z_t' = \frac{\partial H}{\partial \delta} \qquad \delta' = -\frac{\partial H}{\partial z_t} + \frac{P_X(s)}{p_0 c^2}\frac{\partial H}{\partial \delta}$$

$$P_X(s) = c\sum_j u_j \delta(s - s_j),$$

Tracking using (Taylor) maps + radiation

Closed orbit with **energy sawtooth**:

$$\mathrm{X}_0(\mathrm{s}) = \left(x_0(s), y_0(s), z_{t0}(s), p_{x0}(s), p_{y0}(s), \delta_0(s)\right) = \mathrm{X}_0(\mathrm{s}+\mathrm{C})$$

Twiss functions or eigenvectors for coupled synchro-betatron

$$x = x_0(s) + D_x(s)(\delta_s + \varepsilon) + x_\beta,$$

$$\vdots$$

$$\delta = \delta_0(s) + \varepsilon$$

Tracking of oscillations around closed orbit

$$x = x_0(s) + \tilde{x},$$

$$\vdots$$

$$\delta = \delta_s(s) + \varepsilon$$

N.B. Momentum deviation on closed orbit $\delta_s(s)$ reflects systematic radiation losses and gains from RF (sawtooth).

Tracking modes (refer to MAD)

❑Symplectic tracking with no radiation

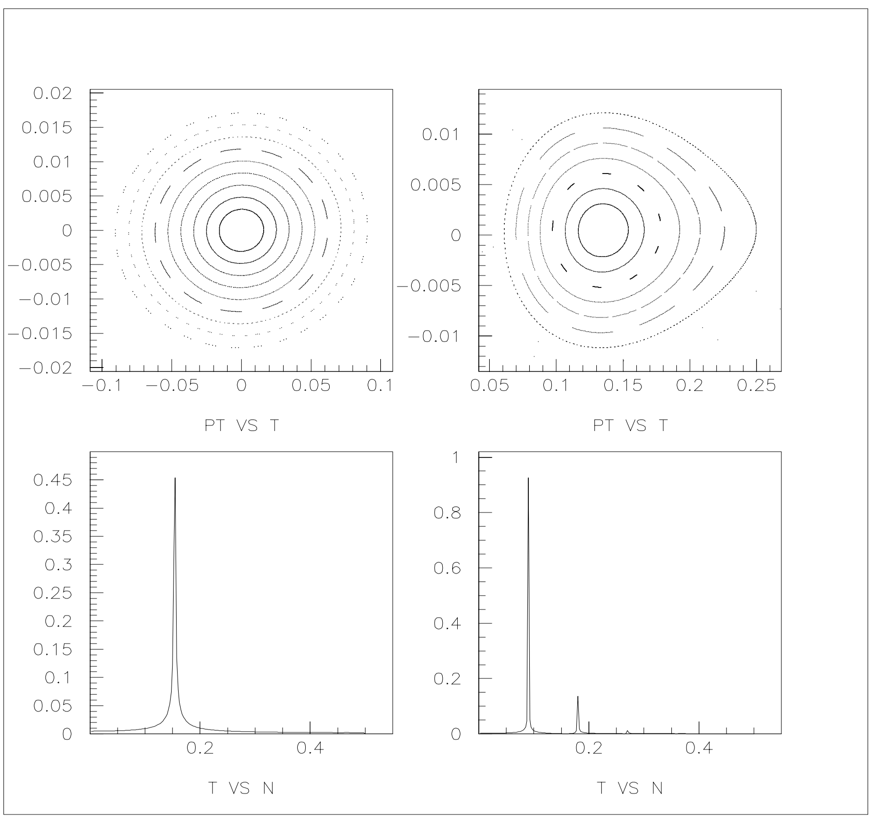
$$\boxed{P_X(s) = 0}$$

Betatron and synchrotron oscillations, tunes, etc. will be incorrect since focussing functions (which include the RF) must be defined on the true closed orbit.

❑Symplectic tracking with radiation

We *proved* that is possible to consider symplectic maps around the closed orbit (i.e. including stable phase angle, energy sawtooth, etc.) determined by radiation:

$$\boxed{P_X(s) = c^2 p_0(s)^2 c_1 b\left(x_0(s), y_0(s), s\right)^2}$$



Symplectic tracking with no radiation | **Symplectic tracking with radiation**

❑ Tracking with radiation damping

Include dependences of classical (deterministic) radiation power on all canonical variables and magnetic elements.

Continuous loss of energy, modification of particle momenta in all magnetic elements.

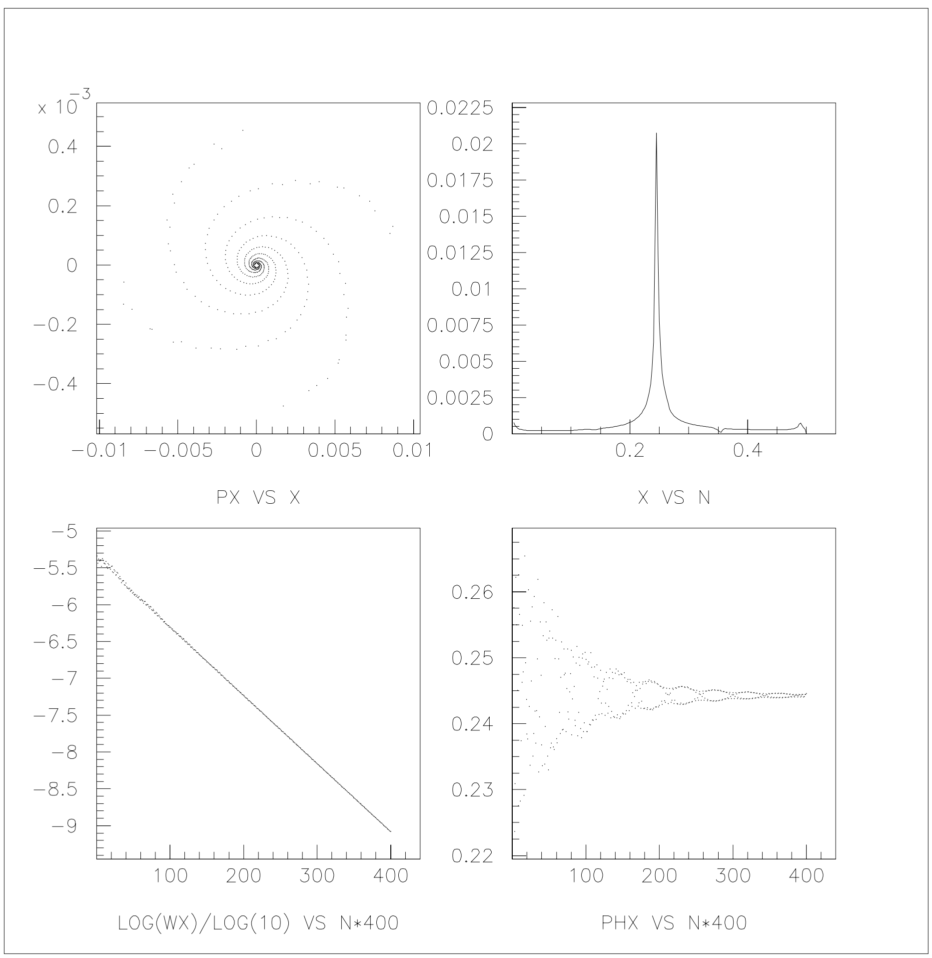
$$P_X(s) = c^2 p^2 c_1 b(x, y, s)^2$$

❑ Radiation damping arises naturally.

❑ Dynamics often becomes relatively simple.

❑ Dynamics is very different from symplectic model!

LEP2 90°/60° optics, 90 GeV, 192 SC + 120 Cu cavities, no betatron amplitudes



Quantum Tracking

- **Emission of individual photons gives quantum fluctuations as well as damping**

$$P_X(s) = c\sum_j u_j \delta(s - s_j)$$

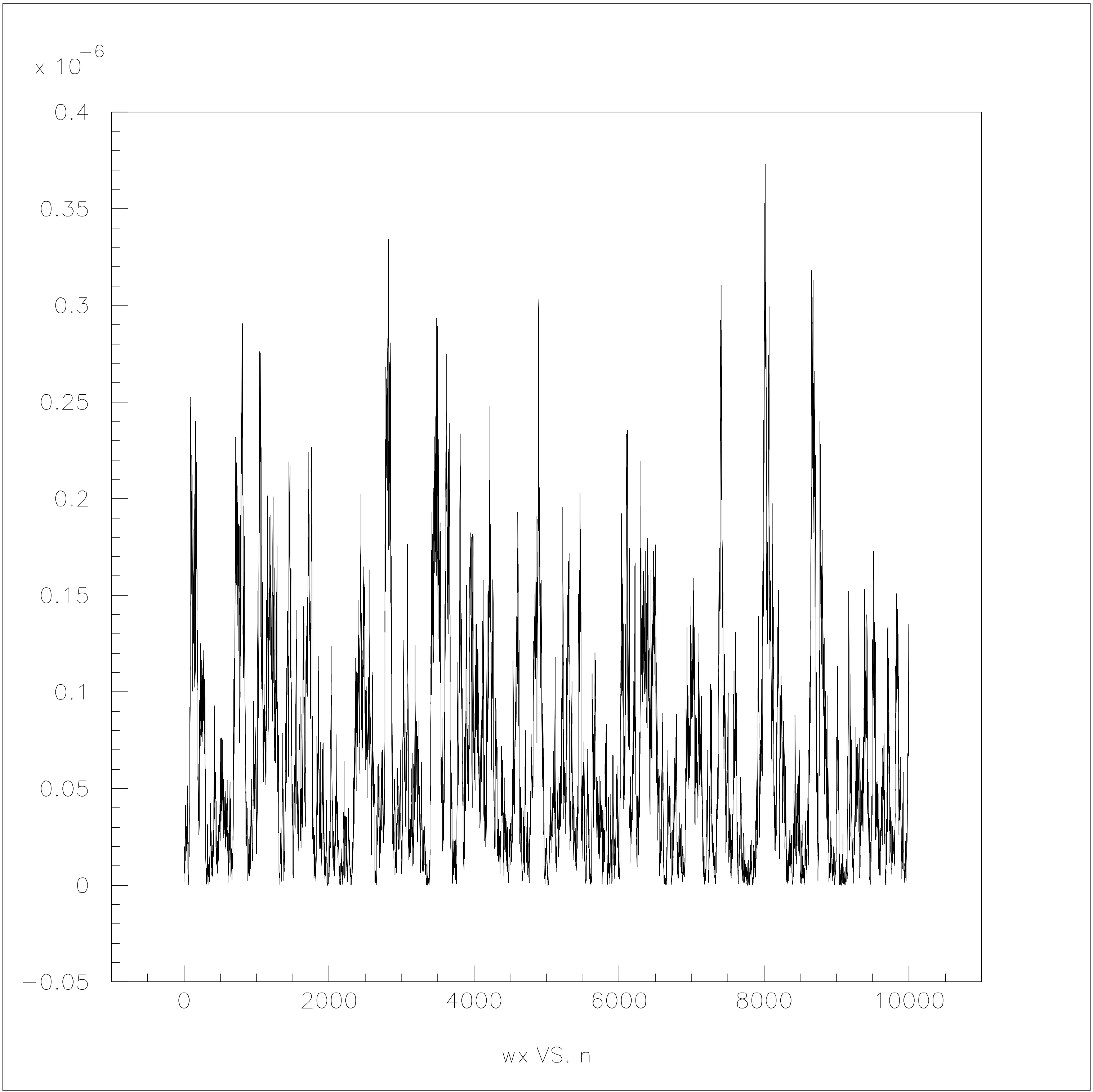
Simulating photon emission

* Decide how many photons to emit in element of length L, field $b_X(s)$, according to Poisson distribution with mean $N_X(s)L/c$
* Generate a random energy for each photon according to the photon distribution for synchrotron radiation (universal form for distribution in units of critical energy which scales with momentum p and magnetic field $b_X(s)$.
* Modify particle momenta (actually done at entrance and exit of each element in MAD)

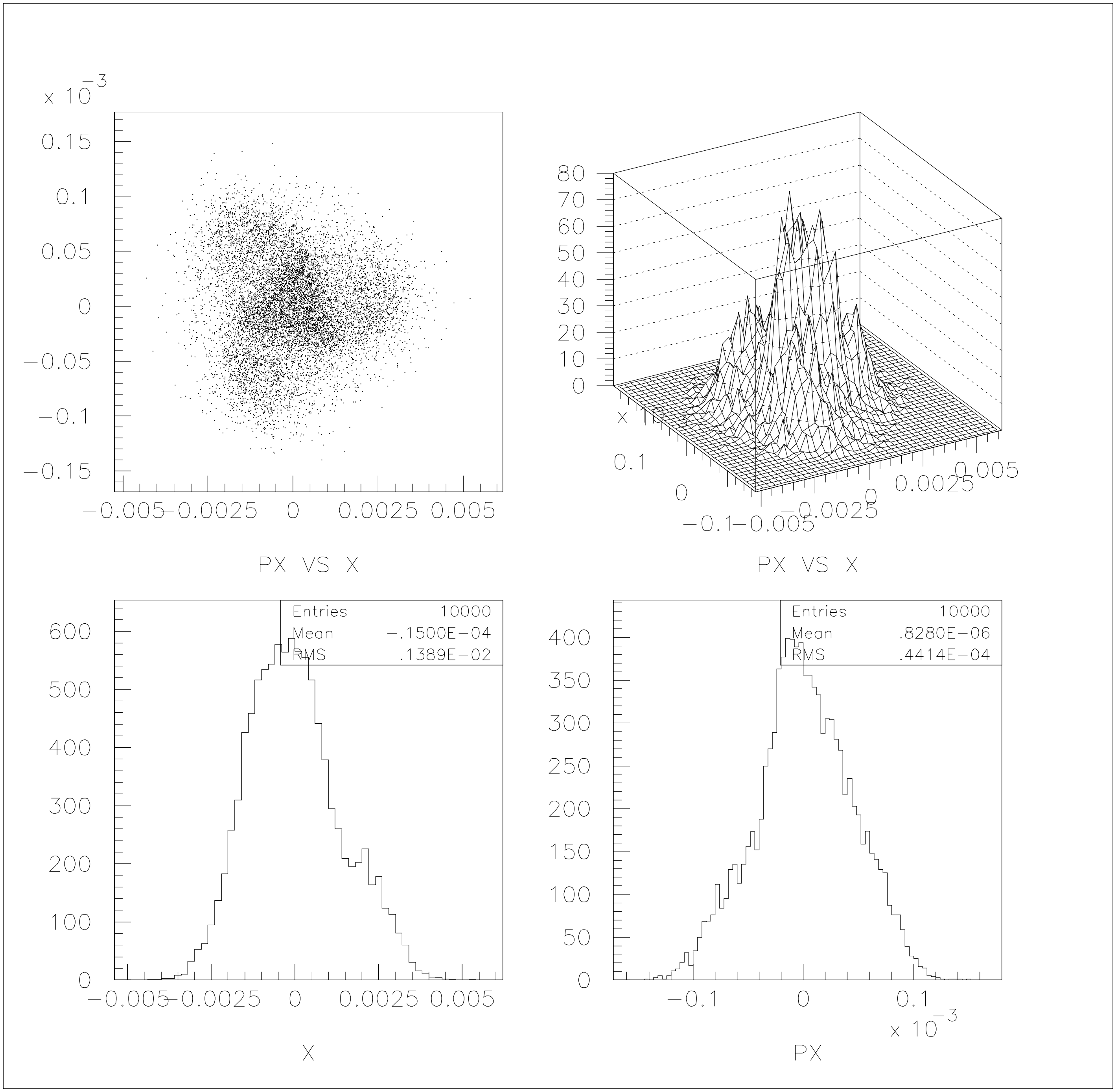
All phenomena related to radiation: closed orbit, damping, energy spread, change of damping partition with f_{RF}, gaussian (or other!) distribution, etc. arise from these photons. Nothing inserted “by hand”!

Amplitude diffusion

❑ "Random walk" of horizontal action variable



LEP, (108°90°) optics, 87 GeV, "bad" imperfect machine
a particle with almost unstable inital horizontal amplitude

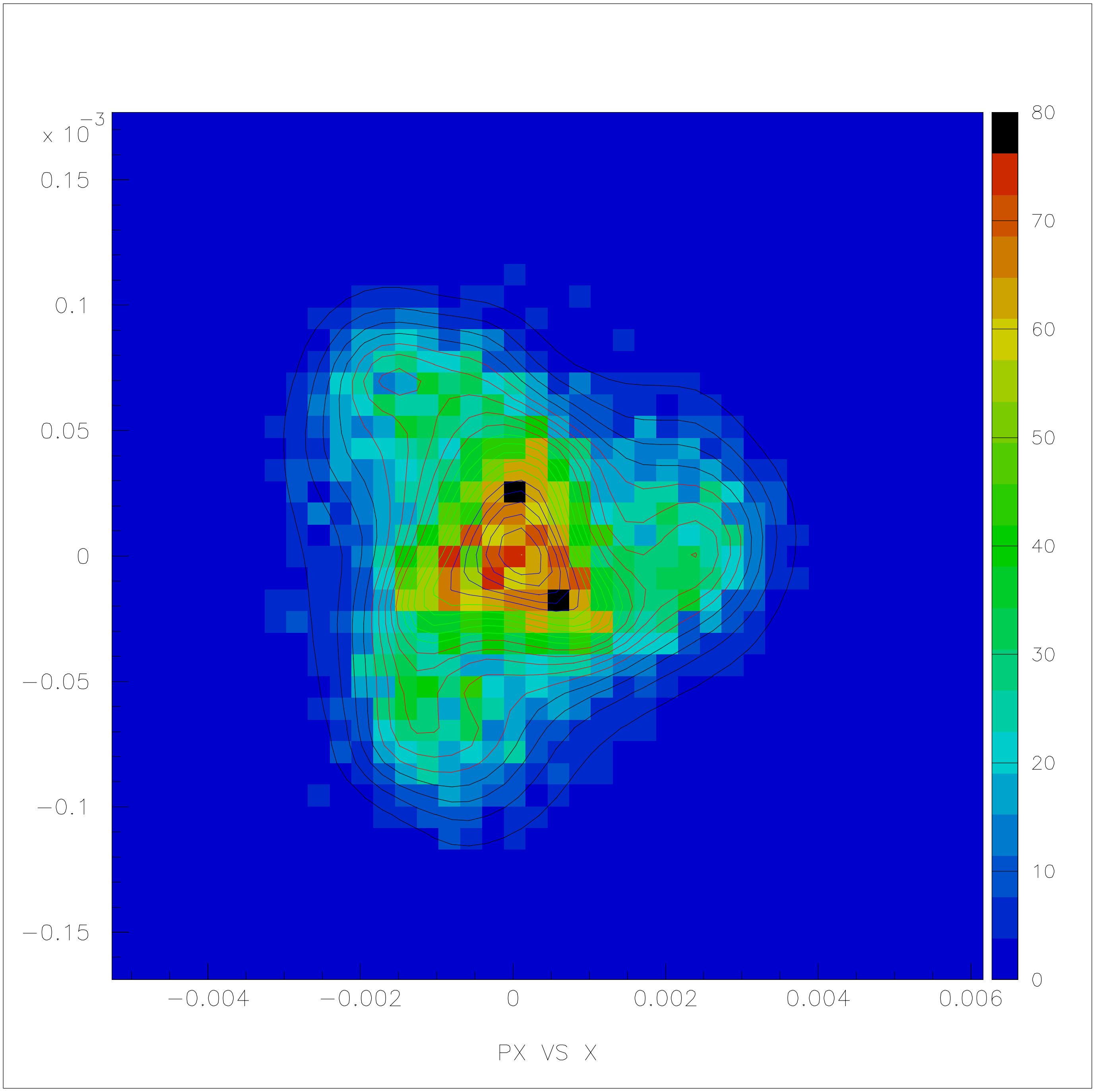


Tracking with quantum fluctuations

LEP2 135°/60° optics, 90 GeV, 192 SC + 120 Cu cavities, starts on closed orbit, 10000 turns, $Q_x \approx 0.35$

Non-gaussian distribution on 3rd order resonance.

Fitted distributions



LEP2 135°/60° optics, 90 GeV, 192 SC + 120 Cu cavities, starts on closed orbit, 10000 turns,

Fitting of contours to distribution.

3rd order resonance islands visible

Instability Mechanisms

Chromatic effects, resonances

Well known, variation of Q_y with momentum/synchrotron amplitude as in hadron machines.

Synchro-betatron couplings and modulations through dispersion in RF, non-linear dispersion, etc.

Non-resonant radiative beta-synchrotron coupling (RBSC)

Particles with large betatron amplitudes make extra energy loss in quadrupoles so their "stable phase angles" change.

This effect is important in determining the **transverse** dynamic aperture at LEP2 energies.

Determine by radiation integrals:

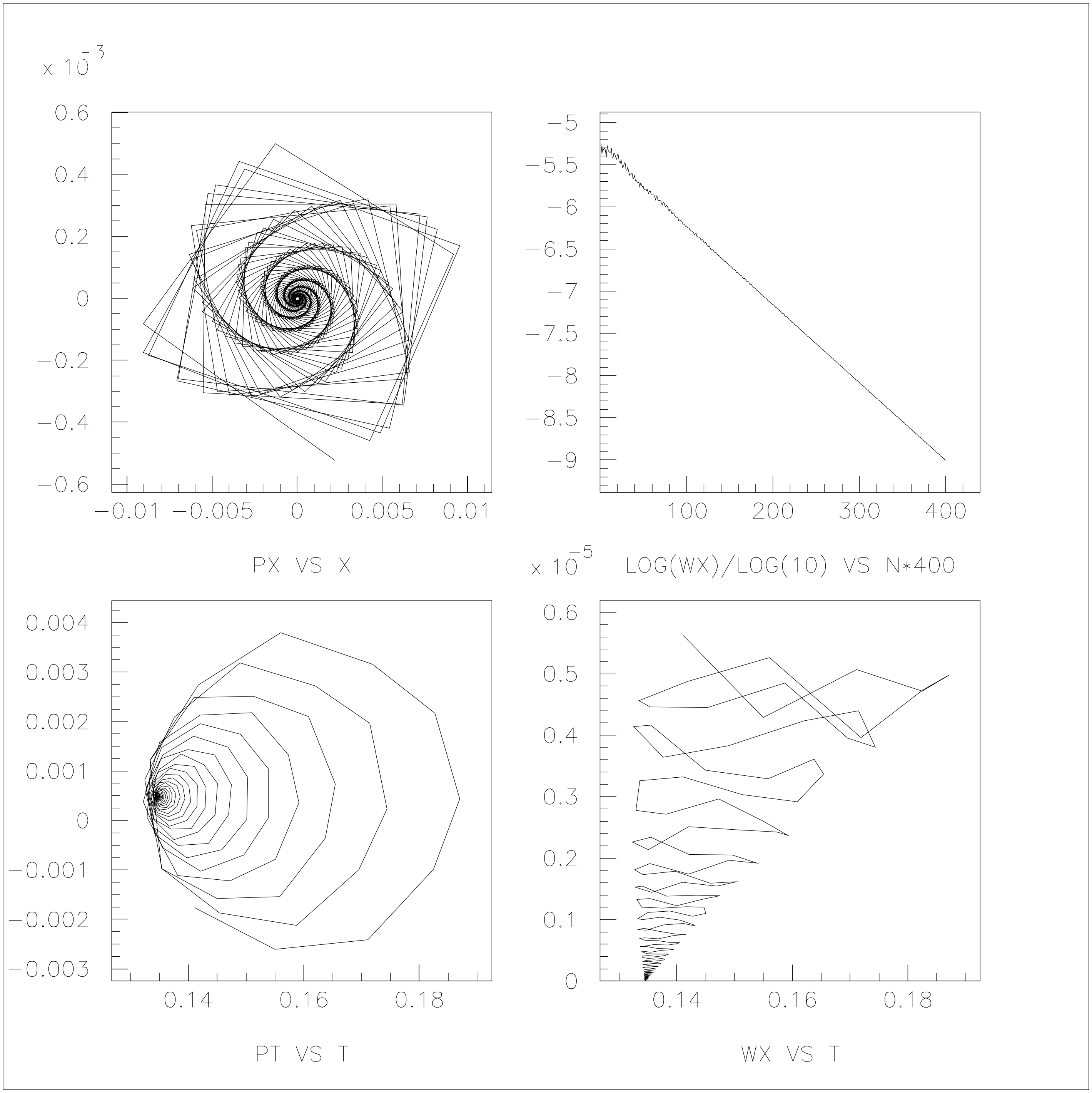
$$I_{6x} = \int K_x^2(s)\beta_x(s)\,ds, \quad I_{6y} = \int K_x^2(s)\beta_y(s)\,ds,$$

$$\Delta\varphi_s \approx \arcsin\left(\sin\varphi_{s0}\frac{U_{\text{dipoles}} + U_{\text{quads}}}{U_{\text{dipoles}}}\right) - \varphi_{s0}$$

$$\approx \tan\varphi_{s0}\frac{U_{\text{quads}}}{U_{\text{dipoles}}} = \tan\varphi_{s0}\frac{I_{6x}W_x + I_{6y}W_y}{I_2} \quad \text{for small } W$$

Comparison of radiation integrals for quadrupoles in LEP2 lattices

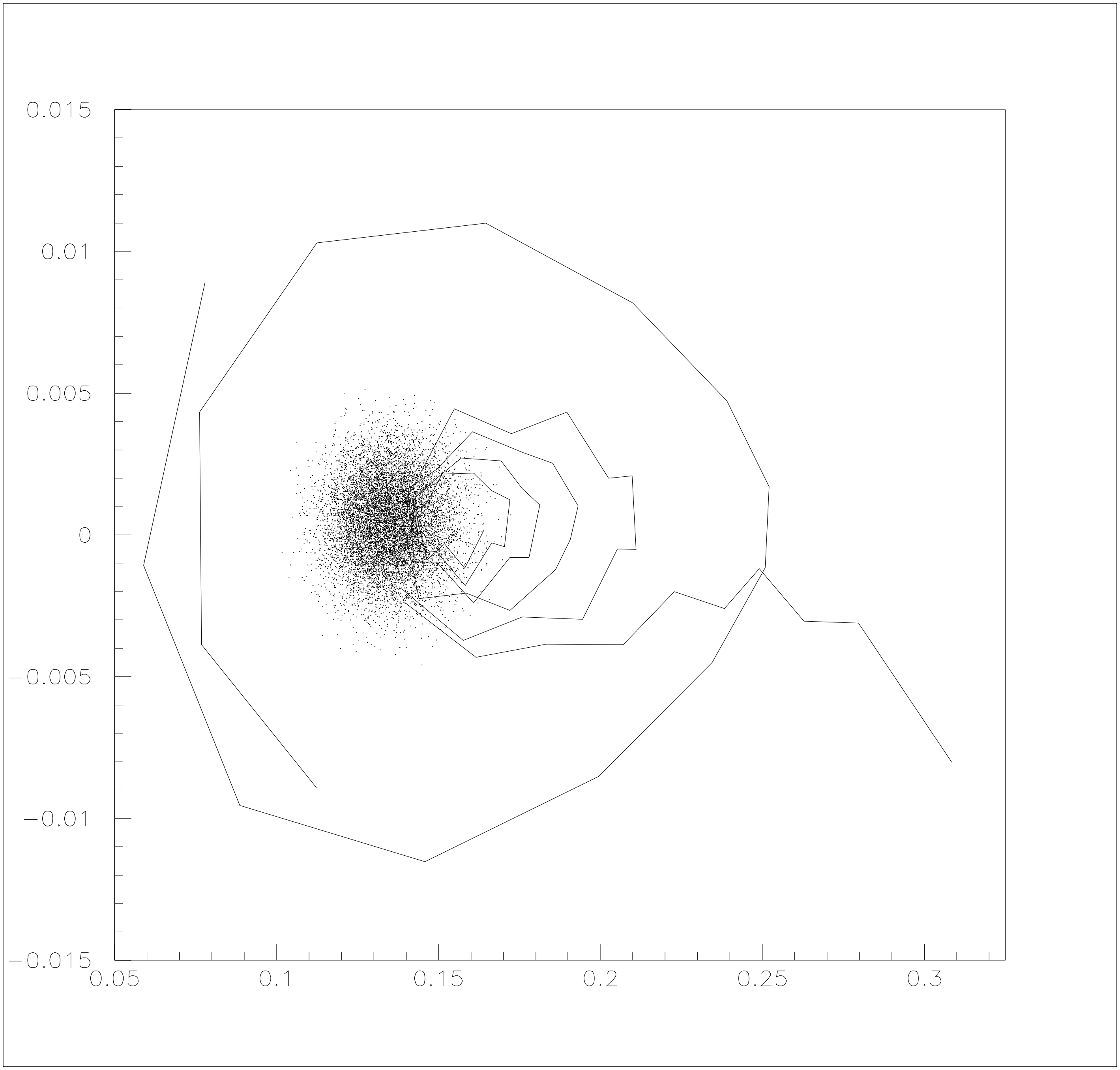
	I6x	I6y
90°/60°	76.81	221.01
135°/60°	106.79	149.96



Tracking with damping

LEP2 90°/60° optics, 90 GeV, etc., fairly large A_x, and initial A_t=0, 400 turns.

Synchrotron motion is generated from betatron motion and both damp away.



Tracking with damping

LEP2 90°/60° optics, 90 GeV, etc., varying A_y, and initial A_t=0, 50 turns. Beam core also shown.

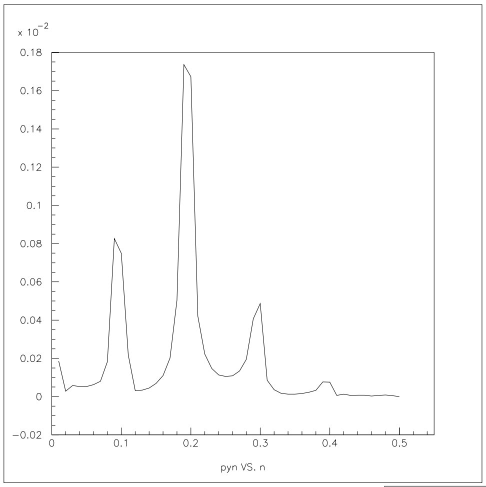
Synchrotron motion is generated from betatron motion and both damp away.

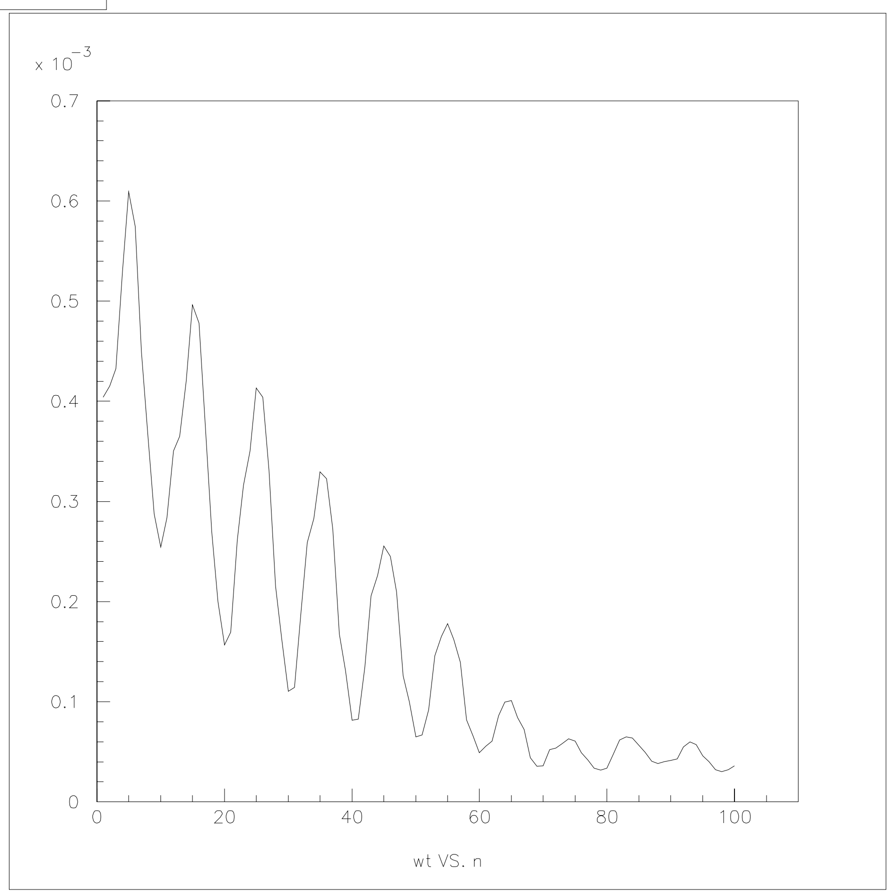
Lost particle has initial $y = 6$ mm

Prevalence of RBSC

- **RBSC tends to amplify any effect leading to amplitude growth in transversed planes**

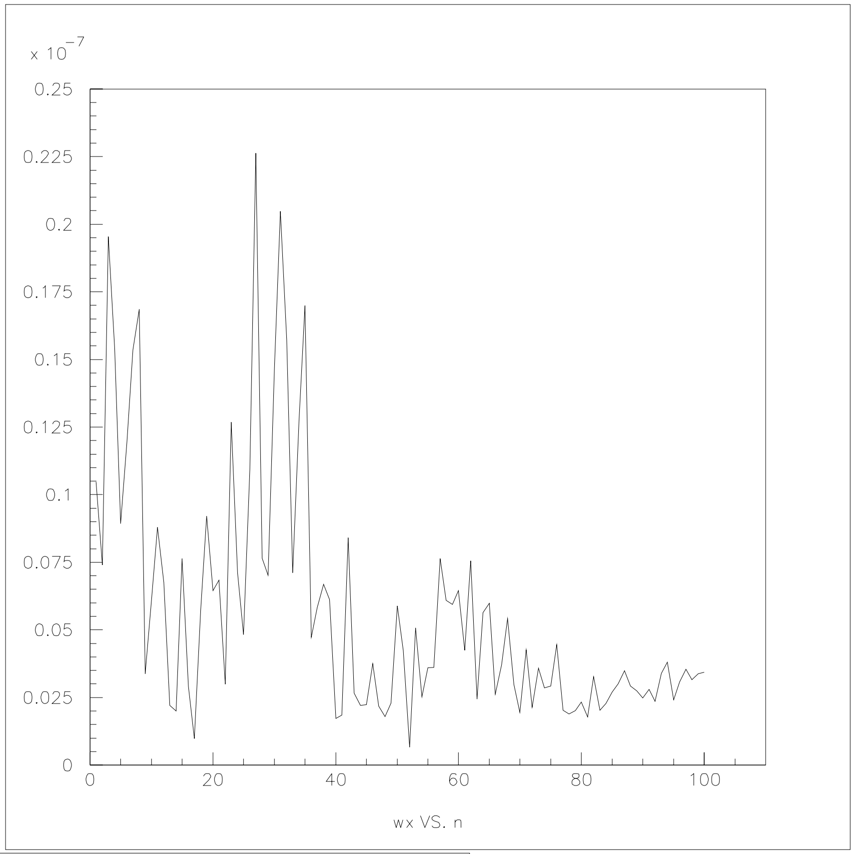
  e.g., classical synchro-betatron resonance driven by vertical dispersion in RF cavities

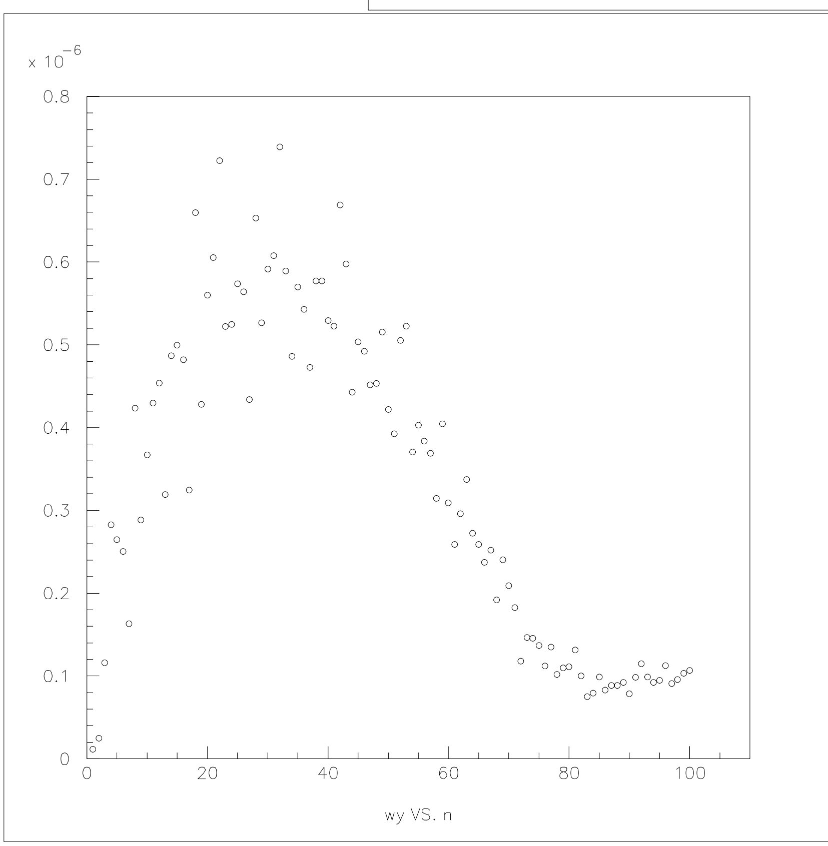




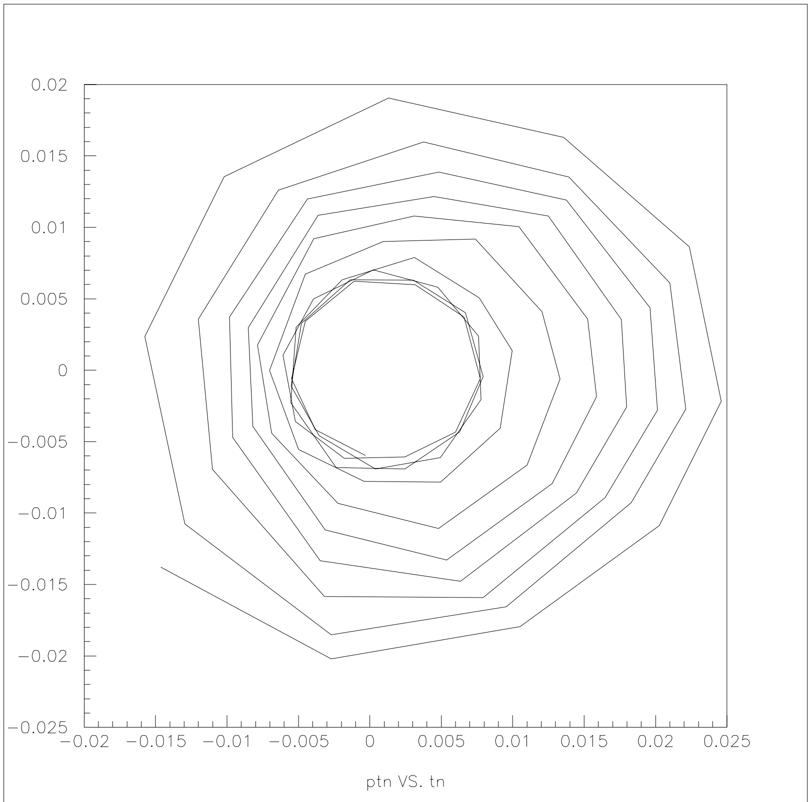
LEP, (108°90°) optics, 87 GeV, "bad" imperfect machine
a particle with almost unstable inital horizontal amplitude

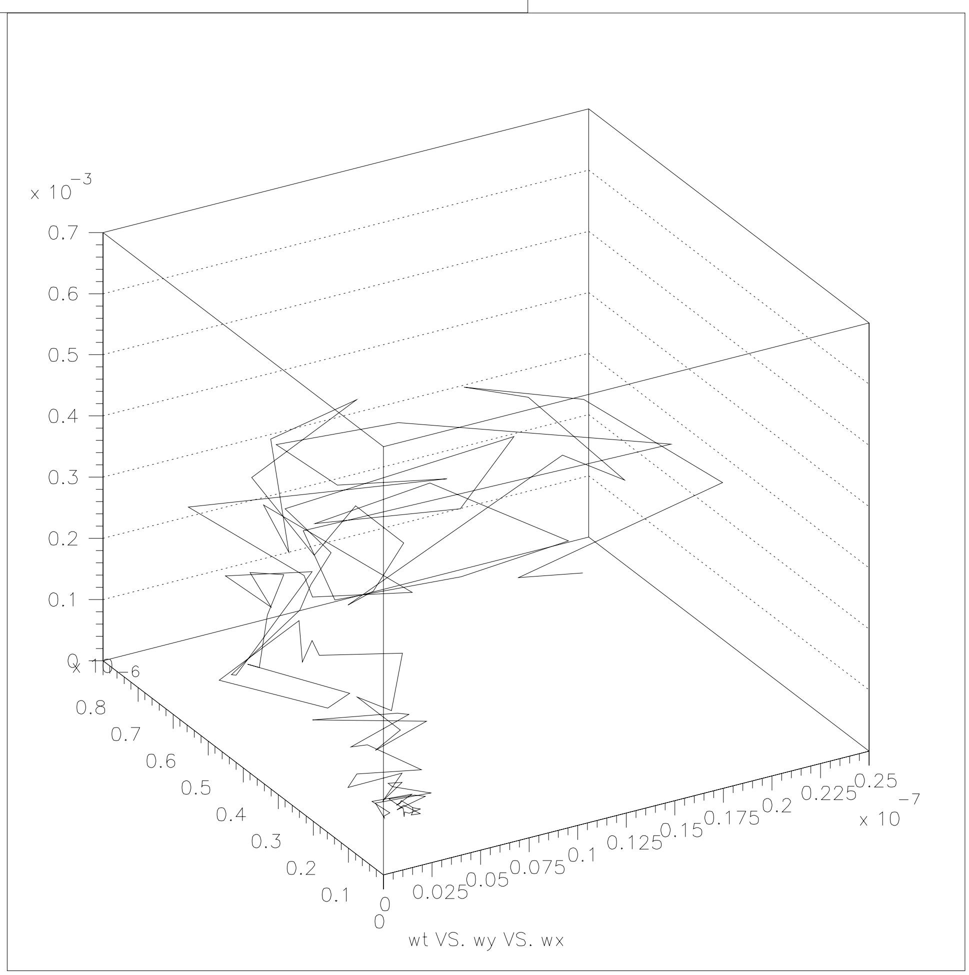
(same example, continued)





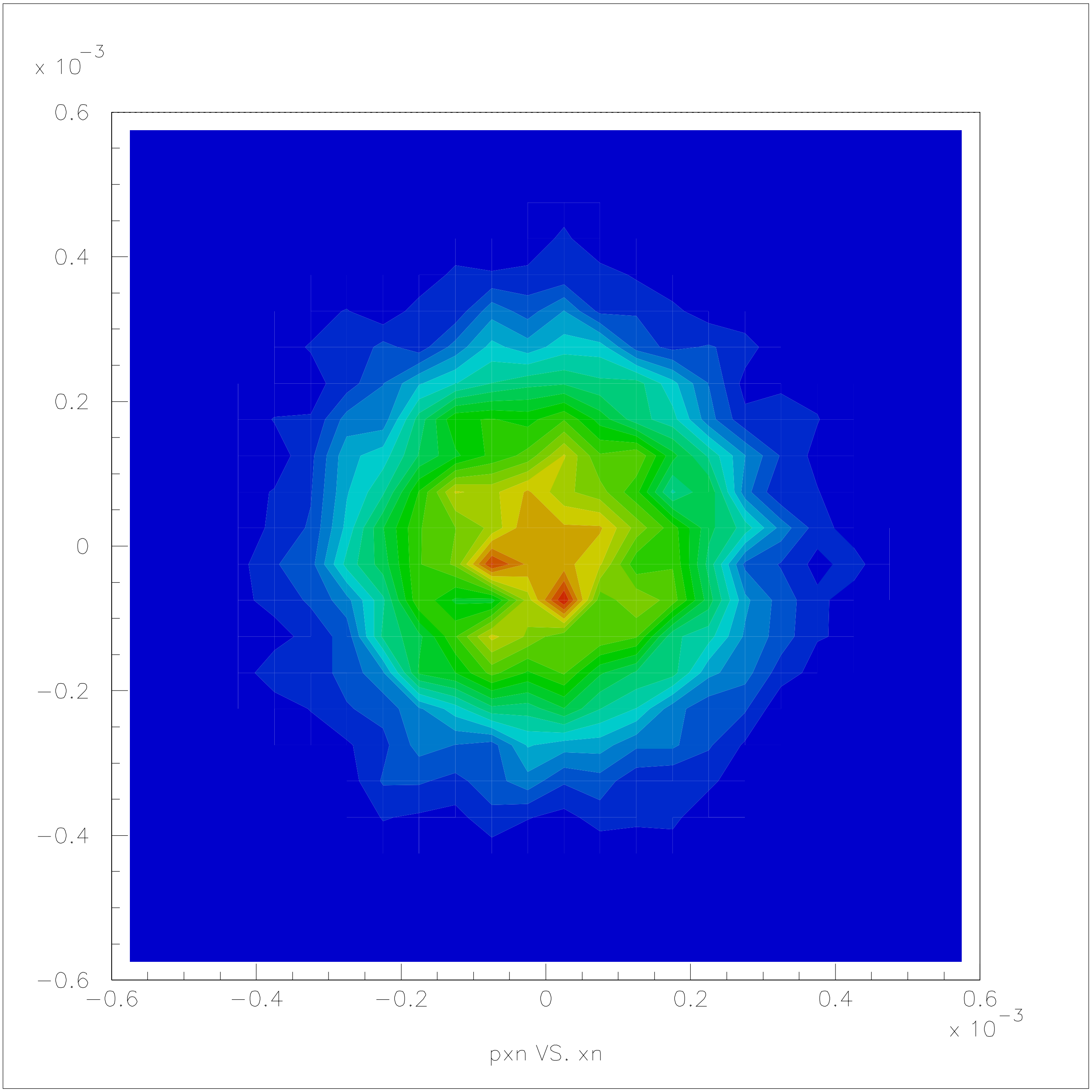
(same example, continued)





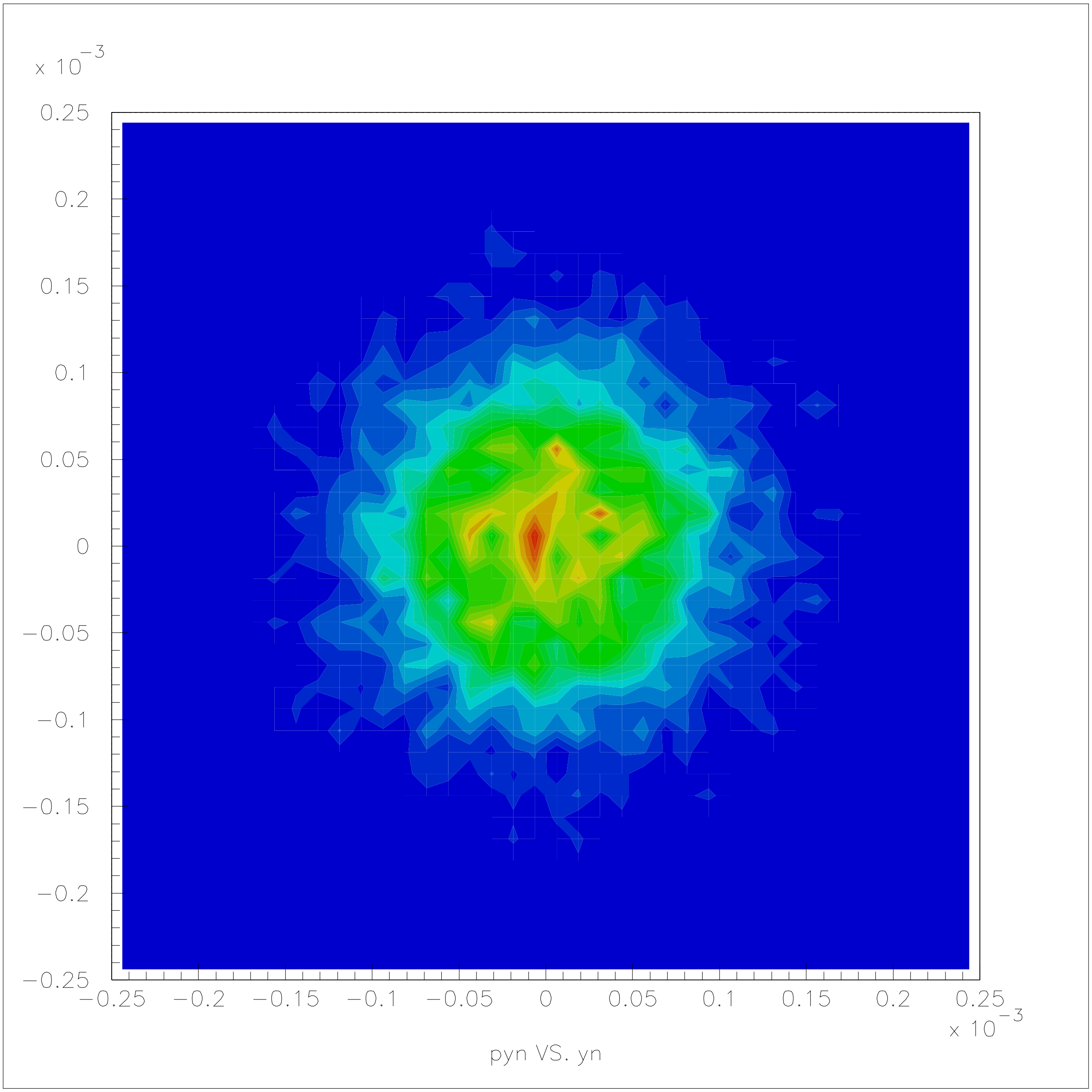
Mode 1 distribution (same ex.)

- Non-linear resonances affect distribution



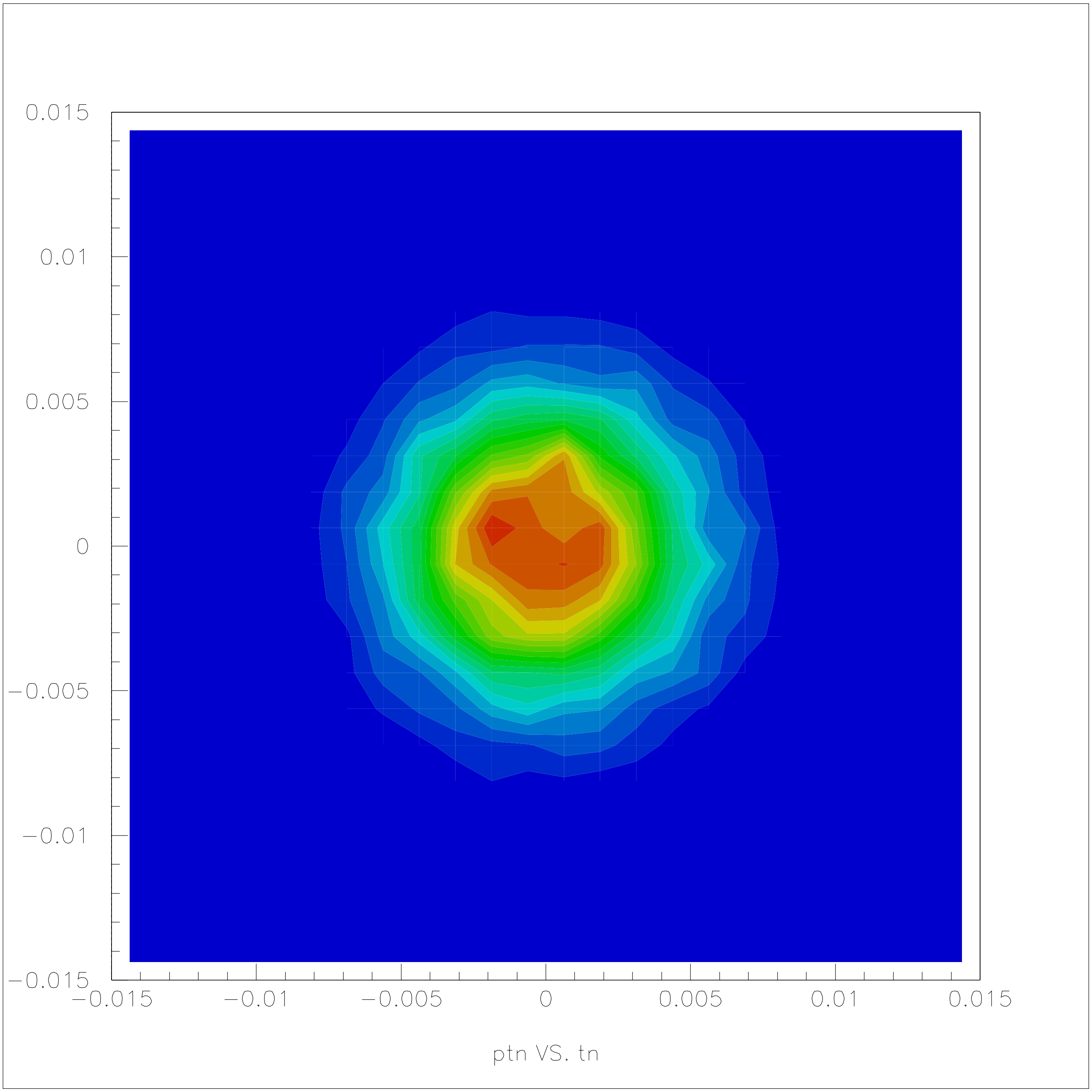
Mode 2 distribution (same ex.)

- Non-linear resonances + SBR affect distribution

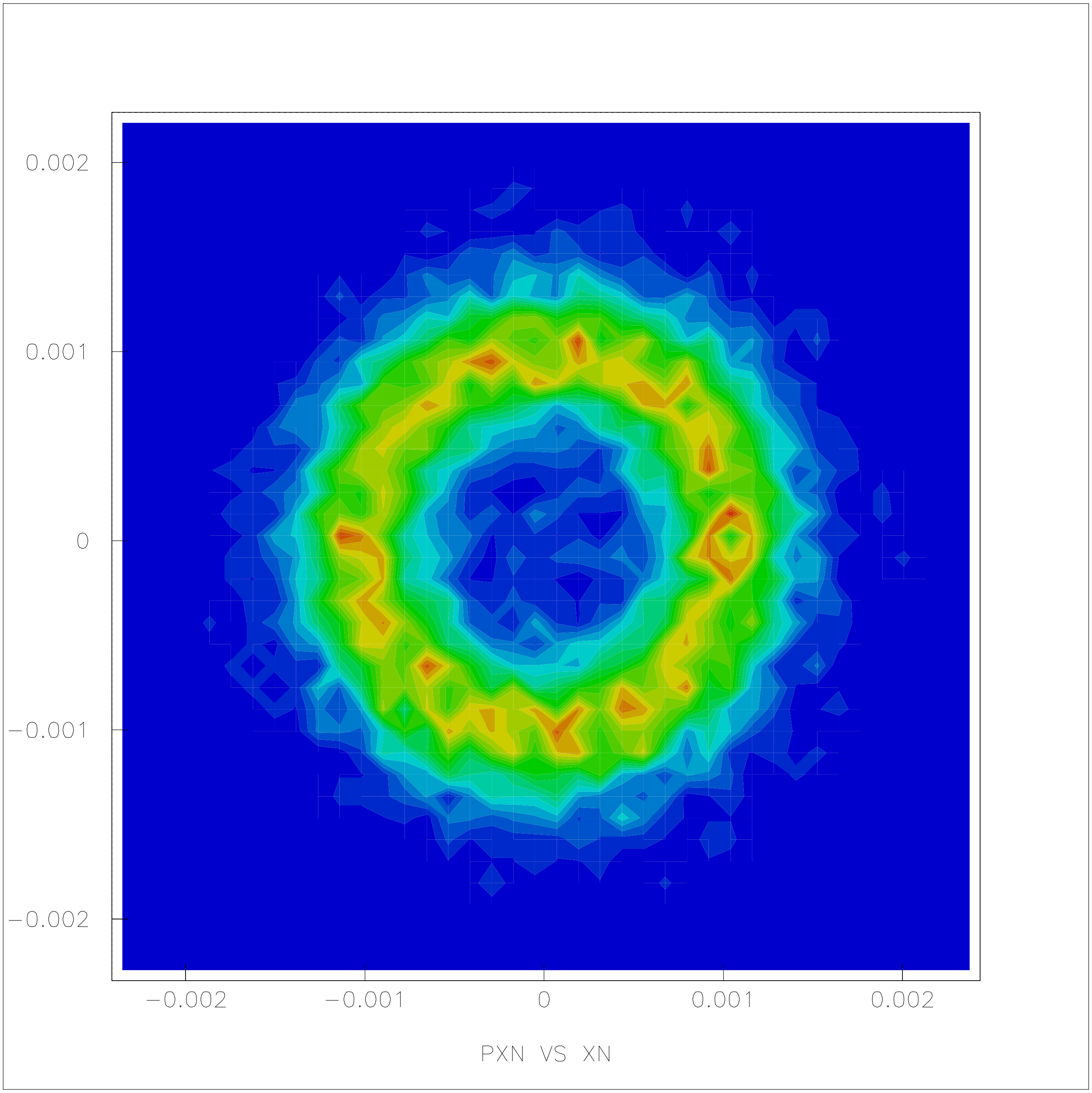


Mode 3 distribution (same ex.)

- **Synchrotron phase-space distribution modified by RBSC**



Coherent excitation of beam



LEP2 90°/60° optics, 90 GeV, 192 SC + 120 Cu cavities, starts on closed orbit, 10000 turns, kicker exciting close to tune.

Hopf bifurcation of closed orbit into limit cycle and further structure (crater distribution).

Tracking Methodology

- **Analysis of tracked orbits ⇒ physics**

  FFT spectra, phase space plots (all projections), evolution of action variables in time, etc.

- **4D dynamic aperture scans**

  In (square roots of) action variables of three normal modes and synchrotron phase

- **Most tracking done with radiation damping**

  Occasionally without, learning to work with quantum fluctuations (tracking becomes a Monte-Carlo).

- **Full LEP2 RF system in real layout**

  120 Cu + 192 SC cavities, typical voltages, in real layout

- **Dynamic aperture independent of number of turns**

  (at high energy with enough damping)

- **Vacuum chamber boundary included**

- **Ensembles of imperfect machines (Monte Carlo)**

- **Correction applied as in control room:**

  closed orbit, tune, optics at IP

- **All calculations repeated for positron and electron**

Summary

❑ Electron dynamics: fluctuating, dissipative system

Add stochastic terms to Hamilton's equations.

Simple problems canbe solved by means of techniques for stochastic systems:

- Fokker-Planck equations
- Integration of truncated moment equations, etc.

❑ Damping and quantum fluctuations

Damping governed by Robinson's Theorem.

Damping rates and quantum excitation rate depend strongly on energy of ring

Distribution functions

❑ Tracking for realistic (complicated) problems

Damping gives rise to new instability mechanisms (RBSC)

Tracking with quantum fluctuations now a feasible means to calculate core of beam distributions, including all relevant single-particle effects.

Tails are more difficult!