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A RENORMALISATION APPROACH TO INVARIANT CIRCLES IN AREA-PRESERVING MAPS

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Kadanoff and Shenker introduced a renormalisation approach to invariant circles in area-preserving maps. This paper makes more precise the connection between invariant circles and the renormalisation operator. Restricting attention to noble rotation numbers, the stability of a simple fixed point of the renormalisation is analysed, corresponding to a linear twist map. It is found to be essentially attracting, so that noble circles persist under perturbation, giving a new view on KAM theory. Shenker and Kadanoff found evidence for another fixed point, corresponding to a map with a non-smooth noble circle. Further evidence is given in this paper. It has essentially only one unstable direction, and its stable manifold is believed to give the boundary of the set of twist maps with a noble circle. Finally, noble circles are shown to be locally most robust, in an important sense.

Contents

1. Area-preserving twist maps

Nonlinear stability in many conservative systems is equivalent to existence of invariant circles for related area-preserving maps. I will motivate this paper with a simple but significant example from plasma physics, namely, flow along magnetic field lines. To a first approximation, charged particles in a magnetic field follow the field lines in tight helices. The idea of fusion devices such as the tokamak is to confine them by a magnetic field which is largely toroidal. If the field lines remain confined within the device, there is a chance that the particles will too.

Field line flow can be reduced to iteration of a *return map F* on a poloidal section, given by following the field lines once around the device. Fig. 1 shows some orbits of such a return map for a real magnetic field (Sinclair et al. [1]) (see also White et al. [2]). Since magnetic flux is conserved, this return map preserves the area form $\boldsymbol{B}\cdot(\xi\times\boldsymbol{\eta})$, where ξ, $\boldsymbol{\eta}$ are tangent vectors to the poloidal section. By Dar-

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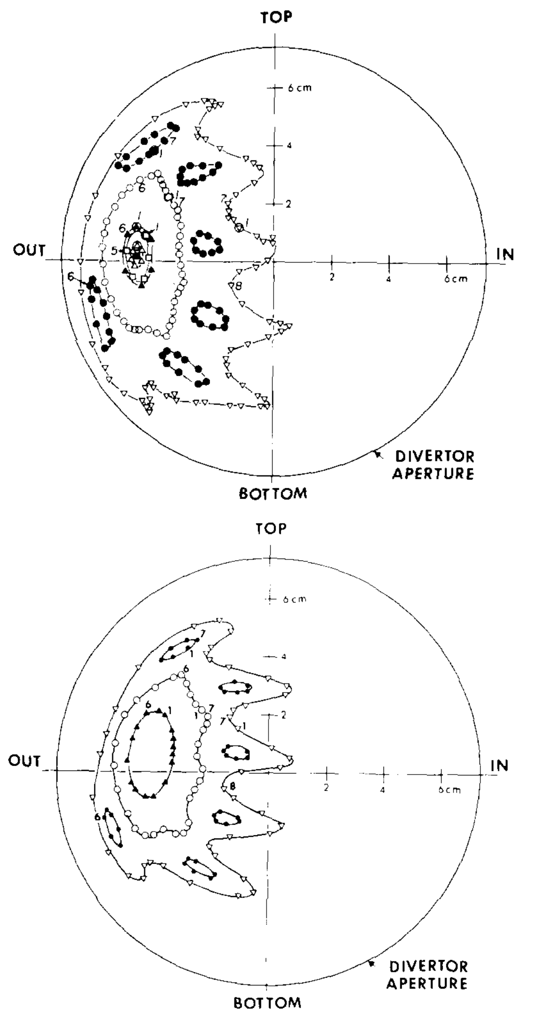

Fig. 1. Some orbits of a return map for a stellarator field (from Sinclair et al. [1]). The upper figure was produced by following a low energy electron beam injected paralel to the field. The lower figure was produced by integrating the field computationally.

boux's theorem [3], coordinates can be chosen to make it preserve the usual area.

Typically, a tokamak has a magnetic axis, that is, a field line which closes after one revolution. This corresponds to a fixed point of the return map. Also, the field is designed to have *rotational transform*, i.e. the other field lines twist around this axis. So in the poloidal section orbits rotate around the fixed point. Finally, tokamaks generally have *magnetic shear*, that is, the rotation rate varies with distance from the magnetic axis.

Thus we are led to consider *area-preserving twist maps*, that is, maps

$$(\theta', z') = F(\theta, z). \tag{1.1}$$

with θ an angle variable, $\det DF = 1$, and $\partial\theta'/\partial z$ of constant sign.

For the field line problem (and many others), (θ, z) are some sort of polar coordinates centred on a fixed point, and there is a coordinate singularity there. Removing the fixed point gives a map which can be regarded as acting on a cylinder. Maps of the cylinder derived in this way, however, have some special properties. Firstly, they are *end-preserving*. Secondly, given an area-preserving map of a cylinder, and a set U containing all points below some level z_1, and no points above some other level z_2, the difference between the areas of U and $F(U)$ is independent of U, and is called the *Calabi invariant* of F. For maps of the cylinder derived from a map of the plane by removing a fixed point, the Calabi invariant is clearly zero. Thus I will restrict attention to the *class A* of end-preserving, area-preserving twist maps of a cylinder with zero Calabi invariant.

Often I will want to consider a *lift* of F, rather than the map F itself. This means θ is regarded as a coordinate on a line rather than a circle, so we get a *periodic* map (of period 1, say), i.e.

$$\theta'(\theta + 1, z) = \theta'(\theta, z) + 1. \tag{1.2}$$

Maps of class A are relevant to many other conservative problems, for instance, all Hamiltonian systems with two degrees of freedom, or with one degree of freedom and periodic time dependence. This includes other examples from plasma physics, such as the motion of a charged particle in a 2-D field, guiding centre motion in 3-D, and ray tracing for waves in 2-D. They also have applications in other fields such as celestial mechanics, and solid-state physics (e.g. Aubry, this volume).

2. Invariant circles

If a map of class A has an invariant circle encircling the cylinder, then the circle traps everything below it. Conversely, Birkhoff (1932) [4] showed that an encircling invariant circle is necessary for confinement of any connected open set containing all points below some level. In all that follows I will restrict attention to this class of invariant circles. Note that zero Calabi invariant is necessary for existence of any such circle.

Next I introduce an important quantity for an invariant circle. Poincaré (Nitecki [5]) showed that for a homeomorphism g of a circle (or really, for a lift of g to a periodic homeomorphism of the line), the limit:

$$\omega = \lim_{q \to \infty} \frac{g^q(\theta_0)}{q} \tag{2.1}$$

exists and is the same for all θ_0. It is called the rotation number of g. In the case that g is the restriction of an area-preserving map to an invariant circle, ω is called the *rotation number* of the circle.

There are systems which have an invariant circle for every rotation number is some range. A map of a surface is said to be *integrable* if it possesses a differentiable invariant function which is not constant on any open set. For example, an axisymmetric magnetic field has a *flux function*. An extremum of the invariant is typically surrounded by many circles on which the invariant is constant. They are invariant under some power of the map, and so is the region between any two. Liouville (Arnold, [3]) showed that if the derivative of the invariant is non-zero on a compact connected invariant set, then there exist *angle–action coordinates* (θ, z), in which the map takes the standard integrable form

$$(\theta', z') = (\theta + \omega(z), z). \tag{2.2}$$

So the set is foliated by invariant circles.

Integrable maps are very special. For example, (2.2) has an invariant circle for every rotation number in a range, including rationals, but generically there are no rational circles. Nevertheless, conditions close to integrable are very common. For example, a map is arbitrarily close to integrable near enough to any typical elliptic point. A remarkable theory, due to Kolmogorov, Arnold and Moser, shows that systems close enough to an integrable one possess an arbitrarily large fraction of the invariant circles of the integrable system. The particular result most relevant to this paper is a corollary (Mather, private communication) of the Moser twist theorem (Moser, [6]). First, let us introduce some terminology. ω is called a *Diophantine number* (Niven, [7]), if

$$\exists C > 0,\ \tau \text{ such that } \left|\omega - \frac{p}{q}\right| \geq \frac{C}{q^\tau}\ \forall p, q \in \mathbb{Z}, q > 0\,. \tag{2.3}$$

An invariant circle is called *smooth* if the motion on it is sufficiently differentiably conjugate to rotation (the number of derivatives depending on τ for a Diophantine rotation number), i.e. if there is a sufficiently differentiable coordinate function ψ on the circle, with differentiable inverse, such that the map sends ψ to $\psi + \omega$. Then the result is that:

> Smooth Diophantine circles persist, for small enough perturbation in class A.

On the other hand, here are maps of class A with no encircling invariant circles. For example, the *standard map*,

$$z' = z - \frac{k}{2\pi} \sin 2\pi\theta,$$
$$\theta' = \theta + z', \tag{2.4}$$

has no invariant circles for $|k| \geq 2\pi$, because then it has an accelerator mode (Chirikov [8]),and even for $|k| > 4/3$ (Mather [9]).

The size of the perturbations allowed by the Moser twist theorem depends only on C and τ in the Diophantine condition, and the local twist. It is largest for τ small and C large. In any interval, the number(s) for which τ can be taken smallest and C largest (excluding a finite set of q) is always a *noble* number (terminology due to Percival [10]). These are the numbers whose *continued fraction expansion*,

$$\omega = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cdots}} \equiv [m_0, m_1, m_2, \ldots],$$
$$m_i \in \mathbb{Z}, m_i \geqslant 1 \text{ for } i \geqslant 1, \tag{2.5}$$

has $m_i = 1$ for all large enough i. They satisfy a Diophantine condition with $\tau = 2$, the smallest possible. The noblest of them all is the *golden ratio*:

$$\gamma = [(1,)^\infty] = \frac{\sqrt{5}+1}{2}, \tag{2.6}$$

which has the largest possible value for C (for $\tau = 2$) of $1/\gamma^2$ (Prasad, [33]). This leads one to suspect that typically noble circles may be the most robust, in the sense that the last circle to break up in any region, as a parameter varies, will be a noble. For this reason I will concentrate almost entirely on nobles.

The proofs in KAM theory generally give unrealistically low estimates of the perturbation sizes sufficient for persistence of invariant circles. In this paper, I develop a new approach to KAM theory which, I believe, gives the boundary of the set of twist maps with an invariant circle of given rotation number.

3. Action representation

Before I describe the renormalisation approach to invariant circles, I will need an important representation for area preserving twist maps. As this representation does not require periodicity, I use coordinates (x, y) in place of (θ, z). Given a function $\tau(x, x')$, with $\tau_{12}(x, x')$ of constant sign, the relations

$$\begin{aligned} y' &= \tau_2(x, x'), \\ y &= -\tau_1(x, x'), \end{aligned} \tag{3.1}$$

generate an area-preserving map T: $(x, y) \to (x', y')$ (where subscript i refers to the derivative with respect to the ith argument). It satisfies the twist condition (section 1), since

$$\frac{\partial x'}{\partial y} = -\frac{1}{\tau_{12}(x, x')}. \tag{3.2}$$

Conversely, every area-preserving twist map can be generated in the above fashion. In a given coordinate system, the generating function is unique up to addition of a constant. I call it the *action generating function*.

The rule for composition of action generating functions (where defined) is that of *stationarity*, i.e. the generating function for the composition TU of two maps T, U with generating functions τ, v, is

$$v \oplus \tau(x, x'') = v(x, x') + \tau(x', x''), \tag{3.3}$$

where $x'(x, x'')$ is chosen to make the sum stationary with respect to variations in x', i.e.

$$0 = v_2(x, x') + \tau_1(x', x''). \tag{3.4}$$

That $v \oplus \tau$ generates TU can be seen immediately from (3.1).

Note for a periodic map, that since $\tau(\theta + 1, \theta' + 1)$ generates the same map as $\tau(\theta, \theta')$, they can differ only by a constant. This constant can easily be shown to be the Calabi invariant.

4. Renormalisation

Now I will motivate the renormalisation. Rotation number can be generalised to other orbits than those on an invariant circle. I say that (the orbit of) (θ, z) has rotation number

$$\lim_{q \to \infty} \frac{\pi_1 F^q(\theta, z)}{q} \tag{4.1}$$

if the limit exists (which it need not), where π_1 is the projection onto the first coordinate. Without loss of generality, consider the orbit of the origin $\mathbf{0}$. If it has rotation number ω, then

$$\pi_1 F^q R^p(\mathbf{0}) = q\omega - p + o(q), \quad \text{as } q \to \infty, \tag{4.2}$$

where

$$R(\theta, z) = (\theta - 1, z). \tag{4.3}$$

If $\mathbf{0}$ belongs to a circle on which F is topologically conjugate to rotation then we have the stronger statement

$$\pi_1 F^{q_n} R^{p_n}(\mathbf{0}) \to 0, \quad \text{if } q_n\omega - p_n \to 0. \tag{4.4}$$

In the case of differentiable conjugacy to rotation, one can say even more:

$$\pi_1 F^{q_n} R^{p_n}(\mathbf{0}) \sim K(q_n\omega - p_n) \quad \text{as } q_n\omega - p_n \to 0. \tag{4.5}$$

This suggests that we consider the sequence of maps

$$B_n F^{q_n} R^{p_n} B_n^{-1}, \tag{4.6}$$

where the B_n are coordinate changes, looking on successively smaller scales.

A choice of p_n, q_n for which $q_n\omega - p_n$ is particularily small is given by the *convergents* of ω. They are the successive truncations

$$\frac{p_n}{q_n} = m_0 + \cfrac{1}{m_1 + \cfrac{1}{\cdots + \cfrac{1}{m_n}}} \equiv [m_0, m_1, \ldots, m_n] \tag{4.7}$$

of its continued fraction expansion. For this choice, there is a systematic way to generate the sequence (4.6). Define the *renormalisation operator* N_m (for $m \in \mathbb{Z}$), acting on the pairs (U, T) of maps, by

$$N_m\colon \begin{cases} U' = BTB^{-1}, \\ T' = BT^mUB^{-1}. \end{cases} \tag{4.8}$$

B is a coordinate change, chosen to *renormalise* the pair (T, T^mU) in some sense, for which I will not make a specific choice here. Then it follows from the definition of convergents that

$$N_{m_n} \ldots N_{m_0}(F, R) = (B_n F^{q_{n-1}} R^{p_{n-1}} B_n^{-1}, B_n F^{q_n} R^{p_n} B_n^{-1}), \tag{4.9}$$

where B_n is the composition of the successive coordinate changes B. This is essentially the same renormalisation as that introduced by Kadanoff [11] and Shenker [12], and applied to the dissipative case by Feigenbaum et al. [13] and Rand et al. [14] (see also the articles by Shenker and by Siggia, in this volume). It is also closely related to the approximate renormalisation of Escande and Doveil [15] for Hamiltonians.

An apparent problem with the renormalisation is that in looking on successively smaller scales one loses the periodicity of the map in θ. In the next section, I will show how the essence of the periodicity can be saved, by generalising the class of periodic maps to that of commuting pairs of maps.

5. Commuting pairs

To say that F is periodic in θ is equivalent to saying that F commutes with R (4.3). So let us generalise the important concepts for periodic maps, in particular, rotation number and Calabi invariant, to commuting pairs (U, T). I use coordinates $\boldsymbol{x} = (x, y)$ in place of (θ, z) to indicate that there is not necessarily any periodicity in x.

Firstly I generalise orbits and invariant circles.

Definitions. The *orbit* of $\boldsymbol{x}$ under a commuting pair (U, T) is $\{U^qT^p\boldsymbol{x} : p, q \in \mathbb{Z}\}$.
A point $\boldsymbol{x}$ is *periodic* if $\exists\ (p, q) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ so that $U^qT^p\boldsymbol{x} = \boldsymbol{x}$. It has *type* (p, q) if these are the smallest such integers $(q \geqslant 0)$.
An *invariant curve* is a curve from $x = -\infty$ to $+\infty$, invariant under both U and T.

Next, I generalise rotation number. If (θ, z) has rotation number ω under a periodic map F, then

$$\frac{\pi_1 F^q R^p(\theta, z)}{q} = \omega - \frac{p}{q} + o(1), \quad \text{as } q \to \infty, \tag{5.1}$$

This tends to zero for a sequence p_n, q_n iff $p_n/q_n \to \omega$. So generalise (and also allow $\omega = \infty$, i.e. consider rotation number as belonging to the projective line):

$\boldsymbol{x}$ has *rotation number* $\omega \in \mathbb{R}P$ under (U, T) if for all sequences $p_n, q_n \in \mathbb{Z}$ so that $r_n = \max(|p_n|, |q_n|) \to \infty$, then

$$\frac{\pi_1 U^{q_n} T^{p_n} \boldsymbol{x}}{r_n} \to 0 \quad \text{iff} \frac{p_n}{q_n} \to \omega. \tag{5.2}$$

Poincaré's theorem (section 2) generalises to invariant curves of a commuting pair, under the condition that

$$\exists m, n \in \mathbb{Z},\ K > 0 \text{ so that } \pi_1 U^m T^n \boldsymbol{x} \leqslant x - K \tag{5.3}$$

for all points $\boldsymbol{x}$ on the curve (cf. $\pi_1 R(\theta, z) = \theta - 1$). So an invariant curve has a rotation number [19].

For the *twist condition*, I want both U and T to have action generating functions, i.e. $\partial x'/\partial y$ should have constant sign. This is probably more restrictive than necessary, and slightly unfortunately so, as R does not satisfy the twist condition. If U, T commute and have generating functions v, τ, then the generating functions for UT and TU can differ by only a constant, so I call it the *Calabi invariant* $C(v, \tau)$.

I call the extension of class A to commuting pairs *class AA*. Presumably Moser twist and other results like Mather's theorem (section 9) would generalise under suitable conditions.

I can now make some nice connections between rotation number and the renormalisation:

$\boldsymbol{x}$ has rotation number ω under (U, T) iff $B\boldsymbol{x}$ has rotation number ω' under $N_m(U, T)$, where ω, ω' are related by $\omega = m + 1/\omega'$.

$\boldsymbol{x}$ has rotation number $\omega_0 = [m_0, m_1, \ldots]$ under (U, T) iff $B_{n-1} \ldots B_0 \boldsymbol{x}$ has rotation number $\omega_n = [m_n, \ldots]$ under $N_{m_{n-1}} \ldots N_{m_0}(U, T)$, where the B_j are the successive coordinate changes.

(U, T) has an invariant curve of rotation number ω_0 iff $N_{m_{n-1}} \ldots N_{m_0}(U, T)$ has an invariant curve of rotation number ω_n.

In particular, a periodic map F has an invariant circle of rotation number ω_0 iff $N_{m_{n-1}} \ldots N_{m_0}(F, R)$ has an invariant curve of rotation number ω_n.

Restricting B to linear diagonal scale changes $B(x, y) = (\alpha x, \beta y)$, N_m induces the following renormalisation on action generating functions:

$$\begin{aligned} v'(x, x') &= \alpha\beta\, \tau\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right), \\ \tau'(x, x') &= \alpha\beta\, v \oplus \tau \oplus \cdots \oplus \tau\left(\frac{x}{\alpha}, \frac{x'}{\alpha}\right). \end{aligned} \tag{5.4}$$

Note that it preserves zero Calabi invariant.

The idea of renormalisation is not new to KAM theory. Most proofs consist in finding successive coordinate changes to make the system look more like a linear twist, restricting attention each time to a narrower annulus (see, for example, Moser [6], Herman [16], Rüssmann [17], Gallavotti [18]). So scale changes are made in the z-direction. I believe that the freedom we have to make scale changes in the θ direction too will make this renormalisation more powerful. The only expense is that at each step one has to change the generators for the group $\{F^q R^p : p, q \in \mathbb{Z}\}$. The main benefit will be that we will probably get the boundary in class A of the maps with an invariant circle of given rotation number.

6. Simple fixed point

Quadratic irrationals have eventually periodic continued fraction expansion. So for maps with an orbit of quadratic irrational rotation number, $[b_0, \ldots b_j, (c_1, \ldots c_k,)^\infty]$, this suggests that one might find asymptotic behaviour under $N_{c_k} \ldots N_{c_1}$, after removing the aperiodic head by applying $N_{b_j} \ldots N_{b_0}$.

I will look at the simplest case, namely, nobles, for which the repeat pattern $[\boldsymbol{c}] = [1]$. N_1 (4.8) has two important fixed points, the main objects of discussion in this paper. I begin with the *simple* *fixed point*:

$$T:\begin{cases}x' = x + y + 1,\\ y' = y,\end{cases} \quad U:\begin{cases}x' = x + \dfrac{y}{\gamma} - \gamma,\\ y' = y,\end{cases} \tag{6.1}$$
$$B:\begin{cases}x' = -\gamma x,\\ y' = -\gamma^2 y.\end{cases}$$

It corresponds to a linear shear, with $y = 0$ as a golden curve.

One can analyse its stability under N_1,

$$\mathrm{D}N_1:\begin{cases}\delta U' = B\delta T B^{-1}\\ \delta T' = B\delta T U B^{-1} + B\mathrm{D}T_{UB^{-1}}\cdot \delta U B^{-1}.\end{cases} \tag{6.2}$$

Here we are using linearity of B to identify $\mathrm{D}B$ with B, for simplicity of notation. Also we are ignoring contributions to $\delta U'$, $\delta T'$ due to variation of B with (U, T), which would depend on the particular prescription for renormalising (T, TU). These contributions are only in the direction of coordinate changes, so they will have no essential effect.

At the simple fixed point, $\mathrm{D}N_1$ *is*

$$\begin{aligned}\delta U_x'(x,y) &= -\gamma\delta T_x\left(-\frac{x}{\gamma}, -\frac{y}{\gamma^2}\right),\\ \delta T_x'(x,y) &= -\gamma\delta U_x\left(-\frac{x}{\gamma}, -\frac{y}{\gamma^2}\right)\\ &\quad -\gamma\delta T_x\left(-\frac{x}{\gamma} - \frac{y}{\gamma^3} - \gamma, -\frac{y}{\gamma^2}\right)\\ &\quad -\gamma\delta U_y\left(-\frac{x}{\gamma}, -\frac{y}{\gamma^2}\right),\\ \delta U_y'(x,y) &= -\gamma^2\delta T_y\left(-\frac{x}{\gamma}, -\frac{y}{\gamma^2}\right),\\ \delta T_y'(x,y) &= -\gamma^2\delta U_y\left(-\frac{x}{\gamma}, -\frac{y}{\gamma^2}\right)\\ &\quad -\gamma^2\delta T_y\left(-\frac{x}{\gamma} - \frac{y}{\gamma^3} - \gamma, -\frac{y}{\gamma^2}\right).\end{aligned} \tag{6.3}$$

Let us define an order of monomials:

$$1 < y < x < y^2 < xy < x^2 < \cdots \tag{6.4}$$

and say a polynomial is of *rank* p, q is its largest monomial, with respect to this order, is $x^p y^q$. Then observe that at the simple fixed point, $\mathrm{D}N_1$ never increases the rank of a polynomial perturbation. Thus, we can put $\mathrm{D}N_1$ into Jordan normal form on the space of polynomial perturbations, with polynomial eigenvectors or generalised eigenvectors. What happens on the rest of the space is discussed in the next section. Expanding δU_x, δT_x, δU_y, δT_y in power series, and ordering the coefficients by rank, and in the above order within rank, we see that $\mathrm{D}N_1$ is block upper triangular, with 4×4 diagonal blocks:

$$\begin{vmatrix} 0 & -\gamma & 0 & 0\\ -\gamma & -\gamma & -\gamma & 0\\ 0 & 0 & 0 & -\gamma^2\\ 0 & 0 & -\gamma^2 & -\gamma^2\end{vmatrix} \times (-\gamma)^{-p}(-\gamma^2)^{-q} \tag{6.5}$$

for rank p, q. This diagonal block has eigenvalues

$$-\gamma^2, 1, -\gamma^3, \gamma \times (-\gamma)^{-p}(-\gamma^2)^{-q} \tag{6.6}$$

with respective eigenvectors

$$\underset{B_{pq},}{\begin{vmatrix}1\\ \gamma\\ 0\\ 0\end{vmatrix}}, \quad \underset{D_{pq},}{\begin{vmatrix}\gamma\\ -1\\ 0\\ 0\end{vmatrix}}, \quad \underset{A_{pq},}{\begin{vmatrix}1/2\gamma\\ \gamma/2\\ 1\\ 1\end{vmatrix}}, \quad \underset{C_{pq}.}{\begin{vmatrix}\gamma\\ -\gamma\\ \gamma\\ -1\end{vmatrix}}. \tag{6.7}$$

For each of these eigenvectors for the block, one can determine coefficients of lower rank to make eigenvectors or generalised eigenvectors for $\mathrm{D}N_1$. They span the space of polynomial perturbations. As there is a lot of degeneracy, the coefficients need not be uniquely determined.

N_1 leaves invariant several important spaces, namely:

i) commuting pairs;
ii) area-preserving pairs;
iii) commuting area-preserving pairs with zero Calabi invariant (class AA);
iv) symmetric commuting pairs;

v) coordinate transforms of the fixed point;
vi) all intersections of the above.

One would like to use the freedom (due to degeneracy) in determining the eigenvectors or generalised eigenvectors, to choose them to respect these invariant subspaces. This can be done, and the result is shown in Table I, where they are labelled by their terms of maximal rank, as in (6.7). The details are described in MacKay [19].

Note with regard to iv) above, that we say T is *symmetric* if $(TS)^2$ is the identity, where

$$S(x, y) = (-x, y). \tag{6.8}$$

Symmetry alone, however, is not preserved by N_1. This symmetry property corresponds to the important class of *reversible* systems (Devaney [20]), but unfortunately there is not room here to discuss them.

7. Compactness of DN_1 at the simple fixed point

Next I show that DN_1 is a compact operator at the simple fixed point, in a suitable norm, and that the error in truncating it at finite degree goes to zero as the degree goes to infinity. Thus, standard results in functional analysis (e.g. Krasnosel'skii et al., [21] section 18) imply that the diagonalisation of table I is complete, apart from a component with eigenvalue 0. It is clear, incidentally, that table I does not cover the whole space, as there are arbitrarily small perturbations of the simple fixed point, in class AA, which are not coordinate transforms of the simple fixed point. For example, for $k = 0$ the standard map is equivalent to the simple fixed point, and has a whole circle of points of type (0, 1), but for $k \neq 0$ there are only two.

To show compactness of DN_1, I show that N_1 is analyticity improving in a neighbourhood of the simple fixed point, on suitable domains. Specifically, if T is analytic on the product of discs $|x| \leqslant X$, $|y| \leqslant Y$, for some $X, Y > 0$ with

$$X > \gamma^3 + \frac{Y}{\gamma} \tag{7.1}$$

and U is analytic on $|x| \leqslant X'$, $|y| \leqslant Y'$, with

$$\frac{X}{\gamma} < X' < \gamma X, \; \frac{Y}{\gamma^2} < Y' < \gamma^2 Y, \tag{7.2}$$

then (U', T') is analytic on larger discs, for (U, T) close enough to the simple fixed point. Close enough means with respect to the l_1 norm for power series expansions in the discs (7.1, 7.2).

Table I
Decomposition of the spectrum of DN_1 at the simple fixed point, according to area preservation (a.p.), commutativity (comm), coordinate transforms of the fixed point (c.c), non-zero Calabi invariant (C.I.), and symmetry (s). Prefix n- stands for non-

Classification				Eigenvector	Eigenvalue	Eigenvalues greater than or equal to 1 in modulus
a.p.	comm	c.c.	s	B_{pq}, p even	$-\gamma^2(-\gamma)^{-p}(-\gamma^2)^{-q}$	$-\gamma^2$, 1, -1
a.p.	comm	c.c.	s	D_{0q}	$(-\gamma^2)^{-q}$	1
a.p.	comm	c.c.	ns	B_{pq}, p odd	$-\gamma^2(-\gamma)^{-p}(-\gamma^2)^{-q}$	γ
a.p.	comm	C.I.	ns	A_{00}	$-\gamma^3$	$-\gamma^3$
a.p.	n-comm		ns	D_{pq}, $p \geqslant 1$	$(-\gamma)^{-p}(-\gamma^2)^{-q}$	
a.p.	n-comm		ns	C_{p0}	$\gamma(-\gamma)^{-p}$	γ, -1
a.p.	n-comm		ns	A_{p0}, $p \geqslant 1$	$-\gamma^3(-\gamma)^{-p}$	γ^2, $-\gamma$, 1
n-a.p.	comm	c.c.	s	A_{pq}, $q \geqslant 2$, p odd	$-\gamma^3(-\gamma)^{-p}(-\gamma^2)^{-q}$	
n-a.p.	comm	c.c.	ns	A_{pq}, $q \geqslant 2$, p even	$-\gamma^3(-\gamma)^{-p}(-\gamma^2)^{-q}$	
n-a.p.	comm	n-c.c.	ns	A_{p1}	$\gamma(-\gamma)^{-p}$	γ, -1
n-a.p.	n-comm		ns	C_{pq}, $q \geqslant 1$	$\gamma(-\gamma)^{-p}(-\gamma^2)^{-q}$	

The error in truncating DN_1 at degree d is less than $C\lambda^d$, where

$$\lambda = \max\left(\frac{X}{\gamma X'}, \frac{X'}{\gamma X}, \frac{Y}{\gamma Y'}, \frac{Y'}{\gamma Y}, \frac{\frac{X}{\gamma} + \frac{Y}{\gamma^3} + \gamma}{X}\right) < 1. \tag{7.3}$$

8. Significance of the simple fixed point

If one restricts attention to class AA, table I shows that all polynomial directions from the simple fixed point are coordinate changes. Taking section 7 into consideration, this implies that, modulo coordinate changes, the simple fixed point attracts a neighbourhood, in fact faster than exponentially. This is, of course, what one should expect from KAM theory. Note also that the simple fixed point is attracting in the space of symmetric commuting pairs, as one expects from the reversible version of Moser's twist theorem [6].

$N_1(U, T)$ possesses a golden curve iff (U, T) does, but I want to show that convergence of $N_1^n(U, T)$ to a pair with a golden curve implies that (U, T) has a golden curve. For convergence to the simple fixed point, this follows from Moser's twist theorem (assuming it generalises to commuting pairs). It would give a $C^{3+\epsilon}$-neighbourhood of the simple fixed point (for any $\epsilon > 0$), in which all commuting pairs have a (smooth) golden curve. Convergence of $N_1^n(U, T)$ to the simple fixed point (in the $C^{3+\epsilon}$ topology) implies that $N_1^{n_0}(U, T)$ is in this neighbourhood for some n_0, and so (U, T) has a (smooth) golden curve (cf. Escande and Doveil [15]).

It may not be necessary, however, to use Moser's twist theorem. Mather [22] proved a necessary and sufficient condition for existence of an invariant circle, to be discussed in the next section. In section 12, I will show how one can probably use this theorem to prove that convergence of $N_1^n(F, R)$ to any fixed point, not just the simple one, implies that F has a golden circle. I suspect that even use of this theorem is not necessary at the simple fixed point.

Most proofs of results in KAM theory restrict one to pretty small perturbations (although Herman (private communication) is obtaining much more realistic results) determining a reasonably large neighbourhood of attraction for the simple fixed point. The results of sections 6 and 7 can be extended to other rotation numbers than nobles. The simple fixed point of N_1 belongs to a simple line, invariant and attracting under all of the N_m. Presumably, the size of the basin of attraction diminishes to zero as $m \to \infty$, thus setting a restriction on the growth rate of m_i for convergence to the simple line.

9. Connections between invariant circles and nearby periodic orbits

Now I discuss some connections between invariant circles and nearby periodic orbits. The first is a conjecture of Greene [23]. The other is Mather's theorem referred to in the previous section. This section will lead us to another fixed point of N_1.

I return to the setting of periodic maps. In the action representation a periodic orbit of type (p, q) corresponds to a sequence

$$\boldsymbol{\theta} = \theta_0, \theta_1, \ldots \theta_q = \theta_0 + p$$

for which the *action*

$$W(\boldsymbol{\theta}) = \sum_{i=0}^{q-1} \tau(\theta_i, \theta_{i+1}) \tag{9.1}$$

is stationary with respect to variations in $\boldsymbol{\theta}$. Birkhoff (1927) [24] showed that a map in class A has at least two periodic orbits of type (p, q) for each rational p/q in lowest terms, in an appropriate interval. If $\tau_{12} < 0$, the periodic orbits respectively minimize and minimaximize the action $W(\boldsymbol{\theta})$ over an appropriate set of $\boldsymbol{\theta}$. For $\tau_{12} > 0$, interchange "max" and "min". I shall restrict attention without loss of generality to the former case.

The linear stability of a periodic orbit can be measured by its *residue* (Greene [25]):

$$R = (2 - \mathrm{Tr}\, DF^q)/4, \tag{9.2}$$

where DF^q is the derivative of F^q at any of its points. In the action representation, considering without loss of generality the case $q = 1$, the residue of a fixed point is

$$R = \frac{\frac{d^2}{d\theta^2}\tau(\theta,\theta)}{4\tau_{12}(\theta,\theta)}. \tag{9.3}$$

Thus, the minimizing periodic orbits found by Birkhoff have non-positive residue R^-, and the minimaximizing orbits have non-negative residue R^+. They give rise to *island chains*. Fig. 2 shows some island chains for the *quadratic map*.

$$\begin{cases} x' = p - y - x^2, \\ y' = x, \end{cases} \tag{9.4}$$

for parameter $p = 2.38216325159$. For purposes of orientation, thin island chains, as when they are born by bifurcation from a periodic orbit, have residue close to zero.

Greene [23, 26, 34] suggested a connection between existence of invariant circles and the stability of nearby periodic orbits. This has also been followed up by Schmidt [27] and Bialek [28]. Given ω irrational, let us restrict attention to the Birkhoff periodic orbits of type (p_n, q_n) with p_n/q_n convergents of ω. Calling their residues $R_n^\pm$, one finds numerically one of three cases:

i) *Subcritical.* $R_n^\pm \to 0$, and it looks as if the island chains converge to a smooth invariant circle of rotation number ω.

ii) *Critical.* $R_n^\pm$ are eventually bounded away from 0 and $\pm\infty$, and it looks as if the island chains converge to a non-smooth invariant circle of rotation number ω.

iii) *Supercritical.* $R_n^\pm \to \pm\infty$, and it looks as if there is no circle of rotation number ω.

The critical case is shown in fig. 2 for $\omega = 1/\gamma^2 = [0, 2, (1,)^\infty]$ for the quadratic map. The convergents are 2/5, 3/8, 5/13, ..., and these island chains can be seen to be converging to the outermost invariant circle. It is non-smooth in the sense that it has thin spots (smoothness refers to the conjugacy, not the graph).

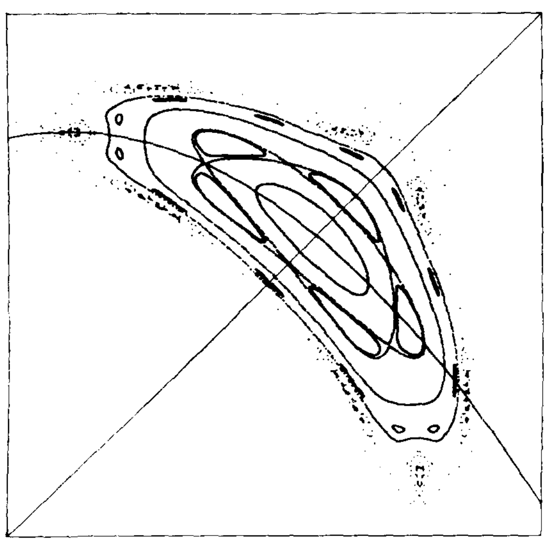

Fig. 2. Some orbits of the quadratic map, for $p = 2.38216325159$, and two symmetry lines.

The conjecture is that under suitable conditions, one could replace "and it looks as if" in the above by "which implies that". A partial result in this direction follows from the Moser twist theorem, namely, if there is a smooth Diophantine circle, then $R_n^\pm \to 0$ (in fact, faster than exponentially) (Mather, private communication), so in the critical and supercritical cases there is no smooth circle (for Diophantine rotation number). The converse, however, is not known.

A necessary and sufficient condition for existence of an invariant circle with irrational rotation number has been proved by Mather [22, 29], based on another property of nearby island chains. If one defines $\Delta W_{p/q}$ to be the difference in actions between the two Birkhoff orbits of type (p, q):

$$\Delta W = W_{\text{minimax}} - W_{\text{min}}, \tag{9.5}$$

then Mather's result states that

There exists an invariant circle of irrational rotation number ω iff $\Delta W_{p/q} \to 0$ as $p/q \to \omega$. In the case that there is no invariant circle then $\Delta W_{p/q}$ has a positive limit, and there is an invariant Cantor set of rotation number ω.

Similar results in this direction were found by Aubry (e.g., this volume).

In the subcritical and critical cases above I find numerically that $\Delta W_{p_n/q_n} \to 0$ (faster than exponentially in the subcritical case, as Moser twist implies for a smooth circle), so there is an invariant circle, and in the supercritical case $\Delta W_{p_n/q_n}$ tends to a positive limit, so there is no circle, but there is a Cantor set. This is shown in fig. 3 for $\omega = 1/\gamma^2$, and four parameter values in the quadratic map, one subcritical, one critical, and two supercritical.

Note that existence of a smooth Diophantine circle is stable to perturbation, by Moser twist. Also extensions of Mather's theorem [22], show that non-existence of an invariant circle of given rotation number is stable to perturbation. So one expects, and finds numerically, the subcritical and supercritical cases to be open sets in class A.

These approaches to finding where there are invariant circles have the advantage that they generalise directly to continuous time systems. There is no need to choose a surface of section or evaluate a return map. Finding periodic orbits and evaluating their residues and actions is a relatively straightforward procedure, especially if one takes advantage of symmetries the system may possess (see, for example, MacKay [19]).

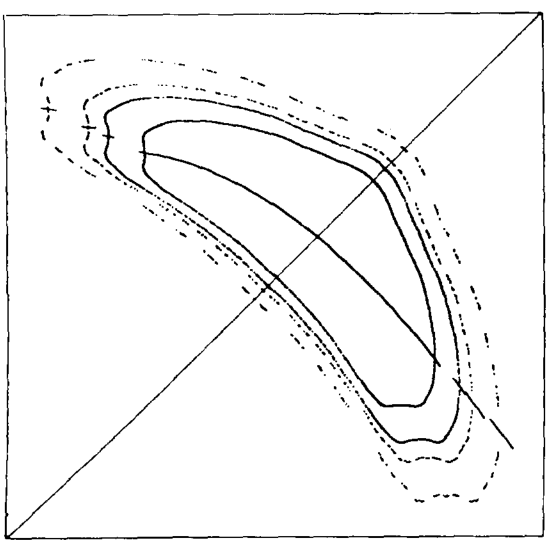

Fig. 3. An orbit of rotation number $1/\gamma^2$ for four parameter values in the quadratic map.

10. Critical noble circles

I now wish to concentrate on the critical case of the previous section. For reversible maps F in class A with a critical quadratic irrational circle, Shenker and Kadanoff [12] found scaling behaviour in the neighbourhood of certain points. The behaviour appears to be the same for all quadratic irrationals with the same repeat pattern, in most maps.

In particular, for nobles one obtains fig. 4 in critical cases, if one looks on a small enough scale and in appropriate coordinates in the neighbourhood of the point where the critical noble circle crosses a "dominant symmetry line". In the picture, the symmetry line has been transformed to the X-axis, and the noble circle crosses it at the origin. Note that everything in the picture repeats itself on a smaller scale and turned over, in the smaller box. Asymptotically, the scaling factors are

$$\begin{aligned} \alpha &= -1.4148360 \quad \text{in } Y, \\ \beta &= -3.0668882 \quad \text{in } X. \end{aligned} \tag{10.1}$$

In summary, it looks as if, for a map with a

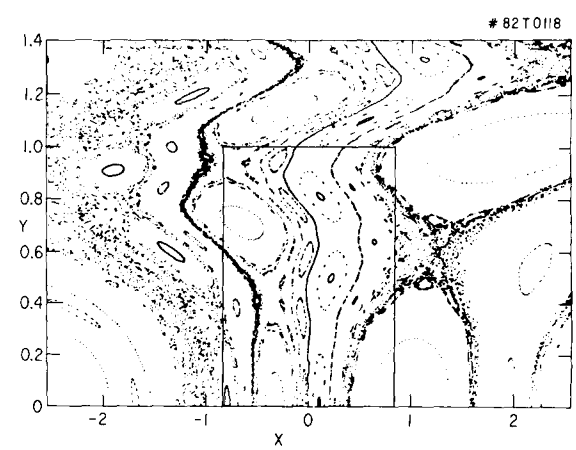

Fig. 4. Some orbits of the universal map F^*.

critical noble there are coordinate changes B_n, such that the maps

$$B_n F^{q_n} R^{p_n} B_n^{-1} \tag{10.2}$$

converge to some universal map F^*, with

$$B_{n+1} \simeq B B_n \quad \text{as } n \to \infty, \quad B(Y, X) = (\alpha Y, \beta X). \tag{10.3}$$

In terms of the renormalisation, it looks as if $N_1^n N_{b_j} \dots N_{b_0}(F, R)$ converges to a fixed point (U^*, T^*), with $T^* = F^*$, $U^* = BF^*B^{-1}$.

Given a one parameter family passing through a critical case, one finds further self-similarity. For example, the parameter values p_n at which $R_n^+ = 1$, converge asymptotically geometrically to the critical value, at rate $1/\delta$

$$\delta = 1.6280 \, . \tag{10.4}$$

This is the way I located critical parameter values. There is a faster way, however. For a critical noble

$$\begin{aligned} R_n^+ &\to 0.2500888, \\ R_n^- &\to -0.255426. \end{aligned} \tag{10.5}$$

The convergence is at rate

$$\delta' = -0.6108. \tag{10.6}$$

Thus the parameter values p_n' where $R_n^+ = 0.2500888$ converge at rate $\delta'/\delta = -0.3752$, faster than $1/\delta$.

The self-similarity can be summarised by saying that it looks as if there is a reparametrisation μ, and (parameter dependent) coordinate changes B_n such that the one-parameter families

$$B_n F^{q_n}_{\mu\delta^{-n}} R^{p_n} B_n^{-1} \tag{10.7}$$

converge to a universal one parameter family F_μ^*, with B_n scaling as in (10.3). In renormalisation language, the fixed point (U^*, T^*) has an unstable manifold of essentially only one dimension, with eigenvalue δ. The dominating attraction rate on its stable manifold is δ'.

Figs. 5 and 6 show some orbits of F_μ^* for $\mu = -0.3$, $+0.3$, subcritical and supercritical cases. The scale in parameter is chosen to make the minimaximizing point of type (1, 1) (or [1], in continued fraction notation) have residue $R_{[1]}^+ = 1$ at $\mu = 1$. Fig. 7 shows how $R_{[1]}^+$ varies with μ. The universal family is the significant object for any renormalisation scheme. Further properties are given in section 13 and MacKay [19], including critical exponents.

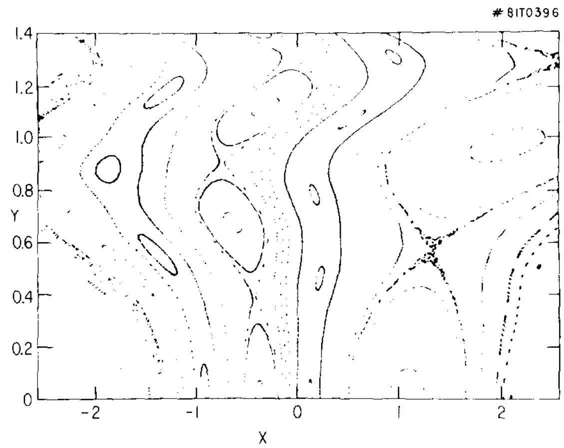

Fig. 5. Some orbits of the universal one parameter family F_μ^*, for $\mu = -0.3$.

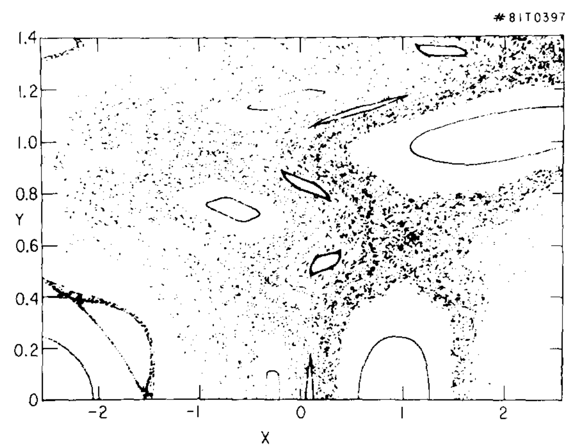

Fig. 6. Some orbits of the universal one parameter family F_μ^*, for $\mu = +0.3$.

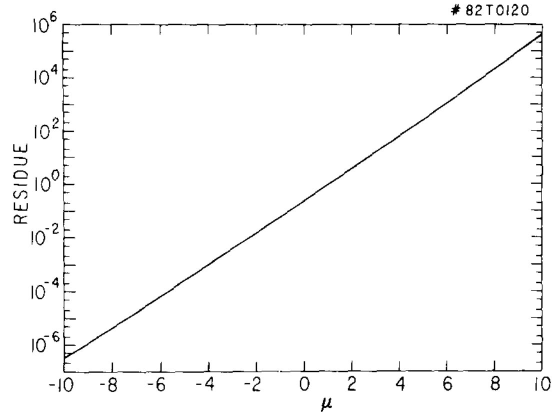

Fig. 7. Dependence of $R_{[1]}^{+}$ on μ.

11. Critical fixed point

The results of the previous section strongly suggest that there is another fixed point of N_1. Following the pioneering work of Kadanoff [30], I worked in the action representation, using the induced renormalisation (5.4). In order for the truncation at finite degree to have vanishing effect as the degree goes to infinity, it is necessary to find domains of expansion in $\mathbb{C}^2$ such that v, τ analytic in their domains implies v', τ' analytic on the same domains and slightly more, for v, τ close enough to the fixed point in the l_1 norm. However, I couldn't find any domain of the form

$$\begin{aligned} &|a_{11}x + a_{12}x' - c| \leqslant 1, \\ &|a_{21}x + a_{22}x' - c'| \leqslant 1, \end{aligned} \tag{11.1}$$

for which this seemed to be satisfied.

From the previous section, we expect the fixed point to be symmetric. So I considered a modified renormalisation:

$$N_{\mathrm{sl}}: \begin{cases} v'(x, x') = \alpha\beta\, \tau\left(-\dfrac{x'}{\alpha}, -\dfrac{x}{\alpha}\right), \\ \tau'(x, x') = \alpha\beta\, \tau \oplus v\left(\dfrac{x}{\alpha}, \dfrac{x'}{\alpha}\right). \end{cases} \tag{11.2}$$

This is the same as N_1, restricted to the space of symmetric commuting pairs, but permits good domains. I determined α and β by

$$\tau_1\left(0, \frac{1}{\alpha}\right) = 0, \quad \frac{\alpha}{\beta} = \tau_{12}\left(0, \frac{1}{\alpha}\right), \tag{11.3}$$

which forces the normalisation

$$v_1'(0, 1) = 0, \quad v_{12}'(0, 1) = 1. \tag{11.4}$$

The domains I used were

$$|x - c| \leqslant r, \quad |x' - c'| \leqslant r', \tag{11.5}$$

with

$$\begin{aligned} c &= 0.050707985, \quad r = 0.502060282, \\ c' &= -0.655406307, \quad r' = 0.329680205, \end{aligned} \tag{11.6}$$

for τ, and its rescaled and reflected version for v. This choice is close to optimal, if (11.2) is considered as one second order equation, and has an analyticity improvement factor of at least 1.1374. If regarded as two first order equations, the domain for v should be diminished a little.

Newton's method was used to find a fixed point of N_{sl}, truncating at various degrees. The results appear to converge as the degree is increased, and are consistent with the findings of section 10. For example, the values of α, β, δ, δ' for several truncation levels are presented in table II. δ nd δ' were found by diagonalising the derivative $\mathrm{D}N_{\mathrm{sl}}$. The eigenvalues of $\mathrm{D}N_{\mathrm{sl}}$ larger than 0.4 in modulus whose eigenvectors are symmetric are shown and interpreted in table III. There is only one relevant eigenvalue not contained inside the unit circle, namely, δ, as expected. For details on the identification of the eigenvalues, see MacKay [19].

This procedure could in principle be carried to arbitrary precision. Also, existence of the fixed point and bounds on its spectrum could be proved in the same way as Lanford [31] and Eckmann et al. [32] did for period doubling in one-dimensional and area-preserving maps, respectively.

Table II
Values of α, β, δ, δ' for the fixed points of N_{s1}, truncated at several degrees

Degree	α	β	δ	δ'
14	−1.414836085	−3.066888192	1.6279496	−0.61083048
15	−1.414836021	−3.066888344	1.6279506	−0.61083021
16	−1.414836072	−3.066888224	1.6279499	−0.61083040
17	−1.414836052	−3.066888269	1.6279502	−0.61083026
18	−1.414836062	−3.066888246	1.6279500	−0.61083028

Table III
Spectrum of DN_1 at the critical fixed point, showing those eigenvalues greater than 0.4 in modulus which have symmetric eigenvectors

Eigenvalue	Compare with	Value	Interpretation
7.0208826	$\gamma\alpha\beta$	7.0208826	Constant terms in action
−3.0668882	β	−3.0668882	Coordinate change
−2.6817385	$-\alpha\beta/\gamma$	−2.6817385	Constant terms in action
1.6279500	δ	1.6280	Relevant direction (10.4)
−1.5320950	β/α^2	−1.5320951	Coordinate change
1.0000001	α/α	1	Scale change
1.0000000	β/β	1	Scale change
−0.7653736	β/α^4	−0.7653736	Coordinate change
−0.6108303	δ'	−0.6108	Essential convergence rate (10.6)
0.4995593	α/α^3	0.4995601	Coordinate change

12. Golden curves for pairs converging to any fixed point

In this section I show how Mather's theorem probably implies that if $N_1^n(F, R)$ converges to any fixed point with $\alpha\beta > 1$, then F has a golden circle. Let (ν^*, τ^*) be the generating functions for the fixed point. Commutation and zero Calabi invariant imply that $\tau^*(x, x)$ has a minimum and maximum, so write $\Delta\tau^*$ for their difference. Then, provided the domains of convergence are large enough to include the relevant Birkhoff periodic points, convergence to the fixed point implies that

$$(\alpha\beta)_n \ldots (\alpha\beta)_1 \Delta W_{[(1,\ )^n]} \to \Delta\tau^* \tag{12.1}$$

from (5.4), where the $(\alpha\beta)_j$ are the successive values of $\alpha\beta$. Convergence to the fixed point also implies that

$$(\alpha\beta)_j \to \alpha\beta > 1, \tag{12.2}$$

so

$$\Delta W_{[(1,\ )^n]} \to 0 \tag{12.3}$$

and F has a golden circle, by Mather's theorem. Note that the convergence need only be C^1 for Mather's theorem to apply. The same argument would imply existence of a circle of any frequency if $N_{m_n} \ldots N_{m_0}(F, R)$ remains in a region with $\Delta\tau$ bounded and $\alpha\beta$ bounded above 1.

13. Robustness of noble circles

One of the most significant features of figs. 4 and 6 is that in the critical case, the noble circle appears to be (locally) the only circle, and in the supercritical case there appear to be no circles at all. The dots all belong to one orbit. As further evidence, I measured residues and differences in actions for other periodic orbits than the convergents of the noble. Given the noble $[\boldsymbol{a}, (1,\ )^\infty]$, where $\boldsymbol{a}$ is a finite sequence of integers, I considered the periodic orbits with rotation number $[\boldsymbol{a}, (1,)^n \boldsymbol{b}]$, for finite sequences $\boldsymbol{b}$. Figs. 8 and 9 show the residues R^+ in the limit as $n \to \infty$, in the critical case. In these figures I have used the natural ambiguity

$$[b_0, \ldots, b_m + 1] = [b_0, \ldots, b_m, 1] \tag{13.1}$$

to group the points into a tree which branches two ways at each point. The point to notice is that they are all bounded away from 0 (assuming that one can extrapolate the trends). For a smooth Diophantine circle, however, the residues for its convergent periodic orbits must tend to zero. Thus a critical noble has a neighbourhood containing no smooth Diophantine circles. Fig. 10 is the universal "fractal diagram" [27] for the neighbourhood of any noble. Since it shows all residues increasing with the parameter μ, there are no smooth Diophantine circles in the supercritical case either.

In the subcritical case, of course, we expect to have a smooth noble circle. Another corollary (Mather, private communication) of Moser twist is that a smooth Diophantine circle has others arbitrarily close. In fact each smooth Diophantine circle is a density point in the set of smooth Diophantine circles. So there are lots of circles in the subcritical case.

Next we consider differences in actions. Fig. 11 shows $(\alpha\beta)^n \Delta W_{[\boldsymbol{a}, (1,)^n \boldsymbol{b}]}$, in the limit as $n \to \infty$, for various $\boldsymbol{b}$ in the critical case. Apart from the sequence $[(1,)^m]$, converging to γ, we plotted points only for $\boldsymbol{b}$ with $b_0 > 1$, as the self-similarity allows one to fill in for $b_0 = 1$. They are all bounded away from 0, so using Mather's result, there are no invariant circles with irrational rotation number with continued fraction expansion $[\boldsymbol{a}, (1,)^n b_0, \ldots]$, $b_0 > 1$]. Thus there are no irrational circles apart from the noble $[\boldsymbol{a}, (1,)^\infty]$. Assuming a conjectured

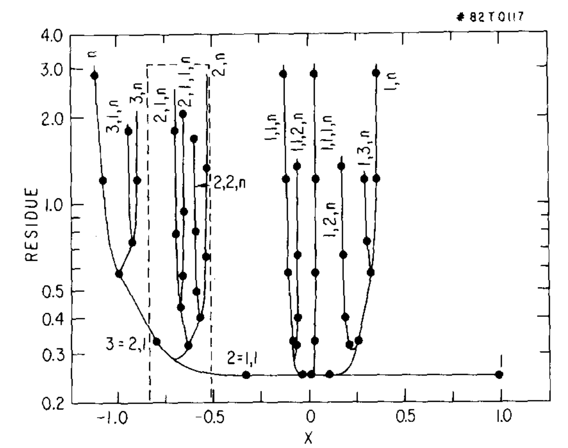

Fig. 8. Residues $R^+_{[\boldsymbol{b}]}$ of periodic orbits of F^*, plotted against position X on the dominant half-line.

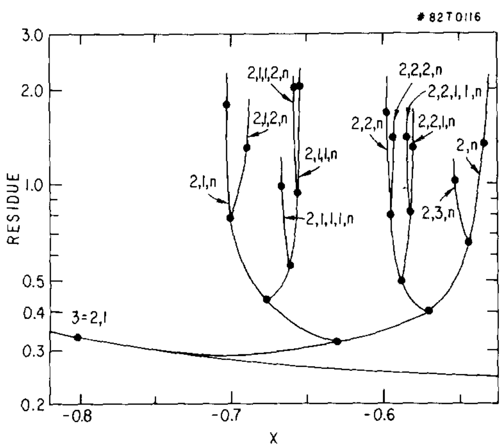

Fig. 9. Inset to fig. 8.

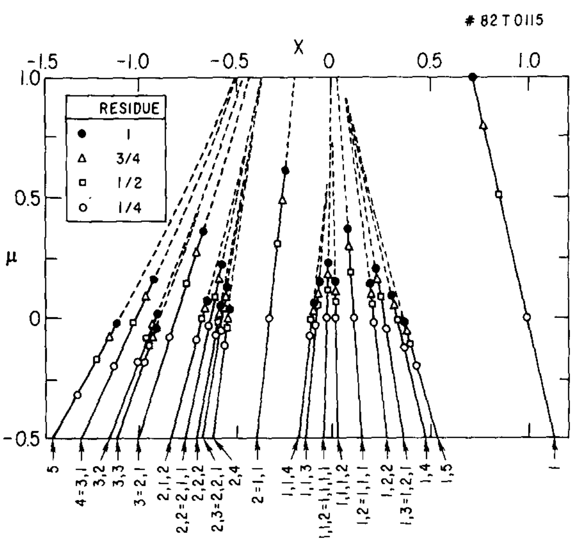

Fig. 10. Position X on the dominant half-line of periodic points of F^*_μ, plotted against μ, and indicating how their residues change.

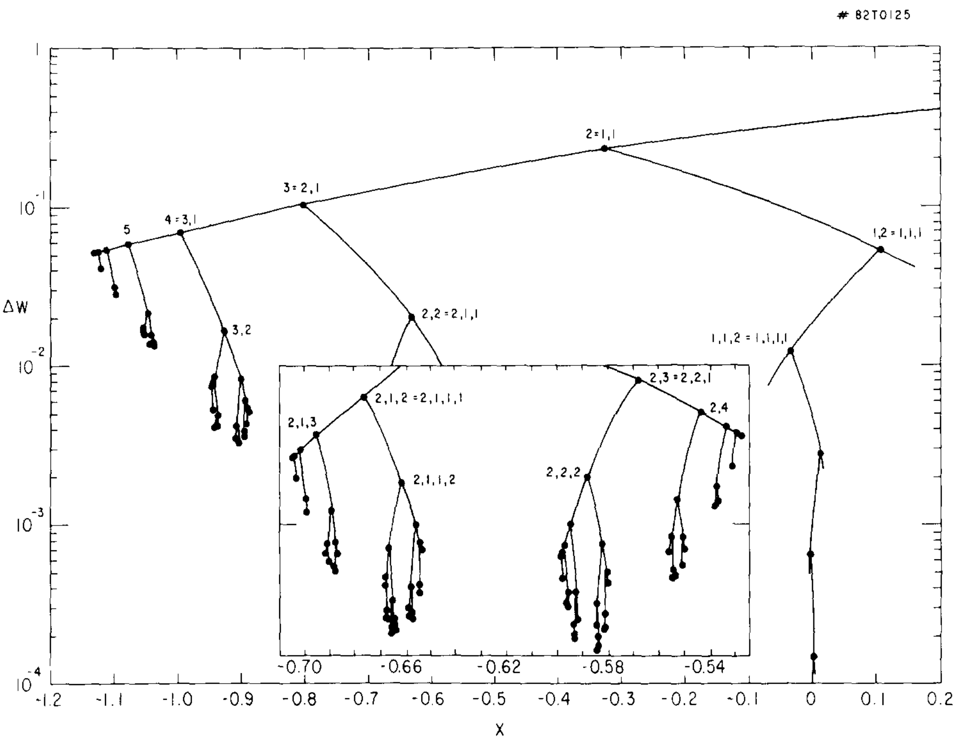

Fig. 11. Values of ΔW for various periodic orbits of F^*, plotted against position X on the dominant half-line.

extension of Mather's work to rational circles, there are no rational circles either.

Thus I conclude that noble circles are robust in an important sense, namely, a critical noble has a neighbourhood containing no other invariant circles, and all narrow enough connected neighbourhoods of a supercritical noble (Cantor set) contain no invariant circles at all. Since nobles are dense, one would like to conjecture a stronger result, namely, that isolated circles are typically nobles, but deducing this from the previous statement would require estimates on the sizes of the neighbourhoods, which I do not have at present.

14. Conclusion

In conclusion, I speculate a picture like fig. 12, in class AA (modulo coordinate changes). The critical fixed point has a codimension 1 stable manifold W^s, which, I believe, separates the space, at least locally, into pairs with a smooth golden curve and those with no golden curve. Any one parameter family crossing W^s transversally (in which I include non-zero speed) will have asymptotically the same behaviour, on a small enough scale in space and parameter, and on a long enough timescale, as the "universal" one parameter family given by a natural parametrisation of the one-dimensional unstable manifold W^u. In some sense there is a fixed point at infinity too, as I find asymptotic behaviour in the supercritical case too. How to express it, however, is not clear, as it has infinite actions and residues, and $\beta = -\infty$ (although $\alpha = -1$).

One could do exactly parallel analysis for any quadratic irrational rotation number, but other irrationals will require a modified treatment. We

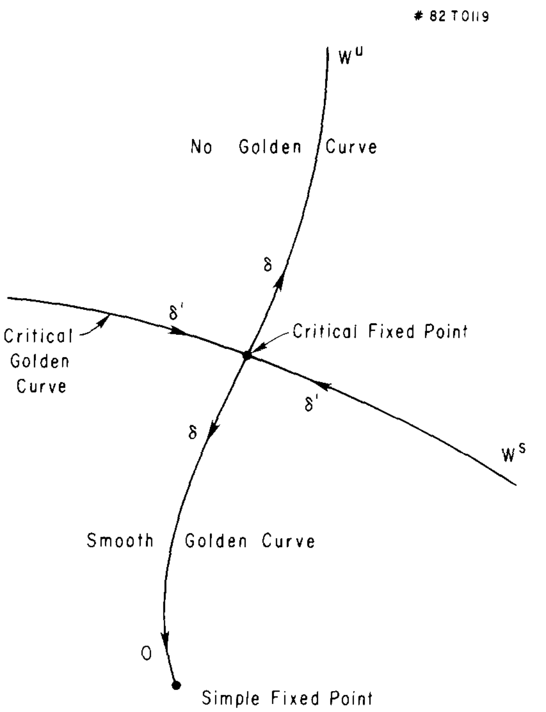

Fig. 12. Schematic of the action of N_1 in the space of commuting pairs in class AA (modulo coordinate changes).

indicated, however, that nobles are probably the most significant.

The ideas of this paper also carry over directly to the problem of existence and breakup of invariant tori in dissipative systems. The simple fixed point is very simple, and analysis of its spectrum much simpler than in the area preserving case (e.g. see Feigenbaum et al., 1982). Construction of a neighbourhood of attraction is also easier [19]. I obtained the critical fixed point to an accuracy of 10^{-7}, working in power series of degree 19 in x^3 on carefully chosen domains [19] (see also Rand et al. [17], and Feigenbaum et al. [13]).

Lastly, the ideas of this paper can be extended to invariant tori of arbitrary dimension. For a $2n$-dimensional symplectic map, for example, one would consider commuting $(n+1)$-tuples of maps. Rotation numbers would lie in $\mathbb{R}P^n$ and the renormalisation would generalise in an obvious way.

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