Accelerator Beam Dynamics

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May 13, 2016

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1 Twiss Parameters

Consider propagation of a particle trajectory through a beam line. The phase space 2-vector is

$$\mathbf{x} = (x, p_x/p) \sim (x, x').$$

x and x' are offset and angle with respect to some reference orbit, where in the small angle approximation $p_x/p \sim x'$. The Jacobian of the mapping from $\mathbf{x}^i$ to $\mathbf{x}^f$, is

$$M = \begin{pmatrix} \frac{\partial x_1^f}{\partial x_1^i} & \frac{\partial x_1^f}{\partial x_2^i} \\ \\ \frac{\partial x_2^f}{\partial x_1^i} & \frac{\partial x_2^f}{\partial x_2^i} \end{pmatrix}$$

The Jacobian M for a Hamiltonian system is symplectic. In a two dimensional phase space a symplectic matrix has unit determinant. If the system is linear, then

$$\mathbf{x}^f = M_{fi}\mathbf{x}^i$$

1.1 Scalar invariant

Define the scalar

$$s = \mathbf{x}^T A \mathbf{x}$$

Then define A, so that

$$s = \begin{pmatrix} x & x' \end{pmatrix} \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = \gamma x^2 + 2\alpha x x' + \beta x'^2$$

A can be any 4 parameters that we like. No loss of generality by setting $A = A^T$ since we only need three numbers to define the most general scalar combination of x and x'. α, β, γ are the twiss parameters. Now let's assume linearity and propagate $\mathbf{x}_b \to \mathbf{x}_e$ with the help of M. Then $\mathbf{x}_e = M\mathbf{x}_b$ and

$$s = \mathbf{x}_b^T M^T (M^T)^{-1} A_b M^{-1} M \mathbf{x}_b = \mathbf{x}_e^T (M^T)^{-1} A_b M^{-1} \mathbf{x}_e = \mathbf{x}_e^T A_e \mathbf{x}_e$$

The matrix s is invariant as long as

$$A_e = (M^T)^{-1} A_b M^{-1} \tag{1}$$

Or

$$\begin{pmatrix} \gamma_e & \alpha_e \\ \alpha_e & \beta_e \end{pmatrix} = (M^T)^{-1} \begin{pmatrix} \gamma_b & \alpha_b \\ \alpha_b & \beta_b \end{pmatrix} M^{-1}$$

We propagate twiss parameters using the transfer matrix. Evidently, once the twiss parameters are selected at one point in the beam line, they are defined everywhere by the mapping that propagates the phase space coordinates. Another thing, from Equation 1 we see that $|A_e| = |A_b|$. The determinant of the twiss matrix is invariant. We set it to unity for convenience. Then $\gamma\beta - \alpha^2 = 1$. It should be clear that except for the unit determinant requirement, the twiss parameters (α, β, γ) are totally unconstrained. We assign them whatever values we like at one location along the beam line and they are determined everywhere else.

The distribution of particles can be characterized in terms of the twiss parameters. The twiss parameters establish how the phase space coordinates are correlated, how x and x' are related. Consider the matrix of second moments. (The average of the first moments is zero).

$$\Sigma = \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}$$

The matrix is constructed as

$$\mathbf{x}\mathbf{x}^T = \begin{pmatrix} x \\ x' \end{pmatrix} \begin{pmatrix} x & x' \end{pmatrix}$$

and the distribution is propagated along the beam line according to

$$\mathbf{x}_e \mathbf{x}_e^t = M \mathbf{x}_b \mathbf{x}_b^T M^T$$

Then

$$\langle \mathbf{x}_e \mathbf{x}_e^t \rangle = \langle M \mathbf{x}_b \mathbf{x}_b^T M^T \rangle = M \langle \mathbf{x}_b \mathbf{x}_b^T \rangle M^T$$

or

$$\begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}_e = M \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}_b M^T$$

which looks almost like the rule for propagating the twiss matrix A. W found that

$$A_e = (M^T)^{-1} A_b M^{-1}.$$

Then

$$A_e^{-1} = M A_b^{-1} M^T$$

and the matrix A_e^{-1} transforms the same as the Σ matrix. The elements of the two matrices are clearly related. In particular

$$A^{-1} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} = \begin{pmatrix} \langle xx \rangle & \langle xx' \rangle \\ \langle xx' \rangle & \langle x'x' \rangle \end{pmatrix}$$

where ϵ remains to be determined.

$$|A^{-1}| = |\Sigma| \rightarrow \epsilon^2 = \langle xx \rangle \langle x'x' \rangle - \langle xx' \rangle^2$$

Since

$$\sigma^2 = \langle xx \rangle, \quad (\sigma')^2 = \langle x'x' \rangle$$

we have that

$$\beta = \frac{\sigma^2}{\epsilon}, \qquad \gamma = \frac{(\sigma')^2}{\epsilon}$$

The twiss parameters are determined by the distribution of the phase space coordinates of the trajectories.

2 Computing transfer matrix with tracking

Sometimes it is difficult to construct the transfer matrix from first principles. The matrix conveys the focusing effect of the element but to build the matrix we essentially need to know all the gradients etc. Alternatively we can determine the Jacobian directly by particle tracking. Remember that the transfer matrix is the Jacobian of the map

$$M = \begin{pmatrix} \frac{\partial x_1^f}{\partial x_1^i} & \frac{\partial x_1^f}{\partial x_2^i} \\ \\ \frac{\partial x_2^f}{\partial x_1^i} & \frac{\partial x_2^f}{\partial x_2^i} \end{pmatrix}$$

The strategy is essentially to compute the derivatives numerically. If we know the reference trajectory (uniquely defined in a circular machine, but not so straightforward in a transfer line like the entrance through the backlog iron and into the inflector), we can calculate trajectories displaced by Δx and $\Delta x'$ from the reference and build M. In principle we need only three non degenerate trajectories to determine the 2X2 matrix for horizontal or vertical motion as well as the reference. Write

$$\begin{aligned} M_{i\rightarrow f}(\mathbf{x}_{in} - \mathbf{x}_{ref}) &= \mathbf{x}_f - \mathbf{x}_{ref} \\ M_{i\rightarrow f}\mathbf{x}_{in} - (M_{i\rightarrow f} - I)\mathbf{x}_{ref} &= \mathbf{x}_f \\ M_{i\rightarrow f}\mathbf{x}_{in} - \mathbf{x}_0 &= \mathbf{x}_f \end{aligned} \tag{2}$$

Next construct

$$N = \begin{pmatrix} M_{i\rightarrow f} & \mathbf{x_0} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & x_0 \\ m_{21} & m_{22} & x_0' \\ 0 & 0 & 1 \end{pmatrix}$$

and Equation 2 becomes

$$N \begin{pmatrix} \mathbf{x}_{in} \\ -1 \end{pmatrix} = \begin{pmatrix} \mathbf{x}_f \\ -1 \end{pmatrix}.$$

The goal remember is to compute $M_{i\to f}$ and $\mathbf{x}_{ref}$. Choose three distinct values for $\mathbf{x}_{in}$, namely $\mathbf{x}^i_{in}$, $i = 1, 2, 3$, track each to $\mathbf{x}^i_f$ and we get

$$N \begin{pmatrix} \mathbf{x}^1_{in} & \mathbf{x}^2_{in} & \mathbf{x}^3_{in} \\ -1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} \mathbf{x}^1_f & \mathbf{x}^2_f & \mathbf{x}^3_f \\ -1 & -1 & -1 \end{pmatrix}$$

Finally

$$N = \begin{pmatrix} \mathbf{x}^1_f & \mathbf{x}^2_f & \mathbf{x}^3_f \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mathbf{x}^1_{in} & \mathbf{x}^2_{in} & \mathbf{x}^3_{in} \\ -1 & -1 & -1 \end{pmatrix}^{-1}$$

Extract M and $\mathbf{x}_{ref}$ from N as per above. The strategy is readily extended to the full 6 dimensional phase space where

$$\mathbf{x} \to \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{pmatrix}$$

where $\delta = \Delta E/E$. So to determine the evolution of the phase space (that is the twiss parameters) through the iron and inflector into the ring we simply compute 7 trajectories. We can in principle use the same 7 trajectories to determine the transfer matrix between any two points along the reference orbit.

3 Full Turn Map

Consider the full turn map M in a closed ring. The stability of the lattice is indicated by multiturn behavior. The initial phase space coordinates $\mathbf{x}_{in}$ are mapped after n-turns to

$$\mathbf{x}_{out} = M^n \mathbf{x}_{in}$$

3.1 Eigenvalues

It is easy to determine stability if we work in an eigen-basis. The eignenvalues of the 2X2 determinant 1 matrix M are

$$\lambda_\pm = e^{\pm i\phi}$$

where $\cos\phi = \frac{1}{2}\text{Tr}M$ and $\sin\phi = \sqrt{1 - \frac{1}{2}\text{Tr}M}$.

In view of the constraint on the trace, the full turn matrix M can be written as

$$M = \begin{pmatrix} \cos\phi + x & y \\ z & \cos\phi - x \end{pmatrix}$$

$$|M| = \cos^2\phi - x^2 - yz = 1 \to -x^2 - yz = \sin^2\phi$$

Next define

$$x = \alpha\sin\phi,\ y = \beta\sin\phi,\ z = -\gamma\sin\phi,\ \alpha^2 - \beta\gamma = -1$$

and the most general unit determinant matrix is

$$M = \begin{pmatrix} \cos\theta + \alpha\sin\theta & \beta\sin\theta \\ -\gamma\sin\theta & \cos\theta - \alpha\sin\theta \end{pmatrix}$$

For the time being, α, β, and γ are arbitrary real numbers. The relationship with the twiss parameters described above remains to be determined. The normalized eigenvectors are

$$\mathbf{v}_{\pm} = \sqrt{\frac{1}{\gamma+\beta}} \begin{pmatrix} \sqrt{\beta} \\ \frac{\pm i-\alpha}{\sqrt{\beta}} \end{pmatrix}$$

Propagation of the phase space vector through n turns is gives

$$\mathbf{v}^n_{\pm} = M^n \mathbf{v}_{\pm} = e^{\pm in\theta} \mathbf{v}_{\pm}.$$

The linear lattice (as represented by the Jacobian of the map M) is stable if θ is real ($|\mathrm{Tr}M| < 1$).

3.2 Decomposition

The similarity transformation to the eigenbasis is

$$U^{-1}MU = \Lambda \equiv \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

where

$$U = \mathbf{v}_{+}\mathbf{v}_{-} = N \begin{pmatrix} \sqrt{\beta} & \sqrt{\beta} \\ \frac{i-\alpha}{\sqrt{\beta}} & -\frac{i+\alpha}{\sqrt{\beta}} \end{pmatrix}$$

It is convenient to work in a real basis. We note that

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = K^{-1}\Lambda K$$

where

$$K = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix}$$

Then

$$K^{-1}U^{-1}MUK = K^{-1}\Lambda K = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \equiv R(\theta)$$

Next define

$$G \equiv UK = \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

where the constant N is chosen so that $\det G = 1$. In summary

$$G^{-1}MG = R(\theta)$$

In an earlier section we defined the parameters α, β, γ in terms of scalar invariant. Are the two definitions consistent? The twiss parameters transform according to

$$M^T A M = A'$$

where

$$A = \begin{pmatrix} \gamma & \alpha \\ \alpha & \beta \end{pmatrix}$$

If the definitions are consistent then the β, α, γ of G are the same as the α, β, γ that are the components of A and

$$M^T A M = (GRG^{-1})^T AGRG^{-1} = (G^{-1})^T R^T \left(G^T A G\right) R G^{-1}$$

But $G^T A G = I$, and we have that

$$A' = GR^T R G^T = A$$

as claimed.

3.3 Propagation of Twiss Parameters

Now suppose that the full turn matrix at point 1 is written as a product of the matrices of each individual element (quadrupole, drift, dipole, etc.) in the ring.

$$M_1 = T_{12}T_{23}\ldots$$

The full turn matrix at point 2 is

$$M_2 = T_{23}T_{34}\ldots T_{12}$$

and

$$M_2 = T_{12}^{-1}M_1T_{12}$$

The twiss parameters at 2 are given by M_2. In particular,

$$\cos\theta = \frac{1}{2}\mathrm{Tr}M,\ \alpha_2 = \frac{1}{2}(m_{11}-m_{22})/\sin\theta,\ \beta_2 = m_{12}/\sin\theta,\ \gamma_2 = \frac{1+\alpha_2^2}{\beta_2}$$

and the same for point 1. We would like to write T_{12} in terms of the twiss parameters at 1 and 2. We have that

$$\begin{aligned} M_2 = T_{12}^{-1}M_1T_{12} &= G_2R(\theta)G_2^{-1} = T_{12}^{-1}G_1RG_1^{-1}T_{12} \\ &\to R(\theta) = \left(G_2^{-1}T_{12}^{-1}G_1\right)R\left(G_1^{-1}T_{12}G_2\right) \\ R &= W^{-1}RW \end{aligned}$$

For an orthogonal matrix R

$$[R,W]=0 \iff W^{-1}=W^T$$

If W is orthogonal we can write

$$\begin{aligned} W &= R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \end{aligned}$$

and then

$$\begin{aligned} T_{12} &= G_2R(\phi)G_1^{-1} && (3) \\ &= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ \frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta_1}} & 0 \\ -\frac{\alpha_1}{\sqrt{\beta_1}} & \sqrt{\beta_1} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\beta_2} & 0 \\ \frac{\alpha_2}{\sqrt{\beta_2}} & \frac{1}{\sqrt{\beta_2}} \end{pmatrix} \begin{pmatrix} \frac{\cos\phi-\alpha_1\sin\phi}{\sqrt{\beta_1}} & \sin\phi\sqrt{\beta_1} \\ \frac{-\sin\phi-\alpha_1\cos\phi}{\sqrt{\beta_1}} & \cos\phi\sqrt{\beta_1} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\phi-\alpha_1\sin\phi) & \sqrt{\beta_2\beta_1}\sin\phi \\ \frac{(\alpha_2-\alpha_1)\cos\phi-(\alpha_2\alpha_1+1)\sin\phi}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}(\alpha_2\sin\phi+\cos\phi) \end{pmatrix} && (4) \end{aligned}$$

Therefore

$$\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = T_{12}\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} \quad (5)$$

$$x(s) = \left(\sqrt{\frac{\beta(s)}{\beta_1}}(\cos\phi_{s0} - \alpha_1 \sin\phi_{s0})\right) x_0 + \left(\sqrt{\beta(s)\beta_1}\sin\phi_{s0}\right) x_0' \tag{6}$$

$$x'(s) = \left(\frac{(\alpha(s) - \alpha_1)\cos\phi_{s0}) - (\alpha(s)\alpha_1 + 1)\sin\phi_{s0}}{\sqrt{\beta(s)\beta_1}}\right) x_0 + \sqrt{\frac{\beta_1}{\beta(s)}}(\alpha(s)\sin\phi_{s0} + \cos\phi_{s0})x_0' \tag{7}$$

where ϕ_{s0} is some phase advance from point 0 to s. How are α and β related? How is ϕ_{s0} defined? Since $x' = \frac{dx}{ds}$ we have that

$$\begin{aligned} x'(s) &= \frac{dx}{ds} = \left[\frac{1}{2}\frac{\beta'(s)}{\sqrt{\beta(s)}}\left(\sqrt{\frac{1}{\beta_1}}(\cos\phi_{s0} - \alpha_1\sin\phi_{s0}\right) + \sqrt{\frac{\beta(s)}{\beta_1}}(-\sin\phi_{s0} - \alpha_1\cos\phi_{s0})\phi_{s0}'\right] x_0 \\ &+ \left[\frac{1}{2}\frac{\beta'(s)}{\sqrt{\beta_1}}\sin\phi_{s0} + \sqrt{\frac{\beta(s)}{\beta_1}}\cos\phi_{s0}\phi_{s0}'\right] x_0' \end{aligned} \tag{8}$$

3.4 Connection to Differential Equation

Comparing Equations 7 and 8 we see that

$$\alpha(s) = \frac{1}{2}\beta' = \frac{1}{2}\frac{d\beta(s)}{ds}, \quad \phi'(s) = \frac{d\phi}{ds} = \frac{1}{\beta}$$

The phase advance

$$\phi_{s0} = \int_0^s \frac{ds}{\beta(s)}$$

3.4.1 Propagation of β-function

As noted in an earlier section there is an invariant

$$S = \gamma x^2 + 2\alpha x x' + \beta x^2 \equiv \vec{\beta}\cdot\vec{X} \tag{9}$$

where

$$\vec{\beta} = \begin{pmatrix}\gamma \\ \alpha \\ \beta\end{pmatrix}, \quad \vec{X} = \begin{pmatrix} x^2 \\ xx' \\ (x')^2\end{pmatrix}$$

Consider propagation from 1 to 2 by matrix M.

$$\begin{aligned} x_2 &= m_{11}x_1 + m_{12}x_1' \\ x_2' &= m_{21}x_1 + m_{22}x_1' \end{aligned}$$

Then

$$\vec{X}_2 = \mathcal{M}\vec{X}_1$$

and

$$\mathcal{M} = \begin{pmatrix} m_{11}^2 & 2m_{11}m_{12} & m_{12}^2 \\ m_{11}m_{21} & m_{12}m_{21} + m_{11}m_{22} & m_{12}m_{22} \\ m_{21}^2 & 2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \tag{10}$$

We would like to determine the corresponding matrix $\mathcal{N}$ that propagates the 3-vector $\vec{\beta}$. Since $\vec{\beta}_2 \cdot \vec{X}_2 = \vec{\beta}_1 \cdot \vec{X}_1 = \vec{\beta}_1 \cdot M^{-1}M\vec{X}_1 = \vec{\beta}_1\mathcal{M}^{-1}\vec{X}_2$, it follows that

$$\begin{aligned} \vec{\beta}_2 &= (\mathcal{M}^{-1})^T\vec{\beta}_1 \\ \mathcal{N} &= (\mathcal{M}^{-1})^T. \end{aligned} \tag{11}$$

$\mathcal{N}$ is constructed by replacing the elements of M with those of its inverse in Equation 10 and transposing, yielding

$$\mathcal{N} = \begin{pmatrix} m_{22}^2 & -m_{22}m_{21} & m_{21}^2 \\ -2m_{22}m_{12} & m_{11}m_{22} + m_{12}m_{21} & -2m_{21}m_{11} \\ m_{12}^2 & -m_{12}m_{11} & m_{11}^2 \end{pmatrix} \tag{12}$$

Consider for example propagation of the twiss parameters through a field free region. The elements of M in a field free region of length s are

$$M = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

Then

$$\mathcal{N} = \begin{pmatrix} 1 & 0 & 0 \\ -2s & 1 & 0 \\ s^2 & -s & 1 \end{pmatrix}.$$

At a waist (minimum β,$\alpha = 0$), $\vec{\beta}_0(\gamma_0, \alpha_0, \beta_0) = (1/\beta_0, 0, \beta_0)$. At a distance s from the waist,

$$\vec{\beta}(s) = \mathcal{N}\vec{\beta}_0 = (1/\beta_0, -2s/\beta_0, \beta_0 + s^2/\beta_0).$$

We find that β increases quadratically with distance from the minimum.

Using 11, we write $\mathcal{N}$ in terms of twiss parameters $\vec{\beta}_0$ and $\vec{\beta}(s)$ and the phase advance $\Delta\phi = \phi(s) - \phi_0$. Suppose that β at 0 is perturbed by a mismatch or quad error, so that

$$\begin{aligned} \vec{\beta}_0 &\rightarrow \begin{pmatrix} \gamma_0 + \Delta\gamma_0 \\ \alpha_0 \\ \beta_0 + \Delta\beta_0 \end{pmatrix} = \begin{pmatrix} \gamma_0 - \gamma_0\frac{\Delta\beta_0}{\beta_0} \\ \alpha_0 \\ \beta_0 + \Delta\beta_0 \end{pmatrix} \\ \Delta\vec{\beta}_0 &= \begin{pmatrix} \gamma_0\frac{\Delta\beta_0}{\beta_0} \\ 0 \\ \Delta\beta_0 \end{pmatrix} \end{aligned}$$

In view of Equations 11 and 12

$$\begin{aligned} \Delta\beta(s) &= -m_{12}^2\gamma_0\frac{\Delta\beta_0}{\beta_0} + m_{11}^2\Delta\beta_0 \\ &= -(\beta(s)\beta_0\sin^2\Delta\phi)\gamma_0\frac{\Delta\beta_0}{\beta_0} + \frac{\beta(s)}{\beta_0}(\cos\Delta\phi - \alpha_0\sin\Delta\phi)^2\Delta\beta_0 \\ &= \beta(s)\frac{\Delta\beta_0}{\beta_0}(\cos^2\Delta\phi - \sin^2\Delta\phi) - 2\alpha_0\cos\Delta\phi\sin\Delta\phi) \\ &= \Delta\beta_0\frac{\beta(s)}{\beta_0}(\cos 2\Delta\phi - \alpha_0\sin 2\Delta\phi) \end{aligned}$$

The β error propagates as the square of the phase advance

3.5 Closed Ring

In a closed ring, where $s = C$, and $\beta(C) = \beta_0$, the change in β on the n^{th} turn is

$$\Delta\beta(n) = \Delta\beta_0(\cos 2\mu - \alpha_0 \sin 2\mu)$$

where $\mu = 2\pi Q$ and Q is the tune.

3.5.1 Quadrupole error

Suppose that there is a focusing error at s, with focal length f . Then

$$M \to MQ$$

where the transfer matrix for a thin lens is

$$Q = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$$

Then

$$\begin{aligned} M &= \begin{pmatrix} \cos\mu + \alpha\sin\mu - \frac{1}{f}\beta\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu + \frac{1}{f}(\cos\mu - \alpha\sin\mu) & \cos\mu - \alpha\sin\mu \end{pmatrix} \\ &= \begin{pmatrix} \cos\mu' + \alpha'\sin\mu' & \beta'\sin\mu' \\ -\gamma'\sin\mu' & \cos\mu' - \alpha'\sin\mu' \end{pmatrix} \end{aligned}$$

Then $\cos\mu' = \cos(\mu + \Delta\mu) = \frac{1}{2}\mathrm{Tr}M = \cos\mu - \frac{1}{2}\frac{\beta}{f}\sin\mu$ If $f \gg \beta$,

$$\cos(\mu + \Delta\mu) = \cos\mu - \Delta\mu\sin\mu = \cos\mu - \frac{\beta}{f}\sin\mu \to \Delta\mu = \frac{\beta}{2f}$$

4 Dispersion

The formalism developed so far describes a 2 dimensional phase space. It can be used to account for motion independently in the horizontal, vertical or longitudinal direction. Now we consider systems with coupling between transverse (horizontal) and longitudinal degrees of freedom. The phase space vector

$$\mathbf{x} \to \begin{pmatrix} x \\ \frac{p_x}{p} \\ z \\ p_z = \frac{p - p_0}{p_0} \end{pmatrix}$$

where in the small angle limit $x' = p_x/p$ and p_0 is the reference energy. The Jacobian of the mapping from $\mathbf{x}^i$ to $\mathbf{x}^f$ is

$$T_{ij} = \frac{\partial \mathbf{x}_j^f}{\partial \mathbf{x}_1^i} \tag{13}$$

As noted above, the Jacobian for a Hamiltonian system is symplectic. A matrix is symplectic if

$$T^T S T = S$$

where

$$S = \begin{pmatrix} s & 0 & \dots \\ 0 & s & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

In a system with no coupling of horizontal and longitudinal motion

$$T = \begin{pmatrix} M_x & 0 \\ 0 & P_z \end{pmatrix}$$

where M_x and P_z are 2×2 and $|M_z| = |P_z| = 1$.

4.1 Coupling of Longitudinal and Transverse Motion

A bending magnet couples changes in energy (p_z) with a change in horizontal angle x' and position x so that m_{12} and m_{22} are in general non zero. The symplectic condition then requires that p_{11} and p_{12} are also finite and the Jacobian will have the form

$$T = \begin{pmatrix} \multicolumn{2}{c}{M_x} & \begin{matrix} 0 & m_{12} \\ 0 & m_{22} \end{matrix} \\ \begin{matrix} p_{11} & p_{12} \\ 0 & 0 \end{matrix} & P_z \end{pmatrix} \tag{14}$$

The dispersion is the dependence of transverse position and angle on energy offset δ, that is

$$\begin{pmatrix} \eta \\ \eta' \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \delta} \\ \frac{\partial x'}{\partial \delta} \end{pmatrix}$$

The dispersion is defined at every point s along the beamline or ring so that the horizontal position can be written in terms of a combination of *betatron* and *energy* components.

$$x(s) = \mathbf{x}_\beta(s) + \mathbf{x}_\delta(s).$$

Suppose that at some location along the beam line there is zero betatron amplitude. Then $\mathbf{x}_1 = \mathbf{x}_{1,\delta} = \boldsymbol{\eta}_1 \delta$ and from 14

$$\begin{aligned} \mathbf{x}_2 &= M\boldsymbol{\eta}_1\delta + \begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} \delta \\ &\Rightarrow \boldsymbol{\eta}_2 = M\boldsymbol{\eta}_1 + m_{12} \\ &\Rightarrow \begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} = \boldsymbol{\eta}_2 - M\boldsymbol{\eta}_1 \end{aligned} \tag{15}$$

Note that in a beam line without RF accelerating cavities there is no mechanism to couple longitudinal offset (z) and transverse position. Equation 14 is therefore a good representation of transport in such a beam line.

It is easy to show that if

$$m = \begin{pmatrix} 0 & m_{12} \\ 0 & m_{22} \end{pmatrix}$$

then the symplectic condition implies that $|M_z| = 1$. We learned all about the formalism for a unit determinant matrix in a previous section and how to can express M_z in terms of $\vec{\beta}$ and $\Delta\phi$. We find

$$\begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} = \boldsymbol{\eta}_2 - T_{12}\boldsymbol{\eta}_1 \tag{16}$$

where T_{12} given in Equation 4. The linear mapping from point 1 to point 2 can be written entirely in terms of $\vec{\beta}$, $\boldsymbol{\eta}$ at the end points and the betatron phase advance $\Delta\phi_{12}$. We learned that the dispersion function is propagated according to Equation 15. If $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$ are the closed ring dispersion at 1 and 2, then dispersion errors, perhaps due to some mismatch, propagate according to

$$\begin{aligned} \Delta\boldsymbol{\eta}_2 &= M\Delta\boldsymbol{\eta}_1 \\ &= \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\phi - \alpha_1 \sin\phi) & \sqrt{\beta_2\beta_1}\sin\phi \\ \frac{(\alpha_2-\alpha_1)\cos\phi-(\alpha_2\alpha_1+1)\sin\phi}{\sqrt{\beta_2\beta_1}} & \sqrt{\frac{\beta_1}{\beta_2}}(\alpha_2\sin\phi + \cos\phi) \end{pmatrix} \Delta\boldsymbol{\eta}_1 \end{aligned}$$

4.2 Closed Ring

In a closed ring with circumference C, $\boldsymbol{\eta}(s) = \boldsymbol{\eta}(s + C)$ and full turn matrix

$$T = \begin{pmatrix} M & m \\ p & P \end{pmatrix}$$

Equation 14 gives the closed ring dispersion

$$\begin{aligned} \boldsymbol{\eta} &= M\boldsymbol{\eta} + \begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} \\ \rightarrow (I - M)\boldsymbol{\eta} &= \begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} \\ \rightarrow \boldsymbol{\eta} &= (I - M)^{-1}\begin{pmatrix} m_{12} \\ m_{22} \end{pmatrix} \end{aligned}$$

5 Symplectic Transformation

Consider the canonical transformation from generalized coordinates $\mathbf{q}, \mathbf{p}$ to $\mathbf{Q}, \mathbf{P}$ where $\mathbf{q} = (q_1, q_2, \ldots, q_N)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_N)$. We aim to show that the Jacobian matrix

$$J = \frac{\partial(\mathbf{Q}, \mathbf{P})}{\partial(\mathbf{q}, \mathbf{p})}$$

is symplectic. We recall that the time evolution operator, (namely the Hamiltonian), has the form of a generator of canonical transformations. Therefore, the transformation of phase space coordinates from point η along a beam line to another point ξ is canonical and the transfer matrix is symplectic.

Suppose that the phase space vectors at two distinct points along the beam line are

$$\vec{\eta} = \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ \vdots \end{pmatrix} \quad \text{and} \quad \vec{\xi} = \begin{pmatrix} X \\ P_x \\ Y \\ P_y \\ \vdots \end{pmatrix}$$

and

$$K(\vec{\xi}) = H(\vec{\eta}) + \frac{\partial F}{\partial t}$$

where H and K are the Hamiltonian in terms of the coordinates $\vec{\eta}$ and K in terms of $\vec{\xi}$. F is the generator of the transformation. The generalized Hamilton's equations are

$$\dot{\eta}_i = S_{ij}\frac{\partial H}{\partial \eta_j} \qquad \text{where} \quad S = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ -1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Similarly

$$\dot{\xi}_i = S_{ij}\frac{\partial K}{\partial \xi_j} = S_{ij}\frac{\partial}{\partial \xi_j}\left(H(\vec{\eta}) + \frac{\partial F}{\partial t}\right) \tag{17}$$

We could also write

$$\begin{aligned} \dot{\xi}_i &= \frac{\partial \xi_i}{\partial \eta_j}\dot{\eta}_j + \frac{\partial \xi_i}{\partial t} \\ &= \frac{\partial \xi_i}{\partial \eta_j}S_{jk}\frac{\partial H}{\partial \eta_k} + \frac{\partial \xi_i}{\partial t} \end{aligned} \tag{18}$$

Combining Equations 17 and 18

$$\begin{aligned} S_{ij}\frac{\partial}{\partial \xi_j}\left(H(\vec{\eta}) + \frac{\partial F}{\partial t}\right) &= \frac{\partial \xi_i}{\partial \eta_j}S_{jk}\frac{\partial H}{\partial \eta_k} + \frac{\partial \xi_i}{\partial t} \\ S_{ij}\left(\frac{\partial H}{\partial \xi_j} + \frac{\partial}{\partial t}\frac{\partial F}{\partial \xi_j}\right) &= \frac{\partial \xi_i}{\partial \eta_j}S_{jk}\frac{\partial H}{\partial \eta_k} + \frac{\partial \xi_i}{\partial t} \\ S_{ij}\left(\frac{\partial H}{\partial \eta_k}\frac{\partial \eta_k}{\partial \xi_j} + \frac{\partial}{\partial t}\frac{\partial F}{\partial \xi_j}\right) &= \frac{\partial \xi_i}{\partial \eta_j}S_{jk}\frac{\partial H}{\partial \eta_k} + \frac{\partial \xi_i}{\partial t} \end{aligned} \tag{19}$$

$$\tag{20}$$

If the generating function F is type 2, then $\frac{\partial F}{\partial \xi_j} = S_{jk}\xi_k$ and Equation 19 becomes

$$\begin{aligned} S_{ij}\left(\frac{\partial H}{\partial \eta_k}\frac{\partial \eta_k}{\partial \xi_j} - S_{jk}\dot{\xi}_k\right) &= \frac{\partial \xi_i}{\partial \eta_j}S_{jk}\frac{\partial H}{\partial \eta_k} + \frac{\partial \xi_i}{\partial t} \\ \to S_{ij}\frac{\partial H}{\partial \eta_k}\frac{\partial \eta_k}{\partial \xi_j} &= \frac{\partial \xi_i}{\partial \eta_j}S_{jk}\frac{\partial H}{\partial \eta_k} \end{aligned}$$

since $S_{ij}J_{jk} = -\delta_{ik}$. Then

$$S_{ij}\frac{\partial \eta_k}{\partial \xi_j} = \frac{\partial \xi_i}{\partial \eta_j}S_{jk} \tag{21}$$

We recognize $\frac{\partial \xi_i}{\partial \eta_j} = M_{ij}$ as the Jacobian of the mapping from η to ξ and $\frac{\partial \eta_k}{\partial \xi_j} = M^{-1}$ as the inverse. Then

$$\begin{aligned} S_{ij}M^{-1}_{kj} &= S_{ij}(M^{-1})^T_{jk} = M_{ij}S_{jk} = M_{ij}S_{jk} \\ S(M^{-1})^T = MS & \\ \to S &= MSM^T \end{aligned}$$

which is the definition of a symplectic matrix. The Jacobian of a canonical transformation is a symplectic matrix.

6 Coupling

Let's begin with transverse coupling. The Jacobian is a 4X4 symplectic matrix M such that $MSM^T = S$. A 4X4 symplectic matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

has the property that

$$|A| + |B| = |C| + |D| = 1(show).$$

Diagonalizing gives us $M = UEU^{-1}$ where E is a 4X4 diagonal symplectic matrix (*show that the diagnonal matrix is symplectic*). Then if

$$E = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

it follows that

$$\lambda_1\lambda_2 = \lambda_3\lambda_4 = 1$$

We know that

$$M\vec{u}_i = \lambda_i\vec{u}_i$$

where $\vec{u}_i$ is the eigenvector with eigenvalue λ_i. Then the phase space evolves after n turns according to

$$M^n\vec{u}_i = \lambda_i^n\vec{u}_i$$

If the magnitude of any of the eigenvalues (λ_i) is greater (or less) than 1, then the motion grows exponentially. The eigenvalues for the full turn coupled matrix for a stable system with dimension $2NX2N$ are unimodular, complex conjugate pairs.

7 Normal Mode Decomposition of $2N \times 2N$ symplectic matrices

Normal mode decomposition of a 4X4 symplectic matrix is a standard technique for analyzing transverse coupling in a storage ring. We generalize the decomposition to any 2NX2N symplectic matrix T and derive the transformation W from lab coordinates to normal mode coordinates U. That is

$$T = WUW^{-1} \tag{22}$$

where U is block diagonal and real and we construct the real matrix W with the form

$$W = \begin{pmatrix} \gamma_1 I & C_1 & C_2 & \dots \\ C'_1 & \gamma_2 I & C_3 & \dots \\ C'_2 & C'_3 & \gamma_3 I & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{23}$$

I is the 2x2 identity, and C_1, C_2, C'_1 etc are 2x2. (If for example, $n = 2$, then $\gamma_1 = \gamma_2$ and $C' = -C^\dagger$).

7.1 Normal Modes

The block diagonal matrix

$$U = \begin{pmatrix} A & 0 & \dots \\ 0 & B & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

can be further decomposed as

$$U = YZY^{-1} \tag{24}$$

where

$$Z(\theta_1, \theta_2, \dots, \theta_n) = \begin{pmatrix} R(\theta_1) & 0 & \dots \\ 0 & R(\theta_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{25}$$

with

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{26}$$

and

$$Y = \begin{pmatrix} G_1 & 0 & \dots \\ 0 & G_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{27}$$

and $G_i = \begin{pmatrix} \sqrt{\beta_i} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{pmatrix}$.

Since standard techniques exist for diagonalizing square matrices and identifying eigenvalues and eigenvectors, we begin by doing just that.

$$T = VDV^{-1}, \tag{28}$$

where T is the $2N$ x $2N$ symplectic matrix, D is the diagonal matrix of eigenvalues, and V is the matrix constructed from the eigenvectors. Since T is symplectic, the eigenvalues and eigenvectors, as noted above, appear as unimodular, complex conjugate pairs, λ_i, λ_i^* and $\vec{v}_i$ and $\vec{v}_i^*$. Then D can be written in the form

$$D = \begin{pmatrix} d(\theta_1) & 0 & 0 & \dots \\ 0 & d(\theta_2) & 0 & \dots \\ 0 & 0 & d(\theta_3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \text{ where } d(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{29}$$

The n columns of the matrix V are the n eigenvectors v_i. The eigenvectors are not unique, but may be multiplied by an arbitrary complex number. That is, $\vec{v}_i \rightarrow \rho_i e^{i\phi_i}\vec{v}_i$ and $\vec{v}_i^* \rightarrow \rho_i e^{-i\phi_i}\vec{v}_i^*$. If $V_0 = \vec{v}_1\ \vec{v}_1^*\ \vec{v}_2\ \vec{v}_2^*\ \dots \vec{v}_n\ \vec{v}_n^*$, then

$$\begin{aligned} V(\vec{\rho}, \vec{\phi}) &= V_0 D(\rho_1, \rho_2, \dots \rho_n, \phi_1, \phi_2, \dots, \phi_n) \\ &= V_0 \begin{pmatrix} \rho_1 d(\phi_1) & 0 & 0 & \dots \\ 0 & \rho_2 d(\phi_2) & 0 & \dots \\ 0 & 0 & \rho_3 d(\phi_3) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

Note that $V(\vec{\rho}, \vec{\phi})$ effects the transformation of Equation 28 for any real numbers ρ_i and ϕ_i.

7.2 Real Basis

We transform from a complex to a real basis with K where the real matrix Z (Equation 25) is related to the complex matrix D (Equation 29) by the similarity transformation

$$Z(\theta_2, \theta_2, \theta_3) = KD(\theta_1, \theta_2, \theta_3)K^{-1} \tag{30}$$

where

$$K = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \tag{31}$$

and

$$k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{32}$$

7.3 W-matrix

To construct W and U from V and D, we use Equations 22, 24 and 30 to write

$$\begin{aligned}
T = WUW^{-1} &= VDV^{-1} \\
&= V_0 D(\vec{\rho}, \vec{\phi}) \left(K^{-1}K\right) D(\vec{\theta}) \left(K^{-1}K\right) D^{-1}(\vec{\rho}, \vec{\phi}) V_0^{-1} \\
&= V_0 \left(K^{-1}K\right) D(\vec{\rho}, \vec{\phi}) \left(K^{-1}K\right) D(\vec{\theta}) \left(KK^{-1}\right) D^{-1}(\vec{\rho}, \vec{\phi}) \left(KK^{-1}\right) V_0^{-1} \\
&= \left(V_0 K^{-1}\right) Z(\vec{\rho}, \vec{\phi}) Z(\vec{\theta}) Z^{-1}(\vec{\rho}, \vec{\phi}) \left(K^{-1} V_0^{-1}\right) \\
&= V'(\vec{\rho}, \vec{\phi}) Z(\vec{\theta}) V'^{-1}(\vec{\rho}, \vec{\phi})
\end{aligned}$$

Now since the columns of V_0 are complex conjugate pairs, $V_0 K^{-1}$ is real. The Z matrices are similarly constructed to be real and therefore V' is real.

So far we have

$$\begin{aligned}
WUW^{-1} &= V'ZV'^{-1} \\
WYZ(\vec{\theta})Y^{-1} &= V'Z(\vec{\theta})V'^{-1} \\
\rightarrow V' = WY &
\end{aligned}$$

where we have used Equation 24.

Next we determine the parameters $\vec{\rho}$ and $\vec{\phi}$. We choose $\vec{\rho}$ so that V' will be symplectic. In particular, if we write V' in terms of the 2X2 matrices V_i^j then

$$\begin{aligned}
V' &= \begin{pmatrix} V_1'^1 & V_1'^2 & \cdots \\ V_2'^1 & V_2'^2 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix} \\
&= \begin{pmatrix} V_{0}{}_1^1 & V_{0}{}_1^2 & \cdots \\ V_{0}{}_2^1 & V_{0}{}_2^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \rho_1 R(\phi_1) & 0 & \cdots \\ 0 & \rho_2 R(\phi_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\
&= \begin{pmatrix} \rho_1 V_{0}{}_1^1 R(\phi_1) & \rho_2 V_{0}{}_1^2 R(\phi_2) & \cdots \\ \rho_1 V_{0}{}_2^1 R(\phi_1) & \rho_2 V_{0}{}_2^2 R(\phi_2) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}
\end{aligned}$$

Symplecticity constrains the sums of determinants of V'^{j}_{i} so that

$$
\begin{aligned}
1 &= \sum_{i=1}^{n} |V'^{j}_{i}| \\
&= \sum_{i=1}^{n} |\rho_j V_0{}^{j}_{i} R(\phi_j)| \\
&= \sum_{i=1}^{n} \rho_j^2 |V_0{}^{j}_{i}| \\
\rightarrow & \;\; \rho_j = \frac{1}{\sqrt{\sum_{i=1}^{n} |V_0{}^{j}_{i}|}}
\end{aligned}
$$

In order to determine the order of the conjugate columns of V', and finally the paramters $\vec{\phi}$ we expand

$$
\begin{aligned}
V' &= WY(\vec{G}) \\
V' &= \begin{pmatrix} V'^{1}_{1} & V'^{2}_{1} & \dots \\ V'^{1}_{2} & V'^{2}_{2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \gamma_1 I & C & \dots \\ C' & \gamma_2 I & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} G_1 & 0 & \dots \\ 0 & G_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \\
\begin{pmatrix} \rho_1 V_0{}^{1}_{1} R(\phi_1) & \rho_2 V_0{}^{2}_{1} R(\phi_2) & \dots \\ \rho_1 V_0{}^{1}_{2} R(\phi_1) & \rho_2 V_0{}^{2}_{2} R(\phi_2) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} &= \begin{pmatrix} \gamma_1 G_1 & C G_2 & \dots \\ C' G_1 & \gamma_2 G_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}
\end{aligned}
$$

Then the diagonal blocks are required to have the form

$$
\begin{aligned}
V'^{i}_{i} &= \gamma_i G_i \\
\rho_i V_0{}^{i}_{i} R(\phi_i) &= \gamma_i \begin{pmatrix} \sqrt{\beta_i} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{pmatrix}
\end{aligned}
$$

A real solution requires that $|V'^{i}_{i}| > 0$. We are free to choose the order of the conjugate columns of V' to ensure that this is true. (Note that if we reverse the order of the columns $V'^{i,j} \rightarrow V'^{j,i}$, then the sign of the determinant of the 2 X 2 blocks is reversed.) If we reverse the order of eigenvectors in V', then we also reverse the order of eigenvalues in $D(\vec{\theta})$ or equivalently $\theta_i \rightarrow 2\pi - \theta_i$. To find $\vec{\phi}$ we proceed with our expansion of $V_0{}^{ii}$ and $R(\phi_i)$ and write

$$
\rho_i \begin{pmatrix} V_0{}^{ii}_{11} & V_0{}^{ii}_{12} \\ V_0{}^{ii}_{21} & V_0{}^{ii}_{22} \end{pmatrix} \begin{pmatrix} \cos\phi_i & \sin\phi_i \\ -\sin\phi_i & \cos\phi_i \end{pmatrix}) = \gamma_i \begin{pmatrix} \sqrt{\beta_i} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{pmatrix}
$$

We choose ϕ_i so that $G^{i}_{22} = 0$, or

$$
\tan\phi_i = \frac{V_0{}^{ii}_{11}}{V_0{}^{ii}_{12}}
$$

The ambiguity in ϕ_i, $(\tan\phi_i = \tan(2\pi - \phi_i))$ is resolved with the condition that $G^{i}_{11} = V_0{}^{ii}_{11} \cos\phi_i - V_0{}^{ii}_{12} \sin\phi > 0$.

7.4 Summary

1. Find eigenvectors and eigenvalues
2. Transform eigenvectors to a real basis
3. Construct V. The columns of V are the eigenvectors. The eigenvectors appear as complex conjugate pairs since T is symplectic.
4. Choose the normalization for each pair of eigenvectors so that W will be symplectic. In particular if

$$V = \begin{pmatrix} c_1 V_{1,1} & c_2 V_{1,2} & \dots \\ c_1 V_{2,1} & c_2 V_{2,2} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

   where $V_{i,j}$ are 2X2 matrices, and $c_i = \rho e^{i\phi_i}$ then choose ρ_1 so that

$$\rho_1^2 \left(|V_{1,1}| + |V_{2,1}| + |V_{3,1}| + \dots \right) = 1$$

5. Adjust the order of complex conjugate pairs so that $|V_{i,i}| > 0$. That is, if $|V_{i,i}| < 0$, than swap the order of the columns.
6. Choose the phases ϕ_i so that

$$\begin{aligned} G_i &= V_{i,i} R\theta) \\ \text{has the form} & \\ &= \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \end{aligned}$$

8 Interpretation of the Coupling Matrix

Consider two dimensional coupling, that is horizontal and vertical, or horizontal and longitudinal. The full turn matrix T is written in terms of normal modes

$$T = VUV^{-1}$$

where

$$T = \begin{pmatrix} M & m \\ n & N \end{pmatrix}$$

and

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

As before (section on propagating twiss parameters), we can write

$$A = G_a^{-1} R(\mu_a) G_a, \quad B = G_b^{-1} R(\mu_b) G_b$$

where

$$G_a = \begin{pmatrix} \sqrt{\beta_a} & 0 \\ \frac{\alpha_a}{\sqrt{\beta_a}} & \frac{1}{\sqrt{\beta_a}} \end{pmatrix}, \qquad R(\mu) = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

Now

$$U^n = G\begin{pmatrix} R(n\mu_a & 0 \\ 0 & R(n\mu_b)\end{pmatrix})G^{-1}$$

where

$$G = \begin{pmatrix} G_a & 0 \\ 0 & G_b\end{pmatrix}$$

The real phase space vector $\vec{x}$ is related to the normalized normal mode vector $\vec{u}$ according to

$$\vec{x} = VG^{-1}\vec{u} = \begin{pmatrix} \gamma I & C \\ -C^{\dagger} & \gamma I\end{pmatrix} G^{-1}\vec{u}$$

where for a 4X4 matrix, V has the form defined in Equation 7, and C is 2X2. Then after propagation through n turns, we have

$$\begin{aligned} \vec{x}^n &= T^n\vec{x}^0 = T^n G^{-1}V\vec{u}^0 = (VG^{-1}RGV^{-1})^n(VG^{-1})\vec{u}^0 \\ &= VG^{-1}R(n\mu_a)\vec{u}^0 = G(G^{-1}VG^{-1})R(n\mu_a)\vec{u}^0 = G\bar{V}\vec{u}^n \end{aligned}$$

where

$$\bar{V} = GVG^{-1}$$

is the normalized coupling matrix.

In the limit of vanishing coupling between horizontal and vertical planes, the normal mode emittances ϵ_a and ϵ_b reduce to ϵ_x, ϵ_y as

$$\begin{array}{ccc} \epsilon_a & \lim_{a\to 0} & \epsilon_x \\ \epsilon_b & \lim_{b\to 0} & \epsilon_y \end{array}$$

In electron storage rings typically $\epsilon_b \ll \epsilon_a$. So let's suppose that $\epsilon_b = 0$. Then in general in the normal mode coordinates

$$\vec{u} = \begin{pmatrix} u_a \\ u_{a'} \\ 0 \\ 0 \end{pmatrix}.$$

The normal mode coordinates propagate from one turn to the next as a rotation.

$$\vec{u}^n = \begin{pmatrix} R(n\mu_a) & 0 \\ 0 & R(n\mu_b)\end{pmatrix}\begin{pmatrix} u_a^0 \\ u_{a'}^0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u_a^0\cos(n\mu) + u_{a'}^0\sin(n\mu) \\ -u_a^0\sin(n\mu) + u_{a'}^0\cos(n\mu) \\ 0 \\ 0 \end{pmatrix}$$

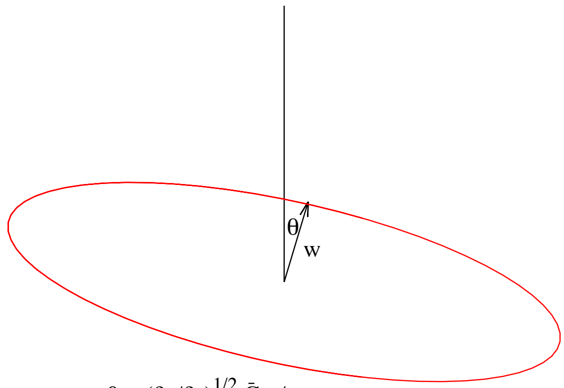
It is convenient to define our starting point so that $u_{a'}^0 = 0$.

Now the laboratory frame phase space coordinates on turn n are related to the normal mode coordinates according to

$$\begin{aligned} \vec{x}^n &= G\bar{V}R^n\vec{u}^0 \\ &= G\bar{V}\begin{pmatrix} \cos(n\mu) \\ -\sin(n\mu) \\ 0 \\ 0 \end{pmatrix} u_a^0 = G\begin{pmatrix} \gamma\cos(n\mu_a) \\ \gamma\sin n\mu_a \\ -\bar{C}_{22}\cos(n\mu_a) - \bar{C}_{12}\sin(n\mu_a) \\ \bar{C}_{21}\cos(n\mu_a) + \bar{C}_{11}\sin(n\mu_a) \end{pmatrix} u_a^0 \end{aligned}$$

Now we have parameteric equations for the real space trajectory

$$\begin{aligned} x^n &= \sqrt{\beta_a}\gamma\cos(n\mu_a)u_a^0 \\ y^n &= \sqrt{\beta_b}(-\bar{C}_{22}\cos(n\mu_a) + \bar{C}_{12}\sin(n\mu_a))u_a^0 \end{aligned}$$



9 Equations of motion

For a relativistic charged particle in a static magnetic field

$$\frac{d\vec{p}}{dt} = e\vec{v}\times\vec{B}$$

The field is static, so energy does not change and γm is independent of time and

$$\begin{aligned} \frac{d\vec{p}}{dt} &= m\gamma\frac{d\vec{v}}{dt} = e\vec{v}\times\vec{B} \\ \rightarrow \dot{\vec{v}} = \frac{e\vec{v}\times\vec{B}}{\gamma m} & \end{aligned} \tag{33}$$

Now we have to compute $\ddot{\vec{R}} = \dot{\vec{v}}$ in the curvilinear coordinate system. We define ρ as the curvature. The coordinates of the particle at any position along the path s are $\vec{R} = (\rho + x)\hat{x} + y\hat{y}$.

$$\begin{aligned} \dot{\vec{R}} &= \dot{x}\hat{x} + r\dot{\hat{x}} + \dot{y}\hat{y} \\ &= \dot{x}\hat{x} + (r\dot{\theta})\hat{s} + \dot{y}\hat{y} \end{aligned}$$

where $\dot{\hat{x}} = \dot{\theta}\hat{s}$. The second derivative

$$\begin{aligned} \ddot{\vec{R}} &= \ddot{x}\hat{x} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} + r\dot{\theta}\dot{\hat{s}} + \ddot{y}\hat{y} \\ &= \ddot{x}\hat{x} + \dot{r}\dot{\theta}\hat{s} + r\ddot{\theta}\hat{s} - r\dot{\theta}^2\hat{x} + \ddot{y}\hat{y} \end{aligned}$$

where $\dot{\hat{s}} = -\hat{x}\dot{\theta}$. We know that $ds = \rho d\theta = \rho \frac{v_s}{r} dt$. So we can write that $\dot{\theta} = \frac{v_s}{r}$ and

$$\begin{aligned} \frac{ds}{dt} &= \rho \frac{v_s}{r} \\ &\text{and} \\ \ddot{\vec{R}} &= (\ddot{x} - \frac{v_s^2}{r})\hat{x} + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{s} + \ddot{y}\hat{y} \end{aligned}$$

Next change dependent variable from t to s.

$$\begin{aligned} \frac{d}{dt} &= \frac{ds}{dt}\frac{d}{ds} = \frac{\rho v_s}{r}\frac{d}{ds} \\ \frac{d^2}{dt^2} &= \left(\rho\frac{v_s}{r}\right)^2 \frac{d^2}{ds^2} \end{aligned}$$

Then

$$\begin{aligned} \ddot{\vec{R}} &= \left(\rho\frac{v_s}{r}\right)^2 \vec{R}'' \\ &= (x''\left(\rho\frac{v_s}{r}\right)^2 - \frac{v_s^2}{r})\hat{x} + (r'\frac{v_s}{r} + r\theta'')\hat{s} + r^2\theta''\left(\rho\frac{v_s}{r}\right)^2 \hat{s} + y''\left(\rho\frac{v_s}{r}\right)^2 \hat{y} \end{aligned}$$

Comparison with equation 33 gives

$$\begin{aligned} x'' - \frac{r}{\rho^2} &= \frac{e}{\gamma m}(v_y B_z - v_s B_y)\left(\frac{r}{v_s\rho}\right)^2 \\ x'' - \frac{r}{\rho^2} &\sim -\frac{e}{\gamma m v_s} B_y \left(\frac{r}{\rho}\right)^2 \\ x'' &\sim -\frac{e}{\gamma m v_s} B_y \left(\frac{r}{\rho}\right)^2 + \frac{r}{\rho^2} \\ x'' &\sim -\frac{1}{B\rho}\left(B_y(0) + y\frac{dB_y}{dy} + x\frac{dB_y}{dx}\right)\left(\frac{r}{\rho}\right)^2 + \frac{r}{\rho^2} \\ x'' &\sim -\left(\frac{1}{\rho} + xK\right)\left(\frac{r}{\rho}\right)^2 + \frac{r}{\rho^2} \\ x'' &\sim -\left(\frac{1}{\rho} + xK\right)\left(\frac{x+\rho}{\rho}\right)^2 + \frac{x+\rho}{\rho^2} \\ x'' &\sim -\left(\frac{x}{\rho^2} + xK\right) \\ y'' &= \frac{e}{\gamma m}(v_s B_x - v_x B_z)\left(\frac{r}{v_s\rho}\right)^2 \\ y'' &\sim \frac{eB_x}{\gamma m v_s}\left(1 + \frac{x}{\rho}\right)^2 \\ y'' &\sim \frac{e}{\gamma m v_s}(B_x(0) + \frac{dB_x}{dx}x + \frac{dB_x}{dy}y)\left(1 + \frac{x}{\rho}\right)^2 \\ y'' &\sim \frac{1}{B\rho}(B_x(0) + \frac{dB_x}{dy}y) \\ y'' &\sim Ky \end{aligned}$$

where we have only kept first order terms in x and y and we assume that $B_x(0) = 0$,and $B_y(0)/B\rho = 1/\rho$.

10 Solution to equation of motion

If $K(s)$ is periodic with period C, such that $K(s) = K(C + s)$ then the general solution to the equations of motion is

$$x = Aw(s)\cos[\psi(s) + \delta]$$

with $w(s)$ periodic in C. Substitution into the equation of motion

$$x'' = -Kx \tag{34}$$

gives

$$\begin{aligned}
0 &= \frac{d}{ds}\big(w'\cos(\psi+\delta) - w\psi'\sin(\psi+\delta)\big) + Kw\cos(\psi+\delta) \\
&= w''\cos(\psi+\delta) - 2w'\psi'\sin(\psi+\delta) - w(\psi')^2\cos(\psi+\delta) - w\psi''\sin(\psi+\delta) + Kw\cos(\psi+\delta)
\end{aligned}$$

If that last is true for any phase δ then

$$0 = 2w'\psi' + w\psi'' \tag{35}$$

Define $\beta \equiv w^2$ and

$$\begin{aligned}
0 &= \beta'\beta^{-\frac{1}{2}}\psi' + \beta^{\frac{1}{2}}\left(\psi''\right) \\
\rightarrow \frac{\beta'}{\beta} &= -\frac{\psi''}{\psi'} \\
& \beta\psi'' + \beta'\psi' = 0 \\
\rightarrow & \quad \frac{1}{\psi'}\frac{d}{ds}\psi' = -\frac{1}{\beta}\frac{d}{ds}\beta \\
\rightarrow & \quad \frac{d\psi'}{\psi'} = -\frac{d\beta}{\beta}
\end{aligned}$$

Also

$$\begin{aligned}
0 &= w'' - w(\psi')^2 + Kw \\
0 &= (\sqrt{\beta})'' - \frac{1}{\beta^{3/2}} + K\beta^{\frac{1}{2}} \\
0 &= \frac{1}{2}\frac{d}{ds}\frac{\beta'}{\beta^{\frac{1}{2}}} - \frac{1}{\beta^{3/2}} + K\beta^{\frac{1}{2}} \\
0 &= \frac{1}{2}\left(\frac{\beta''}{\beta^{\frac{1}{2}}} - \frac{1}{2}\frac{\beta'^2}{\beta^{3/2}}\right) - \frac{1}{\beta^{3/2}} + K\beta^{\frac{1}{2}}
\end{aligned}$$

11 Floquet transformation

Let $\phi = \psi/\nu$, where $\nu = \frac{1}{2\pi}\int \frac{ds}{\beta(s)}$ and $\xi = \frac{x}{\beta^{\frac{1}{2}}}$. Then

$$\begin{aligned} x(s) &= A\beta(s)^{\frac{1}{2}}\cos(\psi(s)+\delta) \\ \rightarrow \frac{x}{\beta^{\frac{1}{2}}} &= A\cos(\nu\phi+\delta) \\ \rightarrow \xi &= A\cos(\nu\phi+\delta) \end{aligned}$$

$$\begin{aligned} \frac{d\xi}{d\phi} &= \frac{d\xi}{ds}\frac{ds}{d\phi} = \nu\beta\left(\frac{x'}{\beta^{1/2}} - \frac{1}{2}\frac{x\beta'}{\beta^{3/2}}\right) \\ &= \nu\left(x'\beta^{1/2} - \frac{1}{2}\frac{x\beta'}{\beta^{1/2}}\right) \\ \frac{d^2\xi}{d\phi^2} &= \nu^2\beta\left(x''\beta^{\frac{1}{2}} + \frac{1}{2}\frac{x'\beta'}{\beta^{1/2}} - \frac{1}{2}\frac{x'\beta'}{\beta^{1/2}} - \frac{1}{2}\frac{x\beta''}{\beta^{1/2}} + \frac{1}{4}\frac{x\beta'^2}{\beta^{3/2}}\right) \\ &= \nu^2\left(x''\beta^{3/2} - \frac{1}{2}x\beta''\beta^{1/2} + \frac{1}{4}\frac{x\beta'^2}{\beta^{1/2}}\right) \\ &= \nu^2\left(x''\beta^{3/2} - \frac{1}{2}x\left(\beta''\beta^{1/2} - \frac{1}{2}\frac{\beta'^2}{\beta^{1/2}}\right)\right) \end{aligned}$$

Substitution from above gives

$$\begin{aligned} \frac{d^2\xi}{d\phi^2} &= \nu^2\left(x''\beta^{3/2} + x\left(K\beta^{3/2} - \frac{1}{\beta^{1/2}}\right)\right) \\ \frac{d^2\xi}{d\phi^2} &= \nu^2\frac{x}{\beta^{1/2}} \\ &= \nu^2\xi \end{aligned}$$

And if there are contributions to the equation of motion, for example from magnetic fields $B(s)$ that do not scale linearly with displacement, then refeq:eom becomes

$$x'' + K(s)x = -\frac{qcB(s)}{B\rho} \tag{36}$$

and

$$\frac{d^2\xi}{d\phi^2} + \nu^2\xi = -\nu^2\beta^{\frac{3}{2}}\frac{\Delta B(\xi,\phi)}{B\rho} \tag{37}$$

12 Field errors

12.1 Linear Quadrupole Resonance

Suppose there is a thin quad error field k at $\phi = \phi_0$. Then Equation 37 becomes

$$\begin{aligned}\frac{d^2\xi}{d\phi^2} + \nu^2\xi &= -\nu^2\beta^{\frac{3}{2}}k'\beta^{1/2}\xi\delta(\phi - \phi_0) \\ &= -\nu^2\beta^2 k'\xi \sum_{n=-\infty}^{\infty} e^{i(n\phi - \phi_0)}\end{aligned}$$

In the limit where the error is small the perturbation expansion is

$$\begin{aligned}&= -A\nu^2\beta^2 k' \cos\nu\phi \sum_{n=-\infty}^{\infty} e^{-i\phi_0} e^{in\phi} \\ &= -A\nu^2\beta^2 k' \sum_{n=-\infty}^{\infty} e^{-i\phi_0} e^{i(n\phi + i\nu\phi)} + e^{i(n\phi - i\nu\phi)}\end{aligned}$$

The system will respond at the frequency of the driving force, which is $n\phi \pm \nu\phi$. Near resonance, that is when $(n \pm \nu) \sim \nu$ the solution is

$$\xi \sim \nu^2\beta^2 k' \frac{A}{\nu^2 \pm (n - \nu)^2}$$

with resonance at $\nu = n/2$, when the tune ν is half integer or integer.

12.2 Matrix Method

Recall the full turn map

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Insert a thin quadrupole with focal length f and matrix representation

$$Q = \begin{pmatrix} 1 & 0 \\ -1/f & 0 \end{pmatrix}$$

Then the perturbed full turn matrix is

$$M' = QM = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\frac{1}{f}(\cos\mu + \alpha\sin\mu) - \gamma\sin\mu & -\frac{1}{f}\beta\sin\mu + \cos\mu - \alpha\sin\mu \end{pmatrix}$$

The perturbed tune

$$\cos\mu' = \frac{1}{2}\left(2\cos\mu - \frac{\beta}{f}\sin\mu\right)$$

Stability requires that

$$|\cos\mu'| = |\cos\mu - \frac{\beta}{2f}\sin\mu| \leq 1$$

12.3 Sexupole resonance

A single sextupole with field proportional to $k_2 = \frac{\partial^2 B_y}{\partial y^2}\frac{1}{B\rho}$ scales quadratically with horizontal displacment leads to the equation of motion

$$\begin{aligned}
\frac{d^2\xi}{d\phi^2} + \nu^2\xi &= -\nu^2\beta^{\frac{5}{2}}k_2\xi^2\delta(\phi-\phi_0) \\
&= -\nu^2\beta^{\frac{5}{2}}k'\xi^2\sum_{n=-\infty}^{\infty}e^{i(n\phi-\phi_0)} \\
&\sim -A^2\nu^2\beta^{\frac{5}{2}}k'\cos^2\nu\phi\sum_{n=-\infty}^{\infty}e^{i(n\phi-\phi_0)} \\
&\sim -A^2\nu^2\beta^{\frac{5}{2}}k'\sum_{n=-\infty}^{\infty}e^{-i\phi_0}\left(e^{i(n\phi+2\phi\nu)}+e^{i(n\phi-2\phi\nu)}+e^{i(n\phi)}\right)
\end{aligned}$$

The system responds resonantly when $\nu^2-(\pm 2\nu+n)^2=0$ or $\nu^2\pm n^2=0$, or $\nu=\frac{1}{3}n, \nu=\frac{2}{3}n$, or $\nu=n$.

13 Hamiltonian dynamics

Hamilton's equations

$$\begin{aligned}
\dot{p} &= -\frac{\partial H}{\partial q} \\
\dot{q} &= \frac{\partial H}{\partial p}
\end{aligned}$$

13.1 Point transformation

Consider the generating function

$$F = F_3(p,Q,t) = \vec{p}\cdot(\rho\hat{x}+x\hat{x}+y\hat{y})$$

Then

$$q = -\frac{\partial F_3}{\partial p}, \qquad P = -\frac{\partial F_3}{\partial Q}$$

So

$$\begin{aligned}
P_s &= -\frac{\partial}{\partial s}F_3 = -\vec{p}\cdot\rho(1+\frac{x}{\rho})\frac{\partial\hat{x}}{\partial s} = -\vec{p}\cdot\left(1+\frac{x}{\rho}\right)\hat{s} = -p_s\left(1+\frac{x}{\rho}\right) \\
P_x &= -\frac{\partial}{\partial x} = -p_x \\
P_y &= -\frac{\partial}{\partial y} = -p_y
\end{aligned}$$

Define canonical vector potential

$$\begin{aligned}
A_s &= \vec{A}\cdot\hat{s}\left(1+\frac{x}{\rho}\right) \\
A_x &= \vec{A}\cdot\hat{x} \\
A_y &= \vec{A}\cdot\hat{y}
\end{aligned}$$

The new hamiltonian is just the original expressed in the new coordinates. The original hamiltonian is

$$H = \sqrt{(\vec{p} - e\vec{A})^2 c^4 + m^2 c^4} + eV$$

The new Hamiltonian ($z \to s$) is

$$H' = c \left[\frac{1}{\left(1 + \frac{x}{\rho}\right)^2} (p_s - eA_s)^2 + (p_x - eA_x)^2 + (p_y - eA_y)^2 + m^2 c^2 \right]^{\frac{1}{2}} + eV$$

13.2 Poincare invariants

Consider a 2-dimensional region of phase space and define the area of any surface in the space

$$J_1 = \sum_i \int_S \int dp_i dq_i$$

is invariant with respect to canonical transformations of the phase space variables q, p. That is

$$J_1 = \sum_i \int_S \int dp_i dq_i = \sum_i \int_S \int dP_i dQ_i \tag{38}$$

where P_i, Q_i are related to p_i, q_i by a canonical transformation. Any point on the surface can be identified by two coordinates u, v. Then on the surface $p_i(u, v)$, and $q_i(u, v)$. The area element $dp_i dq_i$ transforms to the area element $dudv$ by the Jacobian determinant

$$\frac{\partial(q_i, p_i)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial p_i}{\partial u} \\ \frac{\partial q_i}{\partial v} & \frac{\partial p_i}{\partial v} \end{vmatrix} \tag{39}$$

so that

$$dq_i dp_i = \frac{\partial(q_i, p_i)}{\partial(u, v)} dudv$$

Then Equation 38 becomes

$$J_1 = \sum_i \int_S \int \frac{\partial(q_i, p_i)}{\partial u, v} dudv = \sum_i \int_S \int \frac{\partial(Q_i, P_i)}{\partial u, v} dudv \tag{40}$$

Since the surface is arbitrary, Equation 40 and the invariance of J is equivalent to the statement that

$$\sum_i \begin{vmatrix} \frac{\partial q_i}{\partial u} & \frac{\partial q_i}{\partial v} \\ \frac{\partial p_i}{\partial u} & \frac{\partial p_i}{\partial v} \end{vmatrix} = \sum_i \begin{vmatrix} \frac{\partial Q_i}{\partial u} & \frac{\partial Q_i}{\partial v} \\ \frac{\partial P_i}{\partial u} & \frac{\partial P_i}{\partial v} \end{vmatrix} \tag{41}$$

where $(q, p) \to (Q, P)$ is a canonical transformation. (u, v) can be anything. If we choose $u \to q_j$ and $v \to p_j$, then Equation 41 becomes

$$\sum_i \begin{vmatrix} \frac{\partial q_i}{\partial q_j} & \frac{\partial q_i}{\partial p_j} \\ \frac{\partial p_i}{\partial q_j} & \frac{\partial p_i}{\partial p_j} \end{vmatrix} = \sum_i \begin{vmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{vmatrix} = \sum_i \delta_{ij} = 1 \tag{42}$$

Now let's suppose that q_i, p_i are the phase space coordinates of a particle at time t, and Q_i, P_i the coordinates at a later time t'. The two sets of coordinates are related by a canonical transformation. Therefore, the elements of the Jacobian matrix relating (q, p) to (Q, P) satisfy the relationship of Equation 42

There is one other invariance that we will need to demonstrate symplecticity of the Jacobian. It turns out that Equation 38 can be generalized for a 4-dimensional surface, or a 6-dimensional surface, etc. In particular the integral

$$J_2 = \sum_i \int \int_S \int \int dp_i dq_i dp_k dq_k \tag{43}$$

is invariant with respect to canonical transformations. Then as before we conclude that the Jacobian determinant

$$\frac{\partial(q_i, p_i, q_k, p_k)}{\partial u, v, w, z)} \tag{44}$$

is also invariant and that

$$\frac{\partial(Q_i, P_i, Q_k, P_k)}{\partial(q_i, p_i, q_k, p_k)} = 1 \tag{45}$$

14 Transfer Matrices

Consider a quadrupole magnet. The magnet is consists of 4 poles with alternating pole tip field. If the magnet is very long, then far from the ends the field is purely transverse. Expand the vertical component of the field in the horizontal plane.

$$B_y = B_y^0 + x\frac{\partial B_y}{\partial x} + \dots \tag{46}$$

By symmetry $B_x(y = 0) = 0$,

Consider a magnet with 4 poles with alternating pole tip field. Place the center of each pole on the diagonals. By symmetry

$$\begin{aligned} B_y(\theta) &= B_x(\theta + \pi/2) = -B_y(\theta + \pi) = -B_x(\theta + 3\pi/2) \\ B_x(\theta) &= B_y(\theta + \pi/2) = -B_x(\theta + \pi) = -B_y(\theta + 3\pi/2) \end{aligned}$$

In the two-dimensional limit, $B_z = 0$.

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \frac{\partial B_y}{\partial y} = -\frac{\partial B_x}{\partial x}$$
$$\nabla \times \mathbf{B} = 0 \rightarrow \frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x}$$

Then

$$\begin{aligned} B_y &= x\frac{\partial B_y}{\partial x} + \dots \\ B_x &= y\frac{\partial B_x}{\partial y} + \dots \end{aligned}$$

15 Quadrupole

The linear equations of motion for a magnet with quadrupole symmetry are

$$\begin{aligned}
\frac{\partial \vec{p}}{\partial t} &= e\vec{v} \times \vec{B} = \frac{e}{\gamma m}\vec{p} \times \vec{B} \\
\frac{\partial \vec{p}}{\partial s}\frac{\partial s}{\partial t} &= \frac{e}{\gamma m}\vec{p} \times \vec{B} \\
\frac{\partial \vec{p}'_{\perp}}{\partial s} v_z &= \frac{e}{\gamma m} p_z \vec{B}_{\perp}
\end{aligned}$$

The total momentum is constant and

$$\begin{aligned}
p_z &= \sqrt{p^2 - p_x^2 - p_y^2} \sim p - \frac{1}{2}\left(\frac{p_x}{p}\right)^2 - \frac{1}{2}\left(\frac{p_y}{p}\right)^2 \\
&= p - \frac{1}{2}x'^2 - \frac{1}{2}y'^2
\end{aligned}$$

Substituting into the above

$$\vec{x}' \frac{1}{\gamma m}(p - \frac{1}{2}x'^2 - \frac{1}{2}y'^2) = \frac{e}{\gamma m}(p - \frac{1}{2}x'^2 - \frac{1}{2}y'^2)\vec{B}_{\perp}$$

For small angles $\vec{x}' \ll 1$

$$\begin{aligned}
\frac{1}{p}\frac{\partial \vec{p}_{\perp}}{\partial s} v_z &= \frac{e}{\gamma m}\hat{s} \times \vec{B}_{\perp} \\
x''_{\perp} &= \frac{e}{p}\hat{s} \times \vec{B}_{\perp}
\end{aligned}$$

For the quadrupole

$$\begin{aligned}
x'' &= \frac{e}{p}\frac{\partial B_y}{\partial x} x = Kx \\
y'' &= \frac{e}{p}\frac{\partial B_x}{\partial y} y = -Ky
\end{aligned}$$

where $K = ep\frac{\partial B_y}{\partial x}$ and K has dimensions of L^{-2}. The solutions to the equations of motion are

$$\begin{aligned}
x &= A\cos\sqrt{K}s + B\sin\sqrt{K}s \\
y &= C\cos\sqrt{-K}s + D\sin\sqrt{-K}s
\end{aligned}$$

If $x(s=0) = x_0$ and $x'(s=0) = x'_0$ then $A = x_0$ and $B = x'_0/\sqrt{K}$ and similary $C = y_0$ and $D = y'_0/\sqrt{K}$. If $K > 0$ the quadrupole is horizontally focusing and vertically defocusing and The matrix for the transverse phase space becomes

$$M = \begin{pmatrix} \cos\sqrt{K}l & \frac{1}{\sqrt{K}}\sin\sqrt{K}l & 0 & 0 \\ -\sqrt{K}\sin\sqrt{K}l & \cos\sqrt{K}l & 0 & 0 \\ 0 & 0 & \cosh\sqrt{K}l & \frac{1}{\sqrt{K}}\sinh\sqrt{K}l \\ 0 & 0 & \sqrt{K}\sinh\sqrt{K}l & \cosh\sqrt{K}l \end{pmatrix} \tag{47}$$

and

$$M\begin{pmatrix}x_0\\x'_0\\y_0\\y'_0\end{pmatrix}=\begin{pmatrix}x(l)\\x'(l)\\y(l)\\y'(l)\end{pmatrix}$$

15.1 Tilted quadrupoles

The 4×4 transfer matrix for a horizontally focusing quad with strength k and length l can be written as **??**.

$$\mathbf{M}_{\text{quad}}=\left[\begin{pmatrix}\mathbf{K}_f & 0\\ 0 & \mathbf{K}_d\end{pmatrix}\right] \tag{48}$$

where $\mathbf{K}_{f,d}$ are appropriate 2×2 matrices. The matrix for a quad rotated by an angle θ about the z-axis is

$$\mathbf{Q}_{\text{rot}}=\mathbf{R}^{-1}(\theta)\mathbf{M}_{\text{quad}}\mathbf{R}(\theta) \tag{49}$$

where

$$\mathbf{R}(\theta)=\left[\begin{pmatrix}\mathbf{I}\cos\theta & \mathbf{I}\sin\theta\\ -\mathbf{I}\sin\theta & \mathbf{I}\cos\theta\end{pmatrix}\right] \tag{50}$$

15.2 Skew quadrupoles

A skew quad is rotated by $\theta=45°$,

$$\mathbf{Q}_{\text{skew}}=\frac{1}{2}\left[\begin{pmatrix}\mathbf{K}_f+\mathbf{K}_d & -\mathbf{K}_f+\mathbf{K}_d\\ -\mathbf{K}_f+\mathbf{K}_d & \mathbf{K}_f+\mathbf{K}_d\end{pmatrix}\right] \tag{51}$$

For a thin skew quad, $l\to0$ and $\sqrt{k}\sin(\sqrt{k}l)\to\frac{1}{f}$,

$$\mathbf{Q}_{\text{thin}}=\left[\begin{pmatrix}\mathbf{I} & \mathbf{K}_t\\ \mathbf{K}_t & \mathbf{I}\end{pmatrix}\right],\ K_t=\left[\begin{pmatrix}0 & 0\\ \frac{1}{f} & 0\end{pmatrix}\right] \tag{52}$$

16 Solenoids

16.1 Longitudinal fields

In the longitudinal field of a solenoid

$$\frac{1}{p}\frac{\partial\vec{p}_\perp}{\partial s}v_z = \frac{e}{\gamma m}\frac{p_\perp}{p}\hat{s}\times\vec{B}_z \tag{53}$$

$$x'' = y'\frac{e}{p}B_z=k_sy'$$

$$y'' = -x'\frac{e}{p}B_z=-k_sx' \tag{54}$$

The coupled equations are solved by substituting one into the other

$$x''' = -k_s^2x'$$

$$y''' = -k_s^2y'$$

The solutions are

$$\begin{aligned}
x' &= A\cos k_s l + B\sin k_s l \\
\rightarrow x &= \frac{1}{k_s}(-A\sin k_s l + B\cos k_s l) + a \\
y' &= C\cos k_s l + D\sin k_s l \\
\rightarrow y &= \frac{1}{k_s}(-C\sin k_s l + D\cos k_s l) + b
\end{aligned}$$

Substitution into 54 gives

$$\begin{aligned}
-k_s(A\sin k_s l - B\cos k_s l) &= k_s(C\cos k_s l + D\sin k_s l) \\
\rightarrow A &= -D, \quad B = C
\end{aligned}$$

The boundary conditions fix the constants.

$$\begin{aligned}
x_0' &= A, \quad x_0 = \frac{B}{k_s} + a \\
y_0' &= B, \quad y_0 = -\frac{A}{k_s} + b \\
\rightarrow A &= x_0', \; B = y_0', \quad a = x_0 - \frac{y_0'}{k_s}, \quad b = y_0 + \frac{x_0'}{k_s}
\end{aligned}$$

The transfer matrix for the motion through a longitudinal field is

$$\mathbf{M}_{\mathrm{long}} = \left[\begin{pmatrix} \mathbf{M}_1^s & \mathbf{M}_2^s \\ -\mathbf{M}_2^s & \mathbf{M}_1^s \end{pmatrix}\right] \tag{55}$$

where $k_s = \frac{e}{2pc}B_z$ and

$$\mathbf{M}_1^s = \left[\begin{pmatrix} 1 & \frac{1}{k_s}\sin k_s z \\ 0 & \cos k_s z \end{pmatrix}\right] \tag{56}$$

$$\mathbf{M}_2^s = \left[\begin{pmatrix} 0 & \frac{1}{k_s}(\cos k_s z - 1) \\ 0 & -\sin k_s z \end{pmatrix}\right] \tag{57}$$

16.2 Radial fringe

The fringe field of a solenoid is radial, of equal magnitude and opposite direction at each end. The elements of the transfer matrix for the radial fringe are

$$\begin{aligned}
(x|x_0) &= \cos\chi\cosh\chi \\
(x|x_0') &= \tfrac{1}{\sqrt{2k_r}}(\sin\chi\cosh\chi + \sinh\chi\cos\chi) \\
(x|y_0) &= \sin\chi\sinh\chi \\
(x|y_0') &= \tfrac{1}{\sqrt{2k_r}}(\sin\chi\cosh\chi - \sinh\chi\cos\chi) \\
(y|x_0) &= -(x|y_0) \\
(y|x_0') &= -(x|y_0') \\
(y|y_0) &= \cos\chi\cosh\chi \\
(y|y_0') &= \tfrac{1}{\sqrt{2k_r}}(\sin\chi\cosh\chi + \cos\chi\sinh\chi)
\end{aligned} \tag{58}$$

where $\chi = \sqrt{\frac{k_r}{2}}a$, $k_r = \frac{1}{2a}\frac{e}{pc}B_z$, and a is the length (along z) of the pole tips, that is, the effective length of the radial field. $[(x'|x_0), (x'|x'_0)$, etc. are obtained by differentiating $(x|x_0), (x|x'_0)$, etc. with respect to z.] In the limit of a thin radial fringe $a \to z \to 0$ and $k_r z = \frac{k_s z}{2a} \to \frac{k_s}{2}$, the transfer matrix becomes

$$\mathbf{M}_{\text{fringe}} = \left[\begin{pmatrix} \mathbf{I} & \mathbf{K}_s \\ -\mathbf{K}_s & \mathbf{I} \end{pmatrix}\right], \text{ where } \mathbf{K}_s = \left[\begin{pmatrix} 0 & 0 \\ \frac{k_s}{2} & 0 \end{pmatrix}\right] \tag{59}$$

16.3 Symplecticity of solenoid maps

Eqs.(55), (58) are not symplectic, but the solenoid matrix $\mathbf{M}_{\text{sol}} = \mathbf{M}_{\text{fringe}}\mathbf{M}_{\text{long}}\mathbf{M}_{\text{fringe}}^{-1}$ is symplectic. See Eq.(**??**), Sec.**??**.

Solenoid lens $\mathbf{M}_{\text{sol}}$ can also be written as a combination of rotations of an angle $\theta = k_s l/4$ and a thick lens that focuses in both planes with focusing strength $k = (k_s/2)^2$, where $k_s = \frac{e}{pc}B_z$ and l is the solenoid length,

$$\begin{aligned} \mathbf{M}_{\text{sol}} = &\quad \mathbf{R}\left(\frac{k_s}{4}l\right)\mathbf{F}\left(\frac{k_s^2}{4}, l\right)\mathbf{R}\left(\frac{k_s}{4}l\right) \\ \mathbf{F} = &\quad \left[\begin{pmatrix} \mathbf{K}_f(k,l) & 0 \\ 0 & \mathbf{K}_f(k,l) \end{pmatrix}\right] \end{aligned} \tag{60}$$

$\mathbf{K}_f$ is as in Eq.(48).

17 Superimposed solenoid and quadrupole fields

The matrices $\mathbf{M_{QL}}$ and $\mathbf{M_{QR}}$ are given for superimposed quadrupole and longitudinal fields and for superimposed quadrupole and radial fields. The elements of $\mathbf{M}_{QL}$ are

$$\begin{aligned} (x|x_0) &= -\frac{1}{fk_q}(g^+\theta_-^2 \cos\theta_+ z - g^-\theta_+^2 \cosh\theta_- z) \\ (x|x'_0) &= \frac{1}{fk_q}(-g^+\theta_- \sin\theta_+ z + g^-\theta_+ \sinh\theta_- z) \\ (x|y_0) &= \frac{k_q k_s}{f|k_q|}(-\theta_- \sin\theta_+ z + \theta_+ \sinh\theta_- z) \\ (x|y'_0) &= \frac{k_s}{f}(\cos\theta_+ z - \cosh\theta_- z) \\ (y|x_0) &= (x|y_0) \\ (y|x'_0) &= -(x|y'_0) \\ (y|y_0) &= \frac{1}{f}(g^- \cos\theta_+ z - g^+ \cosh\theta_- z) \\ (y|y'_0) &= \frac{1}{fk_q}(g^-\theta_+ \sin\theta_+ z + g^+\theta_- \sinh\theta_- z) \end{aligned} \tag{61}$$

The focusing strength of the quadrupole is k_q, $k_s = \frac{e}{pc}B_z$, $f = \sqrt{k_s^4 + 4k_q^2}$, $\theta_\pm = |\sqrt{\frac{1}{2}(k_s^2 \pm f)}|$, and $g^\pm = k_q - \frac{1}{2}(k_s^2 \pm f)$. Elements $(x'|x_0), (x'|x'_0)$, etc. are obtained by differentiating $(x|x_0), (x|x'_0)$, etc. with respect to z.

If $k_q^2 > k_r^2$ (where $k_r = k_s/(2a)$), elements of $\mathbf{M}_{QR}$ are

$$\begin{aligned}
(x|x_0) = &\ \tfrac{1}{h}(g^+ \cos\phi z - g^- \cosh\phi z) \\
(x|x_0') = &\ \tfrac{2\phi}{h^2}(g^+ \sin\phi z - g^- \sinh\phi z) \\
(x|y_0) = &\ \tfrac{k_r}{h}(\cosh\phi z - \cos\phi z) \\
(x|y_0') = &\ \tfrac{2k_r\phi}{h^2}(\sinh\phi z - \sin\phi z) \\
(y|x_0) = &\ -(x|y_0) \\
(y|x_0') = &\ -(x|y_0') \\
(y|y_0) = &\ \tfrac{1}{h}(g^+ \cosh\phi z - g^- \cos\phi z) \\
(y|y_0') = &\ \tfrac{2\phi}{h^2}(g^+ \sinh\phi z - g^- \sin\phi z)
\end{aligned} \tag{62}$$

If $k_r^2 > k_q^2$, then

$$\begin{aligned}
(x|x_0) = &\ \tfrac{2}{h}(\phi^2 \cos\zeta \cosh\zeta + k_q \sin\zeta \sinh\zeta) \\
(x|x_0') = &\ \tfrac{2\alpha}{h\phi^2}(-g^- \sin\zeta \cosh\zeta + g^+ \sinh\zeta \cos\zeta) \\
(x|y_0) = &\ \tfrac{2k_r}{h}(\sin\zeta \sinh\zeta) \\
(x|y_0') = &\ \tfrac{-2k_r\alpha}{h\phi^2}(\cos\zeta \sinh\zeta - \sin\zeta \cosh\zeta) \\
(y|x_0) = &\ -(x|y_0) \\
(y|x_0') = &\ -(x|y_0') \\
(y|y_0) = &\ \tfrac{2}{h}(\phi^2 \cos\zeta \cosh\zeta - k_q \sin\zeta \sinh\zeta) \\
(y|y_0') = &\ \tfrac{2\alpha}{h\phi^2}(g^+ \cosh\zeta \sin\zeta - g^- \sinh\zeta \cos\zeta)
\end{aligned} \tag{63}$$

where $\phi = |k_q^2 - k_r^2|^{\frac{1}{4}}$, $h = 2\phi^2$, $g^\pm = \pm\phi^2 - k_q$, $\alpha = \phi/\sqrt{2}$ and $\zeta = \alpha z$. Again $(x'|x_0), (x'|x_0')$, etc. are obtained by differentiating $(x'|x_0), (x'|x_0')$, etc. with respect to z. Matrices $\mathbf{M}_{QL}$ and $\mathbf{M}_{QR}$ are not symplectic, but the combination $\mathbf{M}_{QR}^{\text{thin}}\mathbf{M}_{QL}(\mathbf{M}_{QR}^{\text{thin}})^{-1}$ is symplectic in the limit of zero length fringe. ($\mathbf{M}_{QR}^{\text{thin}}$ is $\mathbf{M}_{QR}$ in the limit of zero length.) See Eq.(59).

18 Dipole

Sextupoles When there is a vertical closed orbit y_0 at a sextupole, the sextupole behaves as a skew quad with strength $k = \frac{2eB_0}{pcr_0^2}$, where B_0 and r_0 are the field and the radius at the pole tip. The sensitivity of the luminosity in e^+e^- colliders to the details of the vertical orbit is related to this property.

19 Cyclotron Equations of Motion

In cylindrical coordinates

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2}$$

$$\begin{aligned}
\frac{d\mathbf{r}}{dt} &= \frac{d}{dt} r\hat{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\hat{r}} = \dot{r}\hat{\mathbf{r}} + r\hat{\theta}\dot{\theta} \\
\frac{d^2\mathbf{r}}{dt^2} &= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\hat{r}} + \dot{r}\hat{\theta}\dot{\theta} + r\dot{\hat{\theta}}\dot{\theta} + r\hat{\theta}\ddot{\theta} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\hat{\theta}\dot{\theta} + \dot{r}\hat{\theta}\dot{\theta} - r\hat{\mathbf{r}}\dot{\theta}^2 + r\hat{\theta}\ddot{\theta} \\
&= \ddot{r}\hat{\mathbf{r}} + 2\dot{r}\hat{\theta}\dot{\theta} - r\hat{\mathbf{r}}\dot{\theta}^2 + r\hat{\theta}\ddot{\theta}
\end{aligned}$$

where we used $\dot{\hat{r}} = \dot{\theta}\hat{\theta}$ and $\dot{\hat{\theta}} = -\hat{\mathbf{r}}\dot{\theta}$.

The Lorentz force is

$$\mathbf{F} = q\left(\mathbf{v} \times \mathbf{B} + \mathbf{E}\right)$$

In a cyclotron with uniform vertical magnetic field and electrostatic focusing

$$F_r = -qv_s B + q\frac{\partial E_x}{\partial r}(r - r_0)$$

where $\frac{1}{r_0} = \frac{qB}{mv_s}$, and v_s is the velocity in the azimuthal direction. The radial force

$$F_r = -qv_s B + q\frac{\partial E_r}{\partial r}(r - r_0) = m(\ddot{r} - r\dot{\theta}^2) \tag{64}$$

Let $r = r_0 + x$ and Equation 64 becomes

$$\begin{aligned}
\ddot{x} - r\dot{\theta}^2 &= \frac{1}{m}(-qvB + q\frac{\partial E_r}{\partial r}x) \\
\ddot{x} &= \frac{1}{m}(-qv_s B + q\frac{\partial E_r}{\partial r}x) + r\left(\frac{v_s}{r}\right)^2 \\
\ddot{x} &= \frac{1}{m}(-qv_s B + q\frac{\partial E_r}{\partial r}x) + \frac{v_s^2}{x + r_0} \\
\ddot{x} &\sim -\frac{v_s^2}{r_0} + \frac{q}{m}\frac{\partial E_r}{\partial r}x + \frac{v_s^2}{r_0}(1 - \frac{x}{r_0}) \\
\ddot{x} &\sim -(\frac{v_s^2}{r_0^2} - \frac{q}{m}\frac{\partial E_r}{\partial r})x \\
\rightarrow \omega_x^2 &= \omega^2(1 - n) \\
Q_x &= \frac{\omega_x}{\omega} = \sqrt{1 - n}
\end{aligned}$$

where $\omega = \frac{qv_s B}{mr_0}$ and

$$n = \left(\frac{r_0}{v_s}\right)^2 \frac{q}{m}\frac{\partial E_r}{\partial r} = \frac{r_0}{v_s B}\frac{\partial E_r}{\partial r}.$$

The vertical force

$$\begin{aligned}
F_z &= q\frac{\partial E_z}{\partial z}z = m\ddot{z} \\
&\rightarrow \ddot{z} - \frac{q}{m}\frac{\partial E_z}{\partial z}z = 0 \\
&\rightarrow \omega_z^2 = -\frac{q}{m}\frac{\partial E_z}{\partial z}z
\end{aligned}$$

Since $\nabla \cdot \mathbf{E} = 0$, $\frac{\partial E_z}{\partial z} = -\frac{\partial E_r}{\partial r}$ and

$$\begin{aligned}
\omega_z^2 &= \frac{q}{m}\frac{\partial E_r}{\partial r} \\
&= \left(\frac{v_s}{r_0}\right)^2 n = \omega^2 n \\
\rightarrow Q_z &= \frac{\omega_z}{\omega} = \sqrt{n}
\end{aligned}$$