x-z Tilt Due to Dispersion in RF Cavities

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1 Constructing transport matrix in terms of twiss parameters and dispersion

First construct a matrix T_{12} from point 1 to 2 with $\vec{\eta}_1$ and $\vec{\eta}_2$ and no RF between 1 and 2. The horizontal and longitudinal motion is described in terms of β, α, ϕ. Next write the matrix for the RF cavity with frequency ω and voltage V. Compute the full turn at point 2. $T = T_{21}T_{RF}T_{12}$. The C_{11} element of the C matrix at point 1 will give the x-z coupling in terms of the dispersion $\vec{\eta}_2$ at the observation point and $\vec{\eta}_1$ at the cavity, and the phase advance from cavity (1) to observation point (2).

1.1 Matrix T_{12} from $\vec{\eta}_1$ to $\vec{\eta}_2$

We start with

$$T_{12} = \begin{pmatrix} M_{12} & m_{12} \\ n_{12} & N_{12} \end{pmatrix} \tag{1}$$

and then

$$T_{12} \begin{pmatrix} \eta_1 \\ \eta_1' \\ l \\ 1 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2' \\ l \\ 1 \end{pmatrix} \tag{2}$$

$$\rightarrow \quad M_{12} \begin{pmatrix} \eta_1 \\ \eta_1' \end{pmatrix} + m_{12} \begin{pmatrix} l \\ 1 \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2' \end{pmatrix} \tag{3}$$

The first column of m is zero because $\vec{x}$ is independent of l.

$$m_{12} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \tag{4}$$

Then from Equations 3 and 4 we get

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \eta_2 \\ \eta_2' \end{pmatrix} - M_{12} \begin{pmatrix} \eta_1 \\ \eta_1' \end{pmatrix}$$

or

$$m_{12} = \begin{pmatrix} 0 & \eta_2 \\ 0 & \eta_2' \end{pmatrix} - M_{12} \begin{pmatrix} 0 & \eta_1 \\ 0 & \eta_1' \end{pmatrix} \tag{5}$$

Both M_{12} and N_{12} can be written in terms of twiss parameters for horizontal and longitudinal motion at 1 and 2. We use the symplecticity of T_{12} to determine n_{12}. Since $T^T S T = S$ we can write

$$\begin{aligned}
\begin{pmatrix} M^T & n^T \\ m^T & N^T \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} M & m \\ n & N \end{pmatrix} &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \\
\begin{pmatrix} M^T & n^T \\ m^T & N^T \end{pmatrix} \begin{pmatrix} sM & sm \\ sn & sN \end{pmatrix} &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \\
\begin{pmatrix} M^T sM + n^T sn & M^T sm + n^T sN \\ m^T sM + N^T sn & m^T sm + N^T sN \end{pmatrix} &= \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \\
\rightarrow n^T sN &= -M^T sm \\
\rightarrow n^T &= M^T sm N^{-1} s
\end{aligned}$$

and

$$\rightarrow N^T sn = -m^T sM \tag{6}$$

$$\rightarrow n = s(N^T)^{-1} m^T sM \tag{7}$$

where in the last line we use $s^{-1} = -s$.

Now substitute m_{12} from Equation 4 for m in Equation 7 and we have

$$\begin{aligned}
\rightarrow n^T &= M^T s \left(\begin{pmatrix} 0 & \eta_2 \\ 0 & \eta_2' \end{pmatrix} - M \begin{pmatrix} 0 & \eta_1 \\ 0 & \eta_1' \end{pmatrix} \right) N^{-1} s \\
&= \left(M^T s \begin{pmatrix} 0 & \eta_2 \\ 0 & \eta_2' \end{pmatrix} - s \begin{pmatrix} 0 & \eta_1 \\ 0 & \eta_1' \end{pmatrix} \right) N^{-1} s
\end{aligned}$$

We can write M and N in terms of β and α at 1 and 2 and ϕ_{12}, the phase advance from one to the other. For simplicity let's suppose that $\alpha_1 = \alpha_2 = 0$ and that $\beta_1 = \beta_2$. Then

$$M = \begin{pmatrix} \cos\mu_x & \beta_x \sin\mu_x \\ \frac{-\sin\mu_x}{\beta_x} & \cos\mu_x \end{pmatrix} \tag{8}$$

and for N, $x \rightarrow z$. But we know that N has the form

$$N = \begin{pmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{pmatrix} \tag{9}$$

since as there are no RF cavities between points 1 and 2, there is no change to the energy. Then

$$
\begin{aligned}
\rightarrow n^T &= \left(\begin{pmatrix} \cos\mu_x & -\frac{\sin\mu_x}{\beta_x} \\ \beta_x\sin\mu_x & \cos\mu_x \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \eta_2 \\ 0 & \eta_2' \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \eta_1 \\ 0 & \eta_1' \end{pmatrix}\right)\begin{pmatrix} 1 & -\alpha_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \left(\begin{pmatrix} \frac{\sin\mu_x}{\beta_x} & \cos\mu_x \\ -\cos\mu_x & \beta_x\sin\mu_x \end{pmatrix}\begin{pmatrix} 0 & \eta_2 \\ 0 & \eta_2' \end{pmatrix} - \begin{pmatrix} 0 & \eta_1' \\ 0 & -\eta_1 \end{pmatrix}\right)\begin{pmatrix} \alpha_{12} & 1 \\ -1 & 0 \end{pmatrix} \qquad (10) \\
&= \left(\begin{pmatrix} 0 & \frac{\sin\mu_x}{\beta_x}\eta_2 + \cos\mu_x\eta_2' \\ 0 & -\cos\mu_x\eta_2 + \beta_x\sin\mu_x\eta_2' \end{pmatrix} - \begin{pmatrix} 0 & \eta_1' \\ 0 & -\eta_1 \end{pmatrix}\right)\begin{pmatrix} \alpha_{12} & 1 \\ -1 & 0 \end{pmatrix} \qquad (11) \\
&= \begin{pmatrix} 0 & \frac{\sin\mu_x}{\beta_x}\eta_2 + \cos\mu_x\eta_2' - \eta_1' \\ 0 & -\cos\mu_x\eta_2 + \beta_x\sin\mu_x\eta_2' + \eta_1 \end{pmatrix}\begin{pmatrix} \alpha_{12} & 1 \\ -1 & 0 \end{pmatrix} \qquad (12) \\
&= \begin{pmatrix} -\frac{\sin\mu_x}{\beta_x}\eta_2 - \cos\mu_x\eta_2' + \eta_1' & 0 \\ \cos\mu_x\eta_2 - \beta_x\sin\mu_x\eta_2' - \eta_1 & 0 \end{pmatrix} \qquad (13) \\
\rightarrow \quad n &= \begin{pmatrix} -\frac{\sin\mu_x}{\beta_x}\eta_2 - \cos\mu_x\eta_2' + \eta_1' & \cos\mu_x\eta_2 - \beta_x\sin\mu_x\eta_2' - \eta_1 \\ 0 & 0 \end{pmatrix} \qquad (14)
\end{aligned}
$$

2 RF cavity

The matrix for an RF cavity is

$$N_{RF} = \begin{pmatrix} 1 & 0 \\ e\frac{L\omega}{c}V & 1 \end{pmatrix} \qquad (15)$$

The full turn matrix for the longitudinal motion is

$$N = \begin{pmatrix} 1 & \alpha_p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \tilde{V} & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha_p\tilde{V} & L\alpha_p \\ \tilde{V} & 1 \end{pmatrix} \qquad (16)$$

Then $\cos\mu_z = 1 + \frac{1}{2}\frac{\alpha_p\omega VL}{cE_{beam}}$ and $\mu_z \sim \sqrt{\frac{\alpha_p\omega VL}{cE_{beam}}}$. And

$$
\begin{aligned}
\beta_z\sin\mu_z &= L\alpha_p \\
\rightarrow \beta_z &= \frac{L\alpha_p}{\mu_z}
\end{aligned}
$$

We can evaluate the full turn matrix at an arbitrary point

$$
\begin{aligned}
T_{arb} &= \begin{pmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \tilde{V} & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha_{21} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1+\tilde{V}\alpha_{12} & \alpha_{12} \\ \tilde{V} & 1 \end{pmatrix}\begin{pmatrix} 1 & \alpha_{21} \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1+\tilde{V}\alpha_{12} & (1+\tilde{V}\alpha_{12})\alpha_{21} + \alpha_{12} \\ \tilde{V} & \tilde{V}\alpha_{21} + 1 \end{pmatrix}
\end{aligned}
$$

Since $\alpha_{12} + \alpha_{21} = \alpha$, we see that to first order in α, β_z is about the same.

3 Full turn

$$\begin{aligned}
T &= \begin{pmatrix} M_{12} & m_{12} \\ n_{12} & N_{12} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & N_{RF} \end{pmatrix} \begin{pmatrix} M_{21} & m_{21} \\ n_{21} & N_{21} \end{pmatrix} \\
&= \begin{pmatrix} M_{12} & m_{12} \\ n_{12} & N_{12} \end{pmatrix} \begin{pmatrix} M_{21} & m_{21} \\ N_{RF} n_{21} & N_{RF} N_{21} \end{pmatrix} \\
\begin{pmatrix} M & m \\ n & N \end{pmatrix} &= \begin{pmatrix} M_{12}M_{21} + m_{12}N_{RF}n_{21} & M_{12}m_{21} + m_{12}N_{RF}N_{21} \\ n_{12}M_{21} + N_{12}N_{RF}n_{21} & n_{12}m_{21} + N_{12}N_{RF}N_{21} \end{pmatrix}
\end{aligned}$$

4 C-matrix

The coupling matrix

$$\begin{aligned}
C \;\propto\;& m + n^\dagger = M_{12}m_{21} + m_{12}N_{RF}N_{21} + (n_{12}M_{21} + N_{12}N_{RF}n_{21})^\dagger \\
=\;& \begin{pmatrix} \cos\mu_{12} & \beta_x \sin\mu_{12} \\ -\frac{\sin\mu_{12}}{\beta_x} & \cos\mu_{12} \end{pmatrix} \begin{pmatrix} 0 & a_{21} \\ 0 & b_{21} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ 0 & b_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{V} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_{21} \\ 0 & 1 \end{pmatrix} \\
&+ \left(\begin{pmatrix} c_{12} & d_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\mu_{21} & \beta_x \sin\mu_{21} \\ -\frac{\sin\mu_{21}}{\beta_x} & \cos\mu_{21} \end{pmatrix} + \begin{pmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{V} & 1 \end{pmatrix} \begin{pmatrix} c_{21} & d_{21} \\ 0 & 0 \end{pmatrix} \right)^\dagger \\
=\;& \begin{pmatrix} 0 & a_{21}\cos\mu_{12} + b_{21}\beta_x \sin\mu_{12} \\ 0 & -a_{21}\frac{\sin\mu_{12}}{\beta_x} + b_{21}\cos\mu_{12} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ 0 & b_{12} \end{pmatrix} \begin{pmatrix} 1 & \alpha_{12} \\ \tilde{V} & \tilde{V}\alpha_{21} + 1 \end{pmatrix} \\
&+ \left(\begin{pmatrix} c_{12}\cos\mu_{21} - d_{12}\frac{\sin\mu_{21}}{\beta_x} & c_{12}\beta_x \sin\mu_{21} + d_{12}\cos\mu_{21} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{21} & d_{21} \\ \tilde{V}c_{21} & \tilde{V}d_{21} \end{pmatrix} \right)^\dagger \\
=\;& \begin{pmatrix} 0 & a_{21}\cos\mu_{12} + b_{21}\beta_x \sin\mu_{12} \\ 0 & -a_{21}\frac{\sin\mu_{12}}{\beta_x} + b_{21}\cos\mu_{12} \end{pmatrix} + \begin{pmatrix} a_{12}\tilde{V} & a_{12}(\tilde{V}\alpha_{21} + 1) \\ b_{12}\tilde{V} & b_{12}(\tilde{V}\alpha_{21} + 1) \end{pmatrix} \\
&+ \left(\begin{pmatrix} c_{12}\cos\mu_{21} - d_{12}\frac{\sin\mu_{21}}{\beta_x} & c_{12}\beta_x \sin\mu_{21} + d_{12}\cos\mu_{21} \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_{21} & d_{21} \\ \tilde{V}c_{21} & \tilde{V}d_{21} \end{pmatrix} \right)^\dagger \\
=\;& \begin{pmatrix} a_{12}\tilde{V} & a_{21}\cos\mu_{12} + b_{21}\beta_x \sin\mu_{12} + a_{12}(\tilde{V}\alpha_{21} + 1) \\ b_{12}\tilde{V} & -a_{21}\frac{\sin\mu_{12}}{\beta_x} + b_{21}\cos\mu_{12} + b_{12}(\tilde{V}\alpha_{21} + 1) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & -c_{12}\beta_x \sin\mu_{21} - d_{12}\cos\mu_{21} \\ 0 & c_{12}\cos\mu_{21} - d_{12}\frac{\sin\mu_{21}}{\beta_x} \end{pmatrix} + \begin{pmatrix} c_{12} + \alpha_{12}\tilde{V}c_{21} & d_{12} + \alpha_{12}\tilde{V}d_{21} \\ \tilde{V}c_{12} & \tilde{V}d_{21} \end{pmatrix}^\dagger \\
=\;& \begin{pmatrix} a_{12}\tilde{V} & a_{21}\cos\mu_{12} + b_{21}\beta_x \sin\mu_{12} + a_{12}(\tilde{V}\alpha_{21} + 1) \\ b_{12}\tilde{V} & -a_{21}\frac{\sin\mu_{12}}{\beta_x} + b_{21}\cos\mu_{12} + b_{12}(\tilde{V}\alpha_{21} + 1) \end{pmatrix} \\
&+ \begin{pmatrix} 0 & -c_{12}\beta_x \sin\mu_{21} - d_{12}\cos\mu_{21} \\ 0 & c_{12}\cos\mu_{21} - d_{12}\frac{\sin\mu_{21}}{\beta_x} \end{pmatrix} + \begin{pmatrix} \tilde{V}d_{21} & -d_{12} - \alpha_{12}\tilde{V}d_{21} \\ -\tilde{V}c_{12} & c_{12} + \alpha_{12}\tilde{V}c_{21} \end{pmatrix}
\end{aligned}$$

Finally just picking out C_{11}

$$C_{11} = (a_{12} + d_{21})\tilde{V} \tag{17}$$

We get a_{12} from Equation 5 and d_{21} from Equation 14, where we are again assuming $\beta_1 = \beta_2$ and $\alpha_1 = \alpha_2 = 0$.

$$\begin{aligned} a_{12} &= \eta_2 - \cos\mu_{12}\eta_1 - \beta_x \sin\mu_{12}\eta_1' \\ d_{21} &= \cos\mu_{21}\eta_1 - \beta_x \sin\mu_{21}\eta_1' - \eta_2 \end{aligned}$$

and

$$C_{11} = \big((\cos\mu_{21} - \cos\mu_{12})\eta_1 - \beta_x\eta_1'(\sin\mu_{12} + \sin\mu_{21})\big)\,\tilde{V} \tag{18}$$

5 $\beta_1 \neq \beta_2$

Return to Equation 5 to get a_{12}

$$\begin{aligned} m_{12} &= \begin{pmatrix} 0 & \eta_2 \\ 0 & \eta_2' \end{pmatrix} - M_{12}\begin{pmatrix} 0 & \eta_1 \\ 0 & \eta_1' \end{pmatrix} \\ m_{12} &= \begin{pmatrix} 0 & \eta_2 \\ 0 & \eta_2' \end{pmatrix} - \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}\cos\mu & \sqrt{\beta_1\beta_2}\sin\mu \\ -\frac{\sin\mu}{\beta_1\beta_2} & \sqrt{\frac{\beta_1}{\beta_2}}\cos\mu \end{pmatrix}\begin{pmatrix} 0 & \eta_1 \\ 0 & \eta_1' \end{pmatrix} \\ \rightarrow a_{12} &= \eta_2 - \sqrt{\frac{\beta_2}{\beta_1}}\cos\mu_{12}\eta_1 - \sqrt{\beta_1\beta_2}\sin\mu_{12}\eta_1' \end{aligned}$$

and then to Equaton 14 to get d_{12}

$$d_{21} = \sqrt{\frac{\beta_2}{\beta_1}}\cos\mu_{21}\eta_1 - \sqrt{\beta_1\beta_2}\sin\mu_{21}\eta' - \eta_2 \tag{19}$$

Then

$$C_{11} = \left(\sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu_{21} - \cos\mu_{12})\eta_1 - \sqrt{\beta_1\beta_2}\eta_1'(\sin\mu_{12} + \sin\mu_{21})\right)\tilde{V} \tag{20}$$

6 Both RF Cavities

East and west RF cavities are approximately symmetric with respect to the IP($\mu = 0$). The C_{11} element of each is

$$\begin{aligned} C_{11}^W &= \left(\sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu_{21} - \cos\mu_{12})\eta_1 - \sqrt{\beta_1\beta_2}\eta_1'(\sin\mu_{12} + \sin\mu_{21})\right)\tilde{V} \\ C_{11}^E &\propto \left(\sqrt{\frac{\beta_2}{\beta_1}}(\cos(\mu_{21} - \Delta\mu) - \cos(\mu_{12} + \Delta\mu)\eta_1 + \sqrt{\beta_1\beta_2}\eta_1'(\sin(\mu_{12} + \Delta\mu) + \sin(\mu_{21} - \Delta\mu)\right)\tilde{V} \end{aligned}$$

where $\Delta\mu$ is the phase advance between the cavities along the path that does not include point 2, and $\eta'_E = -\eta'_W$. Then

$$C_{11}(2) \propto \left(\sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu_{21} - \cos\mu_{12} + \cos(\mu_{21} - \Delta\mu) - \cos(\mu_{12} + \Delta\mu))\eta_1 - \sqrt{\beta_1\beta_2}\eta'_1(\sin\mu_{12} + \sin\mu_{21} - \sin(\mu_{12} + \Delta\mu) - \sin(\mu_{21} - \Delta\mu)\right)\tilde{V}$$