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Calculating Jacobian and Closed Orbit

The goal is to compute the closed orbit and the Jacobian of the mapping of the phase space vector with respect to that closed orbit, by tracking.

Suppose that

$$\vec{x}^{out}(\vec{x}^{in})$$

represents the mapping from the initial phase space coordinates $\vec{x}^{in}$, (where $\vec{x} = (x, x', y, y')$) to the final phase space coordinates $\vec{x}^{out}$. Then the Jacobian

$$J_{ij} = \frac{\partial x_i^{out}}{\partial x_j^{in}} \tag{1}$$

Remember that $\vec{x}^{in}$ and $\vec{x}^{out}$ are with respect to the closed orbit. The Jacobian is equivalent to the transfer matrix R about the closed orbit. In general, five independent trajectories are required to determine the closed orbit and the Jacobian. Given initial phase space coordinates z^{in}, tracking yields final coordinates $\vec{z}^{out}$ and we can write

$$R(\vec{z}^{in} - \vec{z}^{0}) = \vec{z}^{out} - \vec{z}^{0} \tag{2}$$

where $\vec{z}^0$ is the phase space vector of the closed orbit.

Construct a 5 vector

$$\vec{W} = \vec{z}, 1 = \begin{pmatrix} x \\ x' \\ y \\ y' \\ 1 \end{pmatrix}$$

and then a 5X5 matrix of the 5 vectors $\vec{W}$.

$$\mathbf{X} = \vec{W}_1\vec{W}_2\vec{W}_3\vec{W}_4\vec{W}_5 = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_1' & x_2' & x_3' & x_4' & x_5' \\ y_1 & y_2 & y_3 & y_4 & y_5 \\ y_1' & y_2' & y_3' & y_4' & y_5' \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Define the 5X5 matrix $\mathbf{T}$

$$\mathbf{T} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & v_1 \\ R_{21} & R_{22} & R_{23} & R_{24} & v_2 \\ R_{31} & R_{32} & R_{33} & R_{34} & v_3 \\ R_{41} & R_{42} & R_{43} & R_{44} & v_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where $\vec{v} = -\mathbf{R}\vec{z}_0 + \vec{z}_0 = (\mathbf{I} - \mathbf{R})\vec{z}_0$. Now we can rewrite Equation 2 for all five trajectories at once as the matrix equation

$$\mathbf{T}\mathbf{X}^{\mathbf{in}} = \mathbf{X}^{\mathbf{out}} \tag{4}$$

We know $\mathbf{X}^{in}$ and $\mathbf{X}^{out}$. $\mathbf{X}^{in}$ are our choice of 5 starting phase space vectors and $\mathbf{X}^{out}$ are determined by tracking each of the five through the machine. Finally solve for

$$\mathbf{T} = \mathbf{X}^{\mathbf{out}}(\mathbf{X}^{\mathbf{in}})^{\mathbf{-1}}$$

Then we can refer to the definition of $\mathbf{T}$ in Equation 3 to extract $\mathbf{R}$ and then the closed orbit

$$\vec{z}_0 = (\mathbf{I} - \mathbf{R})^{-1}\vec{v}$$

Each row of the 4X5 matrix in subroutine get_init_vec, (matr), contains the starting phase space coordinates of one of the 5 trajectories. They are evenly spaced around the beam ellipse at 72 degree intervals with some attempt to mix up horizontal and vertical. In principle any 5 starting points will do as long as the matrices $\mathbf{X}^{\mathrm{in}}$ and $\mathbf{X}^{\mathrm{out}}$ are non singular. In the remainder of the routine, the length of the vectors is adjusted to enable the calculation of the amplitude dependence of the jacobian.