0.1 Measurement and Diagnosis of Coupling and Solenoid Compensation

D. Rubin, Cornell U.

In e^+e^- storage ring colliders, luminosity is inversely proportional to the vertical dimension of the ribbonlike beams. The most important source of vertical beam size is from its coupling to horizontal and longitudinal motion.

0.1.1 Sources of transverse coupling

Tilted quadrupoles The 4×4 transfer matrix for a horizontally focusing quad with strength k and length l can be written as (Sec.**??**)

$$\mathbf{M}_{\text{quad}} = \begin{bmatrix} \mathbf{K}_f & 0 \\ 0 & \mathbf{K}_d \end{bmatrix} \tag{1}$$

where $\mathbf{K}_{f,d}$ are appropriate 2×2 matrices. The matrix for a quad rotated about an angle θ is

$$\mathbf{Q}_{\text{rot}} = \mathbf{R}^{-1}(\theta)\mathbf{M}_{\text{quad}}\mathbf{R}(\theta) \tag{2}$$

where

$$\mathbf{R}(\theta) = \begin{bmatrix} \mathbf{I}\cos\theta & \mathbf{I}\sin\theta \\ -\mathbf{I}\sin\theta & \mathbf{I}\cos\theta \end{bmatrix} \tag{3}$$

Skew quadrupoles A skew quad is a quad rotated by 45°,

$$\mathbf{Q}_{\text{skew}} = \frac{1}{2}\begin{bmatrix} \mathbf{K}_f + \mathbf{K}_d & -\mathbf{K}_f + \mathbf{K}_d \\ -\mathbf{K}_f + \mathbf{K}_d & \mathbf{K}_f + \mathbf{K}_d \end{bmatrix} \tag{4}$$

For a thin skew quad, $l \to 0$ and $\sqrt{k}\sin(\sqrt{k}l) \to \frac{1}{f}$,

$$\mathbf{Q}_{\text{thin}} = \begin{bmatrix} \mathbf{I} & \mathbf{K}_t \\ \mathbf{K}_t & \mathbf{I} \end{bmatrix}, \quad K_t = \begin{bmatrix} 0 & 0 \\ \frac{1}{f} & 0 \end{bmatrix} \tag{5}$$

0.1.2 Solenoids

Longitudinal fields The transfer matrix for the motion through a longitudinal field is

$$\mathbf{M}_{\text{long}} = \begin{bmatrix} \mathbf{M}_1^s & \mathbf{M}_2^s \\ -\mathbf{M}_2^s & \mathbf{M}_1^s \end{bmatrix} \tag{6}$$

where $k_s = \frac{e}{pc} B_z$ and

$$\mathbf{M}_1^s = \begin{bmatrix} 1 & \frac{1}{k_s}\sin k_s z \\ 0 & \cos k_s z \end{bmatrix} \tag{7}$$

$$\mathbf{M}_2^s = \begin{bmatrix} 0 & \frac{1}{k_s}(\cos k_s z - 1) \\ 0 & -\sin k_s z \end{bmatrix} \tag{8}$$

Radial fringe The fringe field of a solenoid is radial, of equal magnitude and opposite direction at each end. The elements of the transfer matrix for the radial fringe are

$$\begin{aligned}
(x|x_0) &= \cos\chi\cosh\chi \\
(x|x'_0) &= \tfrac{1}{\sqrt{2k_r}}(\sin\chi\cosh\chi + \sinh\chi\cos\chi) \\
(x|y_0) &= \sin\chi\sinh\chi \\
(x|y'_0) &= \tfrac{1}{\sqrt{2k_r}}(\sin\chi\cosh\chi - \sinh\chi\cos\chi) \\
(y|x_0) &= -(x|y_0) \\
(y|x'_0) &= -(x|y'_0) \\
(y|y_0) &= \cos\chi\cosh\chi \\
(y|y'_0) &= \tfrac{1}{\sqrt{2k_r}}(\sin\chi\cosh\chi + \cos\chi\sinh\chi)
\end{aligned} \tag{9}$$

where $\chi = \sqrt{\frac{k_r}{2}}z$, $k_r = \frac{1}{2a}\frac{e}{pc}B_z$, and a is the length (along z) of the pole tips. [$(x'|x_0), (x'|x'_0)$, etc. are obtained by differentiating $(x|x_0), (x|x'_0)$, etc. with respect to z.] In the limit of a thin radial fringe $a \to z \to 0$ and $k_r z = \frac{k_s z}{2a} \to \frac{k_s}{2}$, the transfer matrix becomes

$$\mathbf{M}_{\text{fringe}} = \begin{bmatrix} \mathbf{I} & \mathbf{K}_s \\ -\mathbf{K}_s & \mathbf{I} \end{bmatrix}, \text{ where } \mathbf{K}_s = \begin{bmatrix} 0 & 0 \\ \frac{k_s}{2} & 0 \end{bmatrix} \tag{10}$$

Symplecticity of solenoid maps Eqs.(**??**), (**??**) are not symplectic, but the solenoid matrix $\mathbf{M}_{\text{sol}} = \mathbf{M}_{\text{fringe}}\mathbf{M}_{\text{long}}\mathbf{M}_{\text{fringe}}^{-1}$ is symplectic. See Eq.(**??**), Sec.**??**.

Solenoid lens $\mathbf{M}_{\text{sol}}$ can also be written as a combination of rotations of an angle $\theta = k_s l/4$ and a thick lens that focuses in both planes with focusing strength $k = (k_s/2)^2$, where $k_s = \frac{e}{pc}B_z$ and l is the solenoid length,

$$\mathbf{M}_{\text{sol}} = \mathbf{R}\left(\tfrac{k_s}{4}l\right)\mathbf{F}\left(\tfrac{k_s^2}{4}, l\right)\mathbf{R}\left(\tfrac{k_s}{4}l\right) \tag{11}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{K}_f(k,l) & 0 \\ 0 & \mathbf{K}_f(k,l) \end{bmatrix}$$

$\mathbf{K}_f$ is as in Eq.(**??**).

Superimposed solenoid and quadrupole fields The matrices $\mathbf{M_{QL}}$ and $\mathbf{M_{QR}}$ are given for superimposed quadrupole and longitudinal fields and for superimposed quadrupole and radial fields. The elements of $\mathbf{M}_{QL}$ are

$$\begin{aligned}
(x|x_0) = &\quad -\tfrac{1}{fk_q}(g^+\theta_-^2\cos\theta_+ z - g^-\theta_+^2\cosh\theta_- z) \\
(x|x_0') = &\quad \tfrac{1}{fk_q}(-g^+\theta_-\sin\theta_+ z + g^-\theta_+\sinh\theta_- z) \\
(x|y_0) = &\quad \tfrac{k_q k_s}{f|k_q|}(-\theta_-\sin\theta_+ z + \theta_+\sinh\theta_- z) \\
(x|y_0') = &\quad \tfrac{k_s}{f}(\cos\theta_+ z - \cosh\theta_- z) \\
(y|x_0) = &\quad (x|y_0) \\
(y|x_0') = &\quad -(x|y_0') \\
(y|y_0) = &\quad \tfrac{1}{f}(g^-\cos\theta_+ z - g^+\cosh\theta_- z) \\
(y|y_0') = &\quad \tfrac{1}{fk_q}(g^-\theta_+\sin\theta_+ z + g^+\theta_-\sinh\theta_- z)
\end{aligned} \tag{12}$$

The focusing strength of the quadrupole is k_q, $k_s = \frac{e}{pc}B_z$, $f = \sqrt{k_s^4 + 4k_q^2}$, $\theta_\pm = |\sqrt{\frac{1}{2}(k_s^2 \pm f)}|$, and $g^\pm = k_q - \frac{1}{2}(k_s^2 \pm f)$. Elements $(x'|x_0), (x'|x_0')$, etc. are obtained by differentiating $(x|x_0), (x|x_0')$, etc. with respect to z.

If $k_q^2 > k_r^2$ (where $k_r = k_s/(2a)$), elements of $\mathbf{M}_{QR}$ are

$$\begin{aligned}
(x|x_0) = &\quad \tfrac{1}{h}(g^+\cos\phi z - g^-\cosh\phi z) \\
(x|x_0') = &\quad \tfrac{2\phi}{h^2}(g^+\sin\phi z - g^-\sinh\phi z) \\
(x|y_0) = &\quad \tfrac{k_r}{h}(\cosh\phi z - \cos\phi z) \\
(x|y_0') = &\quad \tfrac{2k_r\phi}{h^2}(\sinh\phi z - \sin\phi z) \\
(y|x_0) = &\quad -(x|y_0) \\
(y|x_0') = &\quad -(x|y_0') \\
(y|y_0) = &\quad \tfrac{1}{h}(g^+\cosh\phi z - g^-\cos\phi z) \\
(y|y_0') = &\quad \tfrac{2\phi}{h^2}(g^+\sinh\phi z - g^-\sin\phi z)
\end{aligned} \tag{13}$$

If $k_r^2 > k_q^2$, then

$$\begin{aligned}
(x|x_0) = &\quad \tfrac{2}{h}(\phi^2\cos\zeta\cosh\zeta + k_q\sin\zeta\sinh\zeta) \\
(x|x_0') = &\quad \tfrac{2\alpha}{h\phi^2}(-g^-\sin\zeta\cosh\zeta + g^+\sinh\zeta\cos\zeta)
\end{aligned}$$

$$
\begin{aligned}
(x|y_0) &= \frac{2k_r}{h}(\sin\zeta\sinh\zeta) \\
(x|y_0') &= \frac{-2k_r\alpha}{h\phi^2}(\cos\zeta\sinh\zeta - \sin\zeta\cosh\zeta) \\
(y|x_0) &= -(x|y_0) \\
(y|x_0') &= -(x|y_0') \\
(y|y_0) &= \frac{2}{h}(\phi^2\cos\zeta\cosh\zeta - k_q\sin\zeta\sinh\zeta) \\
(y|y_0') &= \frac{2\alpha}{h\phi^2}(g^+\cosh\zeta\sin\zeta - g^-\sinh\zeta\cos\zeta)
\end{aligned} \tag{14}
$$

where $\phi = |k_q^2 - k_r^2|^{\frac{1}{4}}$, $h = 2\phi^2$, $g^\pm = \pm\phi^2 - k_q$, $\alpha = \phi/\sqrt{2}$ and $\zeta = \alpha z$. Again $(x'|x_0), (x'|x_0')$, etc. are obtained by differentiating $(x'|x_0), (x'|x_0')$, etc. with respect to z. Matrices $\mathbf{M}_{QL}$ and $\mathbf{M}_{QR}$ are not symplectic, but the combination $\mathbf{M}_{QR}^{\text{thin}}\mathbf{M}_{QL}(\mathbf{M}_{QR}^{\text{thin}})^{-1}$ is symplectic. $\mathbf{M}_{QR}^{\text{thin}}$ is $\mathbf{M}_{QR}$ in the limit of zero length fringe. See Eq.(**??**).

Sextupoles When there is a vertical closed orbit y_0 at a sextupole, the sextupole behaves as a skew quad with strength $k = \frac{2eB_0}{pcr_0^2}$, where B_0 and r_0 are the field and the radius at the pole tip. The sensitivity of the luminosity in e^+e^- colliders to the details of the vertical orbit is related to this property.

0.1.3 Coupling matrix analysis

Normal modes The one-turn transfer matrix $\mathbf{T}$ is decomposed into normal modes as [**?**, **?**]

$$\mathbf{T} = \begin{bmatrix} \mathbf{M} & \mathbf{n} \\ \mathbf{m} & \mathbf{N} \end{bmatrix} = \mathbf{V}\mathbf{U}\mathbf{V}^{-1} \tag{15}$$

$$\text{where}\quad \mathbf{U} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix},\ \mathbf{V} = \begin{bmatrix} \gamma\mathbf{I} & \mathbf{C} \\ -\mathbf{C}^\dagger & \gamma\mathbf{I} \end{bmatrix} \tag{16}$$

and $\gamma^2 + \det\mathbf{C} = 1$. $\mathbf{A}$ and $\mathbf{B}$ are the one-turn matrices for each of the two normal modes. Symplectic conjugate $\mathbf{C}^\dagger = -\mathbf{S}\mathbf{C}^t\mathbf{S}$ (see also Sec.**??**). The laboratory phase space coordinates $\mathbf{x} = (x, x', y, y')$ are related to the normal mode coordinates $\mathbf{w} = (w, w', v, v')$ by $\mathbf{x} = \mathbf{V}\mathbf{w}$. The same relation holds for energy displacements so the normal mode dispersions may be calculated from $(D_u, D_u', D_v, D_v')^t = \mathbf{V}^{-1}(D_x, D_x', D_y, D_y')^t$. Given the normal mode Courant-Snyder parameters and dispersions the normal mode emittances can be calculated.

Phase space normalization of the normal mode vectors yields

$$\bar{\mathbf{w}} = \mathbf{G}\mathbf{w} = \begin{bmatrix} \mathbf{G}_A & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_B \end{bmatrix}\mathbf{w} \tag{17}$$

$$\mathbf{G}_{A,B} = \begin{bmatrix} \frac{1}{\sqrt{\beta_{A,B}}} & 0 \\ \frac{\alpha_{A,B}}{\sqrt{\beta_{A,B}}} & \sqrt{\beta_{A,B}} \end{bmatrix}$$

We define

$$\bar{\mathbf{V}} \equiv \mathbf{G}\mathbf{V}\mathbf{G}^{-1} = \begin{bmatrix} \gamma\mathbf{I} & \bar{\mathbf{C}} \\ -\bar{\mathbf{C}}^{\dagger} & \gamma\mathbf{I} \end{bmatrix} \tag{18}$$

Propagating the coupling matrix Elements of the coupling matrix can be propagated through a lattice given the intervening transfer matrices. Consider the one-turn 4×4 matrix at s_1 with $\mathbf{T}_1 = \mathbf{V}_1\mathbf{U}_1\mathbf{V}_1^{-1}$. At another point s_2, we have $\mathbf{T}_2 = \mathbf{V}_2\mathbf{U}_2\mathbf{V}_2^{-1}$. Let $\mathbf{T}_{12} = \begin{bmatrix} \mathbf{M}_{12} & \mathbf{m}_{12} \\ \mathbf{n}_{12} & \mathbf{N}_{12} \end{bmatrix}$ be the matrix that propagates from s_1 to s_2, then

$$\mathbf{U}_2 = \mathbf{W}\mathbf{U}_1\mathbf{W}^{-1} \quad \text{with} \quad \mathbf{W} = \mathbf{V}_2^{-1}\mathbf{T}_{12}\mathbf{V}_1 \tag{19}$$

Since $\mathbf{U}_{1,2}$ are block diagonal, $\mathbf{W}$ is also block diagonal, $\mathbf{W} = \begin{bmatrix} \mathbf{E}^{-1} & 0 \\ 0 & \mathbf{F}^{-1} \end{bmatrix}$. (For very strong coupling, $\mathbf{W}$ may be off block diagonal [**?**].) The coupling matrix at s_2 can be written in terms of the elements of the coupling matrix at s_1 by

$$\mathbf{C}_2 = (\mathbf{M}_{12}\mathbf{C}_1 + \gamma_1\mathbf{m}_{12})\mathbf{F}_{12}^{-1} \tag{20}$$

where

$$\begin{aligned} \gamma_2^2 = &\quad \det(\mathbf{n}_{12}\mathbf{C}_1 + \gamma_1\mathbf{N}_{12}) \\ \mathbf{E}_{12} = &\quad (\gamma_1\mathbf{M}_{12} - \mathbf{m}_{12}\mathbf{C}_1^{\dagger})/\gamma_2 \\ \mathbf{F}_{12} = &\quad (\mathbf{n}_{12}\mathbf{C}_1 + \gamma_1\mathbf{N}_{12})/\gamma_2 \end{aligned} \tag{21}$$

When there are no couplers between s_1 and s_2, $\mathbf{n}$ and $\mathbf{m}$ are zero and

$$\mathbf{C}_2 = \mathbf{M}_{12}\mathbf{C}_1\mathbf{F}_{12}^{-1} = \mathbf{M}_{12}\mathbf{C}_1\mathbf{N}_{12}^{-1} \tag{22}$$

Define $\bar{\mathbf{M}}_{12} = \mathbf{G}_{\mathbf{A}}^2\mathbf{M}_{12}(\mathbf{G}_A^1)^{-1}$ and $\bar{\mathbf{N}}_{12} = \mathbf{G}_B^2\mathbf{N}_{12}(\mathbf{G}_B^1)^{-1}$ where $\mathbf{G}_A^1$ is the normalization matrix for horizontal motion at s_1, $\mathbf{G}_A^2$ is the normalization matrix at s_2, etc. Then Eq.(**??**) gives

$$\mathbf{G}_A^2\mathbf{C}_2(\mathbf{G}_B^2)^{-1} = \bar{\mathbf{C}}_2 = \bar{\mathbf{M}}\bar{\mathbf{C}}_1\bar{\mathbf{N}}^{-1} \tag{23}$$

The normalized transfer matrices $\bar{\mathbf{M}}$ and $\bar{\mathbf{N}}$ are simple rotations in the normalized phase space and $\bar{\mathbf{C}}_2 = \mathbf{R}(\phi_A)\bar{\mathbf{C}}_1\mathbf{R}(\phi_B)^{-1}$, and $\mathbf{R}$ is a rotation matrix. Define complex coupling coefficients

$$\begin{aligned} a &= \bar{c}_{11} - \bar{c}_{22} + i(\bar{c}_{12} + \bar{c}_{21}) \\ b &= \bar{c}_{11} + \bar{c}_{22} + i(\bar{c}_{21} - \bar{c}_{12}) \end{aligned} \tag{24}$$

Then

$$a_2 = e^{-i(\phi_A+\phi_B)}a_1, \quad b_2 = e^{-i(\phi_A-\phi_B)}b_1 \tag{25}$$

where $\phi_A + \phi_B$ is the sum of the normal mode phase advances between 1 and 2, and $\phi_A - \phi_B$ is the difference.

Coupling resonances Suppose that a coupling error $\delta a, \delta b$ is introduced at s_1. Then propagate a and b around the ring and back to s_1 according to

$$e^{-i\Sigma}a = a + \delta a, \quad e^{-i\Delta}b = b + \delta b \tag{26}$$

$\Sigma = 2\pi(\nu_A + \nu_B)$ and $\Delta = 2\pi(\nu_A - \nu_B)$. The solution to Eq.(**??**) for a and b at s_1 is

$$a = \frac{ie^{-i\frac{\Sigma}{2}}}{2\sin\frac{\Sigma}{2}}\delta a, \quad b = \frac{ie^{-i\frac{\Delta}{2}}}{2\sin\frac{\Delta}{2}}\delta b \tag{27}$$

Note that near the sum resonance the fast wave (a) is magnified and near the difference resonance the slow wave (b) tends to dominate.

0.1.4 Measurement of coupling

Tune split If the full turn betatron phase advance for both horizontal and vertical motion can be tuned to the coupling resonance, the splitting of the normal mode tunes can be a useful measure of machine coupling. Eqs.(**??**, **??**) give

$$\mathrm{tr}(\mathbf{A} - \mathbf{B}) = \sqrt{\mathrm{tr}(\mathbf{M} - \mathbf{N})^2 + 4\det(\mathbf{m} + \mathbf{n}^\dagger)}$$

$$\mathbf{m} + \mathbf{n}^\dagger = -\mathrm{tr}(\mathbf{A} - \mathbf{B})\gamma\mathbf{C} \tag{28}$$

At the coupling resonance, where $\nu_h = \nu_v$, $\mathrm{tr}(\mathbf{M} - \mathbf{N}) = 0$, and Eq.(**??**) becomes

$$\cos 2\pi\nu_A - \cos 2\pi\nu_B = \sqrt{\det(\mathbf{m} + \mathbf{n}^\dagger)} \tag{29}$$

Thus the splitting of the normal mode tunes is proportional to det($\mathbf{C}$). If the full turn matrix is block diagonal, then $\mathbf{m} = \mathbf{n} = \mathbf{0}$, and the splitting of normal mode tunes can be reduced to zero. By adjusting coupling correctors, such as skew quads, to minimize the normal mode tune split near the sum resonance we are reducing a and minimizing the tune split at the difference resonance reduces b.

Tune split due to a thin skew quad or solenoid Consider a machine tuned to the difference resonance that is described by the full turn block diagonal matrix at some point s by $\mathbf{F} = \begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{N} \end{bmatrix}$. We can write $\mathbf{M} = \begin{bmatrix} \cos\mu_h & \beta_h \sin\mu_h \\ -\gamma\sin\mu_h & \cos\mu_h \end{bmatrix}$, and similarly for $\mathbf{N}$. Here $\mu_h = 2\pi\nu_h$. At the difference resonance, $\mathrm{tr}(\mathbf{M}-\mathbf{N}) = 0$. Then introduce a thin skew quad $\mathbf{Q}_{\mathrm{thin}}$ as in Eq.(**??**). The perturbed full turn matrix is

$$\mathbf{P} = \mathbf{FQ} = \begin{bmatrix} \mathbf{M} & \mathbf{MK}_t \\ \mathbf{NK}_t & \mathbf{N} \end{bmatrix}$$

From Eq.(**??**) we have

$$\cos\mu_A - \cos\mu_B = \sqrt{\det[\mathbf{MK}_t + (\mathbf{NK}_t)^\dagger]} \tag{30}$$

Since $\mu_h \sim \mu_v$,

$$\Delta\nu = \nu_A - \nu_B \sim \frac{1}{2\pi}\frac{\sqrt{\beta_h\beta_v}}{f} \tag{31}$$

In the case of a thin solenoid, replace $\frac{1}{f}$ with $\frac{1}{g} = \frac{k_s^3 l^2}{8}$, where l is the length of the solenoid and k_s is defined above.

Coupling wave due to thin skew quads and solenoids Eqs.(**??**, **??**) and thin coupler matrices are combined to give a first order expression for the complex coupling parameters a and b. The coupling at j due to skew quads and/or solenoids at k is

$$a_j = \sum_k \rho_k e^{-i(\xi^+_{jk} + \frac{\Sigma}{2})}, \; b_j = \sum_k \chi_k e^{-i(\xi^-_{jk} + \frac{\Delta}{2})} \tag{32}$$

where

$$\rho_k \;=\; \frac{\sqrt{\beta_h^k \beta_v^k}}{\mathrm{tr}(\mathbf{A}-\mathbf{B})\gamma}\left(\frac{2\sin\frac{\Delta}{2}}{f_k} + \frac{2i\cos\frac{\Delta}{2}}{g_k}\right)$$

$$\chi_k = \frac{\sqrt{\beta_h^k\beta_v^k}}{\mathrm{tr}(\mathbf{A}-\mathbf{B})\gamma}\left(\frac{2\sin\frac{\Sigma}{2}}{f_k} + \frac{2i\cos\frac{\Sigma}{2}}{g_k}\right)$$

$$\begin{aligned}
\xi_{jk}^{\pm} &= \phi_h^j - \phi_h^k \pm (\phi_v^j - \phi_v^k) \\
\Sigma &= \mu_h + \mu_v, \quad \Delta = \mu_h - \mu_v
\end{aligned}$$

In the limit of weak coupling $\mathrm{tr}(\mathbf{A}-\mathbf{B}) = \mathrm{tr}(\mathbf{M}-\mathbf{N}) = 2(\cos\mu_h - \cos\mu_v)$, and $\gamma \sim 1$.

Relative amplitude and phase When a normal mode is excited, the relative amplitude and phase of horizontal and vertical motion can be measured at a BPM [**?**]. Consider the motion in the laboratory coordinate system as a consequence of excitation of the A mode. If the initial normalized, normal mode vector is $\mathbf{G}\mathbf{w_0} = \bar{\mathbf{w}}_\mathbf{0} = (\sqrt{\epsilon_A}, 0, 0, 0)$ then after some number of turns when the normal mode A has propagated through some phase advance ϕ, we have

$$\mathbf{x} = \mathbf{V}\mathbf{w}_n = \mathbf{V}\mathbf{U}^n\mathbf{w_0} = \mathbf{G}^{-1}\bar{\mathbf{V}}\bar{\mathbf{U}}^n\bar{\mathbf{w}}_\mathbf{0} \tag{33}$$

$\Longrightarrow$

$$\begin{bmatrix} x \\ x' \\ y \\ y' \end{bmatrix} = \begin{bmatrix} \gamma\sqrt{\epsilon_A\beta_A}\cos\phi \\ x' \\ -\sqrt{\epsilon_A\beta_A}(\bar{c}_{22}\cos\phi + \bar{c}_{12}\sin\phi) \\ y' \end{bmatrix}$$

$\Longrightarrow$

$$\begin{aligned}
x &= \gamma\sqrt{\epsilon_A\beta_A}\cos\phi \\
y &= -\sqrt{\epsilon_A\beta_B}(\bar{c}_{22}\cos\phi + \bar{c}_{12}\sin\phi) \\
&= -\sqrt{\epsilon_A\beta_A}\sqrt{\bar{c}_{22}^2 + \bar{c}_{12}^2}\cos(\phi + \delta\phi)
\end{aligned} \tag{34}$$

where $\delta\phi = \tan^{-1}\frac{\bar{c}_{12}}{\bar{c}_{22}}$. The ratio of the horizontal and vertical amplitudes of the normal mode motion is

$$(y/x)_{\mathrm{Amp}} = \frac{1}{\gamma}\sqrt{\frac{\beta_B}{\beta_A}}\sqrt{\bar{c}_{12}^2 + \bar{c}_{22}^2} \tag{35}$$

The ratios of the component of the vertical motion that is in phase and out of phase with the horizontal motion are

$$(y/x)_{\text{in phase}} = (1/\gamma)\sqrt{\beta_B/\beta_A}\bar{c}_{22} \tag{36}$$

$$(y/x)_{\text{out of phase}} = (1/\gamma)\sqrt{\beta_B/\beta_A}\bar{c}_{12} \tag{37}$$

The in-phase coupled motion corresponds to a tilt of the real space beam ellipse and the out of phase motion to an increase in the height of the ellipse.

Excitation of the orthogonal mode ($\epsilon_B \neq 0$) yields

$$(x/y)_{\text{in phase}} = (1/\gamma)\sqrt{\beta_A/\beta_B}\,\bar{c}_{11} \tag{38}$$

$$(x/y)_{\text{out of phase}} = (1/\gamma)\sqrt{\beta_A/\beta_B}\,\bar{c}_{12} \tag{39}$$

Measurement The elements of the coupling matrix can be measured at BPM's. A magnetic shaker excites one or the other of the normal modes. The transfer function between shaker drive and horizontal and vertical motion at the BPM is measured. The relative phase of the vertical and horizontal motion $\delta\phi = \bar{c}_{12}/\bar{c}_{22}$ and the relative amplitude is $(y/x)_{\text{Amp}} = (1/\gamma)\sqrt{\beta_B/\beta_A}\sqrt{\bar{c}_{12}^2 + \bar{c}_{22}^2}$. See also Sec.**??**.

The measurement of $\bar{c}_{12}$ in CESR is reproducible at the 0.5% level. Coupling errors are typically reduced to $< 1\%$. $\bar{c}_{22}$ is somewhat more difficult to measure since the vertical amplitude is typically $\ll$ the horizontal amplitude.

0.1.5 Solenoid compensation

Large detector solenoids are strong coupling elements. The compensation consists of coupling elements deployed to globally decouple the motion outside the compensation region and to preclude feedthrough of horizontal motion to beam height at the IP.

Compensation with anti-solenoids The coupling of the solenoid can be compensated by anti-solenoids of equal but opposite integrated strength. If a pair of half strength anti-solenoids are placed symmetrically about the main solenoid, then there is vanishing coupling at the mid-point of the main solenoid. The anti-solenoids may be displaced from the ends of the main solenoid by a field free drift. However, the solenoid matrix does not commute with that of a quadrupole. And anti-solenoids are ineffective if there are intervening quadrupole focusing elements.

Compensation with skew quads Alternatively, skew or rotated quadrupoles can be used to effect compensation. In general, four pairs of such coupling elements are required. In a symmetric IR (where the final focus elements are mirror symmetric about the IP), three pairs of coupling elements are sufficient.

Transfer matrix Define the compensation region of the lattice to include the detector solenoid and all of the compensation elements.

1. The coupling is compensated if the 4×4 matrix ($\mathbf{T}^{\text{through}}$) through the compensation region is block diagonal.
2. Vertical displacement at the IP is independent of the horizontal phase space outside the compensation region.

The second criterion imposes constraints on the matrix ($\mathbf{T}^{\text{outside}\to\text{IP}}$) that propagates trajectories from outside the compensation region to the IP, $\mathbf{T}^{\text{outside}\to\text{IP}}_{31} = \mathbf{T}^{\text{outside}\to\text{IP}}_{32} = 0$.

Let $\mathbf{T}_{\text{right}\to\text{IP}}$ and $\mathbf{T}_{\text{left}\to\text{IP}}$ be the matrices that transport trajectories through the right and left halves of the compensation region to the IP. Then $\mathbf{T}_{\text{through}} = \mathbf{T}_{\text{right}\to\text{IP}}\mathbf{J}\mathbf{T}^{-1}_{\text{left}\to\text{IP}}\mathbf{J}$, where

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Compensation with four pairs of rotated quadrupoles It is clear that if $\mathbf{T}_{\text{right}\to\text{IP}}$ and $\mathbf{T}_{\text{left}\to\text{IP}}$ are block diagonal then $\mathbf{T}_{\text{through}}$ is block diagonal and the compensation criteria are satisfied. Symplectic matrices have the property that if one off diagonal 2×2 block is all zero, then so is the other. Therefore four independent coupling elements (skew or rotated quadrupoles) are sufficient to satisfy the compensation criteria.

Compensation with three pairs of rotated quadrupoles But it is not necessary that $\mathbf{T}_{\text{right}\to\text{IP}}$ be block diagonal. Suppose that the focusing quadrupoles are deployed symmetrically about the IP and the compensating rotated quads antisymmetrically, (equal but opposite rotation angles). (Note that the radial fringe at the left end of the solenoid is of opposite sign to the fringe at the east end.) It is shown [?] that if

1. $T_{31} = 0$
2. $T_{32} = 0$
3. $-T_{41}T_{12} + T_{42}T_{11} = 0$

where $\mathbf{T} \equiv \mathbf{T}^{\text{right}\to\text{IP}}$, then $\mathbf{T}_{\text{through}}$ is block diagonal. The three constraints imposed on the matrix can be realized with three independent pairs of rotated quadrupoles. Independence of course implies nondegenerate betatron phase advance between the rotated quads.

In CESR (5 GeV/beam), the solenoid that is centered at the interaction point has a field of 1.5 T and length of 3.5 m. The final focus quadrupoles are rotated antisymmetrically to compensate the coupling of the solenoid. The three pairs of quadrupoles are rotated by $\pm 4.5°, \pm 6.8°$ and $\pm 14.0°$.

Coupling matrix The compensation conditions can be cast in terms of the **C**-matrix elements. According to Eqs.(**??**, **??**), the blowup of the beam cross section is proportional to $\bar{c}_{12}$ and the tilt of the beam is proportional to $\bar{c}_{22}$. The requirement of no vertical enlargement or twist at the IP implies that

$$\bar{\mathbf{C}}_{IP} = \begin{bmatrix} \bar{c}_{11} & 0 \\ \bar{c}_{21} & 0 \end{bmatrix} \tag{40}$$

And the machine is globally decoupled if at any and all points outside the insertion

$$\bar{\mathbf{C}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{41}$$

Eqs.(**??**, **??**) are true if and only if the compensation criteria described above are satisfied.

The three pair compensation scheme can also be described in terms of the **C**-matrix elements. If the compensating elements are placed antisymmetrically about the IP, then by symmetry, the tilt of the beam $\bar{c}_{22}$ at the IP is zero. Similarly, by symmetry, $\bar{c}_{22} = \bar{c}_{11}$ at a symmetry point outside the compensation region, (such as the point diametrically opposite the IP). So, if antisymmetric pairs of rotated or skew quads are used to compensate for the solenoid, then three constraints must be satisfied to ensure that there is no coupling of horizontal motion into the vertical plane at the IP, and that the full turn matrix evaluated everywhere outside the compensation region is block diagonal. They are

1. $\bar{c}^*_{12} = 0$
2. $\bar{c}^{\text{outside}}_{12} = 0$
3. $\bar{c}^{\text{outside}}_{21} = 0$

where $\bar{c}^{\text{outside}}$ refers to coupling parameters at the symmetry point outside of the compensation region. Then Eqs.(**??**, **??**) are satisfied.

Dispersion Vertical dispersion at the IP increases the vertical beam size and is a source via the beam-beam interaction of synchrobetatron coupling. As long as there are no bending magnets and zero horizontal dispersion inside of the compensation region, then the compensation schemes described above will generate no vertical dispersion at the IP. But if there is horizontal dispersion inside of the compensation region then additional constraints will be required.

References

[1] D. Edwards, L. Teng, PAC 73, p.885

[2] M. Billing, Cornell Report CBN 85-2 (1985)

[3] D. Sagan, D. Rubin, Cornell Report CBN 96-20 (1996)

[4] S. Peggs, PA 12 (1982) 219

[5] D. Rubin, CON 84-4 (1984)

[6] D. Rubin, CON 84-5 (1984)

[7] P. Bagley, D. Rubin, PAC 87, p.1301