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Tune Split due to offset sextupole

Suppose we tune to the coupling resonance. Then

$$\begin{aligned}
T = VUV^{-1} &= \begin{pmatrix} \gamma & C \\ -C^\dagger & \gamma \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} \gamma & -C \\ C^\dagger & \gamma \end{pmatrix} \\
\begin{pmatrix} M & m \\ n & N \end{pmatrix} &= \begin{pmatrix} \gamma A & CB \\ -C^\dagger A & \gamma B \end{pmatrix} \begin{pmatrix} \gamma & -C \\ C^\dagger & \gamma \end{pmatrix} \\
&= \begin{pmatrix} \gamma^2 A + CBC^\dagger & -\gamma AC + \gamma CB \\ -\gamma C^\dagger A + \gamma BC^\dagger & C^\dagger AC + \gamma^2 B \end{pmatrix} \qquad (1)
\end{aligned}$$

Now note that

$$\begin{aligned}
m + n^\dagger &= -\gamma AC + \gamma CB + \gamma(-A^\dagger C + CB^\dagger) \qquad (2) \\
&= -\gamma \left((A + A^\dagger)C - C(B + B^\dagger) \right) \\
&= -\gamma(\mathrm{Tr}(A - B))C \\
||m + n^\dagger|| &= || - \gamma(\mathrm{Tr}(A - B))C|| \\
&= \gamma^2(\mathrm{Tr}(A - B))^2 ||C|| \\
&= \gamma^2(\mathrm{Tr}(A - B))^2(1 - \gamma^2) \qquad (3)
\end{aligned}$$

Also

$$\begin{aligned}
\mathrm{Tr}(M - N) &= \gamma^2 \mathrm{Tr}(A - B) + \mathrm{Tr}CBC^\dagger - \mathrm{Tr}C^\dagger AC \\
&= \gamma^2 \mathrm{Tr}(A - B) + ||C|| \mathrm{Tr}(B - A) \\
&= (\gamma^2 - ||C||)\mathrm{Tr}(A - B) \\
&= (\gamma^2 - (1 - \gamma^2))\mathrm{Tr}(A - B) = (2\gamma^2 - 1)\mathrm{Tr}(A - B) \qquad (4)
\end{aligned}$$

Combining Equations 3 and 4 we find

$$\begin{aligned}
\mathrm{Tr}(M - N)^2 + 4|m + n^\dagger| &= \mathrm{Tr}(A - B)^2 \\
\mathrm{Tr}(A - B) &= \left(\mathrm{Tr}(M - N)^2 + 4|m + n^\dagger| \right)^{\frac{1}{2}} \qquad (5)
\end{aligned}$$

In a machine with no transverse coupling, the full turn 4X4 matrix

$$T = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

and on the coupling resonance, $\mathrm{Tr}M = \mathrm{Tr}N$. If we turn on a skew quad, with matrix $K = \begin{pmatrix} 0 & 0 \\ 1/f & 0 \end{pmatrix}$ the full turn matrix becomes

$$= \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix} \begin{pmatrix} I & K \\ K & I \end{pmatrix} = \begin{pmatrix} M & MK \\ NK & N \end{pmatrix}$$

where

$$\begin{aligned}
M &= \begin{pmatrix} \cos\mu_x + \alpha_x \sin\mu_x & \beta_x \sin\mu_x \\ -\gamma_x \sin\mu_x & \cos\mu_x - \alpha_x \sin\mu_x \end{pmatrix} \\
N &= \begin{pmatrix} \cos\mu_y + \alpha_y \sin\mu_y & \beta_y \sin\mu_y \\ -\gamma_y \sin\mu_y & \cos\mu_y - \alpha_y \sin\mu_y \end{pmatrix} \\
m = MK &= \begin{pmatrix} \beta_x \sin\mu_x/f & 0 \\ (\cos\mu_x - \alpha_x \sin\mu_x)/f & 0 \end{pmatrix} \\
n = NK &= \begin{pmatrix} \beta_y \sin\mu_y/f & 0 \\ (\cos\mu_y - \alpha_y \sin\mu_y)/f & 0 \end{pmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
|m + n^\dagger| &= \left| \begin{pmatrix} \beta_x \sin\mu_x/f & 0 \\ \cos\mu_x - \cos\mu_y - (\alpha_x \sin\mu_x + \alpha_y \sin\mu_y)/f & \beta_y \sin\mu_y/f \end{pmatrix} \right| \\
&= \beta_x\beta_y \sin\mu_x \sin\mu_y/f^2
\end{aligned}$$

Finally, on the coupling resonance Equation 5 becomes (where $\mu_x = \mu_y$)

$$\begin{aligned}
2(\cos\mu_A - \cos\mu_B) &= 2\sqrt{|m + n^\dagger|} = 2\sqrt{\beta_x\beta_y \sin^2\mu/f^2} \\
\cos\mu - \cos(\mu - \Delta\mu) &= \sqrt{\beta_x\beta_y \sin^2\mu/f^2} \\
\sim \sin\mu\Delta\mu &= \sqrt{\beta_x\beta_y \sin^2\mu/f^2} \\
\sim \Delta\mu &= \sqrt{\beta_x\beta_y}/f \\
\sim \Delta\nu &= \sqrt{\beta_x\beta_y}/2\pi f
\end{aligned}$$

Then

$$f = sly = \frac{2\pi\Delta\nu}{\sqrt{\beta_x\beta_y}}$$

and

$$y = \frac{2\pi\Delta\nu}{\sqrt{\beta_x\beta_y}sl}$$

where s is the sextupole strength (m^{-3}) and l is the sextupole length.