Normal mode decomposition and AC dispersion

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0.1 Normal modes

The problem is to take a 4X4 or 6X6 symplectic matrix and decompose it into normal modes and to determine the transformation from lab frame to normal mode frame. We assume that we have the machinery to calculate eigenvalues and eigenvectors. Then

$$T = QDQ^{-1} \tag{1}$$

where T is the $2n$ x $2n$ symplectic matrix, D is the diagonal matrix of eigenvalues, and V is the matrix constructed from the eigenvectors. We assume that D can be written in the form

$$D = \begin{pmatrix} d(\theta_1) & 0 & 0 \\ 0 & d(\theta_2) & 0 \\ 0 & 0 & d(\theta_3) \end{pmatrix} \text{ where } d(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{2}$$

We are looking for the transformation to normal mode coordinates

$$T = VUV^{-1} \tag{3}$$

where U is block diagonal and we assume has the form

$$V = \begin{pmatrix} \gamma_1 I & C & D & ... \\ C' & \gamma_2 I & E & ... \\ D' & E' & \gamma_3 I & ... \\ ... & ... & ... & ... \end{pmatrix} \tag{4}$$

I is the 2x2 identity, and C, C', D etc are 2x2. If $n = 2$, then $\gamma_1 = \gamma_2$ and $C' = -C^\dagger$. The matrix U can be further decomposed according to

$$U = G^{-1} Z G \tag{5}$$

where if n=3 Z looks like

$$Z = \begin{pmatrix} R(\theta_1) & 0 & 0 \\ 0 & R(\theta_2) & 0 \\ 0 & 0 & R(\theta_3) \end{pmatrix} \text{ with } R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{6}$$

and

$$G = \begin{pmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{pmatrix}, \quad G = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \tag{7}$$

0.2 Rotation

The eigenvalues and eigenvectors of the rotation matrix R are

$$\begin{aligned} \lambda_\pm &= e^{\pm i\theta} \\ e_\pm &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix} \end{aligned}$$

Then

$$\begin{aligned} R(\theta) &= k d(\theta) k^{-1} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} e^{i\theta} & -ie^{i\theta} \\ e^{-i\theta} & ie^{-i\theta} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{i\theta}+e^{-i\theta} & -ie^{i\theta}+ie^{-i\theta} \\ ie^{i\theta}-ie^{-i\theta} & e^{i\theta}+e^{-i\theta} \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
\end{aligned}
$$

with

$$k = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{8}$$

Then define

$$K = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \tag{9}$$

0.3 V-matrix

Putting the whole deal together we have

$$T = VUV^{-1} = VG^{-1}ZGV^{-1} = VG^{-1}KDK^{-1}GV^{-1} \tag{10}$$

Now we see that

$$Q = VG^{-1}K \rightarrow V = QK^{-1}G \tag{11}$$

Let's make things simple and work it out for n=3. Define

$$Q = \begin{pmatrix} Q_1 & m & n \\ m' & Q_2 & p \\ n' & p' & Q_3 \end{pmatrix} \tag{12}$$

Then

$$
\begin{aligned}
V &= \begin{pmatrix} Q_1 & m & n \\ m' & Q_2 & p \\ n' & p' & Q_3 \end{pmatrix}\begin{pmatrix} K^{-1} & 0 & 0 \\ 0 & K^{-1} & 0 \\ 0 & 0 & K^{-1} \end{pmatrix}\begin{pmatrix} G_1 & 0 & 0 \\ 0 & G_2 & 0 \\ 0 & 0 & G_3 \end{pmatrix} \\
&= \begin{pmatrix} Q_1K^{-1}G_1 & \sim & \sim \\ \sim & Q_2K^{-1}G_2 & \sim \\ \sim & \sim & Q_3K^{-1}G_3 \end{pmatrix}
\end{aligned}
$$

Then using the form for V from Equation 4 we get that $Q_iK^{-1}G_i = \gamma_i I$ and

$$\begin{aligned} \det Q_iK^{-1}G_i &= \gamma_i^2 \\ &= \det Q_i \det K^{-1} \det G_i \\ \rightarrow \gamma_i^2 &= \det Q_i \end{aligned}$$

Now we can compute

$$G_i^{-1} = \frac{1}{\sqrt{|Q_iK^{-1}|}} Q_iK^{-1} \tag{13}$$

We use Equation 7 to construct G from G_i and then Equation 11 to compute $V = QK^{-1}G$

0.3.1 Similarity transformation

The columns of the matrix Q are the eigenvectors of T. But the eigenvectors are not uniquely defined. In the most general case, each vector can be multiplied by an arbitrary complex number and it remains an eigenvector with the same eigenvalue. Because T is symplectic, we know that the eigenvalues and eigenvectors are complex conjugate pairs. So we are free to choose a single phase and amplitude for each pair of conjugate eigenvectors. We choose the phase so that G_i has the form $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, namely with $G(1,2) = 0$, and the amplitude so that Q is symplectic, that is so that

$$\begin{aligned} 1 &= |Q_1| + |m'| + |n'| \\ 1 &= |m'| + |Q_2| + |p| \\ 1 &= |n'| + |p'| + |Q_3| \end{aligned}$$

0.3.2 n=1

As a very simple example let

$$T = \begin{pmatrix} \cos\theta & \beta\sin\theta \\ -\frac{1}{\beta}\sin\theta & \cos\theta \end{pmatrix}$$

Then

$$D = \begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$$

$$
\begin{aligned}
Q &= c\begin{pmatrix} 1 & 1 \\ \frac{i}{\beta} & -\frac{i}{\beta} \end{pmatrix} \\
QK^{-1} &= c\begin{pmatrix} 1 & 1 \\ \frac{i}{\beta} & \frac{-i}{\beta} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\
&= \frac{c}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix} \\
\det QK^{-1} &= \frac{c^2}{2\beta}
\end{aligned}
$$

$$G^{-1} = \sqrt{\beta}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}$$

0.3.3 n=1 and $\alpha \neq 0$

$$T = \begin{pmatrix} \cos\theta + \alpha\sin\theta & \beta\sin\theta \\ -\frac{1+\alpha^2}{\beta}\sin\theta & \cos\theta - \alpha\sin\theta \end{pmatrix}$$

Then

$$
\begin{aligned}
D &= \begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix} \\
Q &= c\begin{pmatrix} 1 & 1 \\ \frac{-\alpha+i}{\beta} & \frac{-\alpha-i}{\beta} \end{pmatrix} \\
QK^{-1} &= c\begin{pmatrix} 1 & 1 \\ \frac{-\alpha+i}{\beta} & \frac{-\alpha-i}{\beta} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\
&= \frac{c}{\sqrt{2}} \begin{pmatrix} 2 & 0 \\ -\frac{2\alpha}{\beta} & \frac{2}{\beta} \end{pmatrix} \\
\det QK^{-1} &= \frac{2c^2}{\beta}
\end{aligned}
$$

$$G^{-1} = \sqrt{\beta}\begin{pmatrix} 1 & 0 \\ -\frac{\alpha}{\beta} & \frac{1}{\beta} \end{pmatrix}$$

0.4 Example with dispersion

Fine. Now let's work it out for energy, horizontal coupling. That's n=2. The full turn matrix excluding RF is

$$T = \begin{pmatrix} \cos\theta_x & \beta_x \sin\theta_x & 0 & \eta(1-\cos\theta_x) - \eta' \beta_x \sin\theta_x \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x & 0 & \eta'(1-\cos\theta_x) + \frac{\eta}{\beta}\sin\theta_x \\ e & f & 1 & L\alpha_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

We have already let $\alpha_x = 0$. Set $\eta' = 0$ for simplicity.

0.5 Symplectic matrices

For the n=2 case the symplectic matrix T has the property that $T^T S T = S$ where $S = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$ and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Define $T = \begin{pmatrix} M & m \\ n & N \end{pmatrix}$ and expanding to get

$$\begin{aligned} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} &= \begin{pmatrix} M^T & n^T \\ m^T & N^T \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} M & m \\ n & N \end{pmatrix} \\ &= \begin{pmatrix} M^T & n^T \\ m^T & N^T \end{pmatrix} \begin{pmatrix} sM & sm \\ sn & sN \end{pmatrix} \\ &= \begin{pmatrix} M^T sM + n^T sn & M^T sm + n^T sN \\ m^T sM + N^T sn & m^T sm + N^T sN \end{pmatrix} \end{aligned} \tag{15}$$

Now further expanding the lower left 2X2 matrix, using T from Equation 14 we get

$$\begin{aligned} 0 &= m^T sM + N^T sn \\ &= \begin{pmatrix} 0 & 0 \\ \eta(1-\cos\theta_x) & \frac{\eta}{\beta_x}\sin\theta_x \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta_x & \beta_x \sin\theta_x \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e & f \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \eta(1-\cos\theta_x) & \frac{\eta}{\beta_x}\sin\theta_x \end{pmatrix} \begin{pmatrix} \frac{1}{\beta_x}\sin\theta_x & -\cos\theta_x \\ \cos\theta_x & \beta_x \sin\theta_x \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ e & f \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 0 & 0 \\ \frac{\eta}{\beta_x}\sin\theta_x(1-\cos\theta_x) + \frac{\eta}{\beta_x}\sin\theta_x\cos\theta_x & -\eta\cos\theta_x(1-\cos\theta_x) + \eta\sin^2\theta_x \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ e & f \end{pmatrix}$$

Then

$$n = -\begin{pmatrix} \frac{\eta}{\beta_x}\sin\theta_x & \eta(1-\cos\theta_x) \\ 0 & 0 \end{pmatrix}$$

Let's try another strategy. Going back to Equation 15, again we examine the lower left 2X2 matrix.

$$\begin{aligned} 0 &= m^T sM + N^T sn \\ \rightarrow N^T sn &= -m^T sM \\ \rightarrow n &= s(N^T)^{-1} m^T sM \\ &= -s(N^T)^{-1} ssm^T sM \\ &= Nsm^T sM \end{aligned}$$

We know N, M, and given the constraint on $|m|$ 3 of 4 elements of m. Let's proceed.

$$\begin{aligned} Nsm^T sM &= \begin{pmatrix} 1 & L\alpha_p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ b & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta_x & \beta_x\sin\theta_x \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x \end{pmatrix} \\ &= \begin{pmatrix} 1 & L\alpha_p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ d & -b \end{pmatrix}\begin{pmatrix} \cos\theta_x & \beta_x\sin\theta_x \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x \end{pmatrix} \\ &= \begin{pmatrix} 1 & L\alpha_p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -d & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta_x & \beta_x\sin\theta_x \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x \end{pmatrix} \\ &= \begin{pmatrix} -d & b \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta_x & \beta_x\sin\theta_x \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x \end{pmatrix} \\ &= \begin{pmatrix} -d\cos\theta_x - \frac{b}{\beta_x}\sin\theta_x & -d\beta_x\sin\theta_x + b\cos\theta_x \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\eta}{\beta}\sin\theta_x\cos\theta_x - \frac{\eta(1-\cos\theta_x)}{\beta_x}\sin\theta_x & -\frac{\eta}{\beta_x}\sin\theta_x\beta_x\sin\theta_x + \eta(1-\cos\theta_x)\cos\theta_x \\ 0 & 0 \end{pmatrix} \\ &= -\begin{pmatrix} \frac{\eta}{\beta_x}\sin\theta_x & \eta(1-\cos\theta_x) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Add the RF and we get

$$\begin{aligned} T &= \begin{pmatrix} \cos\theta_x & \beta_x\sin\theta_x & 0 & \eta(1-\cos\theta_x) \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x & 0 & \frac{\eta}{\beta}\sin\theta_x \\ -\frac{\eta}{\beta_x}\sin\theta_x & -\eta(1-\cos\theta_x) & 1 & L\alpha_p \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\omega}{c}\frac{V}{E} & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_x & \beta_x\sin\theta_x & -\frac{\omega}{c}\frac{V}{E}\eta(1-\cos\theta_x) & \eta(1-\cos\theta_x) \\ -\frac{1}{\beta_x}\sin\theta_x & \cos\theta_x & -\frac{\omega}{c}\frac{V}{E}\frac{\eta}{\beta}\sin\theta_x & \frac{\eta}{\beta}\sin\theta_x \\ -\frac{\eta}{\beta_x}\sin\theta_x & -\eta(1-\cos\theta_x) & 1-\frac{\omega}{c}\frac{V}{E}L\alpha_p & L\alpha_p \\ 0 & 0 & -\frac{\omega}{c}\frac{V}{E} & 1 \end{pmatrix} \end{aligned}$$

We know that

$$\begin{aligned} 2\cos\theta_z &= \mathrm{Tr}N \\ 2(1-\frac{1}{2}\theta_z^2+...) &= 2-\frac{\omega V}{cE}L\alpha_p \\ \rightarrow \theta_z \sim \sqrt{\frac{\omega V L\alpha_p}{cE}} \end{aligned}$$

0.6 Longitudinal transverse coupling

Coupling parameters C and γ are

$$C = \frac{-H\mathrm{sgn}(\mathrm{Tr}[M-N])}{\gamma\sqrt{(\mathrm{Tr}[M-N])^2+4|H|}} \tag{16}$$

where

$$\begin{aligned} H &= m+n^\dagger \\ &= \begin{pmatrix} -\frac{\omega V}{cE}\eta(1-\cos\theta_x) & \eta(1-\cos\theta_x) \\ -\frac{\omega V\eta}{cE\beta_x}\sin\theta_x & \frac{\eta}{\beta_x}\sin\theta_x \end{pmatrix} + \begin{pmatrix} 0 & \eta(1-\cos\theta_x) \\ 0 & -\frac{\eta}{\beta_x}\sin\theta_x \end{pmatrix} \\ &= \begin{pmatrix} -\frac{\omega V}{cE}\eta(1-\cos\theta_x) & 2\eta(1-\cos\theta_x) \\ -\frac{\omega V\eta}{cE\beta_x}\sin\theta_x & 0 \end{pmatrix} \end{aligned}$$

$|H| = -\frac{2\omega V\eta^2}{cE\beta_x}(1-\cos\theta_x)\sin\theta_x$. If we are in the limit where $4|H| \ll (\mathrm{Tr}[M-N])^2$, that is far from the coupling resonance then, $\gamma \sim 1$ and

$$\begin{aligned} C &\sim \frac{H}{\mathrm{Tr}[M-N]} \\ &\sim \frac{H}{2(\cos\theta_x-\cos\theta_z)} \end{aligned}$$

And

$$\begin{aligned} C_{12} &\sim \frac{\eta(1-\cos\theta)}{\cos\theta_x - \cos\theta_z} \\ &\sim \eta \end{aligned}$$

where we have assumed that $\cos\theta_z \sim 1$.

0.7 Real transformation

The problem is that the matrix K defined in Equation 9 does not in general transform the complex matrix formed from the eigenvectors to a real matrix. Let's try working it from the beginning. Start with a 2X2 matrix

$$T = \begin{pmatrix} \cos\theta + \alpha\sin\theta & \beta\sin\theta \\ -\frac{1+\alpha^2}{\beta}\sin\theta & \cos\theta - \alpha\sin\theta \end{pmatrix}$$

The eigenvalues are $\lambda_{\pm} = e^{\pm i\theta}$ To get the eigenvectors

$$\begin{aligned} \begin{pmatrix} \cos\theta + \alpha\sin\theta & \beta\sin\theta \\ -\frac{1+\alpha^2}{\beta}\sin\theta & \cos\theta - \alpha\sin\theta \end{pmatrix} \begin{pmatrix} a \\ 1 \end{pmatrix} &= \lambda \begin{pmatrix} a \\ 1 \end{pmatrix} \\ \rightarrow (\cos\theta + \alpha\sin\theta)a + \beta\sin\theta &= (\cos\theta \pm i\sin\theta)a \\ \rightarrow (a\alpha + \beta) &= \pm ia \\ \rightarrow a &= \frac{\beta}{\pm i - \alpha} \\ \rightarrow v_{\pm} &= N \begin{pmatrix} \frac{\beta}{\pm i - \alpha} \\ 1 \end{pmatrix} \\ &= \sqrt{\frac{\alpha^2+1}{\alpha^2+1+\beta^2}} \begin{pmatrix} \frac{\beta(\mp i - \alpha)}{1+\alpha^2} \\ 1 \end{pmatrix} \end{aligned}$$

We might also have written

$$\begin{aligned} v_{\pm} &= N \begin{pmatrix} \frac{\beta}{1+\alpha^2} \\ \frac{1}{\mp i - \alpha} \end{pmatrix} \\ &= N \begin{pmatrix} \beta \\ \pm i - \alpha \end{pmatrix} \end{aligned}$$

which is what we had come up with above. And

$$\begin{aligned} V &= \sqrt{\frac{\alpha^2+1}{\alpha^2+1+\beta^2}} \begin{pmatrix} \frac{\beta}{1+\alpha^2}(-i-\alpha) & \frac{\beta}{1+\alpha^2}(i-\alpha) \\ 1 & 1 \end{pmatrix} \\ &= \sqrt{\frac{\gamma}{\gamma+\beta}} \begin{pmatrix} \frac{1}{\gamma}(-i-\alpha) & \frac{1}{\gamma}(i-\alpha) \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Transform to real

$$\begin{aligned} VK^{-1} &= \sqrt{\frac{\gamma}{\gamma+\beta}} \begin{pmatrix} \frac{1}{\gamma}(-i-\alpha) & \frac{1}{\gamma}(i-\alpha) \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\sqrt{\frac{\gamma}{\gamma+\beta}} \begin{pmatrix} -\frac{2\alpha}{\gamma} & -\frac{2}{\gamma} \\ 2 & 0 \end{pmatrix} \end{aligned}$$

At least it is real. Now set the determinant to one. Whoa, can't be done if γ is real. Set the determinant to -1.

$$\begin{aligned} |QK^{-1}| &= -1 \\ \rightarrow &= N^2| \begin{pmatrix} -\alpha & -1 \\ \gamma & 0 \end{pmatrix} | \\ &= \begin{pmatrix} \frac{-\alpha}{\sqrt{\gamma}} & -\frac{1}{\sqrt{\gamma}} \\ \sqrt{\gamma} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha\sqrt{\frac{\alpha^2+1}{\beta}} & -\sqrt{\frac{\alpha^2+1}{\beta}} \\ \sqrt{\frac{\beta}{\alpha^2+1}} & 0 \end{pmatrix} \end{aligned}$$

0.8 Alternative form of G?

Perhaps the normalization with respect to β can be accomplished with this alternative transformation.

$$\begin{aligned} H^{1-}R(\theta)H &= \begin{pmatrix} \frac{-\alpha}{\sqrt{\gamma}} & -\frac{1}{\sqrt{\gamma}} \\ \sqrt{\gamma} & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{\gamma}} \\ -\sqrt{\gamma} & \frac{-\alpha}{\sqrt{\gamma}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\alpha}{\sqrt{\gamma}} & -\frac{1}{\sqrt{\gamma}} \\ \sqrt{\gamma} & 0 \end{pmatrix} \begin{pmatrix} -\sqrt{\gamma}\sin\theta & \frac{1}{\sqrt{\gamma}}(\cos\theta - \alpha\sin\theta) \\ -\sqrt{\gamma}\cos\theta & \frac{1}{\sqrt{\gamma}}(-\sin\theta - \alpha\cos\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta + \alpha\sin\theta & \frac{1}{\gamma}(1+\alpha^2)\sin\theta \\ -\gamma\sin\theta & \cos\theta - \alpha\sin\theta \end{pmatrix}) \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} \cos\theta + \alpha\sin\theta & \beta\sin\theta \\ -\gamma\sin\theta & \cos\theta - \alpha\sin\theta) \end{pmatrix} \\
&= W(\theta)
\end{aligned}
$$

By comparison

$$
\begin{aligned}
G^{-1}R(\theta)G &= \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\beta}}(\cos\theta + \alpha\sin\theta) & \sqrt{\beta}\sin\theta \\ -\frac{1}{\sqrt{\beta}}(-\alpha\cos\theta + \sin\theta) & \sqrt{\beta}\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta + \alpha\sin\theta & \beta\sin\theta \\ -\frac{1}{\beta}(\alpha^2+1)\sin\theta) & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta + \alpha\sin\theta & \beta\sin\theta \\ -\gamma\sin\theta & \cos\theta - \alpha\sin\theta \end{pmatrix} \\
&= W(\theta)
\end{aligned}
$$

What happens if $\theta \to 2\pi - \theta$. Then

So it seems that

$$
\begin{aligned}
H^{1-}R(\theta)H &= G^{-1}R(\theta)G \\
R &= HG^{-1}RGH^{-1} \\
HG^{-1} &= \begin{pmatrix} 0 & \frac{1}{\sqrt{\gamma}} \\ -\sqrt{\gamma} & \frac{\alpha}{\sqrt{\gamma}} \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\alpha}{\sqrt{\gamma\beta}} & \frac{1}{\sqrt{\gamma\beta}} \\ -\sqrt{\gamma\beta}(1 - \frac{\alpha^2}{\gamma\beta}) & \frac{\alpha}{\sqrt{\gamma\beta}} \end{pmatrix} \\
&= \frac{1}{\sqrt{\gamma\beta}} \begin{pmatrix} \alpha & 1 \\ -\frac{(1+\alpha^2)}{(1+\alpha^2)} & \alpha \end{pmatrix} \\
&= \frac{1}{\sqrt{\gamma\beta}} \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix}
\end{aligned}
$$

And

$$
\begin{aligned}
R &= JRJ^{-1} \\
&= \frac{1}{\sqrt{\beta\gamma}} \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \frac{1}{\sqrt{\beta\gamma}} \begin{pmatrix} \alpha & -1 \\ 1 & \alpha \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\beta\gamma}\begin{pmatrix}\alpha & 1\\ -1 & \alpha\end{pmatrix}\begin{pmatrix}\alpha\cos\theta+\sin\theta & -\cos\theta+\alpha\sin\theta\\ -\alpha\sin\theta+\cos\theta & \alpha\cos\theta+\sin\theta\end{pmatrix}\\
&= \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}
\end{aligned}
$$

G is not unique. Which makes it hard to identify twiss parameters with matrix elements. The matrix J transforms from one to the other. Note that if $T = G^{-1}R(\theta)G$ then $T = H^{-1}R(\theta)H$ for $H = GJ(\alpha)$ for arbitrary α. Therefore we can use J with suitable α to transform the H to a form with $G(1,2) = 0$ and then identify twiss parameters.

This is all because the complex conjugate pairs of eigenvectors can be multiplied by conjugate phase factors. Therefore there are n arbitrary phases. We choose the phase so that G takes the form with $G(1,2) = 0$. Note that J is a rotation matrix and $J(\alpha) = R(\cos^{-1}\left(\frac{\alpha}{\sqrt{\alpha^2+1}}\right))$

Suppose that

$$
\begin{aligned}
H &= \begin{pmatrix}a & b\\ c & d\end{pmatrix}\\
\rightarrow G &= HR(\phi) = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\\
&= \begin{pmatrix}a\cos\phi - b\sin\phi & a\sin\phi + b\cos\phi\\ c\cos\phi - d\sin\phi & c\sin\phi + d\cos\phi\end{pmatrix}\\
G(1,2) &= 0 \rightarrow \tan\phi = -\frac{b}{a}\\
G(1,1) &> 0 \rightarrow \frac{a}{\cos\phi} > 0
\end{aligned}
$$

$$
\begin{aligned}
\text{If } a > 0,\ -\frac{\pi}{2} < \phi < \frac{\pi}{2}\\
\text{If } a < 0,\ \frac{\pi}{2} < \phi < \frac{3\pi}{2}
\end{aligned}
$$

0.8.1 Swap columns

If

$$
T = \begin{pmatrix}\cos\theta + \alpha\sin\theta & -\beta\sin\theta\\ \gamma\sin\theta & \cos\theta - \alpha\sin\theta\end{pmatrix}
$$

then

$$\begin{aligned}
T &= \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & -\frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{\beta}} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & -\frac{1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} -\frac{\cos\theta}{\sqrt{\beta}} - \frac{\alpha\sin\theta}{\sqrt{\beta}} & \sqrt{\beta}\sin\theta \\ -\frac{\alpha\cos\theta}{\sqrt{\beta}} + \frac{\sin\theta}{\sqrt{\beta}} & \sqrt{\beta}\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} -(\cos\theta + \alpha\sin\theta) & \beta\sin\theta \\ -\gamma\sin\theta & -(\cos\theta - \alpha\sin\theta) \end{pmatrix} \\
&= -\begin{pmatrix} (\cos\theta + \alpha\sin\theta) & -\beta\sin\theta \\ \gamma\sin\theta & (\cos\theta - \alpha\sin\theta) \end{pmatrix}
\end{aligned}$$

It doesn't quite work unless $G \rightarrow iG$. So if the upper right term of T is less than zero, and $\pi < \theta < 0$, then the standard form for G is not good. If on the other hand we swap the order of the eigenvalues and eigenvectors, that is equivalent to $\theta \rightarrow 2\pi - \theta$, $(\theta \rightarrow -\theta)$, and R transforms to T with the standard form of G.

0.9 W matrix in standard form

1. Find eigenvectors and eigenvalues

2. Transform eigenvectors to a real basis

3. Construct Q. The columns of Q are the eigenvectors. The eigenvectors appear as complex conjugate pairs since T is symplectic.

4. Choose the normalization for each pair of eigenvectors so that W will be symplectic. In particular if

$$Q = \begin{pmatrix} c_1 Q_{1,1} & c_2 Q_{1,2} & \ldots \\ c_1 Q_{2,1} & c_2 Q_{2,2} & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix}$$

where $Q_{i,j}$ are 2X2 matrices, then choose c_1 so that

$$c_1^2 \left(|Q_{1,1}| + |Q_{2,1}| + |Q_{3,1}| + \ldots\right) = 1$$

5. Adjust the order of complex conjugate pairs so that $|Q_{i,i} > 0|$. That is, if the determinant is zero, than swap the order of the columns.

6. Choose the phase so that

$$\begin{aligned} G_i^{-1} &= Q_{i,i}R(\phi_i) \\ \text{has the form} \\ &= \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \end{aligned}$$

0.9.1 Measurement of C^3

We write the full turn matrix

$$T = VG^{-1}RGW^{-1}$$

Then

$$\vec{x}_n = T^n x_0 = VG^{-1}R^nGV^{-1}\vec{x}_0$$

Since in the normal mode frame

$$\vec{v} = GV^{-1}\vec{x}$$

then

$$R^n\vec{v}_0 = GV^{-1}\vec{x}^n = \vec{v}^n$$

and

$$\begin{aligned} \vec{x}^n &= VG^{-1}\vec{v}^n \\ &= \begin{pmatrix} \gamma_1 & C^1 & C^2 \\ \sim & \gamma_2 & C^3 \\ \sim & \sim & \gamma_3 \end{pmatrix} \begin{pmatrix} (G^1)^{-1} & & \\ & (G^2)^{-1} & \\ & & (G^3)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v_3^n \end{pmatrix} \\ &= \begin{pmatrix} C^2(G^3)^{-1}v_3^n \\ C^3(G^3)^{-1}v_3^n \\ \gamma_3(G^3)^{-1}v_3^n \end{pmatrix} \end{aligned}$$

Suppose that $\vec{v}_0 = \begin{pmatrix} l_0 \\ 0 \end{pmatrix}$, then $\vec{v}^n = l_0 \begin{pmatrix} \cos nQ \\ -\sin nQ \end{pmatrix}$. We use the fact that

$$\begin{aligned} CG^{-1} &= \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} \sqrt{\beta} & 0 \\ -\frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\beta}C_{11} - \frac{\alpha}{\sqrt{\beta}}C_{12} & \frac{1}{\sqrt{\beta}}C_{12} \\ \sqrt{\beta}C_{21} - \frac{\alpha}{\sqrt{\beta}}C_{22} & \frac{1}{\sqrt{\beta}}C_{22} \end{pmatrix} \end{aligned}$$

Then we can write

$$\begin{aligned} x^n &= l_0\left((\sqrt{\beta}C^2_{11} - \frac{\alpha}{\sqrt{\beta}}C^2_{12})\cos nQ - \frac{1}{\sqrt{\beta}}C^2_{12}\sin nQ\right) \\ y^n &= l_0\left((\sqrt{\beta}C^3_{11} - \frac{\alpha}{\sqrt{\beta}}C^3_{12})\cos nQ - \frac{1}{\sqrt{\beta}}C^3_{12}\sin nQ\right) \\ z^n &= l_0\gamma_3\sqrt{\beta}\cos nQ \end{aligned}$$

Now we can also write

$$\begin{aligned} x^n &= a_x\cos(nQ+\phi_x) \\ y^n &= a_y\cos(nQ+\phi_y) \\ z^n &= a_z\cos(nQ+\phi_z) \end{aligned}$$

Then

$$\begin{aligned} a_x &= l_0^2\left((\sqrt{\beta}C^2_{11} - \frac{\alpha}{\sqrt{\beta}}C^2_{12})^2 + (\frac{1}{\sqrt{\beta}}C^2_{12})^2\right)^{\frac{1}{2}} \\ \phi_x &= \tan^{-1}\frac{-C^2_{12}}{\beta C^2_{11} - \alpha C^2_{12}} \end{aligned}$$

and

$$\begin{aligned} a_z &= l_0\gamma_3\sqrt{\beta} \\ \phi_z &= 0 \end{aligned}$$

Let's compute $\bar{C}$

$$\begin{aligned} \bar{C} &= G_a C G_b^{-1} \\ &= \begin{pmatrix} \frac{1}{\sqrt{\beta_a}} & 0 \\ \frac{\alpha_a}{\sqrt{\beta_a}} & \sqrt{\beta_a} \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} \sqrt{\beta_b} & 0 \\ -\frac{\alpha_b}{\sqrt{\beta_b}} & \frac{1}{\sqrt{\beta_b}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{\beta_a}}C_{11} & \frac{1}{\sqrt{\beta_a}}C_{12} \\ \sqrt{\beta_a}C_{21} + \frac{\alpha_a}{\sqrt{\beta_a}}C_{11} & \frac{\alpha_a}{\sqrt{\beta_a}}C_{12} + \sqrt{\beta_a}C_{22} \end{pmatrix}\begin{pmatrix} \sqrt{\beta_b} & 0 \\ -\frac{\alpha_b}{\sqrt{\beta_b}} & \frac{1}{\sqrt{\beta_b}} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{\frac{\beta_b}{\beta_a}}C_{11} - \frac{\alpha_b}{\sqrt{\beta_a\beta_b}}C_{12} & \sqrt{\frac{1}{\beta_b\beta_a}}C_{12} \\ \sqrt{\beta_a\beta_b}C_{21} + \alpha_a\sqrt{\frac{\beta_b}{\beta_a}}C_{11} - \frac{\alpha_a\alpha_b}{\sqrt{\beta_a\beta_b}}C_{12} - \alpha_b\sqrt{\frac{\beta_a}{\beta_b}}C_{22} & \frac{\alpha_a}{\sqrt{\beta_a\beta_b}}C_{12} + \sqrt{\frac{\beta_a}{\beta_b}}C_{22} \end{pmatrix} \end{aligned}$$

It seems that we can write

$$\begin{aligned} a_x &= l_0\sqrt{\beta_x}\left(\bar{C}_{11}^2+\bar{C}_{12}^2\right)^{\frac{1}{2}} \\ \phi_x-\phi_z &= \tan^{-1}\frac{-\bar{C}_{12}}{\bar{C}_{11}} \end{aligned}$$

Finally

$$\frac{a_x}{a_z}=\sqrt{\frac{\beta_x}{\beta_z}}(\bar{C}_{11}^2+\bar{C}_{12}^2)^{\frac{1}{2}}$$

$$\sin(\phi_x-\phi_z)=-\frac{a_z}{a_x}\sqrt{\frac{\beta_x}{\beta_z}}\bar{C}_{12}$$