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February 8, 2008

Normal Mode Decomposition of $2N \times 2N$ symplectic matrices

Normal mode decomposition of a 4X4 symplectic matrix is a standard technique for analyzing transverse coupling in a storage ring. We generalize the decomposition to any 2nX2n symplectic matrix T and derive the transformation W from lab coordinates to normal mode coordinates U. That is

$$T = WUW^{-1} \tag{1}$$

where U is block diagonal and real and we construct the real matrix W with the form

$$W = \begin{pmatrix} \gamma_1 I & C_1 & C_2 & ... \\ C'_1 & \gamma_2 I & C_3 & ... \\ C'_2 & C'_3 & \gamma_3 I & ... \\ ... & ... & ... & ... \end{pmatrix} \tag{2}$$

I is the 2x2 identity, and C_1, C_2, C'_1 etc are 2x2. (If for example, $n = 2$, then $\gamma_1 = \gamma_2$ and $C' = -C^\dagger$).
The matrix

$$\begin{aligned} U &= \begin{pmatrix} A & 0 & ... \\ 0 & B & ... \\ ... & ... & ... \end{pmatrix} \text{can be decomposed as} \\ U &= YZY^{-1} \end{aligned} \tag{3}$$

where

$$Z(\theta_1, \theta_2, ..., \theta_n) = \begin{pmatrix} R(\theta_1) & 0 & ... \\ 0 & R(\theta_2) & ... \\ ... & ... & ... \end{pmatrix} \tag{4}$$

with

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{5}$$

and

$$Y = \begin{pmatrix} G_1 & 0 & ... \\ 0 & G_2 & ... \\ ... & ... & ... \end{pmatrix}, \tag{6}$$

and $G_i = \begin{pmatrix} \sqrt{\beta_i} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{pmatrix}$.

Since standard techniques exist for diagonalizing square matrices and identifying eigenvalues and eigenvectors, we begin by doing just that.

$$T = VDV^{-1}, \tag{7}$$

where T is the $2n$ x $2n$ symplectic matrix, D is the diagonal matrix of eigenvalues, and V is the matrix constructed from the eigenvectors. Since T is symplectic, the eigenvalues and eigenvectors appear as unimodular, complex conjugate pairs, λ_i, λ_i^* and $\vec{v}_i$ and $\vec{v}_i^*$. Then D can be written in the form

$$D = \begin{pmatrix} d(\theta_1) & 0 & 0 & ... \\ 0 & d(\theta_2) & 0 & ... \\ 0 & 0 & d(\theta_3) & ... \\ ... & ... & ... & ... \end{pmatrix} \text{ where } d(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \tag{8}$$

The n columns of the matrix V are the n eigenvectors v_i. The eigenvectors are not unique, but may be multiplied by an arbitrary complex number. That is, $\vec{v}_i \rightarrow \rho_i e^{i\phi_i} \vec{v}_i$ and $\vec{v}_i^* \rightarrow \rho_i e^{-i\phi_i} \vec{v}_i^*$. If $V_0 = \vec{v}_1\ \vec{v}_1^*\ \vec{v}_2\ \vec{v}_2^*\ ...\vec{v}_n\ \vec{v}_n^*$, then

$$\begin{aligned} V(\vec{\rho}, \vec{\phi}) &= V_0 D(\rho_1, \rho_2, ...\rho_n, \phi_1, \phi_2, ..., \phi_n) \\ &= V_0 \begin{pmatrix} \rho_1 d(\phi_1) & 0 & 0 & ... \\ 0 & \rho_2 d(\phi_2) & 0 & ... \\ 0 & 0 & \rho_3 d(\phi_3) & ... \\ ... & ... & ... & ... \end{pmatrix} \end{aligned}$$

Note that $V(\vec{\rho}, \vec{\phi})$ effects the transformation of Equation 7 for any real numbers ρ_i and ϕ_i.

We transform from a complex to a real basis with K where the real matrix Z (Equation 4) is related to the complex matrix D (Equation 8) by the similarity transformation

$$Z(\theta_2, \theta_2, \theta_3) = KD(\theta_1, \theta_2, \theta_3)K^{-1} \tag{9}$$

where

$$K = \begin{pmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \tag{10}$$

and

$$k = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \tag{11}$$

W-matrix

To construct W and U from V and D, we use Equations 1, 3 and 9 to write

$$\begin{aligned}
T = WUW^{-1} &= VDV^{-1} \\
&= V_0 D(\vec{\rho},\vec{\phi}) \left(K^{-1}K\right) D(\vec{\theta}) \left(K^{-1}K\right) D^{-1}(\vec{\rho},\vec{\phi}) V_0^{-1} \\
&= V_0 \left(K^{-1}K\right) D(\vec{\rho},\vec{\phi}) \left(K^{-1}K\right) D(\vec{\theta}) \left(KK^{-1}\right) D^{-1}(\vec{\rho},\vec{\phi}) \left(KK^{-1}\right) V_0^{-1} \\
&= \left(V_0 K^{-1}\right) Z(\vec{\rho},\vec{\phi}) Z(\vec{\theta}) Z^{-1}(\vec{\rho},\vec{\phi}) \left(K^{-1}V_0^{-1}\right) \\
&= V'(\vec{\rho},\vec{\phi}) Z(\vec{\theta}) V'^{-1}(\vec{\rho},\vec{\phi})
\end{aligned}$$

Now since the columns of V_0 are complex conjugate pairs, V_0K^{-1} is real. The Z matrices are similarly constructed to be real and therefore V' is real.
So far we have

$$\begin{aligned}
WUW^{-1} &= V'ZV'^{-1} \\
WYZ(\vec{\theta})Y^{-1} &= V'Z(\vec{\theta})V'^{-1} \\
\rightarrow V' = WY &
\end{aligned}$$

where we have used Equation 3.
Next we determine the parameters $\vec{\rho}$ and $\vec{\phi}$. We choose $\vec{\rho}$ so that V' will be symplectic. In particular, if we write V' in terms of the 2X2 matrices V_i^j then

$$\begin{aligned}
V' &= \begin{pmatrix} V_1'^1 & V_1'^2 & ... \\ V_2'^1 & V_2'^2 & ... \\ ... & ... & ... \end{pmatrix} \\
&= \begin{pmatrix} V_{01}^1 & V_{01}^2 & ... \\ V_{02}^1 & V_{02}^2 & ... \\ ... & ... & ... \end{pmatrix} \begin{pmatrix} \rho_1 R(\phi_1) & 0 & ... \\ 0 & \rho_2 R(\phi_2) & ... \\ ... & ... & ... \end{pmatrix} \\
&= \begin{pmatrix} \rho_1 V_{01}^1 R(\phi_1) & \rho_2 V_{01}^2 R(\phi_2) & ... \\ \rho_1 V_{02}^1 R(\phi_1) & \rho_2 V_{02}^2 R(\phi_2) & ... \\ ... & ... & ... \end{pmatrix}
\end{aligned}$$

Symplecticity constrains the sums of determinants of $V_i'^j$ so that

$$1 = \sum_{i=1}^{n} |V_i'^j|$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} |\rho_j V_{0i}^{\ j} R(\phi_j)| \\
&= \sum_{i=1}^{n} \rho_j^2 |V_{0i}^{\ j}| \\
\rightarrow \quad & \rho_j = \frac{1}{\sqrt{\sum_{i=1}^{n} |V_{0i}^{\ j}|}}
\end{aligned}
$$

In order to determine the order of the conjugate columns of V', and finally the paramters $\vec{\phi}$ we expand

$$
\begin{aligned}
V' &= WY(\vec{G}) \\
V' &= \begin{pmatrix} V_1'^1 & V_1'^2 & ... \\ V_2'^1 & V_2'^2 & ... \\ ... & ... & ... \end{pmatrix} = \begin{pmatrix} \gamma_1 I & C & ... \\ C' & \gamma_2 I & ... \\ ... & ... & ... \end{pmatrix} \begin{pmatrix} G_1 & 0 & ... \\ 0 & G_2 & ... \\ ... & ... & ... \end{pmatrix} \\
\begin{pmatrix} \rho_1 V_{01}^{\ 1} R(\phi_1) & \rho_2 V_{01}^{\ 2} R(\phi_2) & ... \\ \rho_1 V_{02}^{\ 1} R(\phi_1) & \rho_2 V_{02}^{\ 2} R(\phi_2) & ... \\ ... & ... & ... \end{pmatrix} &= \begin{pmatrix} \gamma_1 G_1 & C G_2 & ... \\ C' G_1 & \gamma_2 G_2 & ... \\ ... & ... & ... \end{pmatrix}
\end{aligned}
$$

Then the diagonal blocks are required to have the form

$$
\begin{aligned}
V_i'^i &= \gamma_i G_i \\
\rho_i V_{0i}^{\ i} R(\phi_i) &= \gamma_i \begin{pmatrix} \sqrt{\beta_i} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{pmatrix}
\end{aligned}
$$

A real solution requires that $|V_i'^i| > 0$. We are free to choose the order of the conjugate columns of V' to ensure that this is true. (Note that if we reverse the order of the columns $V'^{i,j} \rightarrow V'^{j,i}$, then the sign of the determinant of the 2 X 2 blocks is reversed.) If we reverse the order of eigenvectors in V', then we also reverse the order of eigenvalues in $D(\vec{\theta})$ or equivalently $\theta_i \rightarrow 2\pi - \theta_i$. To find $\vec{\phi}$ we proceed with our expansion of $V_0^{\ ii}$ and $R(\phi_i)$ and write

$$
\rho_i \begin{pmatrix} V_{011}^{\ ii} & V_{012}^{\ ii} \\ V_{021}^{\ ii} & V_{022}^{\ ii} \end{pmatrix} \begin{pmatrix} \cos\phi_i & \sin\phi_i \\ -\sin\phi_i & \cos\phi_i \end{pmatrix}) = \gamma_i \begin{pmatrix} \sqrt{\beta_i} & 0 \\ \frac{\alpha_i}{\sqrt{\beta_i}} & \frac{1}{\sqrt{\beta_i}} \end{pmatrix}
$$

We choose ϕ_i so that $G_{22}^i = 0$, or

$$
\tan\phi_i = \frac{V_{011}^{\ ii}}{V_{012}^{\ ii}}
$$

The ambiguity in ϕ_i, $(\tan\phi_i = \tan(2\pi - \phi_i))$ is resolved with the condition that $G^i_{11} = V^{ii}_{011}\cos\phi_i - V^{ii}_{012}\sin\phi > 0$.

Summary

1. Find eigenvectors and eigenvalues

2. Transform eigenvectors to a real basis

3. Construct V. The columns of V are the eigenvectors. The eigenvectors appear as complex conjugate pairs since T is symplectic.

4. Choose the normalization for each pair of eigenvectors so that W will be symplectic. In particular if

$$V = \begin{pmatrix} c_1 V_{1,1} & c_2 V_{1,2} & ... \\ c_1 V_{2,1} & c_2 V_{2,2} & ... \\ ... & ... & ... \end{pmatrix}$$

   where $V_{i,j}$ are 2X2 matrices, and $c_i = \rho e^{i\phi_i}$ then choose ρ_1 so that

$$\rho_1^2\left(|V_{1,1}| + |V_{2,1}| + |V_{3,1}| + ...\right) = 1$$

5. Adjust the order of complex conjugate pairs so that $|V_{i,i}| > 0$. That is, if $|V_{i,i}| < 0$, than swap the order of the columns.

6. Choose the phases ϕ_i so that

$$G_i = V_{i,i}R\theta)$$

has the form

$$= \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$