Effect of gain error on orbit difference measurements

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The position x_i is related to the button intensities $b_{i,j}$ according to

$$x_i = k_x \frac{b_{i,2} + b_{i,4} - b_{i,3} - b_{i,1}}{\sum} \tag{1}$$

where $\sum = \sum_{j=1}^{4} b_{i,j}$ is independent of i as long as we are at fixed beam current and in the linear regime. Consider a gain error on button b_4. Then

$$x'_i = k_x \frac{b_{i,2} + g b_{i,4} - b_{i,3} - b_{i,1}}{\sum'} \tag{2}$$

and

$$\sum' = b_{i,1} + b_{i,2} + b_{i,3} + g b_{i,4} \tag{3}$$

We can rewrite **??** as

$$\sum' = \sum + (g-1) b_{i,4} \tag{4}$$

Then Equation **??** becomes

$$\begin{aligned} x'_i &= k_x \frac{b_{i,2} + b_{i,4} - b_{i,3} - b_{i,1}}{\sum'} + \frac{(g-1) b_{i,4}}{\sum'} && (5) \\ &= k_x \frac{b_{i,2} + b_{i,4} - b_{i,3} - b_{i,1}}{\sum + (g-1) b_{i,4}} + \frac{(g-1) b_{i,4}}{\sum + (g-1) b_{i,4}} && (6) \end{aligned}$$

Then expanding in the limit where $|(g-1)| \ll 1$ we get that

$$\begin{aligned} x'_i &\sim k_x \frac{b_{i,2} + b_{i,4} - b_{i,3} - b_{i,1}}{\sum} \left(1 - \frac{(g-1) b_{i,4}}{\sum}\right) + k_x (g-1) b_{i,4} && (7) \\ &\sim x_i \left(1 - \frac{(g-1) b_{i,4}}{\sum}\right) + k_x \frac{(g-1) b_{i,4}}{\sum} && (8) \\ &\sim x_i + (k_x - x_i) \frac{(g-1) b_{i,4}}{\sum} && (9) \end{aligned}$$

Then an orbit difference gives

$$
\begin{aligned}
x_2' - x_1' &= x_2 - x_1 + (k_x - x_2)\frac{(g-1)b_{2,4}}{\sum} - (k_x - x_1)\frac{(g-1)b_{1,4}}{\sum} \\
\Delta x' &= \Delta x + ((k_x - x_2)b_{2,4} - (k_x - x_1)b_{1,4})\,\frac{(g-1)}{\sum}
\end{aligned}
$$

If $x_2 > x_1$, and $g > 1$, and $x_1, x_2 < k_x$, and $b_{2,4} > b_{1,4}$ then $\Delta_x' = \Delta x +$

Note that

$$
\begin{aligned}
k_x - x &= k_x\left(1 - \frac{b_4 + b_2 - b_3 - b_1}{\sum}\right) \\
&= 2k_x\frac{(b_1 + b_3)}{\sum}
\end{aligned}
$$

so

$$
\Delta x' = \Delta x + ((b_{2,1} + b_{2,3})b_{2,4} - (b_{1,1} + b_{1,3})b_{1,4})\,\frac{2k_x(g-1)}{\sum^2}
$$

If x_1 is at 0, and x_2 on the positive x-axis, then $b_{i,1} = b_{i,3}$, $b_{1,1} = b1,4$ and $b_{2,1} = b_{1,1} + \Delta b$, $b_{2,4} = b_{1,4} - \Delta b$ and

$$
\begin{aligned}
\Delta x' &= \Delta x + 2\left(b_{1,4}^2 - \Delta b^2 - b_{1,4}^2\right)\frac{2k_x(g-1)}{\sum^2} \\
&= \Delta x - 2\Delta b^2\frac{2k_x(g-1)}{\sum^2}
\end{aligned}
$$

But $\Delta x = 2k_x\Delta b/\sum$. So

$$
\begin{aligned}
\Delta x' &= \Delta x - \frac{(\Delta x)^2}{k_x}(g-1) \\
&= \Delta x(1 - \frac{\Delta x}{k_x}(g-1))
\end{aligned}
$$

Button signals as a function of beam position

Let's get back to basics. In a 2-dimensional approximation, where we think of the beam and the buttons as continuous in the z-direction, the charge on the i^{th} button is

$$
b_i = a_i\frac{1}{|\vec{r} - \vec{r}_i|} = \frac{a_i}{\sqrt{(x_i - x)^2 + (y_i - y)^2}}
$$

where x_i and y_i are the x and y coordinates of the button and x and y are the position of the beam. Then in the limit where $x/x_i \ll 1$ and $y/y_i \ll 1$,

$$\begin{aligned} b_i &\approx \frac{a_i}{\sqrt{x_i^2 + y_i^2}} + \frac{a_i(x_i x + y_i y)}{(x_i^2 + y_i^2)^{\frac{3}{2}}} + .. \\ &\approx \frac{a_i}{r_i} + \frac{a_i}{r_i^3}(c_x^i x + c_y^i y) + .. \end{aligned} \tag{10}$$

where $c_x^i = x_i$ and $c_y^i = y_i$. For the CESR arc beam position monitors,

$$\begin{aligned} x_2 &= x_4 = -x_1 = -x_3 \\ y_4 &= y_3 = -y_1 = -y_2 \end{aligned}$$

Therefore, $c_x^i = (-1)^i c_x$, $c_y^i = c_y$, for $i = 3, 4$, and $c_y^i = -c_y, i = 1, 2$. Also all r_i are equal. Then

$$\begin{aligned} b_2 + b_4 - (b_1 + b_3) &= \frac{a}{r_0^3} 4 c_x x \\ b_3 + b_4 - (b_2 + b_1) &= \frac{a}{r_0^3} 4 c_y y \\ \sum_{i=1}^{4} b_i &= 4 \frac{a}{r_0} \end{aligned}$$

and

$$\begin{aligned} \frac{c_x}{r_0^2} x &= \frac{b_2 + b_4 - (b_1 + b_3)}{\sum_{i=1}^4 b_i} \\ &\rightarrow \quad x = k_x \frac{b_2 + b_4 - (b_1 + b_3)}{\sum_{i=1}^4 b_i} \\ \frac{c_y}{r_0^2} y &= \frac{b_3 + b_4 - (b_2 + b_1)}{\sum_{i=1}^4 b_i} \\ &\rightarrow \quad y = k_y \frac{b_3 + b_4 - (b_2 + b_1)}{\sum_{i=1}^4 b_i} \end{aligned}$$

where $k_x = \sqrt{x_i^2 + y_i^2}/|x_i|$, $k_y = \sqrt{x_i^2 + y_i^2}/|y_i|$

0.1 Gain and offset

From Equation **??** we see that three button signals are required to determine x and y if we know the geometry. Or we say that there are 3 degrees of freedom relating button signal to position. A possible paramterization would be as

$$\begin{aligned} g_i &= a_i/a_0 \\ h_i^x &= x_i/x_0 \\ h_i^y &= y_i/y_0 \end{aligned}$$

Then

$$b_i = \frac{g_i}{((h_i^x x_0)^2 + (h_i^y y_0)^2)^{\frac{1}{2}}} \left(1 + \frac{(h_i^x x_0 x + h_i^y y_0 y)}{((h_i^x x_0)^2 + (h_i^y y_0)^2)}\right)$$

g_i corresponds to the overall response of the button. If h_i^x or h_i^y are different from unity, the button is displaced its nominal position. So what if the buttons are not all the same, then our algorithm for computing x gives

$$\begin{aligned} & \frac{b_4 + b_2 - (b_1 + b_3)}{\sum_{i=1}^4 b_i} \\ = & \frac{\sum_{i=1}^4 \frac{a_i}{r_i} + \left(\frac{a_4}{r_4^3} c_4^x + \frac{a_2}{r_2^3} c_2^x - (\frac{a_3}{r_3^3} c_3^x + \frac{a_1}{r_1^3} c_1^x)\right) x + \left(\frac{a_4}{r_4^3} c_4^y + \frac{a_2}{r_2^3} c_2^y - (\frac{a_3}{r_3^3} c_3^y + \frac{a_1}{r_1^3} c_1^y)\right) y}{\sum_{i=1}^4 \frac{a_i}{r_i} + \left(\frac{a_4}{r_4^3} c_4^x + \frac{a_2}{r_2^3} c_2^x + (\frac{a_3}{r_3^3} c_3^x + \frac{a_1}{r_1^3} c_1^x)\right) x + \left(\frac{a_4}{r_4^3} c_4^y + \frac{a_2}{r_2^3} c_2^y + (\frac{a_3}{r_3^3} c_3^y + \frac{a_1}{r_1^3} c_1^y)\right) y} \end{aligned}$$

Finally we define $g_i^y = k_i^x/k_0^x$, and $g_i^y = k_i^y/k_0^y$

I'm not sure this is getting us anywhere.

0.2 Another parameterization

Let

$$\begin{aligned} x_i &= (1 + g_x^i) x_0 \\ y_i &= (1 + g_y^i) y_0 \\ a_i &= (1 + g^i) a_0 \end{aligned}$$

Then

$$\begin{aligned} b_i &= \frac{(1 + g^i) a_0}{((1 + g_x^i)^2 x_0^2 + (1 + g_y^i)^2 y_0^2)^{\frac{1}{2}}} \left(1 + \frac{(1 + g_x^i) x_0 x + (1 + g_y^i) y_0 y)}{(1 + g_x^i)^2 x_0^2 + (1 + g_y^i)^2 y_0^2)}\right) \\ &\approx \frac{(1 + g^i) a_0}{(x_0^2 + y_0^2)^{\frac{1}{2}}} (1 - g_x^i - g_y^i) \left(1 + \frac{(1 + g_x^i) x_0 x + (1 + g_y^i) y_0 y)}{(x_0^2 + y_0^2)} (1 - g_x^i - g_y^i)\right) \\ &\approx \frac{(1 + g^i - g_x^i - g_y^i) a_0}{(x_0^2 + y_0^2)^{\frac{1}{2}}} \left(1 + \frac{(1 - g_y^i)}{k_x^0} x + \frac{(1 - g_x^i)}{k_y^0} y\right) \end{aligned}$$

0.3 BPM skew

The BPM buttons are mounted as a top block and a bottom block, each with two buttons. So the distance between the two buttons in each block is well defined. The two blocks are very nearly parallel. But there may be a horizontal displacement of the top block compared to the bottom block. Then

$$\begin{aligned} x_4 &= -x_1 = x_0 - \Delta x \\ x_2 &= -x_3 = x_0 + \Delta x \\ -y_1 &= -y_2 = y_3 = y_4 = y_0 \end{aligned}$$

Assuming that the four buttons all have the same gain we can write

$$\begin{aligned} b_1 &= \frac{a_0}{\sqrt{x_1^2 + y_0^2}} + \frac{a_0(x_1 x - y_0 y)}{(x_1^2 + y_0^2)^{\frac{3}{2}})} \\ &= \frac{a_0}{r_1} + \frac{a_0}{((-x_0 + \Delta x)^2 + y_0^2)^{\frac{3}{2}}}((-x_0 + \Delta x)x - y_0 y) \\ &\sim \frac{a_0}{r_1} + \frac{a_0}{(x_0^2 - 2x_0\Delta x + y_0^2)^{\frac{3}{2}}}((-x_0 + \Delta x)x - y_0 y) \\ &\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}(1 + \frac{3x_0\Delta x}{r_0^2})((-x_0 + \Delta x)x - y_0 y) \\ &\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}((-x_0 + (1 - \frac{3x_0^2}{r_0^2})\Delta x)x - (1 + \frac{3x_0\Delta x}{r_0^2})y_0 y)) \\ b_2 &= \frac{a_0}{\sqrt{x_2^2 + y_0^2}} + \frac{a_0(x_2 x - y_0 y)}{(x_2^2 + y_0^2)^{\frac{3}{2}})} \\ &= \frac{a_0}{r_2} + \frac{a_0}{((x_0 + \Delta x)^2 + y_0^2)^{\frac{3}{2}}}((x_0 + \Delta x)x - y_0 y) \\ &\sim \frac{a_0}{r_2} + \frac{a_0}{(x_0^2 + 2x_0\Delta x + y_0^2)^{\frac{3}{2}}}((x_0 + \Delta x)x - y_0 y) \\ &\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}(1 - \frac{3x_0\Delta x}{r_0^2})((x_0 + \Delta x)x - y_0 y) \\ &\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}((x_0 + (1 - \frac{3x_0^2}{r_0^2})\Delta x)x - (1 - \frac{3x_0\Delta x}{r_0^2})y_0 y)) \\ b_3 &= \frac{a_0}{\sqrt{x_2^2 + y_0^2}} + \frac{a_0(-x_2 x + y_0 y)}{(x_2^2 + y_0^2)^{\frac{3}{2}})} \\ &= \frac{a_0}{r_2} + \frac{a_0}{((x_0 + \Delta x)^2 + y_0^2)^{\frac{3}{2}}}(-(x_0 + \Delta x)x + y_0 y) \\ &\sim \frac{a_0}{r_2} + \frac{a_0}{(x_0^2 + 2x_0\Delta x + y_0^2)^{\frac{3}{2}}}(-(x_0 + \Delta x)x + y_0 y) \end{aligned}$$

$$
\begin{aligned}
&\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}(1 - \frac{3x_0\Delta x}{r_0^2})(-(x_0 + \Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}((-x_0 + (-1 + \frac{3x_0^2}{r_0^2})\Delta x)x + (1 - \frac{3x_0\Delta x}{r_0^2})y_0 y)) \\
b_4 &= \frac{a_0}{\sqrt{x_1^2 + y_0^2}} + \frac{a_0(-x_1 x + y_0 y)}{(x_1^2 + y_0^2)^{\frac{3}{2}})} \\
&= \frac{a_0}{r_1} + \frac{a_0}{((x_0 - \Delta x)^2 + y_0^2)^{\frac{3}{2}}}((x_0 - \Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{(x_0^2 - 2x_0\Delta x + y_0^2)^{\frac{3}{2}}}((x_0 - \Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}(1 + \frac{3x_0\Delta x}{r_0^2})((x_0 - \Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}((x_0 - (1 - \frac{3x_0^2}{r_0^2})\Delta x)x + (1 + \frac{3x_0\Delta x}{r_0^2})y_0 y)
\end{aligned}
$$

Define

$$
\begin{aligned}
k_i^x &= \frac{a_0 x_i}{(x_i^2 + y_0^2)^{\frac{3}{2}})} \\
k_i^y &= \frac{a_0 y_0}{(x_i^2 + y_0^2)^{\frac{3}{2}}} = \frac{a_0 y_0}{((x_0 + (-1)^i\Delta x)^2 + y_0^2)^{\frac{3}{2}}} \\
&\sim \frac{a_0 y_0}{(x_0^2 + y_0^2)^{\frac{3}{2}}}(1 - 3\frac{\pm\Delta x}{(x_0^2 + y_0^2)})
\end{aligned}
$$

The sum

$$
\begin{aligned}
\sum_{i=1}^{4} b_i &= 2a_0(\frac{1}{r_1} + \frac{1}{r_2}) \\
&= 2\frac{a_0}{r_0}\left((1 + \frac{x_0\Delta x}{r_0^2}) + (1 - \frac{x_0\Delta x}{r_0^2})\right) \\
&= 4\frac{a_0}{r_0}
\end{aligned}
$$

Then

$$
\begin{aligned}
b_4 - b_3 + b_2 - b_1 &= c_1 - c_2 + 2\frac{a_0}{r_0^3}\left(x_0 x + \frac{3x_0\Delta x}{r_0^2}y_0 y\right) + c_2 - c_1 + 2\frac{a_0}{r_0^3}\left(x_0 x + \frac{3x_0\Delta x}{r_0^2}y_0 y\right) \\
&= 4\frac{a_0}{r_0^3}\left(x_0 x + \frac{3x_0\Delta x}{r_0^2}y_0 y\right)
\end{aligned}
$$

and

$$x' = k_x \frac{b_4 - b_3 + b_2 - b_1}{\sum_{i=1}^{4} b_i} = k_x \left(\frac{x_0}{r_0^2}\right)\left(x + \frac{3\Delta x}{r_0^2} y_0 y\right)$$

Evidently $k_x = (r_0^2/x_0)$. Furthermore

$$\begin{aligned} b_4 - b_2 + b_3 - b_1 &= c_1 - c_2 + \frac{a_0}{r_0^3}(-2(1 - \frac{3x_0^2}{r_0^2})\Delta x x + 2y_0 y) + c_2 - c_1 + 2\frac{a_0}{r_0^3}((-1 + \frac{3x_0^2}{r_0^2})\Delta x x + 2y_0 y) \\ &= 4\frac{a_0}{r_0^3}((-1 + \frac{3x_0^2}{r_0^2})\Delta x x + y_0 y) \end{aligned}$$

and

$$y' = k_y \frac{b_4 - b_2 + b_3 - b_1}{\sum_{i=1}^{4} b_i} = k_y \frac{y_0}{r_0^2}((-\frac{1}{y_0} + \frac{3x_0^2}{y_0 r_0^2})\Delta x x + y)$$

We define $k_y = r_0^2/y_0$. Then we can write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{3\Delta x}{r_0^2} y_0 \\ (-1 + \frac{3x_0^2}{r_0^2})\frac{\Delta x}{y_0} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Then in terms of g_x, g_y and ϕ,

$$\begin{aligned} 1 &= g_x \cos(\theta + \phi) \\ \frac{3\Delta x}{r_0^2} y_0 &= g_x \sin(\theta + \phi) \\ (-1 + \frac{3x_0^2}{r_0^2})\frac{\Delta x}{y_0} &= -g_y \sin(\theta - \phi) \\ 1 &= g_y \cos(\theta - \phi) \end{aligned}$$

Then if we assume that θ and ϕ are both small

$$\begin{aligned} 2\theta &\approx \left(\frac{1}{y_0} + \frac{3}{r_0^2}(-\frac{x_0^2}{y_0} + y_0)\right)\Delta x = \left(1 + \frac{3}{r_0^2}(-x_0^2 + y_0^2)\right)\frac{\Delta x}{y_0} \\ \to \theta &\approx \frac{1}{2}(1 - 3\cos 2\beta)\frac{\Delta x}{y_0} \end{aligned}$$

where $\beta = \tan -1(y_0/x_0)$ and

$$\begin{aligned} 2\phi &\approx \left(\frac{-1}{y_0} + \frac{3}{r_0^2}(\frac{x_0^2}{y_0} + y_0)\right)\Delta x = \left(-1 + 3\frac{x_0^2 + y_0^2}{r_0^2}\right)\frac{\Delta x}{y_0} \\ \to \phi &\approx \frac{\Delta x}{y_0} \end{aligned}$$

Conclusion

In the limit where $x_0 = y_0$, the tilt angle $\theta \to \frac{1}{2}\phi$. In the limit of a very wide spacing, $x \gg y_0$, then $\beta \to 0$ and $\theta \to -\frac{1}{2}\phi$. Finally, if the chamber is tall, $\beta \to \frac{\pi}{2}$ and $\theta \to 2\phi$.