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Dispersion in terms of coupling parameters

Consider the full turn 4X4 matrix $T = T_0 T_{RF}$ that couples transverse (either horizontal or vertical) and longitudinal motion. We write

$$T_0 = \begin{pmatrix} M & m \\ n & N \end{pmatrix}, \quad T_{RF} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}, \quad \text{and} \quad A = \begin{pmatrix} 1 & 0 \\ -\frac{\omega}{c}\frac{V}{E} & 1 \end{pmatrix}$$

Furthermore

$$\begin{aligned}
M &= \begin{pmatrix} \cos\theta_x - \alpha\sin\theta_x & \beta_x\sin\theta_x \\ -\gamma_x\sin\theta_x & \cos\theta_x + \alpha\sin\theta \end{pmatrix} \\
N &= \begin{pmatrix} 1 & L\alpha_p \\ 0 & 1 \end{pmatrix} \\
m &= (I - M)\begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} \\
&= \begin{pmatrix} 0 & \eta(1 - \cos\theta_x + \alpha\sin\theta_x) - \eta'\beta_x\sin\theta_x \\ 0 & \gamma\eta\sin\theta_x + \eta'(1 - \cos\theta_x - \alpha\sin\theta_x) \end{pmatrix}
\end{aligned}$$

the coupling matrix

$$C = \frac{-H\text{sgn}(\text{Tr}[M - N])}{\gamma\sqrt{(\text{Tr}[M - N])^2 + 4|H|}}$$

where $H = m + n^\dagger$. In the limit where the accelerating voltage V and synchrotron tune are zero $|H| = 0$ and $\gamma = 1$. Then

$$C \to \frac{H}{2(1 - \cos\theta_x)}$$

We take advantage of the symplecticity of T_0 to write n in terms of m, M, N. Then

$$\begin{aligned}
m + n^\dagger &= m + (Nsm^TsM)^\dagger = m + M^\dagger s(m^T)^\dagger sN^\dagger \\
&= m - M^\dagger mN^\dagger \\
&= (I - M)\eta - M^\dagger(I - M)\eta N^\dagger \\
&= (I - M)\eta - (M^\dagger + I)\eta \quad (\text{since } \eta N^\dagger = \eta)
\end{aligned}$$

$$
\begin{aligned}
&= (2I - (\mathrm{Tr}M)I)\,\eta \\
&= \begin{pmatrix} 0 & 2(1-\cos\theta_x)\eta \\ 0 & 2(1-\cos\theta_x)\eta' \end{pmatrix}
\end{aligned}
$$

Finally

$$
\begin{aligned}
C &= \frac{m + n^\dagger}{2(1-\cos\theta_x)} \\
&= \begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix}
\end{aligned}
$$

If the accelerating voltage and synchrotron tune are non-zero then

$$
\begin{aligned}
T &= T_0 T_{RF} \\
&= \begin{pmatrix} M & m \\ n & N \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \\
&= \begin{pmatrix} M & m' \\ n & N' \end{pmatrix}
\end{aligned}
$$

where $m' = \begin{pmatrix} am_{12} & m_{12} \\ am_{22} & m_{22} \end{pmatrix}$ and $a = -\frac{\omega V}{cE}$ and $N' = NA$.
Then

$$
\begin{aligned}
H &\rightarrow m' + n^\dagger \\
&= \begin{pmatrix} am_{12} & 2(1-\cos\theta_x)\eta \\ am_{22} & 2(1-\cos\theta_x)\eta' \end{pmatrix}
\end{aligned}
$$

Now $|H| = 2a(1-\cos\theta_x)(m_{12}\eta' - m_{22}\eta)$ and $\mathrm{Tr}[M-N] = 2(\cos\theta_x - \cos\theta_z)$, and $\gamma = 1 - |C|$.

$$
C = \frac{\begin{pmatrix} am_{12} & 2(1-\cos\theta_x)\eta \\ am_{22} & 2(1-\cos\theta_x)\eta' \end{pmatrix}}{2\gamma\sqrt{(\cos\theta_x - \cos\theta_z) + 2a(1-\cos\theta_x)(m_{12}\eta' - m_{22}\eta)}}
$$

In the limit that $a \rightarrow 0$, $|H| \rightarrow 0$, $\cos\theta_z \rightarrow 1$ and $\gamma \rightarrow 1$.