Effect of button geometry on tilt and crunch

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Button signals as a function of beam position

In a 2-dimensional approximation, where we think of the beam and the buttons as continuous in the z-direction, the charge on the i^{th} button is

$$b_i = a_i \frac{1}{|\vec{r} - \vec{r}_i|} = \frac{a_i}{\sqrt{(x_i - x)^2 + (y_i - y)^2}}$$

where x_i and y_i are coordinates of the button and x and y are the position of the beam. Then in the limit where $x/x_i \ll 1$ and $y/y_i \ll 1$,

$$\begin{aligned} b_i &\approx \frac{a_i}{\sqrt{x_i^2 + y_i^2}} + \frac{a_i(x_i x + y_i y)}{(x_i^2 + y_i^2)^{\frac{3}{2}}} + .. \\ &\approx \frac{a_i}{r_i} + \frac{a_i}{r_i^3}(c_x^i x + c_y^i y) + .. \end{aligned} \tag{1}$$

where $c_x^i = x_i$ and $c_y^i = y_i$. For the CESR arc beam position monitors,

$$\begin{aligned} x_2 &= x_4 = -x_1 = -x_3 \\ y_4 &= y_3 = -y_1 = -y_2 \end{aligned}$$

Therefore, $c_x^i = (-1)^i c_x$, $c_y^i = c_y$, for $i = 3, 4$, and $c_y^i = -c_y, i = 1, 2$. Also all r_i are equal. Then

$$\begin{aligned} b_2 + b_4 - (b_1 + b_3) &= \frac{a}{r_0^3} 4 c_x x \\ b_3 + b_4 - (b_2 + b_1) &= \frac{a}{r_0^3} 4 c_y y \\ \sum_{i=1}^{4} b_i &= 4 \frac{a}{r_0} \end{aligned}$$

and

$$
\begin{aligned}
\frac{c_x}{r_0^2}x &= \frac{b_2+b_4-(b_1+b_3)}{\sum_{i=1}^4 b_i} \\
&\rightarrow \quad x = k_x \frac{b_2+b_4-(b_1+b_3)}{\sum_{i=1}^4 b_i} \\
\frac{c_y}{r_0^2}y &= \frac{b_3+b_4-(b_2+b_1)}{\sum_{i=1}^4 b_i} \\
&\rightarrow \quad y = k_y \frac{b_3+b_4-(b_2+b_1)}{\sum_{i=1}^4 b_i}
\end{aligned}
$$

where $k_x = \sqrt{x_i^2+y_i^2}/|x_i|$, $k_y = \sqrt{x_i^2+y_i^2}/|y_i|$.

BPM shear

The BPM buttons are mounted as a top block and a bottom block, each with two buttons. So the distance between the two buttons in each block is well defined. The two blocks are very nearly parallel. But there may be a horizontal displacement of the top block compared to the bottom block. Then

$$
\begin{aligned}
x_4 &= -x_1 = x_0 - \Delta x \\
x_2 &= -x_3 = x_0 + \Delta x \\
-y_1 &= -y_2 = y_3 = y_4 = y_0
\end{aligned}
$$

Assuming that the four buttons all have the same gain we can write for $\Delta x/x_0 \ll 1$,

$$
\begin{aligned}
b_1 &= \frac{a_0}{\sqrt{x_1^2+y_0^2}} + \frac{a_0(x_1x - y_0y)}{(x_1^2+y_0^2)^{\frac{3}{2}})} \\
&= \frac{a_0}{r_1} + \frac{a_0}{((-x_0+\Delta x)^2+y_0^2)^{\frac{3}{2}}}((-x_0+\Delta x)x - y_0y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{(x_0^2-2x_0\Delta x+y_0^2)^{\frac{3}{2}}}((-x_0+\Delta x)x - y_0y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}(1+\frac{3x_0\Delta x}{r_0^2})((-x_0+\Delta x)x - y_0y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}((-x_0+(1-\frac{3x_0^2}{r_0^2})\Delta x)x - (1+\frac{3x_0\Delta x}{r_0^2})y_0y)) \\
b_2 &= \frac{a_0}{\sqrt{x_2^2+y_0^2}} + \frac{a_0(x_2x - y_0y)}{(x_2^2+y_0^2)^{\frac{3}{2}})}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{a_0}{r_2} + \frac{a_0}{((x_0+\Delta x)^2+y_0^2)^{\frac{3}{2}}}((x_0+\Delta x)x - y_0 y) \\
&\sim \frac{a_0}{r_2} + \frac{a_0}{(x_0^2+2x_0\Delta x+y_0^2)^{\frac{3}{2}}}((x_0+\Delta x)x - y_0 y) \\
&\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}(1-\frac{3x_0\Delta x}{r_0^2})((x_0+\Delta x)x - y_0 y) \\
&\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}((x_0+(1-\frac{3x_0^2}{r_0^2})\Delta x)x - (1-\frac{3x_0\Delta x}{r_0^2})y_0 y))
\end{aligned}
$$

$$
\begin{aligned}
b_3 &= \frac{a_0}{\sqrt{x_2^2+y_0^2}} + \frac{a_0(-x_2 x + y_0 y)}{(x_2^2+y_0^2)^{\frac{3}{2}})} \\
&= \frac{a_0}{r_2} + \frac{a_0}{((x_0+\Delta x)^2+y_0^2)^{\frac{3}{2}}}(-(x_0+\Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_2} + \frac{a_0}{(x_0^2+2x_0\Delta x+y_0^2)^{\frac{3}{2}}}(-(x_0+\Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}(1-\frac{3x_0\Delta x}{r_0^2})(-(x_0+\Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}((-x_0+(-1+\frac{3x_0^2}{r_0^2})\Delta x)x + (1-\frac{3x_0\Delta x}{r_0^2})y_0 y))
\end{aligned}
$$

$$
\begin{aligned}
b_4 &= \frac{a_0}{\sqrt{x_1^2+y_0^2}} + \frac{a_0(-x_1 x + y_0 y)}{(x_1^2+y_0^2)^{\frac{3}{2}})} \\
&= \frac{a_0}{r_1} + \frac{a_0}{((x_0-\Delta x)^2+y_0^2)^{\frac{3}{2}}}((x_0-\Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{(x_0^2-2x_0\Delta x+y_0^2)^{\frac{3}{2}}}((x_0-\Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}(1+\frac{3x_0\Delta x}{r_0^2})((x_0-\Delta x)x + y_0 y) \\
&\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}((x_0-(1-\frac{3x_0^2}{r_0^2})\Delta x)x + (1+\frac{3x_0\Delta x}{r_0^2})y_0 y)
\end{aligned}
$$

In summary we have that

$$
\begin{aligned}
b_1 &\sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}((-x_0+(1-\frac{3x_0^2}{r_0^2})\Delta x)x - (1+\frac{3x_0\Delta x}{r_0^2})y_0 y)) \\
b_2 &\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}((x_0+(1-\frac{3x_0^2}{r_0^2})\Delta x)x - (1-\frac{3x_0\Delta x}{r_0^2})y_0 y)) \\
b_3 &\sim \frac{a_0}{r_2} + \frac{a_0}{r_0^3}((-x_0+(-1+\frac{3x_0^2}{r_0^2})\Delta x)x + (1-\frac{3x_0\Delta x}{r_0^2})y_0 y))
\end{aligned}
$$

$$b_4 \sim \frac{a_0}{r_1} + \frac{a_0}{r_0^3}((x_0 - (1 - \frac{3x_0^2}{r_0^2})\Delta x)x + (1 + \frac{3x_0\Delta x}{r_0^2})y_0 y)$$

The sum

$$\begin{aligned}\sum_{i=1}^{4} b_i &= 2a_0(\frac{1}{r_1} + \frac{1}{r_2}) \\ &\sim 2\frac{a_0}{r_0}\left((1 + \frac{x_0\Delta x}{r_0^2}) + (1 - \frac{x_0\Delta x}{r_0^2})\right) \\ &\sim 4\frac{a_0}{r_0}\end{aligned}$$

Then to determine the measured position (x', y') we apply the usual algorithm,

$$\begin{aligned}b_4 - b_3 + b_2 - b_1 &= c_1 - c_2 + 2\frac{a_0}{r_0^3}\left(x_0 x + \frac{3x_0\Delta x}{r_0^2}y_0 y\right) + c_2 - c_1 + 2\frac{a_0}{r_0^3}\left(x_0 x + \frac{3x_0\Delta x}{r_0^2}y_0 y\right) \\ &= 4\frac{a_0}{r_0^3}\left(x_0 x + \frac{3x_0\Delta x}{r_0^2}y_0 y\right)\end{aligned}$$

and

$$x' = k_x \frac{b_4 - b_3 + b_2 - b_1}{\sum_{i=1}^4 b_i} = k_x \left(\frac{x_0}{r_0^2}\right)\left(x + \frac{3\Delta x}{r_0^2}y_0 y\right)$$

Evidently $k_x = (r_0^2/x_0)$. Furthermore

$$\begin{aligned}b_4 - b_2 + b_3 - b_1 &= c_1 - c_2 + \frac{a_0}{r_0^3}(-2(1 - \frac{3x_0^2}{r_0^2})\Delta x x + 2y_0 y) + c_2 - c_1 + 2\frac{a_0}{r_0^3}((-1 + \frac{3x_0^2}{r_0^2})\Delta x x + 2y_0 y) \\ &= 4\frac{a_0}{r_0^3}((-1 + \frac{3x_0^2}{r_0^2})\Delta x x + y_0 y)\end{aligned}$$

and

$$y' = k_y \frac{b_4 - b_2 + b_3 - b_1}{\sum_{i=1}^4 b_i} = k_y \frac{y_0}{r_0^2}((-\frac{1}{y_0} + \frac{3x_0^2}{y_0 r_0^2})\Delta x x + y)$$

We define $k_y = r_0^2/y_0$. Then we can write

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & \frac{3\Delta x}{r_0^2}y_0 \\ (-1 + \frac{3x_0^2}{r_0^2})\frac{\Delta x}{y_0} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

In terms of g_x, g_y and ϕ,

$$\begin{aligned}
1 &= g_x \cos(\theta + \phi) \\
\frac{3\Delta x}{r_0^2} y_0 &= g_x \sin(\theta + \phi) \\
(-1 + \frac{3x_0^2}{r_0^2}) \frac{\Delta x}{y_0} &= -g_y \sin(\theta - \phi) \\
1 &= g_y \cos(\theta - \phi)
\end{aligned}$$

Then if we assume that θ and ϕ are both small

$$\begin{aligned}
2\theta &\approx \left(\frac{1}{y_0} + \frac{3}{r_0^2}(-\frac{x_0^2}{y_0} + y_0) \right) \Delta x = \left(1 + \frac{3}{r_0^2}(-x_0^2 + y_0^2) \right) \frac{\Delta x}{y_0} \\
\rightarrow \theta &\approx \frac{1}{2}(1 - 3\cos 2\beta) \frac{\Delta x}{y_0}
\end{aligned}$$

where $\beta = \tan -1(y_0/x_0)$ and

$$\begin{aligned}
2\phi &\approx \left(\frac{-1}{y_0} + \frac{3}{r_0^2}(\frac{x_0^2}{y_0} + y_0) \right) \Delta x = \left(-1 + 3\frac{x_0^2 + y_0^2}{r_0^2} \right) \frac{\Delta x}{y_0} \\
\rightarrow \phi &\approx \frac{\Delta x}{y_0}
\end{aligned}$$

Conclusion

In the limit where $x_0 = y_0$, the tilt angle $\theta \rightarrow \frac{1}{2}\phi$. In the limit of a very wide spacing, $x \gg y_0$, then $\beta \rightarrow 0$ and $\theta \rightarrow -\frac{1}{2}\phi$. Finally, if the chamber is tall, $\beta \rightarrow \frac{\pi}{2}$ and $\theta \rightarrow 2\phi$.

Example

For the CESR vacuum chamber $x_0 = 12.98$mm, $y_0 = 24.92$mm. Consider the shear $\Delta x =$ 1mm. Then

$$\begin{aligned}
\phi &= 0.04\text{rad} \\
\theta &= 0.054\text{rad}
\end{aligned}$$

and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0.094 \\ -0.011 & 1 \end{pmatrix}$$