0.1 Measurement of Dispersion by Resonant Excitation

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0.1.1 Introduction

Dispersion is the dependence of the closed orbit on the beam energy. In a storage ring with horizontal but no vertical bends, the horizontal dispersion characterizes the optics. Absent misalignments and transverse coupling the first order vertical dispersion is zero. Residual vertical dispersion results from vertical kicks due to offset quadrupoles and tilted dipoles, and from coupling of horizontal dispersion via tilted quadrupoles and offset sextupoles. Vertical dispersion is a principle source of vertical emittance. Measurement of vertical dispersion is essential to identifying and correcting its sources.

The traditional technique for determining the dispersion is to measure the difference in closed orbits of beams with different energies. The energy is changed by adjusting the frequency of the RF cavities. Alternatively, we recognize that dispersion represents the coupling of longitudinal and transverse motion. This allows us to exploit the techniques developed for measuring horizontal-vertical coupling, in particular resonant excitation of the normal mode frequencies and then measurement of the relative phase and amplitude of the vertical and horizontal response at each beam position monitor. [1]

0.1.2 Formalism

We begin with an illustration of horizontal dispersion as coupling of longitudinal and horizontal motion. The linear motion is characterized by a 4X4 full turn map T. We suppose that there is a single RF cavity with matrix

$$C_{rf} = \left(I \quad 00 \quad A \right), \quad \text{where } A = \begin{pmatrix} 1 & 0 \\ -\frac{\omega}{c}\frac{V}{E} & 1 \end{pmatrix} \tag{1}$$

Then

$$T = RC_{rf} \quad \text{where} \quad R = \begin{pmatrix} X & X_z \\ Z_x & Z \end{pmatrix} \tag{2}$$

Write

$$X = \left(\cos\theta_x - \alpha\sin\theta_x \quad \beta_x\sin\theta_x - \gamma_x\sin\theta_x \quad \cos\theta_x + \alpha\sin\theta_x \right) \text{ and } Z = \left(1 \quad L\alpha_p 0 \quad 1 \right) \tag{3}$$

Since $\delta(\vec{\eta})$ is the closed orbit for energy offset δ we have that

$$\begin{aligned} R \begin{pmatrix} \eta \\ \eta' \\ l \\ 1 \end{pmatrix} &= \begin{pmatrix} \eta \\ \eta' \\ l' \\ 1 \end{pmatrix} \\ \rightarrow X \begin{pmatrix} \eta \\ \eta' \end{pmatrix} + X_z \begin{pmatrix} l \\ 1 \end{pmatrix} &= \begin{pmatrix} \eta \\ \eta' \end{pmatrix} \end{aligned}$$

We know that the transverse position is independent of l as long as there is no RF. Therefore we can write that

$$X_z = (I - X) \left(0 \quad \eta 0 \quad \eta' \right) = \begin{pmatrix} 0 & \eta(1 - \cos\theta_x + \alpha \sin\theta_x) - \eta' \beta_x \sin\theta_x \\ 0 & \gamma\eta \sin\theta_x + \eta'(1 - \cos\theta_x - \alpha \sin\theta_x) \end{pmatrix} \tag{4}$$

and using the symplecticity of R we find that

$$Z_x = \begin{pmatrix} \eta'(1 - \cos\theta_x + \alpha \sin\theta_x) - \eta\gamma \sin\theta_x & \eta(\cos\theta_x + \alpha \sin\theta_x - 1) - \eta' \beta_x \sin\theta_x \\ 0 & 0 \end{pmatrix}$$

Then if

$$T = \begin{pmatrix} M & m \\ n & N \end{pmatrix}$$

we find that

$$H = m + n^\dagger = \begin{pmatrix} -m_{12} \frac{\omega V}{cE} & 2\eta(1 - \cos\theta_x) \\ -m_{22} \frac{\omega V}{cE} & 2\eta'(1 - \cos\theta_x) \end{pmatrix} \tag{5}$$

The coupling matrix defined as

$$C = \frac{-H \mathrm{sgn}(\mathrm{Tr}[M - N])}{\gamma \sqrt{(\mathrm{Tr}[M - N])^2 + 4|H|}} \tag{6}$$

where $\gamma^2 = 1 - |C|$, so that

$$T = VUV^{-1} \tag{7}$$

with

$$U = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \gamma I & C \\ -C^\dagger & \gamma I \end{pmatrix}$$

(A and B are 2x2 matrices. I is the 2x2 identity). The determinant of H as given in Equation 5 is

$$\begin{aligned} |H| &= -\frac{2\omega V}{cE}(1 - \cos\theta_x)(m_{12}\eta' - m_{22}\eta) \\ &= -2\frac{(2\pi Q_z)^2}{L\alpha_p}(1 - \cos\theta_x)(m_{12}\eta' - m_{22}\eta). \end{aligned}$$

L is the circumference, and α_p the momentum compaction. In the weak coupling limit where $4|H| \ll (\mathrm{Tr}[M-N])^2$, that is far from the coupling resonance, $\gamma \sim 1$ and

$$\begin{aligned} C &\sim \frac{H}{\mathrm{Tr}[M-N]} \\ &\sim \frac{H}{2(\cos\theta_x - \cos\theta_z)} \\ &\sim \frac{1}{2(\cos\theta_x - \cos\theta_z)} \begin{pmatrix} -m_{12}\frac{\omega V}{cE} & 2\eta(1-\cos\theta_x) \\ -m_{22}\frac{\omega V}{cE} & 2\eta'(1-\cos\theta_x) \end{pmatrix} \\ &\sim \begin{pmatrix} 0 & \eta \\ 0 & \eta' \end{pmatrix} \end{aligned} \tag{8}$$

where we have assumed that $\cos\theta_z \sim 1$, that is synchrotron tune near zero. We see that the dispersion can be identified with elements of the coupling matrix. A measurement of the longitudinal-horizontal and longitudinal-vertical coupling yields the dispersion.

0.1.3 Measurement of the coupling matrix

We need to relate the coupling matrix elements to the quantity that we can measure, namely the vertical and horizontal amplitude and phase of the signal modulated at the synchrotron tune, at each bpm. It is convenient to use normalized phase space coordinates. We remember that the phase space 4-vector $\vec{x}$ is related to the normalized, normal mode representation $\vec{w}$ according to

$$\vec{x} = VG^{-1}\bar{U}\vec{w} \tag{9}$$

where

$$U = G^{-1}\bar{U}G, \quad G = \begin{pmatrix} G_a & 0 \\ 0 & G_b \end{pmatrix} \tag{10}$$

and

$$\bar{U} = \begin{pmatrix} R(\theta_a) & 0 \\ 0 & R(\theta_b) \end{pmatrix}, \quad R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \text{and} \quad G_i = \begin{pmatrix} \frac{1}{\sqrt{\beta_i}} & 0 \\ -\frac{\alpha_i}{\sqrt{\beta_i}} & \sqrt{\beta_i} \end{pmatrix} \tag{11}$$

We imagine that the a-mode is horizontal(vertical) motion, and the b-mode is synchrotron motion. The beam is resonantly excited at the synchrotron tune Q_s.

Then

$$\begin{pmatrix} w_a \\ w'_a \\ w_b \\ w'_b \end{pmatrix} = w_0 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{12}$$

where we have chosen without loss of generality to specify $\vec{w}$ at a time when $w'_b = 0$. Then at any later time and at location i,

$$\begin{pmatrix} w_a \\ w'_a \\ w_b \\ w'_b \end{pmatrix} = w_0 \begin{pmatrix} 0 \\ 0 \\ \cos(\theta - \phi_{iz}) \\ \sin(\theta - \phi_{iz}) \end{pmatrix} \tag{13}$$

where $\theta = \omega_s t$, and ω_s is the synchrotron tune. From a measurement of the time dependence of the position signal at the i^{th} beam position monitor we extract the transverse amplitude and phase where

$$x_i \quad = \quad A_{ix} \cos(\theta - \phi_{ix}) \tag{14}$$

(or $x \rightarrow y$). We can similarly write the longitudinal displacement in terms of the longitudinal amplitude and phase,

$$z_i \quad = \quad A_{iz} \cos(\theta - \phi_{iz}) \tag{15}$$

where $\phi_{i(x/y)}$ is the horizontal (vertical) normal mode betatron phase advance and ϕ_{iz} is the longitudinal phase advance at $\theta = 2\pi n Q_s$. From Equation 9 we get that

$$\begin{aligned} G\vec{x} &= GVG^{-1}\vec{w} \\ G\vec{x} &= \begin{pmatrix} \gamma & \bar{C} \\ -\bar{C}^{\dagger} & \gamma \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \cos(\theta - \phi_{iz}) \\ \sin(\theta - \phi_{iz}) \end{pmatrix} \end{aligned}$$

where $\bar{C} = G_a C G_b^{-1}$ and then

$$\begin{pmatrix} \frac{1}{\sqrt{\beta_x}} A_{ix} \cos(\theta + \phi_{ix}) \\ \sim \\ \frac{1}{\sqrt{\beta_z}} A_{iz} \cos(\theta + \phi_{iz}) \\ \sim \end{pmatrix} = \begin{pmatrix} \bar{C}_{11} \cos(\theta - \phi_{iz}) + \bar{C}_{12} \sin(\theta - \phi_{iz}) \\ \sim \\ \gamma \cos(\theta - \phi_{iz}) \\ \gamma \sin(\theta - \phi_{iz}) \end{pmatrix}$$

With some rearrangement we find that

$$\bar{C}_{12} = \sqrt{\frac{\beta_{iz}}{\beta_{ix}}}\frac{A_{ix}}{A_{iz}}\sin(\phi_{ix} - \phi_{iz})$$

$$\bar{C}_{11} = \sqrt{\frac{\beta_{iz}}{\beta_{ix}}}\frac{A_{ix}}{A_{iz}}\cos(\phi_{ix} - \phi_{iz})$$

Finally, $C_{12} = \sqrt{\beta_a\beta_b}\bar{C}_{12}$ and then according to Equation 8

$$\begin{aligned}\eta_{x/y} &= C_{12} = \beta_{iz}\frac{A_{i(x/y)}}{A_{iz}}\sin(\phi_{i(x/y)} - \phi_{iz}) \\ &= \sqrt{\beta_{iz}}\frac{A_{i(x/y)}}{a_z}\sin(\phi_{i(x/y)} - \phi_{iz})\end{aligned}$$

where $A_{iz} = a_z\sqrt{\beta_i z}$. Since there is no good way to measure the longitudinal phase and amplitude of the beam at each of the BPMs, we depend on the model calculation. Since the longitudinal focusing is typically very weak, the longitudinal phase advances very slowly and any errors that might arise due to discrepancy between model and measurement tend to be very small.

The longitudinal amplitude is not measured. In practice we determine the longitudinal amplitude (the amplitude of the energy oscillation) by fitting the measured C_{12} (horizontal dispersion data) to the model dispersion. The fitted amplitude is used to determine vertical dispersion.

In summary, to measure the dispersion by resonant excitation

1. Drive the beam at the synchrotron tune
2. Measure the amplitude and phase of the horizontal and vertical components of the motion at each beam position monitor.
3. Compute the longitudinal beta and phase at each BPM from the machine model.
4. Use the measured horizontal and computed longitudinal phase and amplitude to determine a quantity proportional to the horizontal dispersion and fit to the model dispersion to determine the amplitude of energy oscillations
5. From the measured vertical phase and amplitude, the computed longitudinal phase, and the fitted amplitude of energy oscillations, determine the vertical dispersion.

In the CesrTA at the Cornell Electron Storage Ring, the technique yields a measurement of the vertical dispersion with few millimeter resolution in a few seconds.

References

[1] D. Sagan, Littauer, R. Meller, D. Rubin, **Physical Review Special Topics-Accelerators and Beams**, Betatron phase and coupling measurements at the Cornell Electron/Positron Storage Ring, (092801) 2000.