Spectral Ray Stokes Parameters Applied to Synchrotron Radiation

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1 Introduction

A useful figure of merit for synchrotron radiation is its spectral brightness, a quantity related to the photon flux in phase space at a given frequency and calculated using the Wigner distribution function of the electric field. Often times in calculating spectral brightness, the electric field is treated as a scalar. In order to take into account the polarization of the electric field, a set of quantities labeled the spectral ray Stokes parameters have been introduced [1]. A code has been written to calculate these parameters from first principles, starting from the electric field of a moving charged particle in the time domain and then using its Fourier transform to calculate Wigner distribution functions. This paper discusses the equations and numerical methods used in the code. As an example, the code is applied to radiation from an undulator.

2 Electric Field Calculations

2.1 Equations

Instead of calculating the electric field directly in the frequency domain as many other codes do, the electric field was first calculated in the time domain and then through a Fast Fourier Transform (FFT) was brought into the frequency domain. The equation used for time domain calculations of the electric field from a moving charged particle at observer position $\mathbf{r}$ and observer time t was [2] :

$$\mathbf{E}(\mathbf{r}, t) = \{\frac{q}{4\pi\epsilon_0}\frac{Rc}{(\mathbf{R}\cdot\mathbf{u})^3}[c(1-\beta^2)\mathbf{u} + \mathbf{R}\times(\mathbf{u}\times\dot{\boldsymbol{\beta}})]\}_{ret} \tag{1}$$

where

$$\mathbf{u} = c(\hat{\mathbf{R}} - \boldsymbol{\beta}), \tag{2}$$

$\mathbf{R}$ is the position vector from the electron to the observer, $\boldsymbol{\beta}$ is the relative velocity, q is the charge of the particle, c is the speed of light in vacuum, and ϵ_0 is the permittivity of free space. The *ret* subscript means that all values in the brackets must be evaluated at the retarded time t_r, the time at which the radiation leaves the particle. This is as opposed to the observer time t, the time at which the radiation reaches the observation point $\mathbf{r}$. The relationship between these two quantities is:

$$t = t_r + \frac{R(t_r)}{c} \tag{3}$$

Note that in Equation (1) the first term, referred to as the generalized Coulomb field or the velocity field, falls as $\frac{1}{R^2}$ while the second term, referred to as the radiation field or the acceleration field, falls as $\frac{1}{R}$. This means that at large distances, the radiation field term dominates and the velocity field is often ignored in computations of the electric field. Although this approximation is in most cases valid, this code chooses to include both terms in its calculations.

2.2 Numerical Methods

To calculate the quantities needed in Equation (1), the Bmad subroutine library was utilized (See Reference [3]). Using functions from this library, an arbitrary magnetic field can be set up and a particle's trajectory can be tracked through the field. The electric field of Equation (1) was calculated at equal spacing in retarded time with the quantities obtained from the tracking. To cut down on computation time for the FFT, equal spacing in observer time, not retarded time, is desirable. Equation (3) was used to calculate the observer times corresponding to the retarded times of the tracking and then the electric field was interpolated to acquire the equal spacing in observer time. A zero pad option is included with the FFT to allow for finer spectral resolution.

2.3 Example: Bending Magnet

The code was first tested on one of simplest cases, the bending magnet or a uniform magnetic dipole field. The electric field as a function of frequency from an electron traveling through a bending magnet is [4]:

$$\mathbf{E}(\omega) = \frac{e}{3\sqrt{3}\pi\epsilon_0 cR}\frac{\omega}{\omega_c}\gamma(1+\gamma^2\theta^2)(K_{2/3}(\xi)\mathbf{u}_\sigma - i\frac{\gamma\theta K_{1/3}(\xi)}{\sqrt{1+\gamma^2\theta^2}}\mathbf{u}_\pi) \tag{4}$$

where

$$\xi = \frac{2}{9}\frac{\omega}{\omega_c}(1+\gamma^2\theta^2)^{3/2},$$

ω_c is the critical frequency and is given by:

$$\omega_c = \frac{2}{3}\frac{c\gamma^3}{\rho},$$

ρ is the bending magnet radius, γ is the Lorentz factor, θ is the angle between the z axis and the vector from the origin to the observation point, the functions K_ν are modified Bessel's functions of the second kind, and $\mathbf{u}_\sigma$ and $\mathbf{u}_\pi$ are unit vectors in the direction of the horizontal and vertical polarizations respectively.

Figure 1: Electric Field from an Electron in a Bending Magnet, Analytical Solution

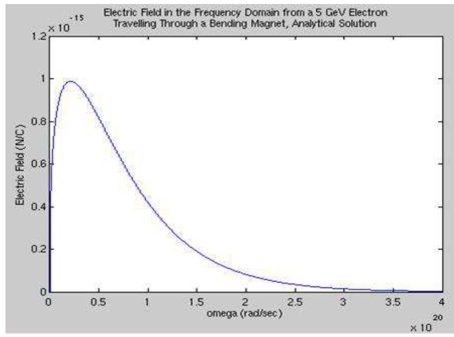
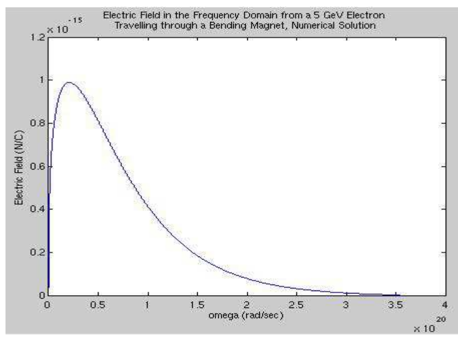


Figure 2: Electric Field from an Electron in a Bending Magnet, Numerical Solution



These two unit vectors are often approximated as the polarization in the x direction and y direction, respectively, as they will be in this code. A plot of the horizontal polarization of Equation (4) is shown in Figure 1 for an observer 100 meters away from an electron with an energy of 5 GeV going through a bending magnet with radius 16.5 meters. The numerical result from Equation (1) for the same situation is shown in Figure 2.

2.4 Example: Undulator

Once it was established that the code worked for a bending magnet, it was tested on the more interesting case of an undulator. The test case used was an undulator of length .15 meters with a magnetic field of strength 1 Tesla and wavelength of 1 centimeter with the observer set 100 meters away. The intensity distribution is shown for the horizontal polarization in Figure 3 and for the vertical polarization in Figure 4.

Figure 3: Horizontal Polarization Intensity from an Undulator

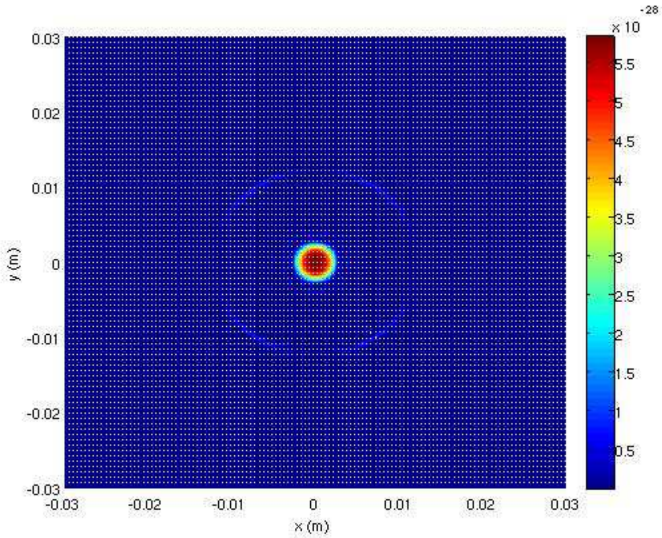
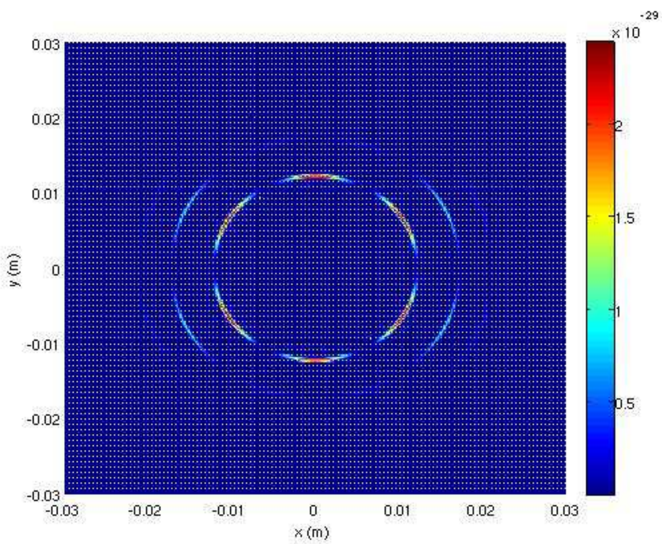


Figure 4: Vertical Polarization Intensity from an Undulator



3 The Wigner Distribution Function

3.1 Theory

In geometric optics, the spectral brightness is defined as the photon flux distribution in phase space at a given frequency (see [5]). This picture of light, however, does not capture wave phenomena such as diffraction and interference. The spectral brightness in wave optics is given by:

$$\mathcal{B}(\mathbf{r}, \mathbf{p}) = (\frac{k}{2\pi})^2 \int_{-\infty}^{\infty} d^2\mathbf{r}' \, \langle E(\mathbf{r} - \mathbf{r}'/2) E^*(\mathbf{r} + \mathbf{r}'/2) \rangle \, e^{ik\mathbf{p}\cdot\mathbf{r}'} \tag{5}$$

where $\mathbf{r}$ represents the Cartesian coordinates in the observer plane, $\mathbf{p}$ represents the directions of propagation, E is the electric field as a function of position at a given frequency, and k is the wave number of the radiation. The integral in Equation (5) is known as a Wigner distribution function and is not itself a physically measurable flux as it can take on negative values which are essential to capture interference and diffraction. To get physically measurable quantities from the Wigner distribution function, Equation (5) can either be integrated over the position variables to yield the angular flux density or the angular variables to yield spatial flux density.

The above definition of spectral brightness treats the electric field as a scalar when in reality it is a vector. To account for the vector nature of the electric field while dealing with intensities, the Stokes parameters were introduced, four parameters each representing the intensity of a different polarization of light as a function of position only. Recently, similar parameters for spectral brightness, named ray Stokes parameters, that are functions of position and propagation direction, have been proposed [1]:

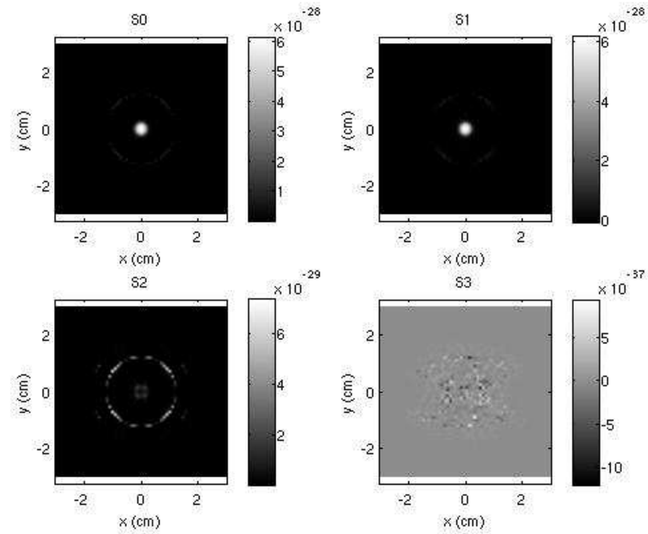
$$\begin{aligned} S_0(\mathbf{r}, \mathbf{p}) &= W_{x,x}(\mathbf{r}, \mathbf{p}) + W_{y,y}(\mathbf{r}, \mathbf{p}) \\ S_1(\mathbf{r}, \mathbf{p}) &= W_{x,x}(\mathbf{r}, \mathbf{p}) - W_{y,y}(\mathbf{r}, \mathbf{p}) \\ S_2(\mathbf{r}, \mathbf{p}) &= W_{x,y}(\mathbf{r}, \mathbf{p}) + W_{y,x}(\mathbf{r}, \mathbf{p}) \\ S_3(\mathbf{r}, \mathbf{p}) &= i[W_{x,y}(\mathbf{r}, \mathbf{p}) - W_{y,x}(\mathbf{r}, \mathbf{p})] \end{aligned} \tag{6}$$

where

$$W_{l,m}(\mathbf{r}, \mathbf{p}) = (\frac{k}{2\pi})^2 \int_{-\infty}^{\infty} d^2\mathbf{r}' \, \langle E_l(\mathbf{r} - \mathbf{r}'/2) E_m^*(\mathbf{r} + \mathbf{r}'/2) \rangle \, e^{ik\mathbf{p}\cdot\mathbf{r}'} \tag{7}$$

and S_0 represents the total spectral brightness, S_1 the amount of linear horizontal and vertical polarization, S_2 the amount of linear 45^o polarization, and S_3 the amount of circular polarization. These parameters provide a complete picture of the light, including its wave nature and polarization. To find the standard intensity Stokes parameters, Equation (6) can be integrated over the angular variables.

Figure 5: Standard Stokes Parameters of Undulator Radiation Calculated from Ray Stokes Parameters



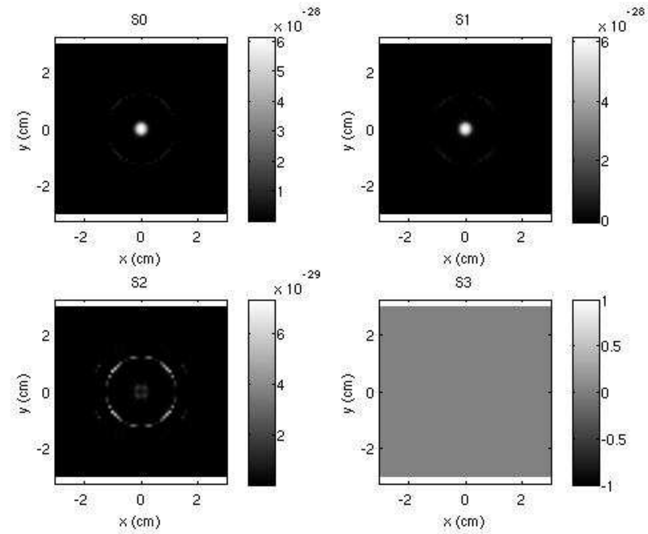
3.2 Example: Undulator

Numerically, the ray Stokes parameters are calculated by shifting the indices of the electric field and taking an FFT. These parameters were calculated for the same example undulator as used to calculate the electric field before. The angular variables were integrated out to yield the standard Stokes parameters as shown in Figure 5. For comparison, the standard Stokes parameters calculated directly from the electric field are show in Figure 6. The presence of S_3 for the Stokes parameters calculated from the Wigner distribution is believed to come from numerical rounding. Note that the ray Stokes parameters are four dimensional quantities and therefore require large amounts of memory to compute. The images in Figures 5 and 6 used the largest sized array that could be handled with the available computing power but this was still fairly small. The pixelation of the images comes from the small number of points used.

4 Conclusion

A code to calculate the spectral ray Stokes parameters, quantities that give a complete picture of light including wave nature and polarization, was presented. The electric field of a charged particle on an arbitrary trajectory was calculated in the time domain without approximation. Its FFT was then used to calculate the Wigner distribution functions required to calculate the spectral ray Stokes parameters. The case of a single electron moving through an undulator was studied. The next step is to extend the code to calculate the ray Stokes pa-

Figure 6: Standard Stokes Parameters of Undulator Radiation Calculated from Electric Fields Directly



rameters for many electrons and to find an efficient method to calculate the ray Stokes parameters to allow calculations with larger arrays.

References

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