Evaluation of a New Cavity Focusing Theory

Robin Bjorkquist
Physics Department, Reed College, Portland, OR, 97202
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This report compares a new derivation of a transport matrix for an accelerating cavity following [1] to a previously published matrix theory [2]. I find that the the two theories are equivalent in the interior section of the cavity. The new theory does not yet fully account for the motion of particles through the fringe field regions at the cavity entrance and exit; by addressing this aspect of the problem, it might be possible to produce a matrix that differs from and improves upon the currently accepted matrix theory. A program is presented that can be used to compare the results of the matrix theories to simulated particle trajectories, in order to test their performance under various initial conditions.

I. INTRODUCTION

An accelerating cavity is a hollow metal structure that supports standing or traveling radio-frequency electromagnetic waves. The longitudinal electric field within the cavity transfers energy to bunches of particles as they pass through, thereby accelerating the beam. However, the cavity also influences the radial motion of particles in the beam. We are interested in the radial motion of particles as they pass through the cavity; only motion in one dimension is considered, in the plane that contains the axis of the cavity.

For the particular mode that is excited in accelerating cavities, the electric field has radial and longitudinal components, and the magnetic field has only an angular component. The longitudinal electric field causes acceleration. The other field components cause a radial force on the beam. The cavity consists of a series of identical cells. The periodic structure of the cavity paired with the oscillations of the fields cause particles to experience an oscillatory radial force as they pass through the cavity. As a result of this force, particles move along oscillatory paths. Particles experience an inward force when they are farther from the axis, and an outward force when they are closer to the axis. Because the radial force is stronger farther from the axis, there is a net inward force on the beam—this produces the cavity focusing effect.

The goal of a cavity focusing theory is to derive a *transport matrix* for the cavity: when the initial coordinates of a particle (before the cavity) are multiplied by the transport matrix, the result is the corresponding coordinates of the particle after it emerges from the other side of the cavity. Such matrices allow for easy calculation of beam characteristics throughout the accelerator structure.

It is important to be able to simulate the behavior of the beam under various conditions because accelerators must be carefully designed to produce needed beam characteristics. A thorough understanding of the effects of all accelerator elements, including the accelerating cavities, improves our ability to control the beam and produce desirable beam qualities, such as small radial size.

II. PREVIOUS CAVITY FOCUSING THEORY

In 1994, J. Rosenzweig and L. Serafini published a cavity focusing theory [2]. Here, I paraphrase their derivation. In places, I have added additional explanation and interpretation beyond what appears in the paper.

The derivation proceeds through the following steps: First, they write the radial force experienced by particles in the cavity in terms of the axial electric field. Then, they present a general form for the axial electric field that can represent both standing and traveling waves. They calculate the average radial force over one cavity cell, following a procedure developed in [3]. Finally, they write and solve an averaged equation of motion, producing a transport matrix.

According to the Lorentz force law, the radial force on a particle in an accelerating cavity, where $\mathbf{E} = E_r\hat{r} + E_z\hat{z}$ and $\mathbf{B} = B_\phi\hat{\phi}$, is given by

$$F_r = q(E_r - v_z B_\phi). \tag{1}$$

However, by making certain assumptions, E_r and B_ϕ may be written in terms of the accelerating field E_z on the axis of the cavity. Those assumptions are that the radial position r is small (and so may be treated only to first order), and that the particle is traveling at relativistic speeds, so that $v \approx v_z \approx c$. Maxwell's equations provide the necessary link between the different field components. In cylindrical coordinates (and in vacuum),

$$\nabla \cdot \mathbf{E} = 0 \Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left[rE_r(r,z,t)\right] + \frac{\partial}{\partial z}\left[E_z(r,z,t)\right] = 0. \tag{2}$$

Integrating the equation with respect to r, and disregarding terms above first order in r, we get

$$rE_r(r,z,t) = \int_0^r -r\frac{\partial}{\partial z}\left[E_z(r,z,t)\right]dr \approx -\frac{1}{2}r^2\frac{\partial}{\partial z}E_z(0,z,t). \tag{3}$$

Similarly,

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \Longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(rB_\phi(r,z,t)) = \frac{1}{c^2}\frac{\partial}{\partial t}E_z(r,z,t) \tag{4}$$

yields the relation

$$B_\phi(r,z,t) = \frac{1}{c^2}\frac{r}{2}\frac{\partial}{\partial t}E_z(0,z,t). \tag{5}$$

In this way, we may write the radial force in terms of the axial accelerating field:

$$F_r \approx q\left(-\frac{r}{2}\frac{\partial}{\partial z}E_z - v_z\frac{1}{c^2}\frac{r}{2}\frac{\partial}{\partial t}E_z\right), \tag{6}$$

where, from now on, E_z is the field at $r = 0$. Applying the chain rule ($\frac{d}{dz} = \frac{\partial}{\partial z} + \frac{1}{v_z}\frac{\partial}{\partial t}$) and then the high-energy approximation that $v_z = c$, we get

$$F_r \approx -\frac{qr}{2}\left(\frac{d}{dz} + \frac{1}{v_z}\left(\frac{v_z^2}{c^2} - 1\right)\frac{\partial}{\partial t}\right)E_z \approx -\frac{qr}{2}\frac{d}{dz}E_z. \tag{7}$$

The accelerating field profile for a traveling wave in an accelerator cavity can be written as the product of a periodic function (dependent on the geometry of the cavity) and a sinusoidal traveling wave with phase velocity $\omega/k = c$, to match the particle velocity:

$$E_z = E_0 \mathrm{Re}\left[\underbrace{\sum_{n=-\infty}^{\infty} b_n e^{-i\frac{2\pi n}{d}z}}_{\text{periodic, with period } d} \quad \overbrace{e^{i(\omega t - kz)}}^{\text{traveling sinusoidal}} \right] \tag{8}$$

where d is the length of one cavity cell. This can be rewritten as

$$E_z = E_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} b_n e^{i(\omega t - k_n z)} \right], \tag{9}$$

where $k_n = k + \frac{2\pi n}{d}$. This is the *Floquet form* used by Rosenzweig and Serafini.

A standing wave may also be represented in Floquet form. In general, a π mode standing wave in an accelerator cavity may be represented by

$$E_z = E_0 A(z) \cos \omega t = E_0 \mathrm{Re}\left[A(z) e^{i\omega t} \right] \tag{10}$$

where $A(z)$ is a real function with periodicity of length $2d$. If we choose to place the origin $z = 0$ at the center of a cavity cell, then the symmetry of the cavity demands that $A(-z) = A(z)$. In addition to periodicity of length $2d$, the accelerating field is opposite in *adjacent* cavity cells (this is what is meant by a π *mode* standing wave): $A(z+d) = -A(z)$. We can write

$$A(z) = \sum_{n=-\infty}^{\infty} a_n e^{-in\frac{\pi}{d}z} \tag{11}$$

with the following restrictions on the a_n:

$$\begin{array}{lll} A(z) \text{ real} & \Longrightarrow & a_{-n} = a_n^* \\ A(z) \text{ even} & \Longrightarrow & a_n \text{ real} \\ A(z+d) = -A(z) & \Longrightarrow & a_n = 0 \text{ for even } n. \end{array} \tag{12}$$

Then, we may rewrite the expression for $A(z)$ with n replaced by $2n+1$, to include only the non-zero terms. The field is thus

$$\begin{aligned} E_z &= E_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} a_{2n+1} e^{-i(2n+1)\frac{\pi}{d}z} e^{i\omega t} \right] = E_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} a_{2n+1} e^{i(\omega t - \frac{\pi}{d}z - 2n\frac{\pi}{d}z)} \right] \\ &= E_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} a_{2n+1} e^{-i\frac{2\pi n}{d}z} e^{i(\omega t - kz)} \right], \end{aligned} \tag{13}$$

noting that $k = \frac{\pi}{d}$ for a π mode standing wave. We see that we have recovered the Floquet form, with $b_n = a_{2n+1}$. Because the a_n are real, so are the b_n, and because $a_{-n} = a_n$, we have

$$b_{-(n+1)} = a_{-(2n+1)} = a_{2n+1} = b_n, \tag{14}$$

the relationship mentioned in Ref. [2]. The b_n are real because we placed $z = 0$ at the center of a cavity cell; similarly, for a traveling wave, the b_n will be real if $z = 0$ is at the center of a cavity cell *and* we specify that the maximally accelerated particle has $\omega t = kz$ (this is equivalent to choosing the placement of the time origin so that at $t = 0$ the maximally accelerated particle is at $z = 0$).

Now that we have an expression for the accelerating field, we calculate the average radial force. For a particle with $\omega t = kz + \Delta\phi$,

$$E_z = E_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} b_n e^{i\left(\Delta\phi - \frac{2\pi n}{d}z\right)}\right]. \tag{15}$$

The radial force experienced by the particle is given by Eq. (7):

$$F_r = -\frac{qr}{2}\frac{d}{dz}E_z = \frac{qE_0\pi}{d} r \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} inb_n e^{i\left(\Delta\phi - \frac{2\pi n}{d}z\right)}\right]. \tag{16}$$

This radial force is oscillatory, with a period equal to one cavity cell. Thus, a particle passing through the cavity undergoes small radial oscillations about an equilibrium position r_0. The cavity has a focusing effect on the beam, so r_0 changes as the beam moves through the cavity. We are more interested in the evolution of r_0 than in the details of the small oscillations. For this reason, it is of interest to calculate the *average* radial force. Ref. [3] details a two-stage process to obtain the average force. Although r_0 is not constant, it changes little over the course of one cavity cell. Thus, we may treat r_0 as constant in order to obtain the trajectory for one period of the small oscillation. Then we average the force over that trajectory. The first step is to replace r by r_0 in the expression for the force, and solve for the motion of the particle. To find the equation of motion, use

$$F_r = \frac{dp_r}{dt} = \frac{d}{dt}(\gamma m v_r). \tag{17}$$

We assume that γ changes little in one cavity cell, so that

$$\frac{d^2r}{dt^2} = \frac{F_r}{\gamma m} \quad \Longrightarrow \quad \frac{d^2r}{dz^2} = \frac{F_r}{\gamma mc^2}, \tag{18}$$

where we have used $v_z = \frac{dz}{dt} \approx c$ to rewrite the time derivative as a space derivative. Using Eq. (16) for F_r, with r_0 in place of r, we have

$$r'' = \frac{qE_0\pi}{\gamma mc^2 d} r_0 \mathrm{Re}\left[\sum_{n=-\infty}^{\infty} inb_n e^{i\left(\Delta\phi - \frac{2\pi n}{d}z\right)}\right]. \tag{19}$$

A particular solution to this differential equation is

$$r = r_0\left(1 - \frac{qE_0 d}{4\pi\gamma mc^2}\mathrm{Re}\left[\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} i\frac{b_n}{n} e^{i\left(\Delta\phi - \frac{2\pi n}{d}z\right)}\right]\right). \tag{20}$$

To get the *general* solution, we would need to add the solution to the corresponding homogeneous differential equation $r'' = 0$; however, here we use this particular solution, with average radial position r_0. Next, we average the force along this path for one period of the cavity structure:

$$
\begin{aligned}
\langle F_r \rangle &= \frac{1}{d}\int_{-d/2}^{d/2} \frac{qE_0\pi}{d} r_0 \left(1 - \frac{qE_0 d}{4\pi\gamma m c^2} \mathrm{Re}\left[\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} i\frac{b_n}{n} e^{i\left(\Delta\phi - \frac{2\pi n}{d}z\right)}\right]\right) \mathrm{Re}\left[\sum_{m=-\infty}^{\infty} imb_m e^{i\left(\Delta\phi - \frac{2\pi m}{d}z\right)}\right] dz \\
&= -\frac{(qE_0)^2}{4\gamma m c^2} r_0 \frac{1}{d}\int_{-d/2}^{d/2} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{b_n}{n} \sin\left(\Delta\phi - \frac{2\pi n}{d}z\right) \sum_{m=-\infty}^{\infty} mb_m \sin\left(\Delta\phi - \frac{2\pi m}{d}z\right) dz \\
&= -\frac{(qE_0)^2}{4\gamma m c^2} r_0 \frac{1}{d}\int_{-d/2}^{d/2} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{b_n}{n} \left[\cos\left(\frac{2\pi n}{d}z\right)\sin(\Delta\phi) - \sin\left(\frac{2\pi n}{d}z\right)\cos(\Delta\phi)\right] \\
&\qquad \times \sum_{m=-\infty}^{\infty} mb_m \left[\cos\left(\frac{2\pi m}{d}z\right)\sin(\Delta\phi) - \sin\left(\frac{2\pi m}{d}z\right)\cos(\Delta\phi)\right] dz \\
&= -\frac{(qE_0)^2}{4\gamma m c^2} r_0 \frac{1}{2} \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \left(\underbrace{b_n^2\left(\sin^2(\Delta\phi) + \cos^2(\Delta\phi)\right)}_{\text{from } m=n} + \underbrace{b_n b_{-n}\left(\cos^2(\Delta\phi) - \sin^2(\Delta\phi)\right)}_{\text{from } m=-n}\right) \\
&= -\frac{(qE_0)^2}{8\gamma m c^2} r_0 \sum_{n=1}^{\infty} \left(b_n^2 + b_{-n}^2 + 2b_n b_{-n}\cos(2\Delta\phi)\right). \qquad (21)
\end{aligned}
$$

For notational simplicity, define

$$\eta(\Delta\phi) = \sum_{n=1}^{\infty} \left(b_n^2 + b_{-n}^2 + 2b_n b_{-n}\cos(2\Delta\phi)\right), \qquad (22)$$

so that

$$\langle F_r \rangle = -\eta(\Delta\phi)\frac{(qE_0)^2}{8\gamma m c^2} r. \qquad (23)$$

Note that $\eta(\Delta\phi)$ is always positive. The minus sign indicates that this is a *focusing* force, as we expected. This average force will be used to solve for the motion we care about—not the small oscillations as the particle passes through the cavity cells, but the more gradual focusing of the particle trajectory.

The next step of the derivation is to write and solve equations for the radial motion of the particle. Now we are interested in the trajectory for more than just one cavity cell, so we no longer treat γ as constant:

$$F_r = \frac{d}{dt}(\gamma m v_r) = c\frac{d}{dz}\left(\gamma m c\frac{dr}{dz}\right) = \gamma' m c^2 r' + \gamma m c^2 r'' \qquad (24)$$

$$r'' + \left(\frac{\gamma'}{\gamma}\right) r' - \frac{F_r}{\gamma m c^2} = 0. \qquad (25)$$

This is an *exact* equation of motion, but we want an equation that takes the *average* force and returns an average trajectory. To accomplish this, average the entire equation over one period:

$$\left\langle r'' + \left(\frac{\gamma'}{\gamma}\right) r' - \frac{F_r}{\gamma m c^2} \right\rangle = 0. \qquad (26)$$

Because Eq. (25) is true, so is Eq. (26). With some further assumptions, however, we may formulate the equation in a more useful form (one that we can solve). The first assumption is that γ changes very little over one cavity cell, so that we can pull it out of the average. The second assumption is that γ increases fairly smoothly, so that we may write $\langle\gamma' r'\rangle$ as $\langle\gamma'\rangle\langle r'\rangle$. Furthermore, we use the approximations $\gamma' \approx \langle\gamma'\rangle$ and $\gamma'' \approx \langle\gamma''\rangle$. Applying these changes and putting in $\langle F_r\rangle$ from Eq. (23), we get

$$r'' + \left(\frac{\langle\gamma'\rangle}{\gamma}\right) r' + \frac{\eta(\Delta\phi)(qE_0)^2}{8(\gamma mc^2)^2} r = 0. \tag{27}$$

Because γmc^2 is the particle energy, $\frac{d}{dz}(\text{energy}) = \text{force}$, and the average force is predominantly longitudinal,

$$\langle\gamma'\rangle = \frac{1}{mc^2}\langle qE_z\rangle = \frac{qE_0\cos(\Delta\phi)}{mc^2}, \tag{28}$$

using $b_0 = 1$, where $\langle E_z\rangle$ comes from averaging Eq. (15). Thus, we may rewrite Eq. (27) as

$$r'' + \left(\frac{\gamma'}{\gamma}\right) r' + \frac{\eta(\Delta\phi)}{8\cos^2(\Delta\phi)}\left(\frac{\gamma'}{\gamma}\right)^2 r = 0, \tag{29}$$

with γ' denoting $\langle\gamma'\rangle$. The solution to Eq. (29) is

$$r = r_i\cos(\alpha) + r_i'\frac{\cos(\Delta\phi)}{\sqrt{\eta(\Delta\phi)/8}}\frac{\gamma_i}{\gamma'}\sin(\alpha), \tag{30}$$

where

$$\alpha \equiv \frac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}\ln\left(\frac{\gamma_f}{\gamma_i}\right). \tag{31}$$

In matrix form, this is

$$\begin{pmatrix} r \\ r' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \dfrac{\cos(\Delta\phi)}{\sqrt{\eta(\Delta\phi)/8}}\dfrac{\gamma_i}{\gamma'}\sin(\alpha) \\ -\dfrac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}\dfrac{\gamma'}{\gamma_f}\sin(\alpha) & \dfrac{\gamma_i}{\gamma_f}\cos(\alpha) \end{pmatrix}\begin{pmatrix} r_i \\ r_i' \end{pmatrix}. \tag{32}$$

III. A NEW CAVITY FOCUSING THEORY

In this section, I present the derivation of a new matrix theory, developed in [1] as an alternative to [2].

Define the coordinate $a = \frac{p_r}{p_0}$. Here, p_r is the radial momentum of the particle, and p_0 is the momentum the particle *would* have if it were moving straight along the axis, with no radial component to its motion. For small r and p_r, $p_0 \approx p_z$. Thus, to first order,

$$r' = \frac{dr}{dz} = \frac{p_r}{p_z} \approx \frac{p_r}{p_0} = a. \tag{33}$$

We seek equations of motion for the coordinates r and a. To that end, take the derivative of a:

$$a' = \frac{d}{dz}\left(\frac{p_r}{p_0}\right) = \frac{1}{p_0}\frac{dp_r}{dz} - \frac{p_r}{p_0^2}\frac{dp_0}{dz} = \frac{1}{p_0}\left(\frac{1}{v_z}\frac{dp_r}{dt} - a\frac{dp_0}{dz}\right). \tag{34}$$

Note that with the added assumption that $v_z = c$, this equation is the same as Rosenzweig and Serafini's equation of motion, Eq. (25). Indeed, the next step in the derivation invokes the high-energy approximation (as well as the linearity of the fields in r) that

$$\frac{dp_r}{dt} = F_r \approx -\frac{qr}{2}\frac{d}{dz}E_z, \tag{35}$$

just as in Eq. (7). A possible area for improvement in this theory is the inclusion of $\frac{1}{\gamma^2}$ terms in the expression for the radial force, but we have not yet incorporated or tested this change. Inserting Eq. (35) into Eq. (34), we can write

$$a' = -\frac{1}{p_0}\left(\frac{qr}{2v_z}\frac{dE_z}{dz} + a\frac{dp_0}{dz}\right). \tag{36}$$

Finally, note that

$$\frac{d^2p_0}{dz^2} = \frac{d}{dz}\left(\frac{q}{v_0}E_z\right) = \frac{q}{v_0}\frac{dE_z}{dz} - \frac{q}{v_0^2}\frac{dv_0}{dz}E_z. \tag{37}$$

Under the assumption that r and a are small, $v_0 = v_z$. Further, $\frac{dv_0}{dz}$ may be neglected in the relativistic limit. Thus, we can rewrite Eq. (36) without explicit reference to the fields:

$$a' = -\frac{1}{p}\left(\frac{r}{2}p'' + ap'\right), \tag{38}$$

where we write p_0 as p for simplicity.

Next, change coordinates from (r, a) to (u, u'), with u defined as

$$u = r\sqrt{p}. \tag{39}$$

The derivatives of u are

$$u' = a\sqrt{p} + \frac{rp'}{2\sqrt{p}} \tag{40}$$

$$u'' = a'\sqrt{p} + \frac{ap'}{\sqrt{p}} - \frac{rp'^2}{4p^{3/2}} + \frac{rp''}{2\sqrt{p}} \tag{41}$$

and a' is given by Eq. (38), so we get

$$u'' = -\frac{1}{\sqrt{p}}\left(\frac{r}{2}p'' + ap'\right) + \frac{ap'}{\sqrt{p}} - \frac{rp'^2}{4p^{3/2}} + \frac{rp''}{2\sqrt{p}} = -\frac{rp'^2}{4p^{3/2}} = -u\left(\frac{p'}{2p}\right)^2. \tag{42}$$

At this point in the derivation, we average the equation over one cavity cell, as Rosenzweig and Serafini did with Eq. (25). Assuming that the energy change in one period is small, we may treat p as a constant and pull it out of the average, yielding

$$u'' = -u\left(\frac{\Delta^2}{4p^2}\right), \quad \Delta \equiv \sqrt{\langle p'^2\rangle}, \tag{43}$$

where u and u'' characterize the averages $\langle u\rangle$ and $\langle u''\rangle$. We make the further assumption that p' is constant, with $p' \approx \langle p'\rangle \equiv \Omega$. Then we may replace the z derivatives with p derivatives, using $\frac{d}{dz} = \Omega\frac{d}{dp}$:

$$\frac{d^2u}{dp^2} = -u\frac{(\Delta/\Omega)^2}{4p^2}. \tag{44}$$

Now change back to r coordinates. Substituting $u = r\sqrt{p}$ into Eq. (44), we get

$$\begin{aligned}
\frac{d^2}{dp^2}\left(r\sqrt{p}\right) &= -r\sqrt{p}\frac{(\Delta/\Omega)^2}{4p^2} \\
\frac{d^2r}{dp^2}\sqrt{p} + \frac{dr}{dp}\frac{1}{\sqrt{p}} - \frac{r}{4p^{3/2}} &= -r\frac{(\Delta/\Omega)^2}{4p^{3/2}} \\
\frac{d^2r}{dp^2} + \frac{1}{p}\frac{dr}{dp} &= -r\frac{\epsilon^2}{p^2},
\end{aligned} \tag{45}$$

where

$$\epsilon \equiv \sqrt{\frac{(\Delta/\Omega)^2 - 1}{4}} \tag{46}$$

is zero for constant p' and therefore characterizes the variations in the cavity's acceleration. Eq. (45) is analogous to Rosenzweig and Serafini's averaged equation of motion, Eq. (29). The solution to Eq. (45) is

$$r = A\cos(\epsilon \ln p) + B\sin(\epsilon \ln p), \tag{47}$$

where A and B are arbitrary constants. In matrix form,

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon\frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos(\epsilon \ln p) & \sin(\epsilon \ln p) \\ -\sin(\epsilon \ln p) & \cos(\epsilon \ln p) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}. \tag{48}$$

Here, p' is still taken to be the constant Ω. However, it may be possible to use the actual p' at a specific location in the first and last cells of a cavity to improve the accuracy of the theory. By inverting this matrix equation, we can solve for A and B in terms of r and a:

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \cos(\epsilon \ln p) & -\sin(\epsilon \ln p) \\ \sin(\epsilon \ln p) & \cos(\epsilon \ln p) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon}\frac{p'}{p} \end{pmatrix} \begin{pmatrix} r \\ a \end{pmatrix}. \tag{49}$$

This is true everywhere, but if we choose to evaluate A and B at the initial position, then we can rewrite Eq. (48) in terms of the initial coordinates:

$$\begin{pmatrix} r \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon\frac{p'}{p} \end{pmatrix} \begin{pmatrix} \cos\left(\epsilon \ln \frac{p}{p_i}\right) & \sin\left(\epsilon \ln \frac{p}{p_i}\right) \\ -\sin\left(\epsilon \ln \frac{p}{p_i}\right) & \cos\left(\epsilon \ln \frac{p}{p_i}\right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\epsilon}\frac{p'}{p_i} \end{pmatrix} \begin{pmatrix} r_i \\ a_i \end{pmatrix}. \tag{50}$$

IV. COMPARISON OF THE TWO THEORIES

We have now derived two transport matrices for the periodic section of an accelerating cavity. The Rosenzweig and Serafini matrix:

$$\begin{pmatrix} \cos(\alpha) & \frac{\cos(\Delta\phi)}{\sqrt{\eta(\Delta\phi)/8}}\frac{\gamma_i}{\gamma'}\sin(\alpha) \\ -\frac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}\frac{\gamma'}{\gamma_f}\sin(\alpha) & \frac{\gamma_i}{\gamma_f}\cos(\alpha) \end{pmatrix}, \quad \alpha = \frac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}\ln\left(\frac{\gamma_f}{\gamma_i}\right) \tag{51}$$

and the Hoffstaetter matrix (with the three component matrices multiplied together, to make the comparison clear):

$$\begin{pmatrix} \cos\left(\epsilon \ln \frac{p_f}{p_i}\right) & \frac{1}{\epsilon}\frac{p_i}{p'} \sin\left(\epsilon \ln \frac{p_f}{p_i}\right) \\ -\epsilon \frac{p'}{p_f} \sin\left(\epsilon \ln \frac{p_f}{p_i}\right) & \frac{p_i}{p_f} \cos\left(\epsilon \ln \frac{p_f}{p_i}\right) \end{pmatrix}. \tag{52}$$

There are two differences between these matrices. First, the Rosenzweig and Serafini matrix uses γ where the Hoffstaetter matrix uses p. However, because p and γ are proportional in the relativistic limit ($p = \gamma m v \approx \gamma m c$), $\frac{\gamma_f}{\gamma_i}$ and $\frac{p_f}{p_i}$ are equivalent for high-energy particles.

The second difference between the two matrices is that the Rosenzweig and Serafini matrix has $\frac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}$ where the Hoffstaetter matrix has ϵ. To compare these two constants, we need to calculate ϵ for the Floquet form of the accelerating field in Eq. (9). The constant ϵ is defined in Eq. (46). Because $p' = \frac{dp}{dt} = \frac{q}{v}E_z$, $\frac{\langle p'^2\rangle}{\langle p'\rangle^2}$ is the same as $\frac{\langle E_z'^2\rangle}{\langle E_z\rangle^2}$. For a particle with phase $\Delta\phi$,

$$\langle E_z \rangle = \left\langle E_0 \sum_{n=-\infty}^{\infty} b_n \cos\left(\Delta\phi - \frac{2\pi n}{d} z\right)\right\rangle = E_0 \cos(\Delta\phi) \tag{53}$$

and, by a calculation much like that in Eq. (21),

$$\begin{aligned} \left\langle E_z^2 \right\rangle &= \left\langle E_0^2 \sum_{n=-\infty}^{\infty} b_n \cos\left(\Delta\phi - \frac{2\pi n}{d} z\right) \sum_{m=-\infty}^{\infty} b_m \cos\left(\Delta\phi - \frac{2\pi m}{d} z\right)\right\rangle \\ &= E_0^2 \left(b_0^2 \cos^2(\Delta\phi) + \frac{1}{2}\sum_{n=1}^{\infty} \left(b_n^2 + b_{-n}^2 + 2 b_n b_{-n} \cos(2\Delta\phi)\right)\right) \\ &= E_0^2 \left(\cos^2(\Delta\phi) + \frac{1}{2}\eta(\Delta\phi)\right), \end{aligned} \tag{54}$$

where $\eta(\Delta\phi)$ is as defined in Eq. (22). Thus we have

$$\epsilon = \sqrt{\frac{1}{4}\left(\frac{\langle E_z^2\rangle}{\langle E_z\rangle^2} - 1\right)} = \sqrt{\frac{1}{4}\left(\frac{E_0^2\left(\cos^2(\Delta\phi) + \frac{1}{2}\eta(\Delta\phi)\right)}{E_0^2 \cos^2(\Delta\phi)} - 1\right)} = \frac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}, \tag{55}$$

showing that the two matrices are identical.

V. THE CAVITY ENTRANCE AND EXIT

The matrices discussed so far propagate the average position and slope through the periodic portion of a cavity array. However, to pass through the entire cavity, a particle must pass through the fringe field regions at the entrance and exit to the cavity—regions in which the central matrix does not apply. In [2], transport matrices for the cavity entrance and exit are developed, and the problem of matching actual coordinates outside the cavity to averaged coordinates inside the cavity is considered. As of this writing, [1] does not completely account for these edge effects; however, this is an area in which it might be possible to improve the theory.

The Rosenzweig and Serafini treatment of the cavity entrance and exit is as follows: the entrance region extends from well outside the cavity to the center of the first cavity cell. The exit region in similarly defined. In making these definitions, we assume that the accelerating field profile of the inside half of the edge cell matches that of the interior cells. Assuming that γ and r are both constant in the edge region, we may integrate Eq. (18) with F_r given by Eq. (7) to find the change in particle angle $\Delta r'$:

$$\Delta r' = -\frac{1}{\gamma mc^2}\frac{qr}{2}\Delta E_z. \tag{56}$$

The axial accelerating field E_z is zero outside the cavity, and $E_m\cos(\Delta\phi)$ at the center of a cell, where E_m is the maximum field at the center of a cell and $\Delta\phi$ is the particle phase. Thus, $\Delta E_z = \pm E_m\cos(\Delta\phi)$ at the entrance (exit) of the cavity, making

$$\Delta r' = \mp\frac{qE_m\cos(\Delta\phi)r}{2\gamma_{i(f)}mc^2} = \mp\frac{\gamma'}{2\gamma_{i(f)}}gr, \tag{57}$$

where γ' is as defined in Eq. (28) and $g = \frac{E_m}{E_0}$ is the ratio of the maximum field at the center of a cell to the average field experienced by a maximally accelerated particle (one with $\Delta\phi = 0$). Returning to our expression for E_z in Eq. (9), and evaluating it at $z = 0$ (the center of a cavity cell) and $t = 0$ (when the field in the center is strongest), we find that

$$E_m = E_0\sum_{n=-\infty}^{\infty} b_n \quad\Longrightarrow\quad g = \sum_{n=-\infty}^{\infty} b_n. \tag{58}$$

Eq. (57) gives the angle at the start of the periodic central section. However, the input to the central matrix needs to be not the *actual* angle, but the corresponding *average* angle. To make the necessary adjustment, Rosenzweig and Serafini calculate the angle of the oscillatory path over which the force was averaged. Taking the derivative of Eq. (20), and evaluating it at $z = 0$, we get

$$-\frac{qE_0\cos(\Delta\phi)r}{2\gamma mc^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} b_n = -\frac{\gamma' r}{2\gamma}(g-b_0) = -\frac{\gamma'}{2\gamma}(g-1)r. \tag{59}$$

This slope adjustment needs to be subtracted from Eq. (57) at the entrance of the cavity, and added to Eq. (57) at the exit. The g terms cancel nicely, resulting in transport matrices of

$$\begin{pmatrix} 1 & 0 \\ \mp\frac{\gamma'}{2\gamma_{i(f)}} & 1 \end{pmatrix} \tag{60}$$

for the edge regions of the cavity. The full transport matrix for the cavity is thus

$$\begin{pmatrix} 1 & 0 \\ \frac{\gamma'}{2\gamma_f} & 1 \end{pmatrix}\begin{pmatrix} \cos(\alpha) & \frac{\cos(\Delta\phi)}{\sqrt{\eta(\Delta\phi)/8}}\frac{\gamma_i}{\gamma'}\sin(\alpha) \\ -\frac{\sqrt{\eta(\Delta\phi)/8}}{\cos(\Delta\phi)}\frac{\gamma'}{\gamma_f}\sin(\alpha) & \frac{\gamma_i}{\gamma_f}\cos(\alpha) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{\gamma'}{2\gamma_i} & 1 \end{pmatrix}. \tag{61}$$

VI. NUMERICAL INTEGRATION

We have compared the two matrix theories to each other. It is also interesting to compare the results of both theories to actual particle trajectories. The derivation of a transport matrix for an accelerating cavity necessarily involves making approximations. Therefore, the resulting matrix will only yield accurate results under certain circumstances—and it is of interest to figure out precisely what those circumstances are.

We used the CLANS program to find the electric and magnetic fields within a 7-cell accelerating cavity; the particular geometry used is a recent version from the design stages for Cornell's Energy Recovery Linac (ERL). With this data as an input, I used *Mathematica* to construct the time-varying fields for a π mode standing wave within the cavity and numerically solve for the trajectory of a particle under the influence of these fields. For a given set of initial coordinates, the program plots the actual particle trajectory, along with the trajectories that result from applying the two matrix theories.

With this program, we have the capability to assess the quality and range of applicability of the transport matrices. A thorough investigations of this sort will be the subject of future work.

VII. CONCLUSION

The new cavity focusing theory results in the same central matrix as the Rosenzweig and Serafini theory, but by means of an alternate derivation. Both derivations rely on the same key assumptions: that r and r' are small and that the particles move at relativistic velocities. In addition, both derivations follow the same basic procedure: first, write equations of motion; then, average those equations over one cavity cell; finally, solve the average equations to obtain a transport matrix. While it does not produce a new result, the Hoffstaetter derivation gives us a new way to approach the same problem—a way which might lead to improvements in the theory.

There are many interesting aspects of this problem still to be addressed.

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