

ASPECTS OF EFFECTIVE SUPERSYMMETRIC THEORIES

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Panteleimon Tziveloglou

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ASPECTS OF EFFECTIVE SUPERSYMMETRIC THEORIES

Panteleimon Tziveloglou, Ph.D.

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This work consists of two parts. In the first part we construct the complete extension of the Minimal Supersymmetric Standard Model by higher dimensional effective operators and then study its phenomenology. These operators encapsulate the effects on LHC physics of any kind of new degrees of freedom at the multiTeV scale. The effective analysis includes the case where the multiTeV physics is the supersymmetry breaking sector itself. In that case the appropriate framework is nonlinear supersymmetry. We choose to realize the nonlinear symmetry by the method of constrained superfields. Beyond the new effective couplings, the analysis suggests an interpretation of the ‘little hierarchy problem’ as an indication of new physics at multiTeV scale.

In the second part we explore the power of constrained superfields in extended supersymmetry. It is known that in $\mathcal{N} = 2$ supersymmetry the gauge kinetic function cannot depend on hypermultiplet scalars. However, it is also known that the low energy effective action of a D-brane in an $\mathcal{N} = 2$ supersymmetric bulk includes the DBI action, where the gauge kinetic function does depend on the dilaton. We show how the nonlinearization of the second SUSY (imposed by the presence of the D-brane) opens this possibility, by constructing the global $\mathcal{N} = 1$ linear + 1 nonlinear invariant coupling of a hypermultiplet with a gauge multiplet. The constructed theory enjoys interesting features, including a novel super-Higgs mechanism without gravity.

BIOGRAPHICAL SKETCH

Pantelis was born in Thessaloniki, Greece, on the 19th of December 1983. In 2001, he enrolled in the Physics department of Aristotle University of Thessaloniki. In 2005, after obtaining the physics degree, he was accepted to Cornell University for postgraduate studies in physics. His academic interests brought him in 2007 to CERN in order to conduct his doctoral research under the guidance of Prof. Ignatios Antoniadis. He stayed there for three years until the completion of his Ph.D. thesis. From October 2010 he is situated in Paris and École Polytechnique for postdoctoral research.

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TABLE OF CONTENTS

| | |
|---|-----------|
| Biographical Sketch | iii |
| Acknowledgements | iv |
| Table of Contents | v |
| List of Figures | viii |
| 1 Introduction | 1 |
| 1.1 The Importance of Supersymmetry | 1 |
| 1.2 Effective and Nonlinear Field Theory in the Minimal Supersymmetric Standard Model | 4 |
| 1.3 Dilaton - DBI couplings in $\mathcal{N} = 2$ supersymmetry | 8 |
| I BEYOND THE MINIMAL SUPERSYMMETRIC STANDARD MODEL | 13 |
| 2 Effective Field Theory | 14 |
| 2.1 Physics is Effective | 14 |
| 2.2 Effective Description of Supersymmetric Theories | 19 |
| 2.2.1 Integrating out Massive Superfields | 19 |
| 2.2.2 Gauge Interactions and Component Analysis | 23 |
| 2.3 Nonlinear Realizations and Constrained Goldstino Superfield | 27 |
| 3 MSSM₅ | 33 |
| 3.1 The Lagrangian | 33 |
| 3.2 Keeping the essential: The irreducible Lagrangian | 36 |
| 4 Phenomenology of MSSM₅ | 43 |
| 4.1 Further Restrictions from Flavor Changing Neutral Currents | 43 |
| 4.2 Phenomenological Implications | 46 |
| 4.3 The MSSM Higgs Sector with Mass Dimension Five Operators | 51 |
| 4.4 Higgs Mass Corrections Beyond MSSM | 53 |
| 4.5 Including Loop Corrections | 61 |
| 5 MSSM Higgs with Operators of Mass Dimension 5 and 6 | 63 |
| 5.1 The Relevant Operators | 63 |
| 5.2 The Scalar Potential | 68 |
| 5.3 Corrections to the MSSM Higgs Masses | 75 |
| 5.4 Analysis of the Leading Corrections and Effective Operators | 81 |
| 6 Nonlinear MSSM | 86 |
| 6.1 The Model | 86 |
| 6.2 New Couplings in the Lagrangian | 89 |
| 6.3 Implications for the Higgs Masses | 92 |

| | | |
|---|--|------------|
| 6.4 | Other Phenomenological Implications | 97 |
| 6.4.1 | Fine Tuning of the Electroweak Scale | 97 |
| 6.4.2 | Limiting Cases and Loop Corrections | 98 |
| 6.4.3 | Invisible Decays of Higgs and Z Bosons | 100 |
| 7 | Summary of Results | 107 |
| II BRANE - BULK INTERACTIONS IN | | |
| N = 2 GLOBAL AND LOCAL SUPERSYMMETRY | | 110 |
| 8 | Preliminaries | 111 |
| 8.1 | The Dirac Born Infeld Action as the Effective Action of a D-brane | 111 |
| 8.2 | Quaternion-Kähler and Hyper-Kähler Manifolds | 114 |
| 8.3 | Superspace Conventions | 116 |
| 9 | The Linear N = 2 Maxwell-Dilaton System | 119 |
| 9.1 | The Single-Tensor Multiplet | 119 |
| 9.2 | The Maxwell Multiplet, Fayet-Iliopoulos Terms | 122 |
| 9.3 | The Chern-Simons Interaction | 125 |
| 9.4 | The Significance of V_1 , X and Y | 127 |
| 9.5 | Chiral N = 2 Superspace | 130 |
| 9.5.1 | Chiral N = 2 Superfields | 131 |
| 9.5.2 | Electric-Magnetic Duality | 134 |
| 10 | Nonlinear N = 2 Supersymmetry and the DBI Action | 138 |
| 10.1 | The N = 2 Super-Maxwell DBI Theory | 138 |
| 10.2 | Coupling the DBI Theory to a Single-Tensor Multiplet: a Super-Higgs Mechanism without Gravity | 141 |
| 10.3 | Hypermultiplet, Double-Tensor and Single-Tensor Dual Formu- lations | 146 |
| 10.4 | The Magnetic Dual | 149 |
| 10.4.1 | The Bosonic Lagrangian | 152 |
| 10.5 | Double-Tensor Formulation and Connection with the String Fields | 154 |
| 10.6 | Nonlinear N = 2 QED | 157 |
| 11 | The Universal Hypermultiplet in Local and Global Supersymmetry | 163 |
| 11.1 | On the Heisenberg Algebra and Global Supersymmetry | 163 |
| 11.1.1 | Lagrangians | 164 |
| 11.1.2 | Hyper-Kähler Metrics with Heisenberg Symmetry | 168 |
| 11.2 | The Universal Hypermultiplet in N = 2 Supergravity | 170 |
| 11.2.1 | The Calderbank-Pedersen Metric with Heisenberg Sym- metry | 171 |
| 11.2.2 | Coupling to N = 2 Supergravity | 174 |
| 11.2.3 | $Sp(2, 4)$ | 175 |

| | | |
|-----------|---|------------|
| 11.2.4 | The Heisenberg Subalgebra of $SU(1,2)$ and $Sp(2,4)$ | 177 |
| 11.2.5 | $N = 2$ Supergravity Scalar Lagrangian | 179 |
| 11.2.6 | Solving the Constraints | 182 |
| 11.3 | Zero-Curvature Hyper-Kähler Limit | 185 |
| 12 | Summary of Results | 191 |
| A | Coefficients for the Higgs Masses | 194 |
| B | The Solution of the Quadratic Constraint | 196 |
| C | Equivalent Descriptions of the Dilaton Multiplet | 199 |
| C.1 | The Single Tensor Formulation | 199 |
| C.2 | Hypermultiplet Formulation | 203 |
| C.3 | Two-Tensor Formulation | 207 |
| D | Obtaining the Taub-NUT Metric from Conformal Supergravity | 210 |
| D.1 | $SU(2,1)/SU(2) \times U(1)$ and its Global Hyper-Kähler Limit | 210 |
| D.2 | Taub-NUT | 219 |
| | Bibliography | 224 |

LIST OF FIGURES

| | | |
|------|---|-----|
| 6.1 | <p>The tree-level Higgs masses (in GeV) and expansion coefficients as functions of \sqrt{f} (in GeV). In (a), (b) $\mu = 900$ GeV, $\tan\beta = 50$, m_A increases upwards from 90 to 150 GeV in steps of 10 GeV. Larger m_A has little impact on m_h for relevant \sqrt{f}. In (c), (d), $m_A = 150$ GeV, and m_h increases as μ varies from 400 to 1200 GeV, in steps of 100 GeV. In (c) $\tan\beta = 50$ while in (d) $\tan\beta = 5$, showing a milder dependence on $\tan\beta$ than in MSSM. For $\tan\beta \geq 10$ there is little difference from (c). In (e), (f) the expansion coefficients are shown, for $m_A = [90, 650]$ GeV with steps of 10 GeV, $\mu = 900$ GeV, $\tan\beta = 50$; they are less than unity (even at larger μ), as required for a convergent expansion.</p> | 94 |
| 6.2 | <p>The partial decay rate of $h^0 \rightarrow \psi_X \chi_1^0$ for (a): $\tan\beta = 50$, $m_{\lambda_1} = 70$ GeV, $m_{\lambda_2} = 150$ GeV, μ increases from 50 GeV (top curve) by a step 50 GeV, $m_A = 150$ GeV. Compare against Figure 6.1 (c) corresponding to a similar range for the parameters. At larger μ, m_h increases, but the partial decay rate decreases. Similar picture is obtained at low $\tan\beta \sim 5$. (b): As for (a) but with $\tan\beta = 5$. Compare against Figure 6.1 (d). Note that the total SM decay rate, for $m_h \sim 114$ GeV, is of order 10^{-3}, thus the branching ratio in the above cases becomes comparable to that of SM Higgs going into $\gamma\gamma$ (see Figure 2 in [81]).</p> | 103 |
| 12.1 | <p>Web of dualities: double arrows indicate duality transformations preserving off-shell $\mathcal{N} = 2$ supersymmetry, simple arrows are $\mathcal{N} = 1$ off-shell dualities only, leading to theories with on-shell $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 1$ superfields and the related equations are indicated.</p> | 192 |

CHAPTER 1

INTRODUCTION

1.1 The Importance of Supersymmetry

Probably the most significant manifestation of the beauty of Supersymmetry is that this simple idea of a symmetry that relates fermions and bosons has proven to be one of the most fruitful proposals in theoretical high energy physics of the last forty years.

At the level of phenomenology, supersymmetry offers a complete or partial solution to almost all shortcomings of the Standard Model (SM). For example, the beautiful properties of SM under renormalization are based on the fact that it is a model of fermions and gauge bosons. However, its cornerstone, the Higgs mechanism, is bound to the existence of a scalar mode. The Higgs scalar seems very unnatural within the framework of the SM. It is the only scalar field and it doesn't share the same renormalization properties with the others. More specifically, the natural value for its mass is at the Planck scale, which would obviously destroy the validity of the model. This puzzle comes with the name "hierarchy problem" and it's believed to be one of the main reasons for leaving SM behind. The solution by supersymmetry is based on treating scalars on equal footing with all other fields. Not only it contains a variety of scalars, degrading them

from the special role they enjoyed in SM, their normalization properties are also no different than all other fields. Their masses scale logarithmically with the cutoff scale which then offers a resolution to the hierarchy problem.

Another source of skepticism towards the SM comes from cosmology. There is a set of cosmological and astrophysical observations that lead to the same conclusion. The stable matter described by the SM, which is the matter that surrounds us, is nothing but a tiny fraction of the full matter content of the universe. ‘Out there’, stable particles exist that we have never observed and that are not described by the SM. The observations can also inform us about the basic properties of these particles. It comes out that they have to be massive and weakly interacting. Once again, supersymmetry has the answer. Supersymmetric models generically come with one stable particle that enjoys the desired properties.

We should also mention that supersymmetry seems to complete the program for unification of gauge interactions. The SM had the striking success of unifying the numerous processes between particles observed in colliders (and seeming extremely complicated in the early years of particle physics) into three fundamental gauge interactions parametrized by three independent coupling constants. The unification would be complete by further unifying into a single gauge group, which would then lead us to a “Grand Unified Theory”. Unfortunately it was calculated that the renormalization group (RG) equations of the SM don’t meet at a single point for unification to occur. New degrees of freedom are needed to shift the RG in a way they meet. It has been shown that the degrees of freedom brought by supersymmetry do the job and the predicted unification occurs at around 10^{16} GeV.

The above arguments favor supersymmetric models as a candidate for departure from the SM. It seems however that it doesn't merely offer a model for a successful replacement of the SM but its basic concepts play a fundamental role in quantum field theory. This can be seen as follows. In a paper of 1967 by S. Coleman and J. Mandula [6] it was shown that the most general Lie algebra of symmetries of the S-matrix is the Poincaré algebra plus a number of Lorentz scalar generators that form the algebra of a compact Lie group. This was a conclusive no-go theorem about the allowed symmetries of the S-matrix and in particular about the impossibility of a nontrivial combination of a spacetime symmetry with an internal one. The Coleman-Mandula theorem was extremely powerful as it was based on generic assumptions that would apply to any quantum field theory. However, it was later discovered that the assumption that the algebras need to be Lie algebras was too restrictive as one could add fermionic generators forming what is called "graded Lie algebras". In a paper by Haag, Sohnius and Lopuszanski seven years later, it was shown that the only graded Lie algebras that generate symmetries allowed by the generic assumptions of quantum field theory are the supersymmetric algebras [7]. In a few words, the exploration of the largest symmetry allowed by the S-matrix has inevitably led us to supersymmetry.

Last but definitely not least, supersymmetry opens a window for the holy grail of theoretical physics, the unification of gravity with the other three forces. The combination of the principle that gravity is the manifestation of the curvature of spacetime, coming from general relativity, and the fact that supersymmetry is a spacetime symmetry, coming from the Haag-Sohnius-Lopuszanski theorem above, implies that a theory with local supersymmetry is a theory of gravity. Such a theory is called "supergravity". Supergravities themselves ap-

pear as the low energy effective theories of various settings of string theory, the only framework where gravity and the other forces are unified into a single and finite theory. In summary, following the path: Global Supersymmetry \rightarrow Local Supersymmetry \rightarrow String Theory we obtain, for the first time, a complete picture of how the unification of particles and interactions works. Furthermore, the principle of supersymmetry is built in string theory. The very first appearance of a symmetry that exchanges bosons and fermions first appeared in the context of dual models [8, 9], which is what was later reinterpreted as string theory. Without supersymmetry, string theory would not be a consistent theory. In a few words, the most basic ingredient of the only known path to a theory where matter and forces are unified, is supersymmetry.

This thesis touches upon both model building in supersymmetric theories and more formal aspects, especially related to string theory. It is then naturally divided in two parts which are weakly related to each other and can be read independently. It is based on publications [1, 2, 3, 4, 5].

1.2 Effective and Nonlinear Field Theory in the Minimal Supersymmetric Standard Model

In the first part we apply the techniques of Effective Field Theory (EFT) on the Minimal Supersymmetric Standard Model (MSSM) and study their phenomenological consequences. The MSSM is the minimal extension of the SM and is used as a prototype model for phenomenological studies of supersymmetry. Our method involves the addition of higher mass dimension terms in the MSSM Lagrangian. From an EFT point of view, the appearance of such terms is

not a sign that the model is sick but rather an indication that it is valid only up to the mass scale that suppresses those terms. Their purpose is to parametrize the effects of any kind of new physics that might exist at a scale that is not approachable by LHC and in the same time not too high, so in the range of a few TeV.

In a few lines, the method of our analysis is as follows. We construct the effective Lagrangian by adding to that of MSSM nonrenormalizable terms of higher mass dimension. These are terms that would appear in a low energy effective model of some UV renormalizable theory by integrating out degrees of freedom above a certain mass scale M . However, in a bottom-up point of view we don't focus on the origin of these terms but rather on a generic analysis of their effects. To this purpose, we choose at a first level to add to MSSM all possible mass dimension five operators that are all allowed by the gauge symmetries and by R-parity. In this way, EFT allows us to draw conclusions that are completely model independent. For a more detailed discussion of supersymmetric EFT, see sec. 2.

Generally this constitutes a huge set of extra free parameters, limiting the predictability of the model. Nevertheless, many of these operators are actually redundant as they can be eliminated by proper field redefinitions. In our analysis, we perform such redefinitions reducing to a model with less parameters and thus more distinct phenomenology. We firstly focus on the Higgs sector because of its special importance in view of the little hierarchy problem and because its extension by effective terms is quite restricted, facilitating drawing clear conclusions. After that we pass on to other couplings and processes that may be interesting for LHC physics. Below we summarize the content of the chapters

of part I.

In chapter 3, we focus on the most general set of R-parity conserving, mass dimension five operators that can exist in the MSSM [1]. We also employ spurion superfields to include any soft supersymmetry breaking effects that these operators parametrize. It turns out that not all of these operators are actually independent. We perform spurion dependent field redefinitions to remove the redundancy thus obtaining the minimal, irreducible set of dimension five operators within MSSM. By incorporating further constraints coming from flavor changing neutral currents (FCNC), we end up with the final model which we call “MSSM₅”.

In chapter 4, we go on to study the phenomenological consequences of MSSM₅ [1]. One consequence is the generation of new effective interactions of the type quark-quark-squark-squark with potentially large effects in squark production compared to those generated in the MSSM, especially for the top/stop quarks. This can be important for LHC supersymmetry searches by direct squark production. Additional “wrong” Higgs couplings, familiar in the MSSM at the loop level [10, 11, 12], are also generated with a coefficient that can be larger than the loop-generated MSSM one. Again, these are largest for the top and also bottom sector at large $\tan\beta$. Furthermore, we study the effect of the new terms in the Higgs potential. It turns out that the mass of the Higgs can be shifted in a way that it alleviates the little hierarchy problem. This implies that we can obtain a novel point of view towards this apparent shortcoming of MSSM. Instead of considering it as a weakness of the theory, we can think of it as an indication for new massive particles at the energy range of few TeV.

Consideration about the stability of the effective potential as well as an ob-

served $\tan\beta$ suppression of the correction to the Higgs mass by five dimensional operators leads to the inevitable inclusion of mass dimension six operators in the Higgs sector [2]. In chapter 5, we perform this analysis insisting on a generic approach, including all possible dimension six operators allowed by the symmetries of the model. In the large $\tan\beta$ region, these two classes of operators can have comparable contributions to the Higgs mass which implies a further alleviation of the little hierarchy.

In chapter 6, we move on to study a different type of EFT, this time realized by nonlinear supersymmetry [3]. In models of low energy SUSY breaking, the gravitino acquires a sub-eV mass and thus it cannot be excluded from the spectrum of the low energy model. If this model is MSSM, we have to study couplings of the gravitino to MSSM. The “equivalence theorem”, which states that in scenaria with very low gravitinos the latter can be effectively replaced by their goldstino component which dominates over the dynamics, greatly simplifies such studies [13]. Nonlinear supersymmetry offers then the most convenient formalism for studying goldstino self interactions and goldstino-matter couplings. We use the method of constrained superfields to realize the nonlinear SUSY algebra and study the most general couplings of the goldstino with MSSM fields.

An important effect of these couplings is the increase in the mass of the Higgs, which can be significant for a SUSY breaking scale at the range of few TeV. This offers one more way for alleviating the little hierarchy. The difference is that in this case we don’t even have to assume some kind of new physics at the high scale. The SUSY breaking mechanism itself brings the correction. In addition, we calculated the invisible decay of Higgs to neutralinos and goldsti-

nos and found that it can be comparable with the standard MSSM decay rate of Higgs to photons. Finally, we found that, in the case that the mass of Z is larger than that of the lightest neutralino, there is a bound on the SUSY breaking scale at around 400 - 700 GeV coming from the invisible Z boson decay.

1.3 Dilaton - DBI couplings in $\mathcal{N} = 2$ supersymmetry

In the second part of the thesis we turn towards aspects of supersymmetry closely related to supergravity and string theory. Our target now is to understand how the coupling of a D-brane to the bulk arises in field theory.

The stage that we choose to focus on is type II strings on $\mathcal{R}_{3,1} \times \mathcal{CY}_3$. The geometry of the Calabi Yau manifold breaks SUSY, giving rise to a 4D $\mathcal{N} = 2$ effective supergravity theory. Generically, the presence of a D-brane in such background spontaneously breaks half supersymmetry on its worldvolume giving rise to an $\mathcal{N} = 1 + 1$ supersymmetric theory where the second supersymmetry is realized nonlinearly. The effective D-brane action is described by a Dirac-Born-Infeld (DBI) theory. It is an effective action for the gauge multiplets of the D-brane as well as for their coupling to the bulk fields. The latter can be described by hypermultiplets, single-tensor multiplets or double-tensor multiplets. All descriptions are Poicaré dual to each other.

Reproducing this action from field theory is the main aim of this second part. This task is nontrivial for two reasons. First, it is known that $\mathcal{N} = 2$ linear supersymmetry, global or local, forbids a dependence of gauge kinetic terms on hypermultiplet scalars. For instance, in $\mathcal{N} = 2$ supergravity, the scalar manifold is the product of a quaternion-Kähler manifold for hypermultiplet scalars

[14] and a Kähler manifold of a special type for vector multiplet scalars [15]. In global $\mathcal{N} = 2$ supersymmetry, the quaternion-Kähler manifold of hypermultiplet scalars is replaced by a Ricci-flat hyper-Kähler space [16]. Second, consistency of compactification of type II strings with D-branes requires the presence of orientifolds necessary for tadpole cancellation. These objects break supersymmetry explicitly globally, although is still preserved locally around the D-branes and away from the orientifold plane. It is then not clear if it is possible to construct from field theory the action that couples the bulk and brane multiplets, even those that would be truncated by the orientifold projection.

The DBI action appearing in D-brane dynamics suggests that the restrictions on the coupling between bulk and brane fields in $\mathcal{N} = 2$ supersymmetry are expected to change if (at least) one of the supersymmetries is nonlinearly realized. This is the path that we follow. In chapter 10, we construct an $\mathcal{N} = 2$ action for the coupling of a single tensor multiplet with a gauge multiplet. This coupling is essentially the supersymmetrization of the Chern-Simons $B \wedge F$ coupling of the antisymmetric NSNS 2-form and the gauge field strength. We then impose nonlinear realization of the second SUSY by applying a supersymmetric constraint on the gauge multiplet. This is the generalization for $\mathcal{N} = 2$ superspace of the constrained superfield method used in the first part of the thesis. The resulting action is invariant under $\mathcal{N} = 1$ linear + 1 nonlinear SUSY and involves the Maxwell goldstino multiplet coupled to a single tensor multiplet [4]. If we remove this multiplet, the action reduces to the standard super-Maxwell DBI theory derived in the past [20, 21, 22].

We have chosen to group the bulk fields in a single tensor multiplet because it is the only one that admits a simple off shell superspace formulation. Hy-

permultiplets also can be formulated off-shell in the context of harmonic superspace but only in the expense of introducing infinite number of auxiliary fields [23]. In any case we can always switch between hyper-, single-tensor and double-tensor multiplets by performing Poicaré dualities.

By appropriate field redefinitions we obtain another equivalent description of the system, in terms of the Higgs phase of $\mathcal{N} = 1+1$ QED [24, 25]. This basis reveals some very interesting features of the system. The goldstino multiplet combines with a chiral superfield to form a $\mathcal{N} = 1$ massive vector multiplet while the other chiral superfield remains massless. This is a novel type of super-Higgs mechanism that does not require a gravitino (which would normally ‘eat’ the goldstino as in the standard super-Higgs mechanism). Also, at one point along the flat direction of the potential, the vector multiplet becomes massless and the $U(1)$ gauge symmetry is restored. This is a known phenomenon from D-brane dynamics, where the $U(1)$ world-volume field becomes generically massive due to the CS coupling.

Having constructed the $\mathcal{N} = 1 + 1$ DBI action, the next step would be to identify its field content in terms of string fields. As already mentioned, the analog of this construction in string theory is that of type IIB strings compactified on a Calabi Yau and interacting with a D-brane. The bulk fields under consideration are the dilaton scalar (associated to the string coupling), the (Neveu-Schwarz) NS-NS antisymmetric tensor and the (Ramond) R-R scalar and two-form. Its natural basis is a double-tensor supermultiplet,¹ having three perturbative isometries associated to the two axionic shifts of the antisymmetric tensors and an extra shift of the R-R scalar. These isometries form a Heisenberg algebra, which

¹This representation of $\mathcal{N} = 2$ global supersymmetry has been only recently explicitly constructed [26]. See also ref. [27].

at the string tree-level is enhanced to the quaternion-Kähler and Kähler space $SU(2,1)/SU(2) \times U(1)$. We can also use an equivalent formulation where the NS-NS and R-R 2-forms are replaced by their Poincaré dual scalars. In this formulation, the aforementioned isometries are realized on the scalar manifold of the four scalars which form a hypermultiplet called the “universal hypermultiplet”.

Therefore, we need to determine the proper ‘global supersymmetry’ limit of the universal hypermultiplet and match it with the hyperKähler scalar manifold of the global action. At the level of global $\mathcal{N} = 2$, imposing the Heisenberg algebra of isometries determines a unique hyperkähler manifold of dimension four, depending on a single parameter. This is in close analogy with the local case of a quaternionic space where the corresponding parameter is associated to the one-loop correction [28]. These similar results suggest a correspondence between the local and global cases which could be studied using a Ricci-flat limit of the quaternion-Kähler manifold preserving the Heisenberg algebra.

Obtaining the global SUSY limit of the universal hypermultiplet is not a trivial task. In $\mathcal{N} = 2$, the scalar curvature comes out to be proportional to the gravitational coupling k so in the global SUSY limit we unavoidably obtain a Ricci-flat manifold. However, if we naively send k to zero we reduce to the trivial case of a flat scalar manifold with canonical kinetic terms. To obtain a non-trivial space, an appropriate limit must be defined, involving a new mass scale that should remain finite as Planck mass goes to infinity. This mechanism has only been explicitly displayed for some particular cases, mostly using the quaternionic quotient method [29, 30]. In chapter 11, we use this procedure to obtain the one-loop effective supergravity of the dilaton hypermultiplet and to then de-

scribe the appropriate zero-curvature limit, using the perturbative Heisenberg symmetry as a guideline [5].

Part I

BEYOND THE MINIMAL SUPERSYMMETRIC STANDARD MODEL

CHAPTER 2

EFFECTIVE FIELD THEORY

2.1 Physics is Effective

The ultimate goal of physics is believed to be the formulation of the theory that will disclose all mysteries of nature. There is a lot of discussion about the kind of truths that will be unveiled to us, however physicists generally agree that this final “Theory of Everything” will provide an exact description of all physical phenomena that occur at any place and any time of the universe. Of course, we don’t have this theory yet. We rather have various theories each one being a good description for some class of physical phenomena while failing for others. “Good” here is used in the sense of being precise enough for our needs. If we want to think in terms of the “parameter space” of nature, where the parameters can be distance, energy, velocity et c., then we can say that our theories are valid in a certain parameter subspace but not outside. For example, in the study of a system that interacts gravitationally, Newtonian gravity is a good description if interactions are non-relativistic but needs replacement by General Relativity if they are relativistic.

Theories that are valid only in a certain region of the full parameter space are called “effective”. This definition might sound redundant since all physi-

cal theories would be effective. Nevertheless, this simple idea has an surprisingly rich structure in quantum field theory (QFT). The most relevant parameter here is distance. After almost a century of experiments in particle physics we have learned that, as we probe smaller distances, nature appears to reveal richer structure. In the context of QFT, this is expressed by the appearance of new degrees of freedom, describing new particles. These are invisible at longer distances either because they are unstable, decaying to known long lived particles, or because they are components of particles that at longer distances seem fundamental. This suggests that a QFT model with a given set of degrees of freedom is valid only at distance scales larger than the threshold for production of new particles, not included in the set. If we agree that the principles of QFT are valid beyond the threshold distance, we will need to exchange the old model with a new one, where the new particles (and the new interactions that they reveal) are included. This process essentially builds a ladder of effective field theory (EFT) models separated by the threshold distances where new particles appear. Various interesting questions arise: How to smoothly switch from one EFT to another, what is their behavior very close to the threshold et c. Another thing that makes EFT nontrivial is the need for regularization. Since regularization involves the behavior of a QFT model at high energies (short distances), it has to be treated with special care ¹.

Let's attempt a discussion motivated by the questions mentioned above. We focus on two neighbor theories, call them the 'UV' and the 'IR' theory, separated by threshold energy M (we prefer to talk in terms of energy than distance). The UV theory contains all the modes of the IR plus those modes with mass of order M that do not appear in the IR. We expect that as we approach M from

¹For a review, see [31].

below, the new physics that the heavy particles bring will become more and more apparent. The way to incorporate these effects in the IR is by integrating out the massive modes. This inevitably introduces a series of higher dimensional, nonrenormalizable operators in the Lagrangian of the IR, suppressed by the threshold scale. From the EFT point of view, the fact that they are nonrenormalizable is not an indication that the model is sick but simply that it is valid up to the threshold scale, as expected [32]. This point, even if it sounds obvious nowadays, was entirely disregarded in the early days of QFT when nonrenormalizable models were considered pathological. In the expansion of the operator series, we choose to cut off at some order in $1/M$ depending on the accuracy we want to achieve. The coefficients of the new terms are determined by matching the S-matrix elements of the UV and the IR models. One might ask why should we bother reducing to an effective IR theory when the full UV theory is known. The reason is that in many cases, calculations in the low energy regime are much simpler in the IR theory where the very massive modes do not appear explicitly.

There are many examples of EFT models. For some of them the UV completion is known while for others it isn't. To mention a few, Fermi theory is an effective theory of weak interactions while chiral perturbation theory and nucleon effective theory are low energy effective descriptions of QCD. On the other hand, the Standard Model (SM) itself is an effective theory (it is renormalizable only when gravity is ignored) but its UV completion is still unknown. The same is true for General Relativity.

In order to elucidate the derivation of an effective theory from a known UV completion, we focus on the popular case of the Fermi theory as an effective

theory for electroweak interactions. In the SM, consider the tree level exchange of a massive Z gauge boson between charged fermions

$$\mathcal{L} \supset i \bar{\psi} \gamma^\mu (\partial_\mu + igZ_\mu) \psi - \frac{M^2}{2} Z_\mu Z^\mu \quad (2.1)$$

By integrating out Z_μ we generate the higher dimensional operator

$$\Delta\mathcal{L} = \frac{g^2}{2M^2} (\bar{\psi} \gamma_\mu \psi)^2 \quad (2.2)$$

which is a nonrenormalizable four-fermion contact term. Similarly, for scalars H :

$$\mathcal{L} \supset |(\partial_\mu + ig Z_\mu) H|^2 - \frac{M^2}{2} Z_\mu Z^\mu \quad (2.3)$$

and

$$\Delta\mathcal{L} = \frac{g^2}{M^2} (H^\dagger \partial_\mu H)^2 \quad (2.4)$$

It is also possible that the effective operator is a higher derivative one. Here, we retrieve such operators by the kinetic mixing of light with heavy states, upon integrating out the latter. For example, from

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 + c \partial^\mu \phi \partial_\mu \chi - \frac{1}{2} M^2 \chi^2 - \frac{1}{2} \lambda' \phi^2 \chi^2 \quad (2.5)$$

one finds after integrating out the massive field χ :

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} (\partial_\mu \phi)^2 + \frac{c^2}{2} \square \phi \frac{1}{M^2 + \square + \lambda' \phi^2} \square \phi \\ &= \frac{1}{2} (\partial_\mu \phi)^2 + \frac{c^2}{2M^2} (\square \phi)^2 + \dots \end{aligned} \quad (2.6)$$

which contains higher derivative terms. In both examples above, the UV completion of the effective theory is known. EFT is then a practical reformulation of the relevant degrees of freedom in the low energy regime. However, does EFT have anything to offer when the UV side is unknown?

This answer is definitely positive. EFT has proved to be a very useful tool for exploring new physics in a bottom-up approach [33, 34, 35, 36]. Since the effects of inaccessible massive states can be incorporated into nonrenormalizable operators, we can simply add such terms in the IR Lagrangian without referring to a particular UV scheme. In a systematic analysis, we include all possible terms up to a given order in $1/M$ that are allowed by the symmetries of the theory, keeping the coefficients arbitrary. This constitutes a model independent way of exploring new physics beyond the validity of the pure IR model. Any possible UV candidate will essentially reduce to a subset of the nonrenormalizable terms with fixed values for the coefficients.

Even at first order in $1/M$, there is usually a long list of terms allowed by the symmetries of the theory, introducing many new arbitrary parameters. Nevertheless, such set is in general highly reducible. This means that we can write the Lagrangian in a way that a smaller number of new operators appears but physics be the same. There are three different methods to perform such reduction. By setting the higher dimensional operators “on shell” [37, 38, 39], by performing field redefinitions [40, 41] and, if the operator is higher derivative, by applying the “unfolding” technique [42, 43]. Since we will be using the first two in the phenomenological analysis of the following chapters, we will briefly present them below in the relevant case of supersymmetric field theories. After restricting to an irreducible set of higher dimensional operators, one can further cut down the parameter space by comparing the model with low energy phenomenology. In the end, the hope is that the effective model will provide concrete testable predictions for the effects that very massive modes can have on low energy observables.

2.2 Effective Description of Supersymmetric Theories

EFT has a lot to offer in the yet unexplored territory of TeV physics. By popular belief, the most promising candidate theory for physics around that scale is supersymmetry. It is then reasonable to construct phenomenological supersymmetric models by means of EFT techniques and this is what we do in the following chapters. In order to familiarize with the concept and tools of EFT in the framework of supersymmetric theories, we present here some representative study cases.

2.2.1 Integrating out Massive Superfields

Consider the following Lagrangian of dimensionful scales M and m with $M \gg m$:

$$\mathcal{L} = \int d^4\theta [\Phi^\dagger \Phi + \chi^\dagger \chi] + \left\{ \int d^2\theta \left[\frac{M}{2} \chi^2 + m \Phi \chi + \frac{\lambda}{3} \Phi^3 \right] + h.c. \right\} \quad (2.7)$$

We want to acquire an effective description by integrating out the heavier mode. We will follow two different paths; either diagonalize the mass matrix and then integrate or directly integrate. Then we will show that the resulting effective theories are all equivalent by using the “field redefinitions” method and the “on shell” method mentioned earlier.

In the first path, we perform the transformation $\Phi = (\cos \theta \Phi_1 - \sin \theta \Phi_2)$ and $\chi = (\sin \theta \Phi_1 + \cos \theta \Phi_2)$. In the diagonal basis of Φ_1 and Φ_2 , one finds

$$\begin{aligned} \mathcal{L} = & \int d^4\theta [\Phi_1^\dagger \Phi_1 + \Phi_2^\dagger \Phi_2] \\ & + \left\{ \int d^2\theta \left[\frac{m_1}{2} \Phi_1^2 + \frac{m_2}{2} \Phi_2^2 + \frac{\lambda}{3} (\cos \theta \Phi_1 - \sin \theta \Phi_2)^3 \right] + h.c. \right\} \end{aligned} \quad (2.8)$$

where

$$\begin{aligned} m_1 &= \frac{M}{2} \left(1 - (1 + 4m^2/M^2)^{1/2} \right) = -\frac{m^2}{M} \left(1 - \frac{m^2}{M^2} \right) + \dots, \\ m_2 &= \frac{M}{2} \left(1 + (1 + 4m^2/M^2)^{1/2} \right) = M \left(1 + \frac{m^2}{M^2} + \dots \right), \end{aligned} \quad (2.9)$$

so Φ_2 is the massive field. Then, we integrate out Φ_2 via its equation of motion

$$-\frac{1}{4} \overline{D}^2 \Phi_2^\dagger + m_2 \Phi_2 - \lambda \sin \theta (\Phi_1 \cos \theta - \Phi_2 \sin \theta)^2 = 0, \quad (2.10)$$

with solution

$$\Phi_2 = \frac{\lambda}{m_2} \cos^2 \theta \sin \theta \Phi_1^2 - \frac{\lambda^2}{4m_2^2} \sin^3 2\theta \Phi_1^3 + \frac{\lambda}{4m_2^2} \cos^2 \theta \sin \theta \overline{D}^2 \Phi_1^{\dagger 2} + \mathcal{O}(M^{-3}). \quad (2.11)$$

The effective Lagrangian that we obtain is:

$$\begin{aligned} \mathcal{L}^{eff} &= \int d^4\theta \Phi_1^\dagger \Phi_1 \\ &+ \left\{ \int d^2\theta \left[\frac{-m^2}{2M} Z \Phi_1^2 + \frac{\lambda}{3} Z^{3/2} \Phi_1^3 - \frac{m^2 \lambda^2}{2M^3} \Phi_1^4 \right] + h.c. \right\} + \mathcal{O}(M^{-4}) \end{aligned} \quad (2.12)$$

where

$$Z = 1 - \frac{m^2}{M^2} + \mathcal{O}(1/M^4). \quad (2.13)$$

This is an effective description of (2.7) where only the light mode propagates.

Alternatively, one can choose to directly integrate out χ from eq. (2.7) without firstly diagonalizing. Its e.o.m. is

$$\overline{D}^2 \chi^\dagger - 4(M\chi + m\Phi) = 0 \quad (2.14)$$

with an iterative solution

$$\chi = \frac{1}{M} \left[-m\Phi - \frac{m}{4M} \overline{D}^2 \Phi^\dagger + \frac{1}{16} \frac{-m}{M^2} \overline{D}^2 D^2 \Phi - \frac{m}{64M^3} \overline{D}^2 D^2 \overline{D}^2 \Phi^\dagger + \dots \right]. \quad (2.15)$$

Plugging this back in (2.7), we find

$$\begin{aligned}
\mathcal{L}^{eff} = & \int d^4\theta \left\{ \left[1 + \frac{m^2}{M^2} \right] \Phi^\dagger \Phi + \frac{m^2}{8M^3} [\Phi D^2 \Phi + h.c.] + \frac{m^2}{16M^4} (\bar{D}^2 \Phi^\dagger) (D^2 \Phi) \right\} \\
& + \left\{ \int d^2\theta \left[\frac{-m^2}{2M} \Phi^2 + \frac{\lambda}{3} \Phi^3 \right] + h.c. \right\} + O(1/M^5)
\end{aligned} \tag{2.16}$$

which, after an appropriate rescaling, is written as

$$\begin{aligned}
\mathcal{L}^{eff} = & \int d^4\theta \left\{ \Phi^\dagger \Phi + \frac{m^2}{8M^3} [\Phi D^2 \Phi + h.c.] + \frac{m^2}{16M^4} (\bar{D}^2 \Phi^\dagger) (D^2 \Phi) \right\} \\
& + \left\{ \int d^2\theta \left[\frac{-m^2}{2M} Z \Phi^2 + \frac{\lambda}{3} Z^{3/2} \Phi^3 \right] + h.c. \right\} + O(1/M^5),
\end{aligned} \tag{2.17}$$

where $Z = 1/(1 + m^2/M^2)$. In this path, we obtained an effective Lagrangian with higher derivative terms. Equations (2.12) and (2.17) look different, however, the physics they describe is the same. We will demonstrate this in two ways.

In the first way, we set ‘‘on shell’’ the higher dimensional operator. By use of the e.o.m.

$$\bar{D}^2 \Phi^\dagger = -\frac{4m^2}{M} \Phi + 4\lambda \Phi^2 + O(1/M^2) \tag{2.18}$$

we can rewrite (2.17). The new Lagrangian will contain the term $\Phi \Phi^{\dagger 2}$ which can be removed by a suitable shift

$$\Phi = \tilde{\Phi} - \frac{\lambda m^2}{2M^3} \tilde{\Phi}^2 \tag{2.19}$$

to find

$$\begin{aligned}
\mathcal{L}^{eff} = & \int d^4\theta \tilde{\Phi}^\dagger \tilde{\Phi} \\
& + \left\{ \int d^2\theta \left[-\frac{m^2}{2M} Z \tilde{\Phi}^2 + \frac{\lambda}{3} \tilde{\Phi}^3 \left(1 - \frac{3m^2}{2M^2} \right) - \frac{\lambda^2 m^2}{2M^3} \tilde{\Phi}^4 \right] + h.c. \right\} + O\left(\frac{1}{M^4}\right)
\end{aligned} \tag{2.20}$$

where $Z = 1/(1 + m^2/M^2)$. It is obvious now that this Lagrangian coincides with that of (2.12) in the approximation $O(1/M^4)$. This confirms that setting the higher

derivative operators “on shell” via equations of motion is a correct procedure, within the approximation considered. We obtained again a higher dimensional operator and a scale dependence acquired classically by the couplings of the low energy effective theory.

In the second way, we perform field redefinitions in eq. (2.17) so as to eliminate the $\Phi D^2 \Phi$ term. We use

$$\Phi = \Phi' + c \bar{D}^2 \Phi'^{\dagger} \quad (2.21)$$

where the dimensionful coefficient c is such that the coefficient of $\Phi D^2 \Phi$ vanish in the new Lagrangian. This gives $c = -m^2/(8M^3)$ and, after some calculations, the Lagrangian in (2.17) becomes

$$\begin{aligned} \mathcal{L}^{eff} = & \int d^4\theta \left[\Phi'^{\dagger} \Phi' + \frac{m^2 \lambda}{2 M^3} (\Phi'^2 \Phi'^{\dagger} + h.c.) \right] \\ & + \left\{ \int d^2\theta \left[-\frac{m^2}{2 M} Z \Phi'^2 + \frac{\lambda}{3} Z^{3/2} \Phi'^3 \right] + h.c. \right\} + \mathcal{O}(1/M^4) \quad (2.22) \end{aligned}$$

By a final shift $\Phi' = \tilde{\Phi} - m^2 \lambda/(2 M^3) \tilde{\Phi}^2$ we obtain an effective Lagrangian identical to that in (2.12) and (2.20).

We have shown that the three apparently different paths to the reduced Lagrangian, leading to either eq. (2.12), (2.17) or (2.22), are actually different formulations of same physics at the expansion order studied. The correction at $1/M$ is solely a wavefunction renormalization while higher dimensional operators appeared only at higher order.

2.2.2 Gauge Interactions and Component Analysis

We proceed to study further examples of effective theories, now with gauge interactions present. We will also verify the superfield analysis at the component level. The effective operators that will be generated are the same with those used in the phenomenological model of the subsequent chapters. Therefore, the analysis here provides us intuition about the kind of UV physics that these effective operators encapsulate.

Consider the Lagrangian of an $\mathcal{N} = 1$ supersymmetric non-Abelian gauge theory²

$$\begin{aligned}
\mathcal{L} = & \int d^4\theta \left[\Phi_1^\dagger e^V \Phi_1 + \Phi_3^\dagger e^V \Phi_3 + \Phi_2 e^{-V} \Phi_2^\dagger + \Phi_4 e^{-V} \Phi_4^\dagger + S^\dagger S \right] \\
& + \int d^4\theta \left[\nu_1 \Phi_1^\dagger e^V \Phi_3 + \nu_2 \Phi_4 e^{-V} \Phi_2^\dagger + h.c. \right] \\
& + \int d^2\theta \left[\mu \Phi_1 \Phi_2 + M \Phi_3 \Phi_4 + \frac{M}{2} S^2 + \lambda S \Phi_1 \Phi_2 \right] + h.c. \quad (2.23)
\end{aligned}$$

where $M \gg \mu$ and V is the standard vector superfield in the Wess-Zumino gauge. The equations of motion for the massive fields $\Phi_{3,4}$ and S give

$$\begin{aligned}
-\frac{\nu_1}{4} \bar{D}^2 (\Phi_1^\dagger e^V) - \frac{1}{4} \bar{D}^2 (\Phi_3^\dagger e^V) + M \Phi_4 &= 0 \\
-\frac{\nu_2}{4} \bar{D}^2 (e^{-V} \Phi_2^\dagger) - \frac{1}{4} \bar{D}^2 (e^{-V} \Phi_4^\dagger) + M \Phi_3 &= 0 \\
-\frac{1}{4} \bar{D}^2 S^\dagger + M S + \lambda \Phi_1 \Phi_2 &= 0 \quad (2.24)
\end{aligned}$$

As in the previous section, we use these equations to integrate out the massive

²For the link to the MSSM, replace $V \rightarrow V_1 \equiv g_2 V_w^i \sigma^i - g_1 V_Y$ with $V_w, (V_Y)$ the $SU(2), (U(1)_Y)$ gauge fields respectively; also $\Phi_2 \rightarrow H_2^T (i\sigma_2), \Phi_1 \rightarrow H_1$ with $\Phi_3, (\Phi_4)$ with same quantum numbers to $\Phi_1, (\Phi_2)$ and $(i\sigma_2) \exp(-\Lambda) = \exp(\Lambda^T) (i\sigma_2)$, then $\Phi_2 e^{-V} \Phi_2^\dagger \rightarrow H_2^\dagger e^{V_2} H_2$, with $V_2 \equiv g_2 V_w^i \sigma^i + g_1 V_Y$.

fields $\Phi_{3,4}$ to find

$$\begin{aligned} \mathcal{L}^{eff} &= \int d^4\theta \left[\Phi_1^\dagger e^V \Phi_1 + \Phi_2 e^{-V} \Phi_2^\dagger + \left(\xi \Phi_1^\dagger e^V \overline{D}^2 e^{-V} \Phi_2^\dagger + h.c. \right) \right] \\ &+ \int d^2\theta \left[\mu \Phi_1 \Phi_2 + \xi' (\Phi_1 \Phi_2)^2 \right] + h.c. + \mathcal{O}(M^{-2}) \end{aligned} \quad (2.25)$$

where $\xi = \frac{\nu_1 \nu_2}{4M}$, $\xi' = -\frac{\lambda^2}{2M}$ and we ignored higher orders in M^{-1} . If the superpotential in (2.23) also contains trilinear couplings of heavy doublets $\Phi_{3,4}$ to quarks and leptons

$$\Delta\mathcal{L} = \int d^2\theta \left[Q \sigma_u U^c \Phi_4 + Q \sigma_d D^c \Phi_3 + L \sigma_e E^c \Phi_3 \right] + h.c. , \quad (2.26)$$

then, following the same procedure, we would get the extra effective terms

$$\begin{aligned} \Delta\mathcal{L}^{eff} &= -\frac{1}{M} \int d^4\theta \left[\nu_1 \Phi_1^\dagger e^V Q \sigma_u U^c + \nu_2 (Q \sigma_d D^c) e^{-V} \Phi_2^\dagger + \nu_2 (L \sigma_e E^c) e^{-V} \Phi_2^\dagger + h.c. \right] \\ &+ \frac{1}{M} \int d^2\theta \left[(Q \sigma_u U^c)(Q \sigma_d D^c) + (Q \sigma_u U^c)(L \sigma_e E^c) \right] + h.c. , \end{aligned} \quad (2.27)$$

where $\sigma_{u,d,e}$ are 3x3 matrices in the families space.

Focusing on (2.25), let us set on shell the higher derivative operator by using the equations of motion for $\Phi_{1,2}$:

$$D^2 [e^V \Phi_1] = 4\mu \Phi_2^\dagger, \quad \overline{D}^2 [e^{-V} \Phi_2^\dagger] = 4\mu \Phi_1. \quad (2.28)$$

We insert these in (2.25) and rescale $\Phi_i \rightarrow \Phi'_i (1 - 2\mu\xi)$, $i = 1, 2$, to find:

$$\begin{aligned} \mathcal{L}^{eff} &= \int d^4\theta \left[\Phi_1^\dagger e^V \Phi_1 + \Phi_2 e^{-V} \Phi_2^\dagger \right] \\ &+ \int d^2\theta \left[\mu(1 - 4\mu\xi) \Phi_1 \Phi_2 + \xi' (\Phi_1 \Phi_2)^2 \right] + h.c. + \mathcal{O}(M^{-2}) \end{aligned} \quad (2.29)$$

It is obvious that the specific operator, when put on shell, brings solely a wave-function renormalization. We now go on to verify at the component level that

both Lagrangians are equivalent. First, we expand (2.25)³:

$$\begin{aligned}
\mathcal{L}^{eff} = & -\phi_1^* \mathcal{D}_\mu \mathcal{D}^\mu \phi_1 + i \bar{\psi}_1 \bar{\sigma}^\mu \mathcal{D}_\mu \psi_1 - \frac{1}{\sqrt{2}} [\bar{\psi}_1 \bar{\lambda} \phi_1 + h.c.] + \phi_1^* \frac{D}{2} \phi_1 + |F_1|^2 \\
& - \phi_2 \mathcal{D}_\mu \mathcal{D}^\mu \phi_2^* + i \psi_2 \sigma^\mu \mathcal{D}_\mu \bar{\psi}_2 + \frac{1}{\sqrt{2}} [\phi_2 \bar{\lambda} \bar{\psi}_2 + h.c.] - \phi_2 \frac{D}{2} \phi_2^* + |F_2|^2 \\
& + \xi^* \left\{ 4 [F_2 \mathcal{D}_\mu \mathcal{D}^\mu \phi_1 + \phi_2 \mathcal{D}_\mu \mathcal{D}^\mu F_1] + 2 \sqrt{2} i [\psi_2 \sigma^\mu \overleftarrow{\mathcal{D}}_\mu \bar{\lambda} \phi_1 + \phi_2 \bar{\lambda} \bar{\sigma}^\mu \mathcal{D}_\mu \psi_1] \right. \\
& + 2 (\phi_2 D F_1 - F_2 D \phi_1) - 2 \sqrt{2} [\psi_2 \lambda F_1 - F_2 (\lambda \psi_1)] - 2 \phi_2 (\bar{\lambda} \bar{\lambda}) \phi_1 \\
& \left. - 4 \psi_2 \sigma^\nu \bar{\sigma}^\mu \mathcal{D}_\nu \mathcal{D}_\mu \psi_1 \right\} + \mu [\phi_1 F_2 + F_1 \phi_2 - \psi_1 \psi_2] \\
& + \xi' \left[-(\phi_1 \psi_2 + \psi_1 \phi_2)^2 + 2 (\phi_1 \phi_2) (\phi_1 F_2 + F_1 \phi_2 - \psi_1 \psi_2) \right] \\
& + h.c. + O(1/M^2) \tag{2.30}
\end{aligned}$$

with

$$\mathcal{D}_\mu = \partial_\mu + i \frac{V_\mu}{2}, \quad \overleftarrow{\mathcal{D}}_\mu = \overleftarrow{\partial}_\mu - i \frac{V_\mu}{2}, \tag{2.31}$$

and the ‘‘h.c.’’ refers to all terms in the last four lines. Notice that in the off shell component form of the Lagrangian we have an interesting tensor coupling $\psi_2 \sigma^\nu \bar{\sigma}^\mu \mathcal{D}_\nu \mathcal{D}_\mu \psi_1$ in spite of the minimal gauge coupling in (2.23). This coupling could be relevant for tree level calculations of the Feynman diagrams. Next, we eliminate the auxiliary fields $F_{1,2}$ using their e.o.m.

$$\begin{aligned}
F_1^* &= -\phi_2 \left(\mu + 2 \xi' (\phi_1 \phi_2) \right) + \xi^* \left(-4 \phi_2 \overleftarrow{\mathcal{D}}_\mu \overleftarrow{\mathcal{D}}^\mu - 4 \phi_2 \frac{D}{2} + 2 \sqrt{2} \psi_2 \lambda \right) \\
F_2^* &= -\phi_1 \left(\mu + 2 \xi' (\phi_1 \phi_2) \right) + \xi^* \left(-4 \mathcal{D}_\mu \mathcal{D}^\mu \phi_1 + 4 \frac{D}{2} \phi_1 - 2 \sqrt{2} \lambda \psi_1 \right) \tag{2.32}
\end{aligned}$$

In the terms proportional to ξ in \mathcal{L}^{eff} we can replace the derivatives of the fermions by their equations of motion, since the error would be of higher or-

³We use $-4 \psi_2 \mathcal{D}_\mu \mathcal{D}^\mu \psi_1 = -4 \psi_2 [\sigma^\nu \bar{\sigma}^\mu - 2 i \sigma^{\mu\nu}] \mathcal{D}_\nu \mathcal{D}_\mu \psi_1 = -4 \psi_2 \sigma^\nu \bar{\sigma}^\mu \mathcal{D}_\nu \mathcal{D}_\mu \psi_1 + 4 \psi_2 \sigma^{\mu\nu} F_{\mu\nu} \psi_1$ and the first term in the rhs is that entering the final expression of \mathcal{L} . Here $F_{\mu\nu} = \partial_\mu V_\nu / 2 - \partial_\nu V_\mu / 2 + i [V_\mu / 2, V_\nu / 2]$.

der. We use

$$\begin{aligned}
i\bar{\sigma}^\mu \mathcal{D}_\mu \psi_1 &= \mu \bar{\psi}_2 + \frac{1}{\sqrt{2}} \bar{\lambda} \phi_1 + \mathcal{O}(\xi), \\
-i\psi_2 \sigma^\mu \overleftarrow{\mathcal{D}}_\mu &= \mu \bar{\psi}_1 - \frac{1}{\sqrt{2}} \phi_2 \bar{\lambda} + \mathcal{O}(\xi).
\end{aligned} \tag{2.33}$$

We then rescale the scalars and Weyl fermions and after neglecting terms $\mathcal{O}(\xi \xi')$ we obtain the on shell Lagrangian

$$\begin{aligned}
\mathcal{L} &= -\phi_1^\dagger \mathcal{D}^2 \phi_1 + i\bar{\psi}_1 \bar{\sigma}^\mu \mathcal{D}_\mu \psi_1 - \frac{1}{\sqrt{2}} [\bar{\psi}_1 \bar{\lambda} \phi_1 + h.c.] + \phi_1^\dagger \frac{D}{2} \phi_1 \\
&- \phi_2 \mathcal{D}^2 \phi_2^\dagger + i\psi_2 \sigma^\mu \mathcal{D}_\mu \bar{\psi}_2 + \frac{1}{\sqrt{2}} [\phi_2 \bar{\lambda} \bar{\psi}_2 + h.c.] - \phi_2 \frac{D}{2} \phi_2^\dagger \\
&- \mu^2 |1 - 4\mu\xi|^2 [\phi_1^\dagger \phi_1 + \phi_2 \phi_2^\dagger] - \mu [(1 - 4\mu\xi) \psi_1 \psi_2 + h.c.] \\
&- 2\xi' \mu [(\phi_1 \phi_2) + h.c.] [\phi_1^\dagger \phi_1 + \phi_2 \phi_2^\dagger], \quad \mathcal{D}^2 = \mathcal{D}^\mu \mathcal{D}_\mu
\end{aligned} \tag{2.34}$$

This Lagrangian is in agreement with that of (2.29), which shows that on shell and in the absence of other interactions, only a wavefunction renormalisation effect is present, giving a new $\mu' = \mu (1 - 4\mu\xi)$. To conclude, integrating out the massive superfields $\Phi_{3,4}$ generated a dimension-five operator $\Phi_2 e^{-V} D^2 e^V \Phi_1$ which however, brings only a (classical) wavefunction renormalisation, in the absence of additional trilinear interactions. Thus this five dimensional operator doesn't bring new physics in the absence of additional interactions. One could ask if this conclusion remains valid when we include soft supersymmetry breaking terms. Also, if additional trilinear interactions were present, other five dimensional operators of type shown in (2.27) could also be generated. All these issues are studied in the subsequent chapters.

2.3 Nonlinear Realizations and Constrained Goldstino Superfield

Consider a field theory invariant under the symmetry group \mathcal{G} . The field content of the theory is divided between fields that are invariant and fields that transform under \mathcal{G} . The latter can transform either linearly under all generators of \mathcal{G} or linearly under a subgroup \mathcal{H} and nonlinearly under the coset \mathcal{G}/\mathcal{H} . In the first case the theory is in its unbroken phase and the classification of all possible transformation laws for the fields is described by representation theory. In the second case the symmetry parametrized by the generators of \mathcal{G}/\mathcal{H} is broken with the breaking scale M_b sent to infinity. In other words, a theory with a nonlinear realization of a symmetry group can be seen as an effective description of the far IR limit of a theory where this group is broken spontaneously [44, 45]. The Goldstone fields that appear are in 1-1 correspondance with the generators of \mathcal{G}/\mathcal{H} .

If \mathcal{G} is the super-Poincaré and \mathcal{H} the Poincaré algebra, we have a nonlinear realization of the supersymmetry algebra and this would describe the far IR regime of a spontaneously broken supersymmetric theory. Since the broken generators are fermionic, the corresponding Goldstone mode has to be a fermion, too. To distinguish it from the standard Goldstone fields, we call it “goldstino”. It is quite surprising that Supersymmetry in four dimensions first appeared in its nonlinear version [46]. The nonlinear transformation of the goldstino $\lambda_\alpha(x)$ can be written as:

$$\delta\lambda_\alpha = f\eta_\alpha + \frac{i}{f}(\lambda\sigma^\mu\bar{\eta} - \eta\sigma^\mu\bar{\lambda})\partial_\mu\lambda_\alpha, \quad (2.35)$$

where η is the supersymmetry transformation parameter and f is a parameter

of mass dimension 2 characterizing the susy breaking scale ($\sqrt{f} = M_b$). The commutator of this transformation

$$[\delta_\eta, \delta_\xi]\lambda_\alpha = 2i(\eta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\eta})\partial_\mu\lambda_\alpha \quad (2.36)$$

is a spacetime translation and proves that the above transformation closes off shell the super-Poincaré algebra.

In order to take advantage of nonlinear realizations we need to know how to construct Lagrangians describing interactions of the goldstino with itself and with other fields. Several strategies have been developed in the past. In the “geometric” method [46, 47, 48] the transformation (2.35) is interpreted as an extension of the standard superspace transformation

$$\begin{aligned} \theta &\rightarrow \theta + \eta, \\ x^\mu &\rightarrow x^\mu + i\theta\sigma^\mu\bar{\eta} - i\eta\sigma^\mu\bar{\theta}, \end{aligned} \quad (2.37)$$

to the chiral goldstino field $\lambda(x)$ by identifying θ with λ/f . The same analogy between θ and λ can be extended to the superspace differentials $d\theta$ and $d\bar{\theta}$ leading to the construction of a volume element invariant (up to total derivative) under the nonlinear transformations. From this we can extract the Lagrangian density

$$\mathcal{L} = -f^2 \det A, \quad \text{with} \quad A_\nu^\mu = \delta_\nu^\mu + \frac{i}{f^2}(\lambda\sigma^\mu\partial_\nu\bar{\lambda} - \partial_\nu\lambda\sigma^\mu\bar{\lambda}). \quad (2.38)$$

It is the Volkov-Akulov Lagrangian describing the dynamics of a single goldstino up to higher derivative terms. By nonlinearly realizing the algebra on matter fields ϕ as well,

$$\delta\phi = -\frac{i}{f}(\lambda\sigma^\mu\bar{\eta} - \eta\sigma^\mu\bar{\lambda})\partial_\mu\phi, \quad (2.39)$$

we can construct goldstino-matter couplings (ϕ denotes any kind of field). For any operator $O(\phi, \partial_\mu\phi)$ we simply have to replace partial derivatives by appro-

appropriate covariant derivatives so that O transforms in the standard way:

$$\delta O = -\frac{i}{f}(\lambda\sigma^\mu\bar{\eta} - \eta\sigma^\mu\bar{\lambda})\partial_\mu O. \quad (2.40)$$

Then any action of the type

$$S = -f^2 \int d^4x \det(A) O \quad (2.41)$$

is invariant under nonlinear transformations. It can be easily shown that, in the geometric method, the lowest order couplings between goldstinos and matter are of the type:

$$\frac{1}{f^2} T^{\mu\nu} t_{\mu\nu}, \quad (2.42)$$

where $T_{\mu\nu}$ and $t_{\mu\nu}$ are the stress energy tensors of the goldstino and matter field.

Another method for constructing goldstino-matter Lagrangians involves promoting the goldstino to a superfield Λ [49, 50]. This is done in a way compliant with the nonlinear supersymmetry transformations of the goldstino so that in the end, the only physical degree of freedom in Λ is simply λ . Since the basic concepts of goldstino Lagrangians have been presented along with the geometric method, we will skip this method and go directly to the next, which is the one used extensively in chapters 6 and 10 (in its $\mathcal{N} = 2$ generalization).

This is the method of constrained superfields [51, 52, 53, 54]. It draws inspiration from a similar technique applied in bosonic symmetries for example in the context of σ models. One starts from the full manifold made up from the linear symmetry transformations and then restrict to a certain submanifold by imposing constraints on the coordinates. This breaks the original symmetry down to the subgroup that is left invariant under the constraints. E.g. in an $O(4)$ σ model of fields $(\sigma, \vec{\pi})$ ($\vec{\pi}$ is a vector of pions), we can break the symmetry down to $O(3)$ by imposing the constraint $\sigma^2 + \vec{\pi} \cdot \vec{\pi} = f^2$. It is the same manifold

that we would obtain by starting from a vacuum state $(f, 0)$ and applying the elements of the coset space $O(4)/O(3)$.

In the context of $\mathcal{N} = 1$ supersymmetry, this technique is realized in the following way. We start from a standard chiral superfield that describes a full supersymmetric multiplet and impose a specific constraint on it. Using the constraint, we eliminate the scalar d.o.f. in favor of the fermion. In particular, the constraint

$$X_{nl}^2 = 0, \quad (2.43)$$

delivers

$$X_{nl} = \phi_X + \sqrt{2} \theta \psi_X + \theta \theta F_X, \quad \text{with} \quad \phi_X = \frac{\psi_X \psi_X}{2 F_X}. \quad (2.44)$$

The simplest possible Lagrangian of X_{nl} :

$$\int d^4\theta X_{nl}^\dagger X_{nl} + \left[\int d^2\theta f X_{nl} + h.c. \right] = |\partial_\mu \phi_X|^2 + F_X^\dagger F_X + \left[\frac{i}{2} \bar{\psi}_X \bar{\sigma}^\mu \partial_\mu \psi_X + f F_X + h.c. \right] \quad (2.45)$$

reproduces the Volkov-Akulov Lagrangian upon integrating out the auxiliary F_X and identifying ψ_X with the goldstino.

The advantage of this method is the use of superfield formalism. For example, the couplings of goldstinos to matter are easily constructed by treating X_{nl} as any other superfield and following the standard rules of superspace. As a demonstration, consider a supersymmetric theory with chiral multiplets $\Phi_i \equiv (\phi_i, \psi_i, F_i)$ and vector multiplets $V \equiv (A_\mu^a, \lambda^a, D^a)$ coupled in a general way to X_{nl} :

$$\begin{aligned} \mathcal{L} = & \int d^4\theta \left[X_{nl}^\dagger X_{nl} + \Phi_i^\dagger (e^V \Phi)_i - (m_i^2/f^2) X_{nl}^\dagger X_{nl} \Phi_i^\dagger (e^V \Phi)_i \right] + \left\{ \int d^2\theta \left[f X_{nl} + W(\Phi_i) \right. \right. \\ & \left. \left. + \frac{B_{ij}}{2f} X_{nl} \Phi_i \Phi_j + \frac{A_{ijk}}{6f} X_{nl} \Phi_i \Phi_j \Phi_k + \frac{1}{4} \left(1 + \frac{2m_\lambda}{f} X_{nl} \right) \text{Tr} W^\alpha W_\alpha \right] + h.c. \right\}, \quad (2.46) \end{aligned}$$

where m_i^2, B_{ij}, A_{ijk} are soft terms for the scalars and m_λ is the gaugino mass. From this, one can find the Goldstino (ψ_X) couplings to ordinary matter and gauge superfields.

Furthermore, this formalism seems to be more general than the geometric method since it can reproduce couplings that were missed by the latter [55, 56]. In particular, from the equivalence theorem of spontaneously broken theories [13], we know that for low energy SUSY breaking, the coupling of the gravitino to matter is dominated by the coupling of its goldstino component and has the form

$$(1/f) \partial^\mu \psi_X J_\mu = -(1/f) \psi_X \partial^\mu J_\mu + (\text{total space-time derivative}), \quad (2.47)$$

Here J_μ is the supercurrent of the theory corresponding to that in (2.46) in which the goldstino is essentially replaced by the spurion, with the corresponding explicit soft breaking terms:

$$\begin{aligned} \mathcal{L}' &= \int d^4\theta [1 - m_i^2 \theta^2 \bar{\theta}^2] \Phi_i^\dagger (e^V \Phi)_i + \int d^2\theta [W(\Phi_i) - \frac{B_{ij}}{2} \theta^2 \Phi_i \Phi_j \\ &\quad - \frac{A_{ijk}}{6} \theta^2 \Phi_i \Phi_j \Phi_k + \frac{1}{4} (1 - 2m_\lambda \theta^2) \text{Tr} W^\alpha W_\alpha] + \text{h.c.} . \end{aligned} \quad (2.48)$$

With this, eq. (2.47) shows that, on shell, all goldstino couplings are proportional to soft terms. Indeed, the supercurrent of (2.48) is given by (with $\mathcal{D}_{\mu,ij} = \delta_{ij} \partial_\mu + i g A_\mu^a T_{ij}^a$)

$$J_\alpha^\mu = -[\sigma^\nu \bar{\sigma}^\mu \psi_i]_\alpha [\mathcal{D}_{\nu,ij} \phi_j]^\dagger + i [\sigma^\mu \bar{\psi}_i]_\alpha F_i - \frac{1}{2\sqrt{2}} [\sigma^\nu \bar{\sigma}^\rho \sigma^\mu \bar{\lambda}^a]_\alpha F_{\nu\rho}^a + \frac{i}{\sqrt{2}} D^a [\sigma^\mu \bar{\lambda}^a]_\alpha$$

so

$$\partial_\mu J_\alpha^\mu = \psi_{i,\alpha} (m_i^2 \phi_j^\dagger + B_{ij} \phi_j + (1/2) A_{ijk} \phi_j \phi_k) + \frac{m_\lambda}{\sqrt{2}} [(\sigma^{\mu\nu})_\alpha^\beta \lambda_\beta^a F_{\mu\nu}^a + D^a \lambda_\alpha^a]. \quad (2.49)$$

From (2.47), (2.49) one then recovers the couplings with one goldstino that are missed in the geometric method.

Finally, in addition to usual SUSY and goldstino couplings, eq. (2.46) brings new goldstino-independent couplings induced by eliminating F_X . Indeed, we get

$$\left(1 - \frac{m_i^2}{f^2} |\phi_i|^2\right) F_X^\dagger = -\left(f + \frac{B_{ij}}{2f} \phi_i \phi_j + \frac{A_{ijk}}{6f} \phi_i \phi_j \phi_k + \frac{m_\lambda}{2f} \lambda \lambda + \dots\right). \quad (2.50)$$

So $|F_X|^2$ generates new couplings in \mathcal{L} , such as quartic scalar terms. As we will see in chapter 6, when applied to MSSM, they bring new corrections to the Higgs scalar potential.

CHAPTER 3

MSSM₅

We apply the methods of EFT on the Minimal Supersymmetric Standard Model (MSSM). Our aim is to study the phenomenological consequences of the complete set of mass dimension five operators that obey the gauge symmetries of MSSM and R-parity. Since not all of them are physically relevant, we will use spurion dependent superfield redefinitions to find the irreducible set of operators. But before getting there, we need to provide the Lagrangian of the model.

3.1 The Lagrangian

We denote the Lagrangian as:

$$\mathcal{L} = \mathcal{L}_{MSSM} + \mathcal{L}^{(5)} \quad (3.1)$$

\mathcal{L}_{MSSM} is the standard Lagrangian of the MSSM. In particular:

$$\begin{aligned} \mathcal{L}_{MSSM} = & \int d^4\theta \left[\mathcal{Z}_1 H_1^\dagger e^{V_1} H_1 + \mathcal{Z}_2 H_2^\dagger e^{V_2} H_2 \right] + \mathcal{L}_K \\ & + \left\{ \int d^2\theta \left[- H_2 Q \lambda_U U^c - Q \lambda_D D^c H_1 - L \lambda_E E^c H_1 + \mu H_1 H_2 \right] + h.c. \right\} \end{aligned} \quad (3.2)$$

Here \mathcal{L}_K accounts for the gauge kinetic part and the kinetic terms of the quark and lepton superfields Q, U^c, D^c, L, E^c as well as their associated soft break-

ing terms obtained using the spurion field formalism. U^c , D^c and E^c denote anti-quark/lepton singlet chiral superfields of components $f_R^c \equiv (f^c)_L$ and \tilde{f}_R^* , $f = u, d, e$, while Q and L denote the left-handed quark and lepton superfields doublets. Furthermore, since the hypercharge of H_1 is -1 and that of H_2 is $+1$, the vector superfields are $V_1 \equiv g_2 V_W^i \sigma^i - g_1 V_Y$ and $V_2 \equiv g_2 V_W^i \sigma^i + g_1 V_Y$. V_Y and V_W are the vector superfields of $U(1)_Y$ and $SU(2)_L$ respectively with g_1 and g_2 being the corresponding couplings. All SUSY breaking terms are included by allowing spurion dependence in the quantities \mathcal{Z}_i, μ and the 3×3 flavor matrices $\lambda_{U,D,E}$:

$$\mathcal{Z}_i \equiv \mathcal{Z}_i(S, S^\dagger), \quad \mu \equiv \mu(S), \quad \lambda_F \equiv \lambda_F(S), \quad F : U, D, E \quad (3.3)$$

where $S \equiv m_0 \theta^2$ is the spurion parametrising the soft supersymmetry breaking and m_0 is the supersymmetry breaking scale in the visible sector (e.g. if ' f ' is the v.e.v. of the auxiliary field that breaks SUSY, m_0 in gravity mediation is f/M_{Planck} and in gauge mediation $f/M_{messenger}$). Since we assume a spontaneously broken effective Lagrangian, consistency of the integration procedure implies the restriction

$$m_0 \ll M. \quad (3.4)$$

\mathcal{L}^5 denotes the complete set of mass dimension five operators that preserve R-parity¹:

$$\begin{aligned} \mathcal{L}^{(5)} &= \frac{1}{M} \left\{ \int d^2\theta \left[Q U^c T_Q Q D^c + Q U^c T_L L E^c + \lambda_H (H_1 H_2)^2 \right] + h.c. \right\} \\ &+ \frac{1}{M} \int d^4\theta \left[H_1^\dagger e^{V_1} Q Y_U U^c + H_2^\dagger e^{V_2} Q Y_D D^c + H_2^\dagger e^{V_2} L Y_E E^c + h.c. \right] \\ &+ \frac{1}{M} \int d^4\theta \left[A(S, S^\dagger) D^\alpha \left(B(S, S^\dagger) H_2 e^{-V_1} \right) D_\alpha \left(\Gamma(S, S^\dagger) e^{V_1} H_1 \right) + h.c. \right] \end{aligned} \quad (3.5)$$

¹For a general discussion of D=5 operators with discrete symmetries see [57].

The notation is such that

$$Q U^c T_Q Q D^c \equiv (Q U^c)^T (i\sigma_2) T_Q Q D^c$$

Similarly,

$$D^\alpha [B(S, S^\dagger) H_2 e^{-V_1}] D_\alpha [\Gamma(S, S^\dagger) e^{V_1} H_1] \equiv D^\alpha [B(S, S^\dagger) H_2^T (i\sigma_2) e^{-V_1}] D_\alpha [\Gamma(S, S^\dagger) e^{V_1} H_1].$$

$T_{Q,L}$ are matrices of parameters both in the up and the down sector, thus they carry four indices. In addition, all SUSY breaking terms are parametrized in the usual way, with spurions:

$$T_Q \equiv T_Q(S), \quad T_L \equiv T_L(S), \quad \lambda_H \equiv \lambda_H(S), \quad Y_F \equiv Y_F(S, S^\dagger), \quad F : U, D, E \quad (3.6)$$

M is the mass scale up to which the effective approach remains valid. It is associated with the mass of the heavy states that have been integrated out in order to obtain the effective operators.

The spurion dependence associated to these operators is the most general one can have. For the kinetic terms it is:

$$\begin{aligned} \mathcal{Z}_1 &= 1 + a_1 S + a_1^* S^\dagger + a_2 S S^\dagger, \\ \mathcal{Z}_2 &= 1 + b_1 S + b_1^* S^\dagger + b_2 S S^\dagger. \end{aligned} \quad (3.7)$$

and for the higher derivative effective operator:

$$\begin{aligned} A(S, S^\dagger) &= \alpha_0 + \alpha_1 S + \alpha_2 S^\dagger + \alpha_3 S S^\dagger \\ B(S, S^\dagger) &= \beta_0 + \beta_1 S + \beta_2 S^\dagger + \beta_3 S S^\dagger \\ \Gamma(S, S^\dagger) &= \gamma_0 + \gamma_1 S + \gamma_2 S^\dagger + \gamma_3 S S^\dagger \end{aligned} \quad (3.8)$$

3.2 Keeping the essential: The irreducible Lagrangian

The parameter space of Lagrangian (3.1) is huge. However, big parts of it are redundant since they describe the same physics. We would like to simplify the Lagrangian by removing this redundancy. One way to do this is by performing appropriate field redefinitions. A familiar set of holomorphic superfield redefinitions is

$$\Phi_i \rightarrow (1 - k_i S) \Phi_i, \quad (3.9)$$

which are commonly used in MSSM in order to restrict the so called “soft” breaking terms. We shall use this freedom later on. Less familiar are the following (super)field transformations²

$$\begin{aligned} H_1 \rightarrow H'_1 &= H_1 - \frac{1}{M} \overline{D}^2 \left[\Delta_1 H_2^\dagger e^{V_2} (i\sigma_2) \right]^T + \frac{1}{M} Q \rho_U U^c \\ H_2 \rightarrow H'_2 &= H_2 + \frac{1}{M} \overline{D}^2 \left[\Delta_2 H_1^\dagger e^{V_1} (i\sigma_2) \right]^T + \frac{1}{M} Q \rho_D D^c + \frac{1}{M} L \rho_E E^c \end{aligned} \quad (3.10)$$

where

$$\rho_F = \rho_F(S); \quad F : U, D, E, \quad \Delta_i = \Delta_i(S, S^\dagger) \quad i = 1, 2 \quad (3.11)$$

are arbitrary functions of the spurion. Also, ρ_F , $F = U, D, E$ are 3×3 matrices. The coefficients of their Taylor expansion in S are free parameters. We are free to fix them in a way to eliminate redundant dimension-five operators. These coefficients should have values smaller than M . The expansion of Δ_i is:

$$\begin{aligned} \Delta_1(S, S^\dagger) &= s_0 + s_1 S + s_2 S^\dagger + s_3 S S^\dagger \\ \Delta_2(S, S^\dagger) &= s'_0 + s'_1 S + s'_2 S^\dagger + s'_3 S S^\dagger \end{aligned} \quad (3.12)$$

²To avoid a complicated index notation, the transformations in (3.10) are written in matrix notation for the Higgs $SU(2)$ doublets. For clarity, $(i\sigma_2)$ appears explicitly even if it is implicit in the superpotential.

Notice that R-parity conservation does not allow for a similar set of transformations (3.10) on quark and lepton superfields. In addition, these field redefinitions, along with mixing operators from \mathcal{L}_{MSSM} and $\mathcal{L}^{(5)}$, generate operators of the type

$$\frac{1}{M^2} \int d^4\theta D^2[H_2 e^{-V_1} \Delta_1^\dagger] e^{V_1} \bar{D}^2[\Delta_1 e^{-V_1} H_2^\dagger] \quad (3.13)$$

plus a similar one for H_1 . Since these operators are of higher-order in $1/M$, their effects are further suppressed with respect to the dimension-five operators considered and we shall neglect them for the time being.

One then finds that the original Lagrangian transforms into:

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_K + \int d^4\theta \left[\mathcal{Z}'_1 H_1^\dagger e^{V_1} H_1 + \mathcal{Z}'_2 H_2^\dagger e^{V_2} H_2 \right] \\ & + \int d^2\theta \left[-H_2 Q \lambda'_U U^c - Q \lambda'_D D^c H_1 - L \lambda'_E E^c H_1 + \mu H_1 H_2 \right] + h.c. \\ & + \frac{1}{M} \int d^2\theta \left[Q U^c T'_Q Q D^c + Q U^c T'_L L E^c + \lambda_H (H_1 H_2)^2 \right] + h.c. \\ & + \frac{1}{M} \int d^4\theta \left[H_1^\dagger e^{V_1} Q Y'_U U^c + H_2^\dagger e^{V_2} Q Y'_D D^c + H_2^\dagger e^{V_2} L Y'_E E^c + h.c. \right] + \Delta\mathcal{L} \end{aligned} \quad (3.14)$$

where

$$\begin{aligned} \Delta\mathcal{L} = & \frac{1}{M} \int d^4\theta \left[-\Delta_1^\dagger H_2 e^{-V_1} D^2(\mathcal{Z}_1 e^{V_1} H_1) - \mathcal{Z}_2 H_2 e^{-V_1} D^2(\Delta_2^\dagger e^{V_1} H_1) + h.c. \right] \\ & + \frac{1}{M} \int d^4\theta \left[A(S, S^\dagger) D^\alpha (B(S, S^\dagger) H_2 e^{-V_1}) D_\alpha (\Gamma(S, S^\dagger) e^{V_1} H_1) + h.c. \right] \end{aligned} \quad (3.15)$$

The relation between primed and unprimed fields is

$$\lambda'_F(S) = \lambda_F(S) + \frac{\mu(S)}{M} \rho_F(S), \quad F : U, D, E \quad (3.16)$$

also

$$\begin{aligned}
Y'_U(S, S^\dagger) &= Y_U(S, S^\dagger) - 4 \Delta_2(S, S^\dagger) \lambda_U(S) + \mathcal{Z}_1(S, S^\dagger) \rho_U(S) \\
Y'_D(S, S^\dagger) &= Y_D(S, S^\dagger) - 4 \Delta_1(S, S^\dagger) \lambda_D(S) + \mathcal{Z}_2(S, S^\dagger) \rho_D(S) \\
Y'_E(S, S^\dagger) &= Y_E(S, S^\dagger) - 4 \Delta_1(S, S^\dagger) \lambda_E(S) + \mathcal{Z}_2(S, S^\dagger) \rho_E(S)
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
T'_Q(S) &= T_Q(S) + \lambda_U(S) \otimes \rho_D(S) + \rho_U(S) \otimes \lambda_D(S) \\
T'_L(S) &= T_L(S) + \lambda_U(S) \otimes \rho_E(S) + \rho_U(S) \otimes \lambda_E(S).
\end{aligned} \tag{3.18}$$

Finally,

$$\begin{aligned}
\mathcal{Z}'_1(S, S^\dagger) &= \mathcal{Z}_1(S, S^\dagger) - \frac{1}{M} (4\mu(S) \Delta_2(S, S^\dagger) + h.c.), \\
\mathcal{Z}'_2(S, S^\dagger) &= \mathcal{Z}_2(S, S^\dagger) - \frac{1}{M} (4\mu(S) \Delta_1(S, S^\dagger) + h.c.).
\end{aligned} \tag{3.19}$$

We perform a second set of field redefinitions to canonically normalize the kinetic terms:

$$H_1 \rightarrow \frac{1}{\sqrt{a'_0}} [1 - k_1 S] H_1, \quad H_2 \rightarrow \frac{1}{\sqrt{b'_0}} [1 - k_2 S] H_2, \quad k_1 \equiv \frac{a'_1}{a'_0}, \quad k_2 \equiv \frac{b'_1}{b'_0} \tag{3.20}$$

with

$$a'_0 \equiv \mathcal{Z}'_1 \Big|_{S, S^\dagger=0}, \quad a'_1 \equiv \mathcal{Z}'_1 \Big|_S, \quad b'_0 \equiv \mathcal{Z}'_2 \Big|_{S, S^\dagger=0}, \quad b'_1 \equiv \mathcal{Z}'_2 \Big|_S \tag{3.21}$$

which can be directly computed using the definition of $\mathcal{Z}'_{1,2}$, $\mathcal{Z}_{1,2}$ and $\Delta_{1,2}$ given above. The Lagrangian then becomes

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_K + \Delta\mathcal{L} + \int d^4\theta \left[\left(1 - \frac{m_1^2}{m_0^2} S S^\dagger\right) H_1^\dagger e^{V_1} H_1 + \left(1 - \frac{m_2^2}{m_0^2} S S^\dagger\right) H_2^\dagger e^{V_2} H_2 \right] \\
&+ \int d^2\theta \left[-H_2 Q \lambda'_U U^c - Q \lambda'_D D^c H_1 - L \lambda'_E E^c H_1 + \mu' H_1 H_2 \right] + h.c.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{M} \int d^2\theta \left[Q U^c T'_Q Q D^c + Q U^c T'_L L E^c + \lambda'_H (H_1 H_2)^2 \right] + h.c. \\
& + \frac{1}{M} \int d^4\theta \left[H_1^\dagger e^{V_1} Q Y''_U U^c + H_2^\dagger e^{V_2} Q Y''_D D^c + H_2^\dagger e^{V_2} L Y''_E E^c + h.c. \right] \quad (3.22)
\end{aligned}$$

Double primed quantities are given by

$$\begin{aligned}
\lambda''_U(S) &= \frac{1}{\sqrt{b'_0}} (1 - k_2 S) \lambda'_U(S) = (1 - b_1 S) \lambda_U(S) + \mathcal{O}(1/M), \\
\lambda''_F(S) &= \frac{1}{\sqrt{a'_0}} (1 - k_1 S) \lambda'_F(S) = (1 - a_1 S) \lambda_F(S) + \mathcal{O}(1/M), \quad F \equiv D, E. \\
\mu'(S) &= \frac{1}{\sqrt{a'_0 b'_0}} [1 - (k_1 + k_2)S] \mu(S) = (1 - (a_1 + b_1)S) \mu(S) + \mathcal{O}(1/M). \quad (3.23)
\end{aligned}$$

Since a'_0, b'_0 are M -dependent, the couplings $\lambda''_{U,D,E}(S)$ and also $\mu'(S)$ have acquired, already at the classical level, a dependence on the scale M of the higher dimensional operators. This is denoted above by $\mathcal{O}(1/M)$ and can be easily computed using (3.19) and (3.21). Note that this $\mathcal{O}(1/M)$ correction is relevant for the Lagrangian (3.22). Similar considerations apply to $m_{1,2}$ that appear in the same Lagrangian. Their exact expressions in terms of initial parameters can be computed in a similar way. Further

$$\begin{aligned}
\lambda''_H(S) &= (1 - 2(a_1 + b_1)S) \lambda_H(S), & Y''_U(S, S^\dagger) &= (1 - a_1^* S^\dagger) Y'_U(S, S^\dagger) \\
Y''_D(S, S^\dagger) &= (1 - b_1^* S^\dagger) Y'_D(S, S^\dagger), & Y''_E(S, S^\dagger) &= (1 - b_1^* S^\dagger) Y'_E(S, S^\dagger) \quad (3.24)
\end{aligned}$$

where we ignored terms which bring $\mathcal{O}(1/M^2)$ corrections to (3.22). Finally, $\Delta\mathcal{L}$ in (3.22) is that of (3.15) after applying transformations (3.20). Its component expansion up to $1/M$ order is:

$$\Delta\mathcal{L} = -\frac{1}{M} \int d^4\theta t_0 H_2 e^{-V_1} D^2 [e^{V_1} H_1]$$

$$\begin{aligned}
& + \frac{m_0}{M} \left[4 [t_1 + t_2 + t_0(a_1 + b_1)] h_2 \mathcal{D}_\mu \mathcal{D}^\mu h_1 - 2 [t_1 - t_2 + t_0(b_1 - a_1)] h_2 D_1 h_1 \right. \\
& + 2\sqrt{2} (t_1 + b_1 t_0) h_2 \lambda_1 \psi_{h_1} - 2\sqrt{2} (t_2 + a_1 t_0) \psi_{h_2} \lambda_1 h_1 - 4 t_3 F_{h_2} F_{h_1} \left. \right] \\
& + \frac{m_0^2}{M} \left[-4 (t_4 - b_1 t_3) h_2 F_{h_1} - 4 (t_5 - a_1 t_3) F_{h_2} h_1 + 2 t_6 \psi_{h_2} \psi_{h_1} \right] \\
& + \frac{m_0^3}{M} \left[-4 (t_7 - a_1 t_4 - b_1 t_5 + a_1 b_1 t_3) h_2 h_1 \right] + h.c. \tag{3.25}
\end{aligned}$$

where D_1 and λ_1 are components of the vector superfield V_1 and we also used the component notation $H_i = (h_i, \psi_{h_i}, F_{h_i})$. We also replaced $k_1, (k_2)$ by $a_1, (b_1)$ respectively, which is correct in the approximation of ignoring $1/M^2$ terms in the Lagrangian. The coefficients t_i are given by

$$\begin{aligned}
t_0 &= \alpha_0 \beta_0 \gamma_0 + s_0^* + s_0'^*, & t_4 &= d_4 - s_3^* - a_1^* s_2^* - b_2 s_0'^* - b_1 s_1'^*, \\
t_1 &= d_1 - s_2^* - b_1 s_0'^*, & t_5 &= d_5 - a_2 s_0^* - a_1 s_1^* - s_3'^* - b_1^* s_2'^*, \\
t_2 &= d_2 - a_1 s_0^* - s_2'^*, & t_6 &= d_6, \\
t_3 &= d_3 - s_1^* - a_1^* s_0^* - s_1'^* - b_1^* s_0'^*, & t_7 &= d_7 - a_2 s_2^* - a_1 s_3^* - b_1 s_3'^* - b_2 s_2'^* \tag{3.26}
\end{aligned}$$

with d_i being combinations of input parameters $\alpha_i, \beta_i, \gamma_i$ of eq. (3.8)

$$\begin{aligned}
d_1 &\equiv -\beta_1 \alpha_0 \gamma_0 - \alpha_1 \beta_0 \gamma_0 / 2, & d_4 &\equiv -\beta_3 \alpha_0 \gamma_0 - \beta_1 \alpha_2 \gamma_0 - \alpha_0 \beta_1 \gamma_2 \\
d_2 &\equiv -\gamma_1 \beta_0 \alpha_0 - \alpha_1 \beta_0 \gamma_0 / 2, & d_5 &\equiv -\gamma_3 \beta_0 \alpha_0 - \gamma_1 \alpha_2 \beta_0 - \alpha_0 \beta_2 \gamma_1, \\
d_3 &\equiv -\alpha_2 \beta_0 \gamma_0 - \alpha_0 \beta_2 \gamma_0 - \alpha_0 \beta_0 \gamma_2, & d_6 &\equiv \alpha_3 \gamma_0 \beta_0 + \alpha_1 \beta_2 \gamma_0 + \alpha_1 \beta_0 \gamma_2 \\
& & d_7 &\equiv -\gamma_3 \beta_1 \alpha_0 - \gamma_1 \beta_3 \alpha_0 - \gamma_1 \beta_1 \alpha_2. \tag{3.27}
\end{aligned}$$

A suitable choice of coefficients s_0, s_0', s_2', s_2 entering in transformation (3.10) allows us to set

$$t_i = 0, \quad i = 0, 1, 2, 3. \tag{3.28}$$

This ensures that the nonstandard terms in the first, second and third lines of $\Delta\mathcal{L}$ above are not present. The remaining terms proportional to m_0^2 and m_0^3 bring solely a renormalisation of soft terms, which are present anyway in Lagrangian (3.22) and can be ignored. Finally, the term $t_6 \psi_{h_2} \psi_{h_1}$ brings a renormalisation of the supersymmetric μ' term ($\mu' H_1 H_2$) of (3.22) and is invariant under the general field transformations (3.10). In principle one could set additional coefficients of the last two lines in $\Delta\mathcal{L}$ to vanish by a suitable choice of remaining $s_{1,3}, s'_{1,3}$; we choose not to do so and instead save these remaining coefficients for additional conditions that can be used to simplify the Lagrangian even further.

We have finally obtained the minimal set of dimension-five operators beyond the MSSM Lagrangian:

$$\begin{aligned}
\mathcal{L} = & \mathcal{L}_K + \int d^4\theta \left[\left(1 - \frac{m_1^2}{m_0^2} S^\dagger S\right) H_1^\dagger e^{V_1} H_1 + \left(1 - \frac{m_2^2}{m_0^2} S^\dagger S\right) H_2^\dagger e^{V_2} H_2 \right] \\
& + \int d^2\theta \left[-H_2 Q \lambda''_U(S) U^c - Q \lambda''_D(S) D^c H_1 - L \lambda''_E(S) E^c H_1 + \mu''(S) H_1 H_2 \right] + h.c. \\
& + \frac{1}{M} \int d^2\theta \left[Q U^c T'_Q(S) Q D^c + Q U^c T'_L(S) L E^c + \lambda'_H(S) (H_1 H_2)^2 \right] + h.c. \\
& + \frac{1}{M} \int d^4\theta \left[H_1^\dagger e^{V_1} Q Y''_U(S, S^\dagger) U^c + H_2^\dagger e^{V_2} Q Y''_D(S, S^\dagger) D^c + H_2^\dagger e^{V_2} L Y''_E(S, S^\dagger) E^c + h.c. \right]
\end{aligned} \tag{3.29}$$

\mathcal{L}_K stands for gauge kinetic terms and kinetic terms of MSSM fields other than $H_{1,2}$, together with their spurion dependence. Also, μ'' here includes the renormalisation due to t_6 (not shown). As explained above, there is still some remaining freedom to further reduce the parameter space and we will use it in the next section. The couplings that appear are given in equations (3.16), (3.17), (3.18), (3.23) and (3.24) in terms of those in the original Lagrangian. The couplings $\lambda''_{U,D,E}(S)$ acquired a threshold correction $O(1/M)$, which can be obtained

from (3.23). The dimension-five operator that was present in the last line of (3.5) is completely “gauged away” in the new fields basis, up to effects which renormalised the soft terms or the supersymmetric μ term. Since physics is independent of the fields basis we choose, in this new basis it is manifest that the last operator in (3.5) cannot affect the relations among physical masses of the Higgs sector. We discuss this in detail in section 4.4.

CHAPTER 4

PHENOMENOLOGY OF $MSSM_5$

4.1 Further Restrictions from Flavor Changing Neutral Currents

The couplings in Lagrangian (3.29) can have dramatic implications if the scale M is not too high, in particular due to FCNC effects. Indeed, if $T'_{Q,L}$ and $Y''_{U,D,E}$ have arbitrary family dependent couplings, one expects stringent limits from FCNC bounds [58]. It is possible however, under some mild assumptions for the original \mathcal{L} of (3.1), to remove the dangerous couplings in (3.29). For example, assume that the flavor matrices in (3.5) and the $\rho_{U,D,E}$ in (3.10), (3.11) are proportional to the ordinary Yukawa couplings¹:

$$\begin{aligned}
 T_Q(S) &= c_Q(S) \lambda_U(0) \otimes \lambda_D(0) \\
 T_L(S) &= c_L(S) \lambda_U(0) \otimes \lambda_E(0) \\
 \rho_F(S) &= c_F(S) \lambda_F(0), \quad F : U, D, E
 \end{aligned} \tag{4.1}$$

and, as usual

$$\lambda_F(S) = \lambda_F(0) (1 + A_F S), \quad F : U, D, E. \tag{4.2}$$

¹The ansatz is motivated by the discussion in subsection 2.2.2, eq. (2.27) where a similar structure of $T_{Q,L}$ and ρ_F is generated by integrating out massive $SU(2)$ superfields doublets.

Here $c_{Q,L}(S)$ are some arbitrary input functions of S ; $\lambda_F(S)$ are 3×3 matrices, while A_F are trilinear couplings. In the following $c_F(S) \equiv c_0^F + S c_1^F$, $F = U, D, E$ are considered free parameters which can be adjusted, together with the remaining $s_{1,3}, s'_{1,3}$, to remove some of the couplings in (3.29). Indeed, if

$$c_U(S) = -c_L(S) - c_E(S), \quad c_D(S) = -c_Q(S) + c_L(S) + c_E(S) \quad (4.3)$$

while $c_E(S)$ remains arbitrary, one obtains

$$T'_Q(S) = 0, \quad T'_L(S) = 0 \quad (4.4)$$

We can therefore remove the associated couplings in (3.29), that is the first two terms in the third line. Finally, let us assume that in (3.5) we also have

$$Y_F(S, S^\dagger) = f_F(S, S^\dagger) \lambda_F(0), \quad F : U, D, E \quad (4.5)$$

where f_F are spurion dependent but family independent functions of arbitrary coefficients:

$$f_F(S, S^\dagger) = f_0^F + S f_1^F + S^\dagger f_2^F + S S^\dagger f_3^F \quad (4.6)$$

Using (3.24), we find that the couplings in (3.29) are

$$Y''_F(S, S^\dagger) = \lambda_F(0) \left[x_0^F + x_1^F S + x_2^F S^\dagger + x_3^F S S^\dagger \right], \quad F = U, D, E \quad (4.7)$$

One finds

$$\begin{aligned} x_0^U &= f_0^U - 4s'_0 + c_0^U \\ x_1^U &= f_1^U - 4s'_1 + c_1^U + a_1 c_0^U \\ x_2^U &= f_2^U - 4s'_2 + a_1^* c_0^U - a_1^* x_0^U \\ x_3^U &= f_3^U - 4s'_3 + a_1^* c_1^U + a_2 c_0^U - a_1^* x_1^U \end{aligned} \quad (4.8)$$

Similar equations exist for the fields in the D and E multiplets. We just need to replace $U \rightarrow D$ (or E), $s'_i \rightarrow s_i$ and $a_i \rightarrow b_i$.

Let us examine if the form of $Y''_F(S, S^\dagger)$ can be simplified using the free parameters that we are left with: these are $s_{1,3}, s'_{1,3}$ from general transformations $\Delta_{1,2}$ and $c_E(S) = c_0^E + S c_1^E$, a total of 6 free parameters. We can use $s'_{1,3}$ ($s_{1,3}$) to eliminate S and $S S^\dagger$ parts of Y''_U (Y''_D), respectively. Using c_0^E and c_1^E we can also eliminate the S and $S S^\dagger$ of Y''_E . In conclusion, we used the remaining 6 free parameters to bring Y''_F to the form

$$Y''_F(S^\dagger) \equiv Y''_F(0, S^\dagger) = \lambda_F(0)(x_0^F + x_2^F S^\dagger), \quad F : U, D, E \quad (4.9)$$

The coefficients $x_{0,2}^F$ depend on the arbitrary coefficients $f_i^F, i = 0, 1, 2, 3, a_i, b_i, c_i$ of the original Lagrangian (3.1). Other simplifications can occur if we ignore the couplings Y of the first two families. With these considerations, the Lagrangian in (3.29) takes the form

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_K + \int d^4\theta \left[\left(1 - \frac{m_1^2}{m_0^2} S^\dagger S\right) H_1^\dagger e^{V_1} H_1 + \left(1 - \frac{m_2^2}{m_0^2} S^\dagger S\right) H_2^\dagger e^{V_2} H_2 \right] \\ & + \int d^2\theta \left[-H_2 Q \lambda''_U(S) U^c - Q \lambda''_D(S) D^c H_1 - L \lambda''_E(S) E^c H_1 + \mu''(S) H_1 H_2 \right] + h.c. \\ & + \frac{1}{M} \int d^4\theta \left[H_1^\dagger e^{V_1} Q Y''_U(S^\dagger) U^c + H_2^\dagger e^{V_2} Q Y''_D(S^\dagger) D^c + H_2^\dagger e^{V_2} L Y''_E(S^\dagger) E^c + h.c. \right] \\ & + \frac{1}{M} \int d^2\theta \lambda'_H(S) (H_1 H_2)^2 + h.c. \end{aligned} \quad (4.10)$$

with couplings (4.9) and (3.23)². This Lagrangian defines MSSM₅; the extension

² $\lambda''_F(S)$ acquired a threshold correction in M : $\lambda''_U(0) = \lambda_U(0) \left[1 + 1/M (\mu(0) c_U(0) + 2(\mu(0) s_0 + \mu^*(0) s_0^*)) \right]$ with similar relations for D, E obtained by $s_0 \rightarrow s'_0$ and $U \rightarrow D, (U \rightarrow E)$. In terms of original parameters, $s_0 = -[-4\alpha_0^* \beta_0^* \gamma_0^* b_1 - 4d_3^* + (f_1^U + f_1^D + c_1^U + c_1^D + a_1 c_0^U + b_1 c_0^D)]/4(a_1 - b_1)$ with d_3 as in (3.27); for the D, E sectors we use $s'_0 = -\alpha_0^* \beta_0^* \gamma_0^* - s_0$. Similar relations exist for non-supersymmetric counterparts, see (3.23), (3.24).

of MSSM by mass dimension five operators.

4.2 Phenomenological Implications

In the following we explore the new couplings that MSSM_5 brings with respect to standard MSSM. We begin with couplings proportional to m_0 . Part of these are coming from the terms in the second-last line of (4.10). These include non-analytic Yukawa couplings [11]

$$\begin{aligned}
& \frac{m_0}{M} x_2^U (\lambda_0^U)_{ij} (h_1^\dagger q_{Li}) u_{Rj}^c + h.c. \\
& \frac{m_0}{M} x_2^D (\lambda_0^D)_{ij} (h_2^\dagger q_{Li}) d_{Rj}^c + h.c. \\
& \frac{m_0}{M} x_2^E (\lambda_0^E)_{ij} (h_2^\dagger l_{Li}) e_{Rj}^c + h.c., \quad \lambda_0^F \equiv \lambda_F(0), \quad F : U, D, E. \quad (4.11)
\end{aligned}$$

These couplings are not soft in the sense of [59], but “hard” supersymmetry breaking terms in the sense of [11, 12]. They are less suppressed than those listed in [11] where they were generated at order m_0^2/M^2 . Such couplings can bring about a $\tan\beta$ enhancement of a prediction for a physical observable, such as the bottom quark mass relative to bottom quark Yukawa coupling [10, 60]. This effect is also present in the electroweak scale effective Lagrangian of the MSSM alone, after integrating out massive squarks at one loop level, with a result for bottom quark mass [10, 60, 61, 62, 63]

$$m_b = \frac{v \cos\beta}{\sqrt{2}} (\lambda_b + \delta\lambda_b + \Delta\lambda_b \tan\beta) \quad (4.12)$$

where λ_b is the ordinary bottom quark Yukawa coupling, $\delta\lambda_b$ its one loop correction and $\Delta\lambda_b$ is a “wrong” Higgs bottom quark Yukawa coupling, generated

by integrating out massive squarks. In our case, $\Delta\lambda_b$ receives an additional contribution from the second line in (4.11). The size of this extra contribution due to higher dimensional operators, can be comparable and even substantially larger than the one generated in the MSSM at one loop level (for a suitable value for $x_2^D m_0/M$ - recall that x_2^D is not fixed). Such contributions can bring a $\tan\beta$ enhanced correction of the Higgs decay rate to bottom quark pairs. Similar considerations apply to the U and E sectors.

Other similar couplings derived from (4.10) and proportional to m_0 are

$$\begin{aligned} & \frac{m_0}{M} x_2^U (\lambda_0^{U\dagger} \lambda_0^U)_{ij} (h_1^\dagger h_2^\dagger) \tilde{u}_{Ri} \tilde{u}_{Rj}^* + h.c. \\ & \frac{m_0}{M} x_2^U (\lambda_0^U \lambda_0^{U\dagger})_{ij} (h_1^\dagger \tilde{q}_{Li}) (h_2^\dagger \tilde{q}_{Lj}^\dagger) + h.c. \end{aligned} \quad (4.13)$$

where we used that $\lambda_0^{F''}$ and λ_0^F are equal up to $O(1/M)$ corrections, see (3.16) and (3.23). The above terms are strongly suppressed due to the square of the Yukawa coupling, in addition to $m_0/M \ll 1$, so their effects are expected to be small, except for the third generation. Their counterparts in the down (D) sector are

$$\begin{aligned} & \frac{m_0}{M} x_2^D (\lambda_0^{D\dagger} \lambda_0^D)_{ij} (h_2^\dagger h_1^\dagger) \tilde{d}_{Ri} \tilde{d}_{Rj}^* + h.c. \\ & \frac{m_0}{M} x_2^D (\lambda_0^D \lambda_0^{D\dagger})_{ij} (h_2^\dagger \tilde{q}_{Li}) (h_1^\dagger \tilde{q}_{Lj}^\dagger) + h.c. \end{aligned} \quad (4.14)$$

In the lepton sector similar couplings are present, obtained from eq. (4.14) with $Q \rightarrow L$, $D \rightarrow E$. All the quartic couplings listed above are renormalisable, but naively they would seem to break supersymmetry in a hard way if inserted into loops with a cutoff larger than M . This, of course, is just an artifact of using a cutoff larger than the energy scale of the heavy states that we integrated out.

It is interesting to note that there is no “wrong” Higgs-gaugino-higgsino coupling generated [11], even though the original Lagrangian in eq. (3.5) included it, see eq. (3.25) where

$$\frac{m_0}{M} (\psi_{h_2} \lambda_1 h_1 + h_2 \lambda_1 \psi_{h_1}) + h.c. \quad (4.15)$$

was present. Such a coupling can be generated at one loop level [10] but in our case it was removed by the Higgs fields transformation (3.10). This shows that not all “wrong” Higgs couplings are actually independent (this may also apply when such couplings are generated at the loop level).

Note that in the MSSM₅ defined by eq. (4.10), couplings proportional to m_0 involving “wrong” Higgs A-terms are not present, given our ansatz (4.1) and (4.5) leading to (4.9). If this ansatz is not imposed on the third generation, then one could have such terms from (3.29)

$$\frac{m_0^2}{M} \left[y_{u,3} h_1^\dagger \tilde{q}_{L,3} \tilde{u}_{R,3}^* + y_{d,3} h_2^\dagger \tilde{q}_{L,3} \tilde{d}_{R,3}^* + y_{e,3} h_2^\dagger \tilde{l}_{L,3} \tilde{e}_{R,3}^* \right] \quad (4.16)$$

where $y_{f,3}$, $f = u, d, e$ are the coefficients of component $S S^\dagger$ of $Y''(S, S^\dagger)$ of third generation.

There are also new, and perhaps most important, supersymmetric couplings that affect the amplitude of processes like quark + quark \rightarrow squark + squark or similar with (s)leptons. These are

$$\begin{aligned} & \frac{1}{M} x_0^U (\lambda_0^D)_{ij} (\lambda_0^U)_{kl} \tilde{q}_{Li} \tilde{d}_{Rj}^* q_{Lk} u_{Rl}^c + h.c. \\ & \frac{1}{M} x_0^D (\lambda_0^U)_{ij} (\lambda_0^D)_{kl} \tilde{q}_{Li} \tilde{u}_{Rj}^* q_{Lk} d_{Rl}^c + h.c. \\ & \frac{1}{M} x_0^U (\lambda_0^E)_{ij} (\lambda_0^U)_{kl} \tilde{l}_{Li} \tilde{e}_{Rj}^* q_{Lk} u_{Rl}^c + (L \leftrightarrow Q, E \leftrightarrow U) + h.c. \end{aligned} \quad (4.17)$$

They can be important particularly for the third generation. The largest effect would be for squarks pair production from a pair of quarks; the process could be comparable to the MSSM tree level contribution to the amplitude of the same process [64]. Indeed, let us focus on the $q\bar{q} \rightarrow \tilde{q}\tilde{q}^*$ in MSSM generated by a tree-level gluon exchange. The MSSM amplitude behaves as

$$A_{q\bar{q} \rightarrow g \rightarrow \tilde{q}\tilde{q}^*} \sim \frac{g_3^2}{\sqrt{s}}, \quad (4.18)$$

where s is the Mandelstam variable. On the other hand, the operators (4.17) generate a contact term contributing

$$A_{q\bar{q} \rightarrow \tilde{q}\tilde{q}^*}^{MSSM_5} \sim \frac{\lambda_0^U \lambda_0^D}{M}. \quad (4.19)$$

The dimension-five operator for the third generation has therefore a comparable contribution to the MSSM diagrams for energies $E \geq g_3^2 M$, which can be in the TeV range. In MSSM there are other diagrams contributing to this process, in particular Higgs exchange. It can be checked however that at energies above the CP-even Higgs masses, the MSSM amplitude decreases in energy whereas the contact term coming from the dimension-five operators gives a constant contribution which is sizeable for high energy. Of course, at energies above M we should replace the contact term by the corresponding tree-level diagram with exchange of massive $SU(2)$ doublets (or whatever other physics generates this effective operator).

Note that couplings similar to (4.17) could also be generated by the term $\int d^2\theta (QU) T_Q(QD)$ of (3.29). This term is not present in $MSSM_5$ of (4.10) due to our FCNC ansatz (4.1), (4.4); however, the ansatz could be relaxed for the third

generation. Therefore the above process of squark production can have an even larger amplitude from contributions in the third line of (3.29).

The Lagrangian (4.10) also contains other supersymmetric couplings involving gauge interactions which can be important for phenomenology. They arise from any dimension-five D-term in (4.10) giving

$$\begin{aligned} \mathcal{L} \supset & \frac{(\lambda_0^U)_{ij} x_0^U}{M} \left[-h_1^\dagger \mathcal{D}_\mu \mathcal{D}^\mu (\tilde{q}_{Li} \tilde{u}_{Rj}^*) - \frac{1}{\sqrt{2}} h_1^\dagger \lambda_1 (\tilde{q}_{Li} u_{Rj}^c + q_{Li} \tilde{u}_{Rj}^*) - \frac{1}{\sqrt{2}} \bar{\psi}_{h_1} \bar{\lambda}_1 \tilde{q}_{Li} \tilde{u}_{Rj}^* \right. \\ & \left. + \frac{1}{2} h_1^\dagger D_1 \tilde{q}_{Li} \tilde{u}_{Rj}^* + i \bar{\psi}_{h_1} \bar{\sigma}^\mu \mathcal{D}_\mu (\tilde{q}_{Li} u_{Rj}^c + q_{Li} \tilde{u}_{Rj}^*) \right] \\ & + (U \rightarrow D, H_1 \rightarrow H_2, V_1 \rightarrow V_2) + (Q \rightarrow L, H_1 \rightarrow H_2, V_1 \rightarrow V_2, U \rightarrow E) + h.c. \end{aligned} \quad (4.20)$$

where D_1, λ_1 are the auxiliary and gaugino components of V_1 vector superfield, and

$$\begin{aligned} D_1 \equiv & -\frac{g_2^2}{2} \left[h_1^\dagger \bar{\sigma} h_1 + h_2^\dagger \bar{\sigma} h_2 + \tilde{q}_{Li}^\dagger \bar{\sigma} \tilde{q}_{Li} + \tilde{l}_{Li}^\dagger \bar{\sigma} \tilde{l}_{Li} \right] \\ & + \frac{g_1^2}{2} \left[-h_1^\dagger h_1 + h_2^\dagger h_2 + \frac{1}{3} \tilde{q}_{Li}^\dagger \tilde{q}_{Li} - \frac{4}{3} \tilde{u}_{Ri} \tilde{u}_{Ri}^* + \frac{2}{3} \tilde{d}_{Ri} \tilde{d}_{Ri}^* - \tilde{l}_{Li}^\dagger \tilde{l}_{Li} + 2 \tilde{e}_{Ri} \tilde{e}_{Ri}^* \right] \end{aligned} \quad (4.21)$$

Here \mathcal{D}_μ is the covariant derivative, $\mathcal{D}_\mu = \partial_\mu + i/2 V_{1,\mu}$, where $V_{1,\mu}$ is the gauge field of the vector superfield $V_1 \equiv g_2 V_W^i \sigma^i - g_1 V_Y$, introduced in eq. (3.2). Couplings similar to those above are generated by the substitutions shown in (4.20). Some of them can be phenomenologically important, e.g. those involving 2 particles and 2 sparticles such as Higgs-quark-squark-gaugino or gauge-quark-higgsino-squark, arising from (4.20). Also, we notice a term with a ‘‘wrong’’ Higgs-squark-squark derivative coupling.

Yukawa interactions also generate supersymmetric couplings of structure similar to some of those in (4.20), involving 4 squarks and a higgs or 2 squarks

and 3 higgses or 2 squarks, 2 sleptons and a higgs. However, these arise at order λ_F^3 , where $\lambda_F, F : U, D, E$ are Yukawa couplings entering (4.10). Therefore they are suppressed both by the scale M and, relative to the above gauge counterparts, by an extra Yukawa coupling. This is due to the presence of an extra Yukawa coupling in the third line of (4.10) relative to ordinary D-terms. The strength of these interactions is also sub-leading to other Yukawa interactions listed so far which also involved fewer (s)particles.

Finally, supersymmetric couplings with 3 higgses and 2 squarks or 2 sleptons arise from $(H_1 H_2)^2$ of (4.10), suppressed by two Yukawa couplings and by the scale M . Also, there exist potentially larger couplings of 2 higgses and 2 higgsinos, being suppressed only by $\lambda_H(0)$ and the scale M . In addition, there are non-supersymmetric couplings with 4 higgs fields whose effects are discussed in section 4.4. This concludes our discussion of all the new couplings generated by dimension-five operators in the MSSM₅.

4.3 The MSSM Higgs Sector with Mass Dimension Five Operators

In the following we restrict the analysis to the MSSM Higgs sector extended by mass dimension five operators and analyse their implications. In this sector there are in general two dimension-five operators that affect the Higgs fields masses, shown in eq. (4.22) below. According to our previous discussion the last operator in (4.22) is redundant and can be “gauged away”. However, in this section we choose to keep it, in order to show explicitly that it does not bring new physics of its own. The relevant part of MSSM Higgs Lagrangian

with dimension-five operators is

$$\begin{aligned}
\mathcal{L}_1 &= \int d^4\theta \left[\mathcal{Z}_1(S, S^\dagger) H_1^\dagger e^{V_1} H_1 + \mathcal{Z}_2(S, S^\dagger) H_2^\dagger e^{V_2} H_2 \right] \\
&+ \int d^2\theta \left[\tilde{\mu} (1 + c_1 S) H_1 H_2 + \frac{c_3}{M} (1 + c_2 S) (H_1 H_2)^2 \right] + h.c. \\
&+ \frac{1}{M} \int d^4\theta \left\{ A(S, S^\dagger) D^\alpha \left[B(S, S^\dagger) H_2 e^{-V_1} \right] D_\alpha \left[\Gamma(S, S^\dagger) e^{V_1} H_1 \right] + h.c. \right\}
\end{aligned} \tag{4.22}$$

Additional spurion dependence arises from the dimension-five operators considered. For the definitions of $A(S, S^\dagger)$, $B(S, S^\dagger)$, $\Gamma(S, S^\dagger)$ see eq. (3.8). After elimination of the auxiliary fields and a rescaling of scalar fields, the scalar part of \mathcal{L}_1 in (4.22) becomes:

$$\begin{aligned}
\mathcal{L}_{1,scalar} &= -\frac{1}{8} (g_1^2 + g_2^2) (|h_1|^2 - |h_2|^2)^2 + \frac{m_0}{M} (g_1^2 + g_2^2) (|h_1|^2 - |h_2|^2) (\delta_1 h_1 h_2 + h.c.) \\
&+ \frac{2c_3}{M} (|h_1|^2 + |h_2|^2) (\tilde{\mu}^* h_1 h_2 + h.c.) - \frac{m_0}{M} c_3 (\delta_2 (h_1 h_2)^2 + h.c.) \\
&- (|\tilde{\mu}|^2 + m_1^2) |h_1|^2 - (|\tilde{\mu}|^2 + m_2^2) |h_2|^2 - (h_1 h_2 B m_0 \mu + h.c.) - h_1^* \mathcal{D}^2 h_1 - h_2^* \mathcal{D}^2 h_2
\end{aligned} \tag{4.23}$$

where

$$\begin{aligned}
m_1^2 &= m_0^2 (|a_1|^2 - a_2) + \mathcal{O}(m_0/M) \\
m_2^2 &= m_0^2 (|b_1|^2 - b_2) + \mathcal{O}(m_0/M) \\
B m_0 \mu &= \tilde{\mu} m_0 (c_1 - a_1 - b_1) + \mathcal{O}(m_0/M)
\end{aligned} \tag{4.24}$$

The $\mathcal{O}(m_0/M)$ corrections in (4.24) are not shown explicitly since they only renormalise $m_{1,2}$ and $B m_0 \mu$ which are anyway unknown parameters of the MSSM. We denoted

$$\delta_1 = -\beta_1 \alpha_0 \gamma_0 + \gamma_1 \beta_0 \alpha_0 - \alpha_0 \beta_0 \gamma_0 (a_1 - b_1), \quad \delta_2 = c_2 + 2(a_1 + b_1), \tag{4.25}$$

We notice the presence of three contributions in the scalar potential, introduced by our dimension-five operators. The contributions proportional to c_3 are due to $(H_1 H_2)^2$ in (4.22) and were discussed in [65] (also [67, 68]; for a review see [69]). The one proportional to δ_1

$$(|h_1|^2 - |h_2|^2)(h_1 h_2 + h.c.), \quad (4.26)$$

was introduced by the dimension-five operator in the last line of (4.22). This is a new contribution to the scalar potential, and is vanishing if $\alpha_0 = \beta_0 = \gamma_0$. An interesting feature is that its one loop contribution to $h_{1,2}$ self energy remains soft (no quadratic divergences) despite its higher dimensional origin.

4.4 Higgs Mass Corrections Beyond MSSM

Let us consider the implications of (4.23) for the Higgs masses. The scalar potential is

$$\begin{aligned} V = & \tilde{m}_1^2 |h_1|^2 + \tilde{m}_2^2 |h_2|^2 + (B m_0 \mu h_1 h_2 + h.c.) + \frac{g^2}{8} (|h_1|^2 - |h_2|^2)^2 \\ & + (|h_1|^2 - |h_2|^2)(\eta_1 h_1 h_2 + h.c.) + (|h_1|^2 + |h_2|^2)(\eta_2 h_1 h_2 + h.c.) \\ & + \frac{1}{2} (\eta_3 (h_1 h_2)^2 + h.c.) \end{aligned} \quad (4.27)$$

where the definition of $\eta_{1,2,3} \sim 1/M$ can be read from eq. (4.23). We take for simplicity η_i real, and therefore $\eta_3 \geq 0$, $|\eta_2| \leq \eta_3/4$. Also

$$\tilde{m}_1^2 \equiv m_1^2 + |\tilde{\mu}|^2, \quad \tilde{m}_2^2 \equiv m_2^2 + |\tilde{\mu}|^2, \quad g^2 \equiv g_1^2 + g_2^2 \quad (4.28)$$

Consider quantum fluctuations of h_i around a vacuum expectation value

$$h_i = \frac{1}{\sqrt{2}} (v_i + \tilde{h}_i + i\tilde{\sigma}_i), \quad i = 1, 2 \quad (4.29)$$

From the two minimum conditions for the scalar potential V of eq. (4.27) one can express $\tilde{m}_{1,2}$ in terms of $Bm_0\mu$, v_1 , v_2 to find:

$$\begin{aligned}\tilde{m}_1^2 &= -Bm_0\mu \frac{v_2}{v_1} - \frac{1}{8} g^2 (v_1^2 - v_2^2) - \frac{\eta_1}{2} \frac{v_2}{v_1} (3v_1^2 - v_2^2) - \frac{\eta_2}{2} \frac{v_2}{v_1} (3v_1^2 + v_2^2) - \frac{\eta_3}{2} v_2^2 \\ \tilde{m}_2^2 &= -Bm_0\mu \frac{v_1}{v_2} + \frac{1}{8} g^2 (v_1^2 - v_2^2) - \frac{\eta_1}{2} \frac{v_1}{v_2} (v_1^2 - 3v_2^2) - \frac{\eta_2}{2} \frac{v_1}{v_2} (3v_2^2 + v_1^2) - \frac{\eta_3}{2} v_1^2\end{aligned}\quad (4.30)$$

which shall be used in the following. The mass matrix is

$$\mathcal{M}_{ij} = \left. \frac{1}{2} \frac{\partial^2 V}{\partial h_i \partial h_j} \right|_{h_i=v_i/\sqrt{2}, \sigma_i=0} = X_{ij} + Z_{ij} \quad (4.31)$$

where

$$X_{ij} = \frac{1}{2} \begin{pmatrix} 2\tilde{m}_1^2 + \frac{1}{4} g^2 (3v_1^2 - v_2^2) & 2Bm_0\mu - \frac{1}{2} g^2 v_1 v_2 \\ 2Bm_0\mu - \frac{1}{2} g^2 v_1 v_2 & 2\tilde{m}_2^2 - \frac{1}{4} g^2 (v_1^2 - 3v_2^2) \end{pmatrix} \quad (4.32)$$

and

$$Z_{ij} = \frac{1}{2} \begin{pmatrix} 6(\eta_1 + \eta_2) v_1 v_2 + \eta_3 v_2^2 & 3(\eta_1 + \eta_2) v_1^2 + 3(\eta_2 - \eta_1) v_2^2 + 2\eta_3 v_1 v_2 \\ 3(\eta_1 + \eta_2) v_1^2 + 3(\eta_2 - \eta_1) v_2^2 + 2\eta_3 v_1 v_2 & 6(\eta_2 - \eta_1) v_1 v_2 + \eta_3 v_1^2 \end{pmatrix} \quad (4.33)$$

The mass eigenvalues $m_{h,H}^2$ of \mathcal{M}_{ij} are

$$\begin{aligned}m_{h,H}^2 &= M_{h,H}^2 \mp \frac{6\eta_1}{\sqrt{w}} \left[Bm_0\mu (v_1^2 - v_2^2) + v_1 v_2 (\tilde{m}_1^2 - \tilde{m}_2^2 + \frac{g^2}{4} (v_1^2 - v_2^2)) \right] \\ &+ 3\eta_2 \left[v_1 v_2 \pm \frac{1}{2\sqrt{w}} (v_1^2 + v_2^2) (-4Bm_0\mu + g^2 v_1 v_2) \right] \\ &+ \frac{\eta_3}{4} \left[v_1^2 + v_2^2 \pm \frac{1}{\sqrt{w}} (2(\tilde{m}_1^2 - \tilde{m}_2^2)(v_1^2 - v_2^2) + g^2 (v_1^2 + v_2^2)^2 \right. \\ &\left. - 16Bm_0\mu v_1 v_2) \right]\end{aligned}\quad (4.34)$$

where upper (lower) signs correspond to the lighter m_h^2 (heavier m_H^2) Higgs field and $M_{h,H}^2$ expresses the pure MSSM part:

$$M_{h,H}^2 \equiv \frac{1}{2} \left[\tilde{m}_1^2 + \tilde{m}_2^2 + \frac{g^2}{4} (v_1^2 + v_2^2) \mp \frac{1}{2} \sqrt{w} \right] \quad (4.35)$$

Also,

$$w \equiv (4Bm_0\mu - g^2 v_1 v_2)^2 + 4\left(\tilde{m}_1^2 - \tilde{m}_2^2 + \frac{g^2}{2}(v_1^2 - v_2^2)\right)^2 \quad (4.36)$$

With the values of $\tilde{m}_{1,2}$ expressed in terms of $v_{1,2}$ and $Bm_0\mu$ from minimum conditions (4.30), one can express $m_{h,H}^2$ of (4.34) as follows

$$m_{h,H}^2 = \frac{m_Z^2}{2} - \frac{Bm_0\mu(u^2 + 1)}{2u} \mp \frac{\sqrt{w'}}{2} + v^2 \left[\eta_1 q_1^\pm + \eta_2 q_2^\pm + \eta_3 q_3^\pm \right] \quad (4.37)$$

with

$$\begin{aligned} q_1^\pm &= \frac{u^2 - 1}{4u} \pm \frac{(u^2 - 1)}{4u^2(1 + u^2)^2 \sqrt{w'}} \left[m_Z^2 u(1 - 6u^2 + u^4) + Bm_0\mu(1 + u^2)(1 + 18u^2 + u^4) \right] \\ q_2^\pm &= -\frac{1 - 6u^2 + u^4}{4u(1 + u^2)} \mp \frac{m_Z^2 u(1 - 14u^2 + u^4) + Bm_0\mu(1 + u^2)(1 + 10u^2 + u^4)}{4u^2(1 + u^2)\sqrt{w'}} \\ q_3^\pm &= \mp \frac{2u}{(1 + u^2)^2 \sqrt{w'}} \left[Bm_0\mu(1 + u^2) - m_Z^2 u \right] \end{aligned} \quad (4.38)$$

where

$$w' \equiv m_Z^4 + [Bm_0\mu(1 + u^2)^3 + 2m_Z^2 u(1 - 6u^2 + u^4)] \frac{Bm_0\mu}{u^2(1 + u^2)} \quad (4.39)$$

and where we also used $v_1 = v \cos \beta$, $v_2 = v \sin \beta$, $u = \tan \beta$, $m_Z^2 = g^2 v^2/4$ and $Bm_0\mu < 0$.

Similar considerations apply for the pseudoscalar Higgs/Goldstone boson sector. The mass matrix in this case is

$$N_{ij} = \left. \frac{\partial^2 V}{\partial \tilde{\sigma}_i \partial \tilde{\sigma}_j} \right|_{h_i=v_i/\sqrt{2}, \tilde{\sigma}_i=0} \quad (4.40)$$

with entries

$$\begin{aligned} N_{11} &= \tilde{m}_1^2 + \frac{g^2}{8}(v_1^2 - v_2^2) + (\eta_1 + \eta_2)v_1 v_2 - \frac{\eta_3}{2}v_2^2 \\ N_{12} &= -\frac{\eta_1}{2}(v_1^2 - v_2^2) - \frac{\eta_2}{2}(v_1^2 + v_2^2) - \eta_3 v_1 v_2 - \text{Re}(Bm_0\mu) \\ N_{22} &= \tilde{m}_2^2 - \frac{g^2}{8}(v_1^2 - v_2^2) + (\eta_2 - \eta_1)v_1 v_2 - \frac{\eta_3}{2}v_1^2 \end{aligned} \quad (4.41)$$

The eigenvalues of N are

$$\begin{aligned}
m_{G,A}^2 &= \frac{1}{2} (\tilde{m}_1^2 + \tilde{m}_2^2) \mp \frac{1}{8} \sqrt{\kappa} \\
&\mp \frac{4\eta_1}{\sqrt{\kappa}} \left[Bm_0\mu(v_1^2 - v_2^2) + v_1 v_2 \left(\tilde{m}_1^2 - \tilde{m}_2^2 + \frac{g^2}{4}(v_1^2 - v_2^2) \right) \right] \\
&+ \eta_2 \left[v_1 v_2 \mp \frac{4Bm_0\mu}{\sqrt{\kappa}} (v_1^2 + v_2^2) \right] \\
&+ \eta_3 \left[-\frac{1}{4} (v_1^2 + v_2^2) \mp \frac{1}{\sqrt{\kappa}} \left(8Bm_0\mu v_1 v_2 + (v_1^2 - v_2^2)(\tilde{m}_1^2 - \tilde{m}_2^2) + \frac{g^2}{4} (v_1^2 - v_2^2)^2 \right) \right]
\end{aligned} \tag{4.42}$$

with

$$\kappa = 16 \left[4(Bm_0\mu)^2 + \left(\tilde{m}_1^2 - \tilde{m}_2^2 + \frac{g^2}{4} (v_1^2 - v_2^2) \right)^2 \right] \tag{4.43}$$

where the upper sign corresponds to the Goldstone m_G and the lower sign to m_A^2 . One can use (4.30) to replace $\tilde{m}_{1,2}$ in terms of $v_{1,2}$ and m_A . Using (4.30) one shows that $m_G = 0$ and

$$\begin{aligned}
m_A^2 &= -\frac{v_1^2 + v_2^2}{2v_1 v_2} \left[2 Bm_0\mu + \eta_1 (v_1^2 - v_2^2) + \eta_2 (v_1^2 + v_2^2) + 2\eta_3 v_1 v_2 \right] \\
&= -\frac{1+u^2}{u} Bm_0\mu + \frac{u^2-1}{2u} \eta_1 v^2 - \frac{1+u^2}{2u} \eta_2 v^2 - \eta_3 v^2
\end{aligned} \tag{4.44}$$

By eliminating $Bm_0\mu$ between (4.37) and (4.44), one obtains the masses $m_{h,H}$:

$$\begin{aligned}
m_{h,H}^2 &= \frac{1}{2} \left[m_A^2 + m_Z^2 \mp \sqrt{w''} \right] \mp \frac{4 m_A^2 \eta_1 u (u^2 - 1) v^2}{(1+u^2)^2 \sqrt{w''}} \\
&+ \frac{2\eta_2 u v^2}{1+u^2} \left[1 \pm \frac{m_A^2 + m_Z^2}{\sqrt{w''}} \right] + \frac{\eta_3 v^2}{2} \left[1 \mp \frac{(m_A^2 - m_Z^2)(u^2 - 1)^2}{\sqrt{w''} (1+u^2)^2} \right]
\end{aligned} \tag{4.45}$$

where the upper (lower) signs correspond to h (H) respectively and

$$w'' \equiv m_A^4 + m_Z^4 - 2 m_A^2 m_Z^2 \frac{1 - 6u^2 + u^4}{(1+u^2)^2} = (m_A^2 + m_Z^2)^2 - 4 m_A^2 m_Z^2 \cos^2 2\beta \tag{4.46}$$

Replacing $u = \tan\beta$ in $m_{h,H}$ one obtains an equivalent form of $m_{h,H}$

$$m_{h,H}^2 = \frac{1}{2} \left[m_A^2 + m_Z^2 \mp \sqrt{w''} \right] \pm \eta_1 v^2 \sin 4\beta \frac{m_A^2}{\sqrt{w''}} \\ + \eta_2 v^2 \sin 2\beta \left[1 \pm \frac{m_A^2 + m_Z^2}{\sqrt{w''}} \right] + \frac{\eta_3 v^2}{2} \left[1 \mp \frac{(m_A^2 - m_Z^2) \cos^2 2\beta}{\sqrt{w''}} \right] \quad (4.47)$$

For $\eta_2 = \eta_3 = 0$ one finds from (4.47):

$$m_h^2 + m_H^2 = m_A^2 + m_Z^2 \quad (4.48)$$

which is independent of η_1 . Then η_1 does not affect the relation among physical masses which is consistent with the result of section 4.3, where the last term in (4.22), responsible for the η_1 term in V , could be removed by a suitable field redefinition.

In the limit of large $\tan\beta$ with m_A fixed at a value $m_A > m_Z$ one finds:

$$m_h^2 = m_Z^2 + \frac{4m_A^2 v^2}{m_A^2 - m_Z^2} (\eta_2 - \eta_1) \cot\beta \\ - \frac{4m_A^2 m_Z^2}{m_A^2 - m_Z^2} \left[1 - \eta_3 v^2 \frac{m_A^4 + m_Z^4}{2m_A^2 m_Z^2 (m_A^2 - m_Z^2)} \right] \cot^2\beta + \mathcal{O}(\cot^3\beta) \quad (4.49)$$

and

$$m_H^2 = m_A^2 + \eta_3 v^2 + \frac{4(m_A^2 \eta_1 - m_Z^2 \eta_2) v^2}{m_A^2 - m_Z^2} \cot\beta \\ + \frac{4m_A^2 m_Z^2}{m_A^2 - m_Z^2} \left[1 - \eta_3 v^2 \frac{m_A^4 + m_Z^4}{2m_A^2 m_Z^2 (m_A^2 - m_Z^2)} \right] \cot^2\beta + \mathcal{O}(\cot^3\beta) \quad (4.50)$$

Therefore

$$\delta m_h^2 = \frac{4m_A^2 v^2}{m_A^2 - m_Z^2} (\eta_2 - \eta_1) \cot\beta + \mathcal{O}(\cot^2\beta) \\ \delta m_H^2 = \eta_3 v^2 + \frac{4(m_A^2 \eta_1 - m_Z^2 \eta_2) v^2}{m_A^2 - m_Z^2} \cot\beta + \mathcal{O}(\cot^2\beta) \quad (4.51)$$

in agreement with [65] for $\eta_1 = 0$. The above expansions for large $\tan\beta$ should be regarded with due care since they are the result of a double series expansion in η_i and $1/\tan\beta$. Assuming $\eta_3 = 0$ (then $\eta_2 = 0$, too), the term proportional to $\cot\beta$ in (4.49) is larger than the sub-leading one ($\cot^2\beta$), giving $m_h^2 - m_Z^2 > 0$ if $|\eta_1/g^2| \geq 1/(4\tan\beta)$. This bound is however outside the validity of the perturbative expansion in η_1 as we shall see shortly and then the large $\tan\beta$ expansion is not useful. If $\eta_{1,2} = 0$ and $\eta_3 > 0$ one could obtain $m_h > m_Z$ if also the square bracket in (4.49) is negative, which is more easily satisfied (for a small η_3) if m_A is very close to m_Z , but then the above large $\tan\beta$ expansion is not reliable.

Let us therefore analyse the validity of the corrections to $m_{h,H}^2$ from eq. (4.47) in the approximation used. For our perturbative expansion in η_i to be accurate we require that the η_i -dependent entries in the mass matrix \mathcal{M}_{ij} (4.31) be much smaller than the corresponding values of these matrix elements in the MSSM case. From this condition one finds

$$\begin{aligned}
\left| 3(\eta_1 + \eta_2)v_1^2 + 3(\eta_2 - \eta_1)v_2^2 + 2\eta_3 v_1 v_2 \right| &\ll \frac{1}{2} g^2 v_1 v_2 \\
\left| 6(\eta_2 - \eta_1)v_1 v_2 + \eta_3 v_1^2 \right| &\ll \frac{1}{4} g^2 \left| v_1^2 - 3v_2^2 \right| \\
\left| 6(\eta_2 + \eta_1)v_1 v_2 + \eta_3 v_2^2 \right| &\ll \frac{1}{4} g^2 \left| 3v_1^2 - v_2^2 \right|
\end{aligned} \tag{4.52}$$

Similar conditions are derived from the pseudoscalar Higgs mass matrix elements N_{ij} (4.40). One may find this condition too restrictive; in principle it may not be necessary to impose the leading $\eta_i \sim \mathcal{O}(1/M)$ contribution to the mass matrix entries be suppressed relative to the MSSM zeroth order and that one should instead ask that the $\mathcal{O}(1/M)$ correction dominate over the higher order terms $\mathcal{O}(1/M^2)$ [70]. However, at the quantitative level this leads, for the present case, to results which are similar or even stronger (for example for η_3) than those

derived here from comparing the MSSM zeroth order against the $O(1/M)$ terms. From these one can obtain upper bounds for each η_i . Having imposed these bounds, we can examine if the dimension-five operators bring a significant contribution to the higgs mass and in particular if we can surpass the tree level bound $m_h \leq m_Z$.

That would mean to also impose some lower bounds in order to achieve the desired increase. In the approximation considered, these bounds are derived from (4.45) with (4.52) and give

$$\begin{aligned} \frac{(\sqrt{\omega} + 1 - \rho)(1 + u^2)^2 \sqrt{\omega}}{32 u (u^2 - 1)} &\leq -\frac{\eta_1}{g^2} \ll \min \left\{ \frac{u}{6(u^2 - 1)}, \frac{3u^2 - 1}{24u}, \frac{|u^2 - 3|}{24u} \right\} \\ \frac{(\sqrt{\omega} + 1 - \rho)(1 + u^2)^2 \sqrt{\omega}}{4[(1 + u^2)^2 \sqrt{\omega} - (\rho - 1)(1 - u^2)^2]} &\leq \frac{\eta_3}{g^2} \ll \min \left\{ \frac{1}{4}, \frac{|u^2 - 3|}{4u^2}, \frac{u^2 - 1}{4u^2}, \frac{u^2 - 1}{4} \right\} \end{aligned} \quad (4.53)$$

with $\omega \equiv (\rho - 1)^2 + 16u^2\rho/(1 + u^2)^2$ and $\rho \equiv m_A^2/m_Z^2$.

Assuming $\eta_2 = 0$, then $m_h > m_Z$ is possible if one or both eqs in (4.53) are respected. On the other hand, it has no solution for η_1 within $1 \leq \tan\beta \leq 50$ and $m_A/m_Z \geq 1$; η_1 alone cannot change the MSSM bound $m_h \leq m_Z$ within our approximation. If $1 \leq m_A^2/m_Z^2 \leq 2.43$ there is a somewhat ‘‘marginal’’ solution for η_3 , with m_A/m_Z close to unity and large $\tan\beta$ preferred, to enforce the ‘‘ \ll ’’ inequalities in (4.52) and (4.53). For example, if $m_A = m_Z$ and $\tan\beta = 50$, the lower bound on η_3/g^2 is $\eta_3/g^2 \geq 0.02$ while $\eta_3/g^2 \ll 0.25$ is also required. In this case, for $\tan\beta = 50$ the increase of m_h^2 relative to m_Z^2 , $\delta_r = (m_h^2 - m_Z^2)/m_Z^2$ equals $\delta_r = -100/2501 + 2\eta_3/g^2$. Therefore $\delta_r = 12\%$ or $m_h \approx 102$ GeV if $\eta_3/g^2 = 0.08$, corresponding to $\eta_3 = 4.4 \times 10^{-2}$. Larger values for m_h should be regarded with care, since they would correspond to cases when ‘‘ \ll ’’ of (4.53) is not comfortably respected; if $\eta_3/g^2 \approx 0.04$ then $\delta_r \approx 4\%$ or $m_h \approx 95$ GeV. Further, if we now increase m_A even by a small amount relative to m_Z , $m_A^2 = 1.5 m_Z^2$ and $\tan\beta = 50$,

the lower bound on η_3/g^2 is 0.118 which is difficult to comply by a good margin with an upper bound unchanged at $\eta_3/g^2 \ll 0.25$. Even so, the relative difference would be only $\delta_r = 2 \times 10^{-3}\%$, ($\eta_3/g^2 = 0.118$), therefore the increase of m_h is negligible. So far we took $\eta_2 = 0$. If we allow a non-zero value for η_2 , which also requires non-zero η_3 , their combined effect on increasing m_h is not larger and the above results remain valid. Note also that for large $\tan\beta$ regions $1/M^2$ -suppressed operators can be important and can affect the results [65].

From this analysis we see that η_1 alone cannot change the MSSM tree level bound $m_h \leq m_Z$ within the approximation we discuss. This is consistent with section 4.3, where it was shown that the operator which induced the η_1 term could be removed by a general field redefinition of suitable coefficients³. However, η_3 can increase m_h to values $\approx 95 - 100$ GeV if $m_A \approx m_Z$, with the higher values close to the limit of our approximation. Therefore it is the susy breaking term associated to $(H_1 H_2)^2$ that could relax the MSSM tree level bound. This increase brings a small improvement. To conclude, adding the quantum corrections is still needed [65] to bring m_h above the LEP II bound of 114 GeV [66].

These findings show that the MSSM Higgs sector is rather stable under the addition of dimension-five operators, in the approximation we considered (expansion in $1/M$) of integrating out a massive singlet or a pair of massive $SU(2)$ doublets which generated the $\eta_{1,2,3}$ contributions. If M is low enough, the approximation used by integrating out these massive fields becomes unreliable, and one should recompute the full spectrum keeping all fields dynamical. Then the quartic interactions that the initial massive fields brought can be larger or of similar order to their MSSM counterparts and in principle they can change the

³To see this one can also start from (4.22) and perform a “smaller” version of redefinition (3.10), with $\rho_F = 0$.

above conclusions.

4.5 Including Loop Corrections

It is worth mentioning the value of m_h in the presence of one loop corrections from top - stop and dimension five operators [71], mentioned in the text:

$$m_h^2 = \frac{1}{2} [m_A'^2 + m_Z^2 - \sqrt{\tilde{w}'} + \xi] + (2\zeta_{10}\mu_0) v^2 \sin 2\beta \left[1 + \frac{m_A'^2 + m_Z^2}{\sqrt{\tilde{w}'}} \right] + \frac{(-2\zeta_{11}m_0) v^2}{2} \left[1 - \frac{(m_A'^2 - m_Z^2) \cos^2 2\beta}{\sqrt{\tilde{w}'}} \right] \quad (4.54)$$

where

$$\begin{aligned} \tilde{w}' &\equiv [(m_A'^2 - m_Z^2) \cos 2\beta + \xi]^2 + \sin^2 2\beta (m_A'^2 + m_Z^2)^2 \\ m_A'^2 &= \tilde{m}_1^2 + \tilde{m}_2^2 + \xi/2 + (2\zeta_{10}\mu_0) v^2 \sin 2\beta + \zeta_{11} m_0 v^2; \quad \xi \equiv \delta m_Z^2 \sin^2 \beta \end{aligned} \quad (4.55)$$

where δ is the one-loop correction from top-stop Yukawa sector to λ_2^0 of (5.38) which changes according to $\lambda_2^0 \rightarrow \lambda_2^0 (1 + \delta)$ where [72, 73]

$$\begin{aligned} \delta &= \frac{3h_t^4}{g^2\pi^2} \left[\ln \frac{M_{\tilde{t}}}{m_t} + \frac{X_t}{4} + \frac{1}{32\pi^2} (3h_t^2 - 16g_3^2) (X_t + 2 \ln \frac{M_{\tilde{t}}}{m_t}) \ln \frac{M_{\tilde{t}}}{m_t} \right], \\ X_t &\equiv \frac{2(A_t m_0 - \mu \cot \beta)^2}{M_{\tilde{t}}^2} \left[1 - \frac{(A_t m_0 - \mu \cot \beta)^2}{12 M_{\tilde{t}}^2} \right]. \end{aligned} \quad (4.56)$$

with $M_{\tilde{t}}^2 \equiv m_{\tilde{t}_1} m_{\tilde{t}_2}$, and g_3 the QCD coupling. The combined effect of $d = 5$ operators and top Yukawa coupling h_t is that m_h can reach values of 130 GeV for $\tan \beta \leq 7$ with a small fine-tuning $\Delta \leq 10$ [71] and with the supersymmetric coefficient ζ_{10} giving a larger effect than the non supersymmetric one, ζ_{11} . Even for a modest increase of m_h from $d = 5$ operators alone of order $\mathcal{O}(10\text{GeV})$, their impact on the effective quartic coupling of the Higgs field is significant (due to

the small value of the MSSM gauge couplings), and this explains the reduction of fine-tuning by the effective operators.

CHAPTER 5

MSSM HIGGS WITH OPERATORS OF MASS DIMENSION 5 AND 6

We generally expect that corrections to observables from higher order operators will be subdominant to those from the leading, dimension five ones. Nevertheless, we saw in section 4.4 that in the limit of large $\tan\beta$, the correction to the mass of the Higgs due to mass dimension five operators is $\tan\beta$ suppressed. In that limit, corrections from dimension six operators can become comparable to dimension five since $1/M^2 \sim 1/(M \tan\beta)$. Therefore, in order to complete the study of the leading Higgs mass corrections from effective operators, we need to include the contribution from dimension six operators. Since the latter is not $\tan\beta$ suppressed, the sequence ends here, as dimension seven or further will always be subdominant.

5.1 The Relevant Operators

We focus on the Higgs sector of the complete Lagrangian. This is comprised of the MSSM higgs sector \mathcal{L}_0 and the complete set of mass dimension-five and six operators. For \mathcal{L}_0 we have

$$\mathcal{L}_0 = \int d^4\theta \sum_{i=1,2} \mathcal{Z}_i(S, S^\dagger) H_i^\dagger e^{V_i} H_i + \left\{ \int d^2\theta \mu (1 + B m_0 \theta\theta) H_1 \cdot H_2 + h.c. \right\} \quad (5.1)$$

in standard notation. Here $\mathcal{Z}_i(S, S^\dagger) = 1 - c_i m_0^2 \theta\theta\bar{\theta}\bar{\theta}$ with $i = 1, 2$, $c_i = \mathcal{O}(1)$ and m_0 is the supersymmetry breaking scale as presented in the previous chapter.

We extend this Lagrangian by higher dimensional operators. In dimension-five we have the usual contributions studied in the previous chapter:

$$\begin{aligned}\mathcal{L}_1 &= \frac{1}{M} \int d^2\theta \zeta(S) (H_2 \cdot H_1)^2 + h.c. \\ &= 2\zeta_{10} (h_2 \cdot h_1)(h_2 \cdot F_1 + F_2 \cdot h_1) + \zeta_{11} m_0 (h_2 \cdot h_1)^2 + h.c., \\ \mathcal{L}_2 &= \frac{1}{M} \int d^4\theta \left\{ A(S, S^\dagger) D^\alpha \left[B(S, S^\dagger) H_2 e^{-V_1} \right] D_\alpha \left[\Gamma(S, S^\dagger) e^{V_1} H_1 \right] + h.c. \right\} \quad (5.2)\end{aligned}$$

where¹

$$\frac{1}{M} \zeta(S) = \zeta_{10} + \zeta_{11} m_0 \theta\theta, \quad \zeta_{10}, \zeta_{11} \sim 1/M, \quad (5.3)$$

with $S = \theta\theta m_0$ the spurion superfield. We assume that

$$m_0 \ll M \quad (5.4)$$

so that the effective theory approach is reliable.

\mathcal{L}_2 is eliminated by generalised, spurion-dependent field redefinitions as it was shown in detail in the previous chapter. For this reason we keep only \mathcal{L}_1 for the discussion below. These redefinitions bring however a renormalisation of the usual MSSM soft terms and of the μ term as well as additional corrections of order $1/M^2$. Since in the following we will write down and study the full set of $d = 6$ operators, the latter will be automatically included.

The list of $d = 6$ operators is [74]

$$\mathcal{O}_j = \frac{1}{M^2} \int d^4\theta Z_j(S, S^\dagger) (H_j^\dagger e^{V_j} H_j)^2, \quad j \equiv 1, 2.$$

¹We switch to a notation best suited for the analysis here. The dictionary is: $\eta_2 = 2\zeta_{10}\mu^*$, $\eta_3 = -2m_0\zeta_{11}$. With respect to the literature: In [71] $\eta_2 \rightarrow \zeta_1$, $\eta_3 \rightarrow \zeta_2$ and in [65] $\eta_2 \rightarrow 2\epsilon_{1r}$, $\eta_3 \rightarrow 2\epsilon_{2r}$.

$$\begin{aligned}
O_3 &= \frac{1}{M^2} \int d^4\theta Z_3(S, S^\dagger) (H_1^\dagger e^{V_1} H_1) (H_2^\dagger e^{V_2} H_2), \\
O_4 &= \frac{1}{M^2} \int d^4\theta Z_4(S, S^\dagger) (H_2 \cdot H_1) (H_2 \cdot H_1)^\dagger, \\
O_5 &= \frac{1}{M^2} \int d^4\theta Z_5(S, S^\dagger) (H_1^\dagger e^{V_1} H_1) H_2 \cdot H_1 + h.c. \\
O_6 &= \frac{1}{M^2} \int d^4\theta Z_6(S, S^\dagger) (H_2^\dagger e^{V_2} H_2) H_2 \cdot H_1 + h.c. \\
O_7 &= \frac{1}{M^2} \int d^2\theta Z_7(S, 0) \frac{1}{16 g^2 \kappa} \text{Tr} W^\alpha W_\alpha (H_2 H_1) + h.c. \\
O_8 &= \frac{1}{M^2} \int d^4\theta [Z_8(S, S^\dagger) (H_2 H_1)^2 + h.c.] \tag{5.5}
\end{aligned}$$

where $W^\alpha = (-1/4)\bar{D}^2 e^{-V} D^\alpha e^V$ is the chiral field strength of $SU(2)_L$ or $U(1)_Y$ vector superfields V_w and V_Y respectively. Also $V_{1,2} = V_w^\alpha(\sigma^a/2) \mp 1/2 V_Y$ with the upper sign for V_1 . The remaining $d = 6$ operators are:

$$\begin{aligned}
O_9 &= \frac{1}{M^2} \int d^4\theta Z_9(S, S^\dagger) H_1^\dagger \bar{\nabla}^2 e^{V_1} \nabla^2 H_1 \\
O_{10} &= \frac{1}{M^2} \int d^4\theta Z_{10}(S, S^\dagger) H_2^\dagger \bar{\nabla}^2 e^{V_2} \nabla^2 H_2 \\
O_{11} &= \frac{1}{M^2} \int d^4\theta Z_{11}(S, S^\dagger) H_1^\dagger e^{V_1} \nabla^\alpha W_\alpha^{(1)} H_1 \\
O_{12} &= \frac{1}{M^2} \int d^4\theta Z_{12}(S, S^\dagger) H_2^\dagger e^{V_2} \nabla^\alpha W_\alpha^{(2)} H_2 \\
O_{13} &= \frac{1}{M^2} \int d^4\theta Z_{13}(S, S^\dagger) H_1^\dagger e^{V_1} W_\alpha^{(1)} \nabla^\alpha H_1 \\
O_{14} &= \frac{1}{M^2} \int d^4\theta Z_{14}(S, S^\dagger) H_2^\dagger e^{V_2} W_\alpha^{(2)} \nabla^\alpha H_2 \tag{5.6}
\end{aligned}$$

Also $\nabla_\alpha H_i = e^{-V_i} D_\alpha e^{V_i} H_i$ and $W_\alpha^{(i)}$ is the field strength of V_i . In the most generic case, the above operators should actually include spurion dependence of arbitrary coefficients under any ∇_α , in order to include supersymmetry breaking effects associated to them. The wavefunction coefficients introduced above have the structure

$$\frac{1}{M^2} Z_i(S, S^\dagger) = \alpha_{i0} + \alpha_{i1} m_0 \theta\theta + \alpha_{i1}^* m_0 \bar{\theta}\bar{\theta} + \alpha_{i2} m_0^2 \theta\theta\bar{\theta}\bar{\theta}, \quad \alpha_{ij} \sim 1/M^2. \tag{5.7}$$

Regarding the origin of these operators: $O_{1,2,3}$ can be generated in MSSM with an additional, massive $U(1)'$ gauge boson or $SU(2)$ triplets integrated out [65]. O_4 can be generated by a massive gauge singlet or $SU(2)$ triplet while $O_{5,6}$ can be generated by a combination of $SU(2)$ doublets and massive gauge singlet. O_7 is essentially a threshold correction to the gauge coupling with a moduli field replaced by the Higgs. O_8 exists only in broken supersymmetry but is generated when redefining away the $d = 5$ derivative operator, thus we keep it.

Let us consider for a moment the operators $O_{9,\dots,14}$ in the exact supersymmetry case. We can use the equations of motion to set some of them on shell²:

$$-\frac{1}{4}\bar{D}^2(H_2^\dagger e^{V_2}) + \mu H_1^T(i\sigma_2) = 0, \quad \frac{1}{4}\bar{D}^2(H_1^\dagger e^{V_1}) + \mu H_2^T(i\sigma_2) = 0 \quad (5.8)$$

We find that in the supersymmetric case³:

$$O_9 \sim \int d^4\theta H_1^\dagger \bar{\nabla}^2 e^{V_1} \nabla^2 H_1 = 16|\mu|^2 \int d^4\theta H_1^\dagger e^{V_1} H_1 \quad (5.9)$$

and similar for O_{10} . Regarding $O_{11,12}$, they vanish in the supersymmetric case, following the definition of ∇^α and an integration by parts. Furthermore, $O_{13,14}$ are similar to $O_{9,10}$ which can be seen by using the definition of $W_\alpha^{(i)}$ and the relation between $\nabla^2, (\bar{\nabla}^2)$ and $D^2, (\bar{D}^2)$.

Summarizing, in the exact supersymmetry case the operators $O_{9,\dots,14}$ give at most wavefunction renormalisations of operators already included. The supersymmetry breaking terms also bring simply soft terms and μ term renormalization. Since these terms are anyway renormalised by $O_{1,\dots,8}$, where spurion

²Superpotential convention: $\int d^2\theta \mu H_1 \cdot H_2 = \int d^2\theta \mu H_1^T(i\sigma_2) H_2 \equiv \int d^2\theta \mu \epsilon^{ij} H_1^i H_2^j$; $\epsilon^{12} = 1 = -\epsilon^{21}$.

³Also using $(i\sigma_2)e^{-\Lambda} = e^{\Lambda^T}(i\sigma_2)$; $\Lambda \equiv \Lambda^a T^a$; $(i\sigma_2)^T = -(i\sigma_2)$; $(i\sigma_2)^2 = -1_2$

dependence is included with *arbitrary* coefficients, then for what follows there is no loss of generality in ignoring the supersymmetry breaking effects associated to $\mathcal{O}_{9,\dots,14}$. In other words, this discussion shows that $\mathcal{O}_{9,\dots,14}$ are not relevant for the analysis of the Higgs potential performed below. Finally, there can be an additional operator of $d = 6$ from the gauge sector, $\mathcal{O}_{15} = (1/M^2) \int d^2\theta W^\alpha \square W_\alpha$ which could affect the Higgs potential⁴. Using the equations of motion for the gauge field it can be shown that \mathcal{O}_{15} gives a renormalisation of $\mathcal{O}_{1,2,3}$, so its effects are ultimately included, since the coefficients $Z_{1,2,3}$ are arbitrary.

In conclusion, the list of $d = 6$ operators that remain for our study of the Higgs sector beyond MSSM is that of (5.5). Let us stress that not all these operators are necessarily present or generated in a detailed model. Symmetries and details of the “new physics” beyond the MSSM that generated them, may forbid or favour the presence of some of them. Therefore, we regard these remaining operators as independent of each other, although in specific models correlations may exist among their coefficients Z_i . It is important to keep all these operators in the analysis, for the purpose of identifying which of them has the largest individual contribution to the Higgs mass, one of the main interests of this analysis. Finally, some of the $d = 6$ operators can in principle be present even in the absence of the $d = 5$ operators, if these classes of operators are generated by integrating different “new physics”. In specific UV completions, one simply keeps the terms generated by the model and sets all the rest to zero.

⁴Its complete gauge invariant form is $\int d^4\theta Tr e^V W^\alpha e^{-V} D^2(e^V W_\alpha e^{-V})$.

5.2 The Scalar Potential

Following the previous discussion, the overall Lagrangian of the model is

$$\mathcal{L}_H = \mathcal{L}_0 + \mathcal{L}_1 + \sum_{i=1}^8 \mathcal{O}_i \quad (5.10)$$

with the MSSM Higgs Lagrangian \mathcal{L}_0 of eq. (5.1), \mathcal{L}_1 of eq. (5.2) and $\mathcal{O}_{1,2,\dots,8}$ of eq. (5.5).

In order to calculate the scalar potential we need the bosonic expansion of the Lagrangian. For the dimension-six operators we have:

$$\begin{aligned} \mathcal{O}_1 &= \frac{1}{M^2} \int d^4\theta Z_1(S, S^\dagger) (H_1^\dagger e^{V_1} H_1)^2 \\ &= 2\alpha_{10} \left[(h_1^\dagger h_1) [(\mathcal{D}_\mu h_1)^\dagger (\mathcal{D}^\mu h_1) + h_1^\dagger \frac{D_1}{2} h_1 + F_1^\dagger F_1] + |h_1^\dagger F_1|^2 + (h_1^\dagger \mathcal{D}^\mu h_1)(h_1^\dagger \overleftarrow{\mathcal{D}}_\mu h_1) \right] \\ &\quad + \left[2\alpha_{11} m_0 (h_1^\dagger h_1)(F_1^\dagger h_1) + h.c. \right] + \alpha_{12} m_0^2 (h_1^\dagger h_1)^2 + \text{fermionic part} \end{aligned} \quad (5.11)$$

$$\begin{aligned} \mathcal{O}_2 &= \frac{1}{M^2} \int d^4\theta Z_2(S, S^\dagger) (H_2^\dagger e^{V_2} H_2)^2 \\ &= 2\alpha_{20} \left[(h_2^\dagger h_2) [(\mathcal{D}_\mu h_2)^\dagger (\mathcal{D}^\mu h_2) + h_2^\dagger \frac{D_2}{2} h_2 + F_2^\dagger F_2] + |h_2^\dagger F_2|^2 + (h_2^\dagger \mathcal{D}^\mu h_2)(h_2^\dagger \overleftarrow{\mathcal{D}}_\mu h_2) \right] \\ &\quad + \left[2\alpha_{21} m_0 (h_2^\dagger h_2)(F_2^\dagger h_2) + h.c. \right] + \alpha_{22} m_0^2 (h_2^\dagger h_2)^2 + \text{fermionic part} \end{aligned} \quad (5.12)$$

$$\begin{aligned} \mathcal{O}_3 &= \frac{1}{M^2} \int d^4\theta Z_3(S, S^\dagger) (H_1^\dagger e^{V_1} H_1) (H_2^\dagger e^{V_2} H_2), \\ &= \alpha_{30} \left\{ (h_1^\dagger h_1) [(\mathcal{D}_\mu h_2)^\dagger (\mathcal{D}^\mu h_2) + h_2^\dagger \frac{D_2}{2} h_2 + F_2^\dagger F_2] + (h_1^\dagger F_1)(F_2^\dagger h_2) + (1 \leftrightarrow 2) \right\} \\ &\quad + \alpha_{30} \left[(h_1^\dagger \mathcal{D}_\mu h_1)(h_2^\dagger \overleftarrow{\mathcal{D}}^\mu h_2) + h.c. \right] + \left\{ \alpha_{31} m_0 \left[(h_1^\dagger h_1)(F_2^\dagger h_2) + (h_2^\dagger h_2)(F_1^\dagger h_1) \right] + h.c. \right\} \\ &\quad + \alpha_{32} m_0^2 (h_1^\dagger h_1)(h_2^\dagger h_2) + \text{fermionic part} \end{aligned} \quad (5.13)$$

$$\begin{aligned} \mathcal{O}_4 &= \frac{1}{M^2} \int d^4\theta Z_4(S, S^\dagger) (H_2 \cdot H_1) (H_2 \cdot H_1)^\dagger, \\ &= \alpha_{40} \partial_\mu (h_2 \cdot h_1) \partial^\mu (h_2 \cdot h_1)^\dagger + \left[\alpha_{41} m_0 (h_2 \cdot h_1) (h_2 \cdot F_1 + F_2 \cdot h_1)^\dagger + h.c. \right] \\ &\quad + \alpha_{42} m_0^2 (h_2 \cdot h_1) (h_2 \cdot h_1)^\dagger + \alpha_{40} |h_2 \cdot F_1 + F_2 \cdot h_1|^2 + \text{fermionic part} \end{aligned} \quad (5.14)$$

$$\begin{aligned}
O_5 &= \frac{1}{M^2} \int d^4\theta Z_5(S, S^\dagger) (H_1^\dagger e^{V_1} H_1) H_2 \cdot H_1 + h.c. \\
&= \alpha_{50} \left\{ [(\mathcal{D}_\mu h_1)^\dagger (\mathcal{D}^\mu h_1) + h_1^\dagger \frac{D_1}{2} h_1 + F_1^\dagger F_1] (h_2 \cdot h_1) + (h_1^\dagger \overleftarrow{\mathcal{D}}_\mu h_1) \partial^\mu (h_2 \cdot h_1) \right\} \\
&+ [\alpha_{50} (F_1^\dagger h_1) + \alpha_{51}^* m_0 (h_1^\dagger h_1)] (h_2 \cdot F_1 + F_2 \cdot h_1) + m_0 [\alpha_{51} (F_1^\dagger h_1) + \alpha_{51}^* (h_1^\dagger F_1)] (h_2 \cdot h_1) \\
&+ \alpha_{52} m_0^2 (h_1^\dagger h_1) (h_2 \cdot h_1) + \text{h.c. of all} + \text{fermionic part} \tag{5.15}
\end{aligned}$$

$$\begin{aligned}
O_6 &= \frac{1}{M^2} \int d^4\theta Z_6(S, S^\dagger) (H_2^\dagger e^{V_2} H_2) H_2 \cdot H_1 + h.c. \\
&= \alpha_{60} \left\{ [(\mathcal{D}_\mu h_2)^\dagger (\mathcal{D}^\mu h_2) + h_2^\dagger \frac{D_2}{2} h_2 + F_2^\dagger F_2] (h_2 \cdot h_1) + (h_2^\dagger \overleftarrow{\mathcal{D}}_\mu h_2) \partial^\mu (h_2 \cdot h_1) \right\} \\
&+ [\alpha_{60} (F_2^\dagger h_2) + \alpha_{61}^* m_0 (h_2^\dagger h_2)] (h_2 \cdot F_1 + F_2 \cdot h_1) + m_0 [\alpha_{61} (F_2^\dagger h_2) + \alpha_{61}^* (h_2^\dagger F_2)] (h_2 \cdot h_1) \\
&+ \alpha_{62} m_0^2 (h_2^\dagger h_2) (h_2 \cdot h_1) + \text{h.c. of all} + \text{fermionic part} \tag{5.16}
\end{aligned}$$

$$\begin{aligned}
O_7 &= \frac{1}{M^2} \frac{1}{16g^2\kappa} \int d^2\theta Z_7(S, 0) \text{Tr} W^\alpha W_\alpha (H_2 H_1) + h.c. \\
&= \frac{1}{2} (D_w^2 + D_Y^2) [\alpha_{70} (h_2 \cdot h_1) + \alpha_{70}^* (h_2 \cdot h_1)^\dagger] + \text{fermionic part} \tag{5.17}
\end{aligned}$$

$$\begin{aligned}
O_8 &= \frac{1}{M^2} \int d^4\theta [Z_8(S, S^\dagger) [(H_2 H_1)^2 + h.c.]] \\
&= 2\alpha_{81}^* m_0 (h_2 \cdot h_1) (h_2 \cdot F_1 + F_2 \cdot h_1) + m_0^2 \alpha_{82} (h_2 \cdot h_1)^2 + h.c. + \text{fermionic part} \tag{5.18}
\end{aligned}$$

The notation is as follows: $\mathcal{D}^\mu h_i = (\partial^\mu + i/2 V_i^\mu) h_i$, $h_i^\dagger \overleftarrow{\mathcal{D}}^\mu = (\mathcal{D}^\mu h_i)^\dagger$. Further, $D_1 \equiv \vec{D}_w \vec{T} + (-1/2) D_Y$ and $D_2 \equiv \vec{D}_w \vec{T} + (1/2) D_Y$, $T^a = \sigma^a/2$. Finally, one rescales in all O_i ($i \neq 7$): $V_w \rightarrow 2g_2 V_w$, $V_y \rightarrow 2g_1 V_y$. Then $V_{1,2} = 2g_2 \vec{V}_w \vec{T} + 2g_1 (\mp 1/2) V_y$ with the upper sign (minus) for V_1 , where $V_{1,2}$ enter the definition of $O_{1,2}$. Other notations used above: $H_1 \cdot H_2 = \epsilon^{ij} H_1^i H_2^j$. Also $|h_1 \cdot h_2|^2 = |h_1^i \epsilon^{ij} h_2^j|^2 = |h_1|^2 |h_2|^2 - |h_1^\dagger h_2|^2$; $\epsilon^{ij} \epsilon^{kj} = \delta^{ik}$; $\epsilon^{ij} \epsilon^{kl} = \delta^{ik} \delta^{jl} - \delta^{il} \delta^{jk}$, $\epsilon^{12} = 1$, with

$$h_1 = \begin{pmatrix} h_1^0 \\ h_1^+ \\ h_1^- \end{pmatrix} \equiv \begin{pmatrix} h_1^1 \\ h_1^2 \end{pmatrix}, \quad Y_{h_1} = -1; \quad h_2 = \begin{pmatrix} h_2^+ \\ h_2^1 \\ h_2^0 \end{pmatrix} \equiv \begin{pmatrix} h_2^1 \\ h_2^2 \end{pmatrix}, \quad Y_{h_2} = +1 \tag{5.19}$$

With these results we find the following contributions to the scalar potential:

$$V_F = \frac{\partial^2 K}{\partial h_i \partial h_j^*} F_i F_j^* = |F_1|^2 + |F_2|^2 + \frac{\partial^2 K_6}{\partial h_i \partial h_j^*} F_i F_j^* \tag{5.20}$$

where K_6 is the contribution of $\mathcal{O}(1/M^2)$ to the Kähler potential due to $\mathcal{O}_{1,\dots,8}$.

Also,

$$\begin{aligned} F_1^{*q} &= -\{\epsilon^{qp} h_2^p [\mu + 2 \zeta_{10} (h_1 \cdot h_2) + \rho_{11}] + h_1^{*q} \rho_{12}\} \\ F_2^{*q} &= -\{\epsilon^{pq} h_1^p [\mu + 2 \zeta_{10} (h_1 \cdot h_2) + \rho_{21}] + h_2^{*q} \rho_{22}\} \end{aligned} \quad (5.21)$$

where ρ_{ij} are functions of $h_{1,2}$:

$$\begin{aligned} \rho_{11} &= -(2\alpha_{10} \mu + \alpha_{40} \mu + \alpha_{51}^* m_0) |h_1|^2 - (\alpha_{30} \mu + \alpha_{40} \mu + \alpha_{61}^* m_0) |h_2|^2 \\ &\quad - (\alpha_{41}^* m_0 + \alpha_{50}^* \mu) (h_2 \cdot h_1)^* + [(\alpha_{60} + 2 \alpha_{50}) \mu + 2 \alpha_{81}^* m_0] (h_1 \cdot h_2) \\ \rho_{12} &= (2\alpha_{11}^* m_0 + \alpha_{50}^* \mu) |h_1|^2 + (\alpha_{31}^* m_0 + \alpha_{50}^* \mu) |h_2|^2 \\ &\quad - [(2\alpha_{10} + \alpha_{30}) \mu + \alpha_{51}^* m_0] (h_1 \cdot h_2) + \alpha_{51}^* m_0 (h_2 \cdot h_1)^* \end{aligned} \quad (5.22)$$

$$\begin{aligned} \rho_{21} &= -(2\alpha_{20} \mu + \alpha_{40} \mu + \alpha_{61}^* m_0) |h_2|^2 - (\alpha_{30} \mu + \alpha_{40} \mu + \alpha_{51}^* m_0) |h_1|^2 \\ &\quad - (\alpha_{41}^* m_0 + \alpha_{60}^* \mu) (h_2 \cdot h_1)^* + [(\alpha_{50} + 2 \alpha_{60}) \mu + 2 \alpha_{81}^* m_0] (h_1 \cdot h_2) \\ \rho_{22} &= (2\alpha_{21}^* m_0 + \alpha_{60}^* \mu) |h_2|^2 + (\alpha_{31}^* m_0 + \alpha_{60}^* \mu) |h_1|^2 \\ &\quad - [(2\alpha_{20} + \alpha_{30}) \mu + \alpha_{61}^* m_0] (h_1 \cdot h_2) + \alpha_{61}^* m_0 (h_2 \cdot h_1)^* \end{aligned} \quad (5.23)$$

The first two terms in the rhs of (5.20) give (h_i denote $SU(2)_L$ doublets, $|h_i|^2 \equiv h_i^\dagger h_i$):

$$\begin{aligned} V_{F,1} &\equiv |F_1|^2 + |F_2|^2 \\ &= |\mu + 2 \zeta_{10} h_1 \cdot h_2|^2 (|h_1|^2 + |h_2|^2) \\ &\quad + \left[\mu^* (|h_1|^2 \rho_{21} + |h_2|^2 \rho_{11} + (h_1 \cdot h_2)^\dagger (\rho_{22} + \rho_{12})) + h.c. \right] \end{aligned} \quad (5.24)$$

The nontrivial field dependent Kähler metric gives for the last term in V_F of eq. (5.20):

$$\begin{aligned}
V_{F,2} &= |\mu|^2 \left[2(\alpha_{10} + \alpha_{20} + \alpha_{40}) |h_1|^2 |h_2|^2 + (\alpha_{30} + \alpha_{40}) (|h_1|^4 + |h_2|^4) \right. \\
&\quad + 2(\alpha_{10} + \alpha_{20} + \alpha_{30}) |h_1 \cdot h_2|^2 + (|h_1|^2 + 2|h_2|^2)(\alpha_{50} h_2 \cdot h_1 + h.c.) \\
&\quad \left. + (2|h_1|^2 + |h_2|^2)(\alpha_{60} h_2 \cdot h_1 + h.c.) \right] \tag{5.25}
\end{aligned}$$

so that $V_F = V_{F,1} + V_{F,2}$. Furthermore, from the gauge part we have:

$$\begin{aligned}
D_w^a &= -g_2 \left[h_1^\dagger T^a h_1 (1 + \tilde{\rho}_1) + h_2^\dagger T^a h_2 (1 + \tilde{\rho}_2) \right], \quad T^a = \sigma^a / 2 \\
D_Y &= -g_1 \left[h_1^\dagger \frac{-1}{2} h_1 (1 + \tilde{\rho}_1) + h_2^\dagger \frac{1}{2} h_2 (1 + \tilde{\rho}_2) \right] \tag{5.26}
\end{aligned}$$

with notation:

$$\begin{aligned}
\tilde{\rho}_1(h_{1,2}) &\equiv 2\alpha_{10} |h_1|^2 + \alpha_{30} |h_2|^2 + [(\alpha_{50} - \alpha_{70}) h_2 \cdot h_1 + h.c.] \\
\tilde{\rho}_2(h_{1,2}) &\equiv 2\alpha_{20} |h_2|^2 + \alpha_{30} |h_1|^2 + [(\alpha_{60} - \alpha_{70}) h_2 \cdot h_1 + h.c.] \tag{5.27}
\end{aligned}$$

This gives

$$\begin{aligned}
D_w^a D_w^a &= \frac{g_2^2}{4} \left[((1 + \tilde{\rho}_1) |h_1|^2 - (1 + \tilde{\rho}_2) |h_2|^2)^2 + 4(1 + \tilde{\rho}_1)(1 + \tilde{\rho}_2) |h_1^\dagger h_2|^2 \right] \\
D_Y^2 &= \frac{g_1^2}{4} ((1 + \tilde{\rho}_1) |h_1|^2 - (1 + \tilde{\rho}_2) |h_2|^2)^2 \tag{5.28}
\end{aligned}$$

So the gauge part of the scalar potential is written as:

$$\begin{aligned}
V_{gauge} &= \frac{1}{2} (D_w^2 + D_Y^2) [1 + (\alpha_{70} h_2 \cdot h_1 + h.c.)] \\
&= \frac{g_1^2 + g_2^2}{8} (|h_1|^2 - |h_2|^2) [(1 + f_1(h_{1,2})) |h_1|^2 - (1 + f_2(h_{1,2})) |h_2|^2] \\
&\quad + \frac{g_2^2}{2} (1 + f_3(h_{1,2})) |h_1^\dagger h_2|^2 \tag{5.29}
\end{aligned}$$

obtained with (5.26) and where $f_{1,2,3}$ are functions of $h_{1,2}$:

$$\begin{aligned}
f_1(h_{1,2}) &\equiv 4\alpha_{10}|h_1|^2 + [(2\alpha_{50} - \alpha_{70})h_2.h_1 + h.c.] \\
f_2(h_{1,2}) &\equiv 4\alpha_{20}|h_2|^2 + [(2\alpha_{60} - \alpha_{70})h_2.h_1 + h.c.] \\
f_3(h_{1,2}) &\equiv \tilde{\rho}_1 + \tilde{\rho}_2 + (\alpha_{70}h_2.h_1 + h.c.)
\end{aligned} \tag{5.30}$$

The scalar potential also has corrections V_{SSB} from supersymmetry breaking, due to spurion dependence in higher dimensional operators. In addition we also have the usual soft breaking term from the MSSM. As a result

$$\begin{aligned}
V_{SSB} = & -m_0^2 [\alpha_{12}|h_1|^4 + \alpha_{22}|h_2|^4 + \alpha_{32}|h_1|^2|h_2|^2 + \alpha_{42}|h_2.h_1|^2 \\
& + (\alpha_{52}|h_1|^2(h_2.h_1) + h.c.) + (\alpha_{62}|h_2|^2(h_2.h_1) + h.c.)] \\
& - [m_0^2\alpha_{82}(h_1.h_2)^2 + \zeta_{11}m_0(h_2.h_1)^2 + \mu B m_0(h_1.h_2) + h.c.] + m_0^2(c_1|h_1|^2 + c_2|h_2|^2)
\end{aligned} \tag{5.31}$$

Finally, in $\mathcal{O}_{1,\dots,8}$ there are non standard kinetic terms that can contribute to V when the scalar singlet components (denoted h_i^0) of h_i acquire a vev. The relevant terms are:

$$\mathcal{L}_H \supset (\delta_{ij^*} + g_{ij^*}) \partial_\mu h_i^0 \partial^\mu h_j^{0*}, \quad i, j = 1, 2. \tag{5.32}$$

where the field dependent metric is:

$$\begin{aligned}
g_{11^*} &= 4\alpha_{10}|h_1^0|^2 + (\alpha_{30} + \alpha_{40})|h_2^0|^2 - 2(\alpha_{50}h_1^0h_2^0 + h.c.) \\
g_{12^*} &= (\alpha_{30} + \alpha_{40})h_1^{0*}h_2^0 - \alpha_{50}^*h_1^{0*2} - \alpha_{60}h_2^{02}, \quad g_{21^*} = g_{12^*}^* \\
g_{22^*} &= 4\alpha_{20}|h_2^0|^2 + (\alpha_{30} + \alpha_{40})|h_1^0|^2 - 2(\alpha_{60}h_1^0h_2^0 + h.c.)
\end{aligned} \tag{5.33}$$

For simplicity we only included the $SU(2)$ higgs singlets contribution, that we actually need in the following, but the discussion can be extended to the general case. The metric g_{ij^*} is expanded about a background value $\langle h_i^0 \rangle = v_i/\sqrt{2}$, then field redefinitions are performed to obtain canonical kinetic terms. They are:

$$\begin{aligned}
h_1^0 &\rightarrow h_1^0 \left(1 - \frac{\tilde{g}_{11}^*}{2}\right) - \frac{\tilde{g}_{21}^*}{2} h_2^0 \\
h_2^0 &\rightarrow h_2^0 \left(1 - \frac{\tilde{g}_{22}^*}{2}\right) - \frac{\tilde{g}_{12}^*}{2} h_1^0, \quad \tilde{g}_{ij}^* \equiv g_{ij}^* \Big|_{h_i^0 \rightarrow v_i/\sqrt{2}}
\end{aligned} \tag{5.34}$$

These bring further corrections to the scalar potential.

Since the metric has corrections which are $O(1/M^2)$, after (5.34) only the MSSM soft breaking terms and the MSSM quartic terms are affected. The other terms in the scalar potential, already suppressed by one or more powers of the scale M are affected only beyond the approximation $O(1/M^2)$ considered here. Following (5.34) the correction terms $O(1/M^2)$ induced by the MSSM quartic terms and by soft breaking terms in V_{SSB} are:

$$\begin{aligned}
V_{k.t.} &= \tilde{m}_1^2 (-\tilde{g}_{11}^*) |h_1^0|^2 + \tilde{m}_2^2 (-\tilde{g}_{22}^*) |h_2^0|^2 - \frac{1}{2} (\tilde{m}_1^2 + \tilde{m}_2^2) (\tilde{g}_{21}^* h_1^{0*} h_2^0 + h.c.) \\
&+ \frac{1}{2} \left[B m_0 \mu \left((\tilde{g}_{11}^* + \tilde{g}_{22}^*) h_1^0 h_2^0 + \tilde{g}_{12}^* h_1^{02} + \tilde{g}_{21}^* h_2^{02} \right) + h.c. \right] \\
&- \frac{g^2}{8} (|h_1^0|^2 - |h_2^0|^2) (\tilde{g}_{11}^* |h_1^0|^2 - \tilde{g}_{22}^* |h_2^0|^2 + h.c.)
\end{aligned} \tag{5.35}$$

Using equations (5.20), (5.29), (5.31) and (5.35), we find the full scalar potential.

With notation $\tilde{m}_i^2 \equiv c_i m_0^2 + |\mu|^2$, $i = 1, 2$ one finally has:

$$\begin{aligned}
V &= V_{F,1} + V_{F,2} + V_G + V_{SSB} + V_{k.t.} \\
&= V_{k.t.} + \tilde{m}_1^2 |h_1|^2 + \tilde{m}_2^2 |h_2|^2 - [\mu B m_0 h_1 \cdot h_2 + h.c.] \\
&+ \frac{\lambda_1}{2} |h_1|^4 + \frac{\lambda_2}{2} |h_2|^4 + \lambda_3 |h_1|^2 |h_2|^2 + \lambda_4 |h_1 \cdot h_2|^2 \\
&+ \left(\frac{\lambda_5}{2} (h_1 \cdot h_2)^2 + \lambda_6 |h_1|^2 (h_1 \cdot h_2) + \lambda_7 |h_2|^2 (h_1 \cdot h_2) + h.c. \right) \\
&+ \frac{g^2}{8} (|h_1|^2 - |h_2|^2) (f_1(h_{1,2}) |h_1|^2 - f_2(h_{1,2}) |h_2|^2) + 4 |\zeta_{10}|^2 |h_1 \cdot h_2|^2 (|h_1|^2 + |h_2|^2) \\
&+ \frac{g_2^2}{2} f_3(h_{1,2}) |h_1^\dagger h_2|^2
\end{aligned} \tag{5.36}$$

where $g^2 = g_1^2 + g_2^2$, and $f_{1,2,3}(h_{1,2})$ are all quadratic in h_i , see eq. (5.30). Except $V_{k.t.}$, all other fields are in the SU(2) doublets notation. λ_i are given by

$$\lambda_1/2 = \lambda_1^0/2 - |\mu|^2 (\alpha_{30} + \alpha_{40}) - m_0^2 \alpha_{12} - 2m_0 \text{Re}[\alpha_{51} \mu] \quad (5.37)$$

$$\lambda_2/2 = \lambda_2^0/2 - |\mu|^2 (\alpha_{30} + \alpha_{40}) - m_0^2 \alpha_{22} - 2m_0 \text{Re}[\alpha_{61} \mu]$$

$$\lambda_3 = \lambda_3^0 - 2 |\mu|^2 (\alpha_{10} + \alpha_{20} + \alpha_{40}) - m_0^2 \alpha_{32} - 2m_0 \text{Re}[(\alpha_{51} + \alpha_{61}) \mu]$$

$$\lambda_4 = \lambda_4^0 - 2 |\mu|^2 (\alpha_{10} + \alpha_{20} + \alpha_{30}) - m_0^2 \alpha_{42} - 2m_0 \text{Re}[(\alpha_{51} + \alpha_{61}) \mu]$$

$$\lambda_5/2 = -m_0 \mu (\alpha_{51} + \alpha_{61}) - m_0 \zeta_{11} - m_0^2 \alpha_{82}$$

$$\lambda_6 = |\mu|^2 (\alpha_{50} + 2\alpha_{60}) + m_0^2 \alpha_{52} + m_0 \mu (2\alpha_{11} + \alpha_{31} + \alpha_{41}) + 2m_0 \mu^* \alpha_{81}^* + 2\zeta_{10} \mu^*$$

$$\lambda_7 = |\mu|^2 (\alpha_{60} + 2\alpha_{50}) + m_0^2 \alpha_{62} + m_0 \mu (2\alpha_{21} + \alpha_{31} + \alpha_{41}) + 2m_0 \mu^* \alpha_{81}^* + 2\zeta_{10} \mu^*$$

where

$$\lambda_1^0/2 = \frac{1}{8} (g_2^2 + g_1^2), \quad \lambda_2^0/2 = \frac{1}{8} (g_2^2 + g_1^2), \quad \lambda_3^0 = \frac{1}{4} (g_2^2 - g_1^2), \quad \lambda_4^0 = -\frac{1}{2} g_2^2, \quad (5.38)$$

denote the pure MSSM contribution. One can include MSSM loop corrections by replacing λ_i^0 with radiatively corrected values [73].

Equations (5.36) and (5.37) show the effects of various higher dimensional operators on the scalar potential. As a reminder, note that all $\alpha_{ik} \sim \mathcal{O}(1/M^2)$ while $\zeta_{11}, \zeta_{10} \sim \mathcal{O}(1/M)$. In principle, the dimension-five pieces are the dominant. However, as we will see later, when $\tan\beta$ is large the effect on a physical observable of dimension-five and six terms can be of similar size. In specific models correlations exist among these coefficients. The above remarks apply to the case when the $d = 5$ and $d = 6$ operators considered are generated by the same “new physics” beyond the MSSM (i.e. are suppressed by the same scale). However, as mentioned earlier, this may not always be the case; in various models contributions from some $d = 6$ operators can be independent of those from $d = 5$ operators (and present even in the absence of the latter), if generated by different “new physics”. A case by case study is then needed for a thorough analysis of all possible scenarios beyond the MSSM higgs sector.

The overall sign of the h^6 terms depends on the relative size of α_{j0} , $j = 1, 2, 5, 6, 7$, and cannot be fixed even locally, in the absence of the exact values of these coefficients. ζ_{10} also contributes to the overall sign, however this alone cannot fix it. At large fields' values higher and higher dimensional operators become relevant and contribute to it. We therefore do not impose that V be bounded from below at large fields. For a discussion of stability with $d = 5$ operators only see [75].

Eq. (5.36) is the main result of this section. For simplicity, one can take \tilde{g}_{12^*} and \tilde{g}_{21^*} to be real, possible if for example α_{50} and α_{60} are real and there is no vev for $\text{Im}h_i$. $Bm_0\mu$ can also be taken to be real. In the next section we shall adopt these simplifications.

5.3 Corrections to the MSSM Higgs Masses

Having obtained the general expression for the scalar potential, we proceed with the computation of the mass spectrum. The general expression for the mass of the CP-even Higgs fields h, H is:

$$m_{h,H}^2 \equiv \frac{1}{2} \frac{\partial^2 V}{\partial h_i^0 \partial h_j^0} \Big|_{\langle h_i \rangle = v_i / \sqrt{2}, \langle \text{Im} h_i \rangle = 0} \quad (5.39)$$

In the leading order $O(1/M)$ one has (upper signs for m_h):

$$m_{h,H}^2 = \frac{m_Z^2}{2} + \frac{B m_0 \mu (u^2 + 1)}{2u} \mp \frac{\sqrt{w}}{2} + v^2 \left[(2 \zeta_{10} \mu) q_1^\pm + (-2 m_0 \zeta_{11}) q_2^\pm \right] + \delta m_{h,H}^2 \quad (5.40)$$

with

$$\begin{aligned}
q_1^\pm &= \frac{1}{4u^2(1+u^2)\sqrt{w}} \\
&\times \left[-(1-6u^2+u^4)u\sqrt{w} \mp (m_Z^2 u(1-14u^2+u^4) - Bm_0\mu(1+u^2)(1+10u^2+u^4)) \right] \\
q_2^\pm &= \mp \frac{2u}{(1+u^2)^2\sqrt{w}} \left[-Bm_0\mu(1+u^2) - m_Z^2 u \right] \tag{5.41}
\end{aligned}$$

where

$$w \equiv m_Z^4 + [-Bm_0\mu(1+u^2)^3 + 2m_Z^2 u(1-6u^2+u^4)] \frac{(-Bm_0\mu)}{u^2(1+u^2)}, \quad u \equiv \tan\beta \tag{5.42}$$

In eq. (5.40)

$$\delta m_{h,H}^2 = O(1/M^2) \tag{5.43}$$

and we also used that $m_Z = gv/2$. One also shows that the Goldstone mode has $m_G = 0$ and the pseudoscalar A has a mass:

$$m_A^2 = \frac{1+u^2}{u} Bm_0\mu - \frac{1+u^2}{u} \zeta_{10}\mu v^2 + 2m_0\zeta_{11}v^2 + \delta m_A^2, \quad \delta m_A^2 = O(1/M^2) \tag{5.44}$$

These results agree with the independent calculation up to order $O(1/M)$ of the previous chapters.

Ignoring for the moment the corrections $O(1/M^2)$, one eliminates $Bm_0\mu$ between (5.40) and (5.44) to obtain:

$$\begin{aligned}
m_{h,H}^2 &= \frac{1}{2} \left[m_A^2 + m_Z^2 \mp \sqrt{\tilde{w}} \right] \\
&+ (2\zeta_{10}\mu)v^2 \sin 2\beta \left[1 \pm \frac{m_A^2 + m_Z^2}{\sqrt{\tilde{w}}} \right] + \frac{(-2\zeta_{11}m_0)v^2}{2} \left[1 \mp \frac{(m_A^2 - m_Z^2) \cos^2 2\beta}{\sqrt{\tilde{w}}} \right] \\
&+ \delta' m_{h,H}^2, \quad \delta' m_{h,H}^2 = O(1/M^2) \tag{5.45}
\end{aligned}$$

where the upper (lower) signs correspond to h (H) respectively and

$$\tilde{w} \equiv (m_A^2 + m_Z^2)^2 - 4m_A^2 m_Z^2 \cos^2 2\beta \tag{5.46}$$

This is important if one considers m_A as an input; it is also needed if one considers the limit of large $\tan\beta$ at fixed m_A (see later).

The $O(1/M^2)$ corrections $\delta m_{h,H}^2$, δm_A^2 and $\delta' m_{h,H}^2$ of equations (5.40), (5.44) and (5.45) in the general case of including all operators and their associated supersymmetry breaking, have a rather complicated form. For most purposes, an expansion in $1/\tan\beta$ is accurate enough. The reason for this is that it is only at large $\tan\beta$ that $d = 6$ operators bring corrections comparable to those of $d = 5$. The relative $\tan\beta$ enhancement of $O(1/M^2)$ operators compensates for the extra suppression factor $1/M$ that these operators have relative to $O(1/M)$ operators (which involve both h_1 and h_2 and thus are not enhanced in this limit).

If we neglect supersymmetry breaking effects of $d = 6$ operators (i.e. $\alpha_{j1} = \alpha_{j2} = 0$, $\alpha_{j0} \neq 0$, $j = 1, \dots, 8$) and with $d = 5$ operators contribution, one has⁵ for the correction $\delta m_{h,H}^2$ in eq. (5.40) (upper signs correspond to δm_h^2)

$$\delta m_{h,H}^2 = \sum_{j=1}^7 \gamma_j^\pm \alpha_{j0} + \gamma_x^\pm \zeta_{10} \zeta_{11} + \gamma_z^\pm \zeta_{10}^2 + \gamma_y^\pm \zeta_{11}^2 \quad (5.47)$$

The expressions of the coefficients γ^\pm are provided in Appendix A and can be used for numerical studies. While these expressions are exact, they are complicated and not very transparent. It is then instructive to analyse an approximation of the $O(1/M^2)$ correction as an expansion in $1/\tan\beta$. We present in this limit the correction $\delta m_{h,H}^2$ of eq. (5.40), which also includes all supersymmetry breaking effects associated with all $d = 5, 6$ operators, (i.e. $\alpha_{j1} \neq 0$, $\alpha_{j2} \neq 0$, $\zeta_{11} \neq 0$, $j = 1, \dots, 8$) in addition to the MSSM soft terms. This has a simple expression:

$$\delta m_h^2 = -2v^2 \left[\alpha_{22} m_0^2 + 2\alpha_{61} m_0 \mu + (\alpha_{30} + \alpha_{40}) \mu^2 - \alpha_{20} m_Z^2 \right]$$

⁵In the case of including the supersymmetry breaking effects from effective operators, associated with coefficients α_{j1} , α_{j2} $j = 1, 2, \dots, 8$, the exact formula is very long and is not included here.

$$\begin{aligned}
& + \frac{v^2}{\tan\beta} \left[4\alpha_{62} m_0^2 + 4\mu m_0 (2\alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81}) + 4\mu^2 (2\alpha_{50} + \alpha_{60}) \right. \\
& \quad \left. - m_Z^2 (2\alpha_{60} - 3\alpha_{70}) - \frac{v^2}{(Bm_0\mu)} (2\zeta_{10}\mu)^2 \right] + \mathcal{O}(1/\tan^2\beta) \quad (5.48)
\end{aligned}$$

which is obtained with $Bm_0\mu$ kept fixed. The result is dominated by the first line, including both SUSY and non-SUSY terms from the effective operators. This correction can be comparable to linear terms in ζ_{10}, ζ_{11} from $d = 5$ operators for $(2\zeta_{10}\mu) \approx 1/\tan\beta$ (see later). Not all $\mathcal{O}_{1,2\dots,8}$ are necessarily present, so in some models some $\alpha_{ij}, \zeta_{10}, \zeta_{11}$ could vanish. Also:

$$\begin{aligned}
\delta m_H^2 &= -\frac{1}{4}(Bm_0\mu) v^2 \alpha_{60} \tan^2\beta + \frac{v^2 \tan\beta}{8} \left[-8Bm_0\mu \alpha_{20} - 4\alpha_{62} m_0^2 \right. \\
&\quad \left. - 4\mu m_0 (2\alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81}) - 4\mu^2 (2\alpha_{50} + \alpha_{60}) + (2\alpha_{60} - \alpha_{70}) m_Z^2 \right] \\
&\quad + \frac{3}{4} Bm_0\mu v^2 (\alpha_{50} + \alpha_{60}) + \frac{v^2}{8 \tan\beta} \left[-8Bm_0\mu \alpha_{10} + (12\alpha_{52} - 16\alpha_{62}) m_0^2 \right. \\
&\quad \left. - 4\mu m_0 (-6\alpha_{11} + 8\alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81}) - 4\mu^2 (5\alpha_{50} - 2\alpha_{60}) \right. \\
&\quad \left. + (6\alpha_{50} + 20\alpha_{60} - 13\alpha_{70}) m_Z^2 + \frac{8v^2}{Bm_0\mu} (2\zeta_{10}\mu)^2 \right] + \mathcal{O}(1/\tan^2\beta) \quad (5.49)
\end{aligned}$$

which is obtained for $(Bm_0\mu)$ fixed. Note the $\mathcal{O}(1/M^2)$ effects from $d = 5$ operators (ζ_{10}^2).

Similar expressions exist for the neutral pseudoscalar A . The results are simpler in this case and we present the exact expression of δm_A^2 of (5.44) in the most general case, that includes all supersymmetry breaking effects from the operators of $d = 5, 6$ and from the MSSM. One finds

$$\begin{aligned}
\delta m_A^2 &= \frac{v^2}{8 \tan^2\beta (1 + \tan^2\beta)} \left[-2Bm_0\mu \alpha_{50} + [-(4\alpha_{31} + 4\alpha_{41} + 8\alpha_{81} + 8\alpha_{11}) m_0\mu \right. \\
&\quad \left. - 4\alpha_{52} m_0^2 - 8Bm_0\mu \alpha_{10} - 4(\alpha_{50} + 2\alpha_{60})\mu^2 + (2\alpha_{50} - \alpha_{70}) m_Z^2 \right] \tan\beta \\
&\quad + [2Bm_0\mu (10\alpha_{50} + 3\alpha_{60}) + 16\alpha_{82} m_0^2 + 16(\alpha_{51} + \alpha_{61}) m_0\mu] \tan^2\beta \\
&\quad + 2[-4Bm_0\mu (\alpha_{10} + \alpha_{20} + 2\alpha_{30} + 2\alpha_{40}) - 6(\alpha_{50} + \alpha_{60})\mu^2 - (\alpha_{50} + \alpha_{60} - \alpha_{70}) m_Z^2]
\end{aligned}$$

$$\begin{aligned}
& - 2(\alpha_{62} + \alpha_{52})m_0^2 - 4(\alpha_{11} + \alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81})m_0\mu] \tan^3 \beta \\
& + [2Bm_0\mu(3\alpha_{50} + 10\alpha_{60}) + 16\alpha_{82}m_0^2 + 16(\alpha_{51} + \alpha_{61})m_0\mu] \tan^4 \beta \\
& - [8Bm_0\mu\alpha_{20} + 4(2\alpha_{50} + \alpha_{60})\mu^2 - (2\alpha_{60} - \alpha_{70})m_Z^2 + 4\alpha_{62}m_0^2 \\
& + 4(2\alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81})m_0\mu] \tan^5 \beta - 2Bm_0\mu\alpha_{60} \tan^6 \beta \Big] \quad (5.50)
\end{aligned}$$

We also showed that $\delta m_G = 0$ so the Goldstone mode remains massless in $\mathcal{O}(1/M^2)$, which is a good consistency check. A result similar to that in eq. (5.48) is found from an expansion of (5.50) in the large $\tan\beta$ limit:

$$\begin{aligned}
\delta m_A^2 &= -\frac{1}{4}(Bm_0\mu)\alpha_{60}v^2 \tan^2 \beta + \frac{\tan\beta}{8}v^2 \Big[-8Bm_0\mu\alpha_{20} - 4\alpha_{62}m_0^2 \\
& - (8\alpha_{21} + 4\alpha_{31} + 4\alpha_{41} + 8\alpha_{81})m_0\mu - (8\alpha_{50} + 4\alpha_{60})\mu^2 + 2\alpha_{60}m_Z^2 - \alpha_{70}m_Z^2 \Big] \\
& + \frac{v^2}{4} \Big[Bm_0\mu(3\alpha_{50} + 11\alpha_{60}) + 8m_0^2\alpha_{82} + 8m_0\mu(\alpha_{51} + \alpha_{61}) \Big] \\
& + \frac{v^2}{8\tan\beta} \Big[-8Bm_0\mu(\alpha_{10} + 2\alpha_{30} + 2\alpha_{40}) - 4(2\alpha_{11} + \alpha_{31} + \alpha_{41} + 2\alpha_{81})m_0\mu \\
& - 4\alpha_{52}m_0^2 - (4\alpha_{50} + 8\alpha_{60})\mu^2 - (2\alpha_{50} + 4\alpha_{60} - 3\alpha_{70})m_Z^2 \Big] + \mathcal{O}(1/\tan^2 \beta) \quad (5.51)
\end{aligned}$$

We emphasise that the large $\tan\beta$ limits presented so far were done with $(Bm_0\mu)$ fixed. While this is certainly an interesting case, a more natural expression to consider at large $\tan\beta$ is that in which one keeps m_A fixed and $Bm_0\mu$ arbitrary. We present below the correction $\mathcal{O}(1/M^2)$ to $m_{h,H}^2$ for the case m_A is kept fixed to an appropriate value. The result is (assuming $m_A > m_Z$, otherwise $\delta' m_h^2$ and $\delta' m_H^2$ are exchanged):

$$\begin{aligned}
\delta' m_h^2 &= -2v^2 \Big[\alpha_{22}m_0^2 + (\alpha_{30} + \alpha_{40})\mu^2 + 2\alpha_{61}m_0\mu - \alpha_{20}m_Z^2 \Big] - \frac{(2\xi_{10}\mu)^2 v^4}{m_A^2 - m_Z^2} \\
& + \frac{v^2}{\tan\beta} \Big[\frac{1}{(m_A^2 - m_Z^2)} \Big(4m_A^2 \Big((2\alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81})m_0\mu + (2\alpha_{50} + \alpha_{60})\mu^2 + \alpha_{62}m_0^2 \Big) \\
& - (2\alpha_{60} - 3\alpha_{70})m_A^2 m_Z^2 - (2\alpha_{60} + \alpha_{70})m_Z^4 \Big) + \frac{8(m_A^2 + m_Z^2)(\mu m_0 \xi_{10} \xi_{11})v^2}{(m_A^2 - m_Z^2)^2} \Big] \\
& + \mathcal{O}(1/\tan^2 \beta) \quad (5.52)
\end{aligned}$$

A similar formula exists for the correction to m_H :

$$\begin{aligned}
\delta' m_H^2 = & \left[-2(m_0\mu(\alpha_{51} + \alpha_{61}) + \alpha_{82} m_0^2) v^2 + \frac{(2\zeta_{10}\mu)^2 v^4}{m_A^2 - m_Z^2} \right] \\
& + \frac{v^2}{\tan\beta} \left[\frac{1}{m_A^2 - m_Z^2} (2m_A^2 (2(\alpha_{11} - \alpha_{21}) m_0\mu + (\alpha_{60} - \alpha_{50}) \mu^2 + (\alpha_{52} - \alpha_{62}) m_0^2 - \alpha_{60} m_A^2) \right. \\
& - [4(\alpha_{11} + \alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81}) m_0\mu + 6(\alpha_{50} + \alpha_{60}) \mu^2 + 2(\alpha_{52} + \alpha_{62}) m_0^2 \\
& - (\alpha_{50} + 5\alpha_{60} - 2\alpha_{70}) m_A^2] m_Z^2 - (\alpha_{50} - \alpha_{60}) m_Z^4) - \frac{8(m_A^2 + m_Z^2)(\mu m_0 \zeta_{10} \zeta_{11}) v^2}{(m_A^2 - m_Z^2)^2} \left. \right] \\
& + \mathcal{O}(1/\tan^2\beta) \tag{5.53}
\end{aligned}$$

Corrections (5.52) and (5.53) must be added to the rhs of eq. (5.45) to obtain the value of $m_{h,H}^2$ expressed in function of m_A . The corrections in equations (5.47) to (5.53) extend those of the previous chapter to include all $\mathcal{O}(1/M^2)$ terms.

From equations (5.48) and (5.52) we are able to identify the effective operators of $d = 6$ that give the leading contributions to m_h^2 , which is important for model building. These are $\mathcal{O}_{2,3,4}$ in the absence of supersymmetry breaking and $\mathcal{O}_{2,6}$ when this is broken. It is however preferable to increase m_h^2 by supersymmetric rather than supersymmetry-breaking effects of the effective operators, because the latter are less under control in the effective approach and one would favour a supersymmetric solution to the fine-tuning problem associated with increasing the MSSM Higgs mass above the LEP II bound. Therefore $\mathcal{O}_{2,3,4}$ are the leading operators, with the remark that \mathcal{O}_2 has a smaller effect, of order $(m_Z/\mu)^2$ relative to $\mathcal{O}_{3,4}$ (for similar α_{j0} , $j = 2, 3, 4$). At smaller $\tan\beta$, $\mathcal{O}_{5,6}$ can also give significant contributions while \mathcal{O}_7 has a relative suppression factor $(m_Z/\mu)^2$.

5.4 Analysis of the Leading Corrections and Effective Operators

One expects that when in the Lagrangian appear effective operators of mass dimension five and six, coming from the same UV physics, those of dimension six will be subleading. However, this is not the case when an extra suppression makes the two classes comparable. In our case, some dimension five operators are suppressed by $1/(M \tan\beta)$ but dimension six only have $1/M^2$. Thus, in the limit of large $\tan\beta$ these two classes can be comparable.

In the particular case of the Higgs mass, by comparing $O(1/M)$ terms in eq. (5.45) against $O(1/M^2)$ terms in equations (5.52) and (5.53), one identifies the situation when these two classes of operators give comparable corrections:

$$\begin{aligned} \frac{4m_A^2}{m_A^2 - m_Z^2} \frac{|\zeta_{10}\mu|}{\tan\beta} &\approx \left| \alpha_{22}m_0^2 + (\alpha_{30} + \alpha_{40})\mu^2 + 2\alpha_{61}m_0\mu - \alpha_{20}m_Z^2 + \frac{2(\zeta_{10}\mu)^2 v^2}{m_A^2 - m_Z^2} \right| \\ \left| \zeta_{11} m_0 + \frac{4m_Z^2}{m_A^2 - m_Z^2} \frac{\zeta_{10}\mu}{\tan\beta} \right| &\approx \left| (m_0\mu(\alpha_{51} + \alpha_{61}) + \alpha_{82} m_0^2) - \frac{2(\zeta_{10}\mu)^2 v^2}{m_A^2 - m_Z^2} \right| \end{aligned} \quad (5.54)$$

In this case $O(1/(M \tan\beta))$ and $O(1/M^2)$ corrections are approximately equal (for $M \approx m_0 \tan\beta$). Similar relations can be obtained by comparing (5.40) and (5.44) against $\delta m_{h,H}^2$ of (5.48), (5.49) and (5.50).

Note that we don't have to consider operators of dimension > 6 since they do not receive any $\tan\beta$ enhancement in order to become comparable with $d = 6$ and will always be subleading.

Let us now examine more closely the corrections to the Higgs masses due to $d = 6$ operators. The interest is to maximise the correction to the MSSM classi-

cal value of m_h . From equations (5.48) and (5.52) and ignoring SUSY breaking corrections ($\alpha_{jk}, k \neq 0$), we saw that at large $\tan\beta$ $O_{3,4}$ bring the largest correction and also O_2 to a lower extent. At smaller $\tan\beta$, $O_{5,6,7}$ can have significant corrections. All this can be seen from the relative variation:

$$\epsilon_{rel} \equiv \frac{m_h - m_Z}{m_Z} = \sqrt{\delta_{rel}} - 1, \quad (5.55)$$

with

$$\begin{aligned} \delta_{rel} \equiv & 1 - \frac{4m_A^2}{m_A^2 - m_Z^2} \frac{1}{\tan^2\beta} + \frac{v^2}{m_Z^2} \left\{ \frac{2\zeta_{10}\mu}{\tan\beta} \frac{4m_A^2}{m_A^2 - m_Z^2} + \frac{(-2\zeta_{11}m_0)}{\tan^2\beta} \frac{2(m_A^4 + m_Z^4)}{(m_A^2 - m_Z^2)^2} \right. \\ & - \left[2(\alpha_{22}m_0^2 + (\alpha_{30} + \alpha_{40})\mu^2 + 2\alpha_{61}m_0\mu - \alpha_{20}m_Z^2) + \frac{(2\zeta_{10}\mu)^2 v^2}{m_A^2 - m_Z^2} \right] \\ & + \frac{1}{\tan\beta} \frac{1}{m_A^2 - m_Z^2} \left[4m_A^2\mu \left((2\alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81})m_0 + (2\alpha_{50} + \alpha_{60})\mu \right) \right. \\ & \left. + 4\alpha_{62}m_0^2 m_A^2 - (2\alpha_{60} - 3\alpha_{70})m_A^2 m_Z^2 - (2\alpha_{60} + \alpha_{70})m_Z^4 + 8\zeta_{10}\zeta_{11}\mu m_0 v^2 \frac{m_A^2 + m_Z^2}{m_A^2 - m_Z^2} \right] \left. \right\} \\ & + O(1/\tan^4\beta) + O(\tilde{m}/(M \tan^3\beta)) + O(\tilde{m}^2/(M^2 \tan^2\beta)) \end{aligned} \quad (5.56)$$

where \tilde{m} is some generic mass scale of the theory such as μ , m_Z , m_0 or v . The arguments of the functions O in the last line show explicitly the origin of these corrections (MSSM, $d = 5$ and $d = 6$ operators, respectively). Eq. (5.55) gives the overall relative change of the classical value of m_h in the presence of all possible higher dimensional operators of $d = 5$ and $d = 6$ beyond the MSSM Higgs sector, for large $\tan\beta$ with m_A fixed. Depending on the signs of coefficients α_{jk} , ζ_{10} and ζ_{11} this relative variation can be positive and increase m_h above the MSSM classical upper bound m_Z . The accuracy of the expansion at intermediate $\tan\beta$ depends on \tilde{m}/M ; in any case one can use the exact $\delta m_{h,H}^2$ in (5.47).

The same expansion in large $\tan\beta$ can also be computed keeping $Bm_0\mu$ fixed, instead of m_A . Then

$$\begin{aligned}
\delta_{rel} \equiv & 1 - \frac{4}{\tan^2 \beta} + \frac{v^2}{m_Z^2} \left\{ \frac{4(2\zeta_{10}\mu)}{\tan \beta} + \frac{2}{\tan^2 \beta} \left((-2\zeta_{11}m_0) + \frac{2m_Z^2(2\zeta_{10}\mu)}{Bm_0\mu} \right) \right. \\
& - 2 \left[\alpha_{22}m_0^2 + 2\alpha_{61}m_0\mu + (\alpha_{30} + \alpha_{40})\mu^2 - \alpha_{20}m_Z^2 \right] + \frac{1}{\tan \beta} \left[\frac{(2\zeta_{10}\mu)^2 v^2}{-Bm_0\mu} \right. \\
& + 4(2\alpha_{21} + \alpha_{31} + \alpha_{41} + 2\alpha_{81})m_0\mu + 4(2\alpha_{50} + \alpha_{60})\mu^2 + 4\alpha_{62}m_0^2 - (2\alpha_{60} - 3\alpha_{70})m_Z^2 \left. \right\} \\
& + O(1/\tan^4 \beta) + O(\tilde{m}/(M \tan^3 \beta)) + O(\tilde{m}^2/(M^2 \tan^2 \beta)) \tag{5.57}
\end{aligned}$$

In (5.55) and (5.57), the $d = 6$ operators (α_{ij} dependence) give contributions which are dominated by $\tan\beta$ -independent terms. One particular limit to consider for δm_h^2 or $\delta' m_h^2$ is that in which the effective operators of $d = 6$ have coefficients such that these contributions add up to maximise δ_{rel} . Since coefficients α_{ij} are not known, we can choose them equal in absolute value

$$-\alpha_{22} = -\alpha_{61} = -\alpha_{30} = -\alpha_{40} = \alpha_{20} > 0 \tag{5.58}$$

In this case, at large $\tan\beta$:

$$\delta m_h^2 \approx 2v^2\alpha_{20}[m_0^2 + 2m_0\mu + 2\mu^2 + m_Z^2] \tag{5.59}$$

and similar for $\delta' m_h^2$. A simple numerical example is illustrative. For $m_0 = 1$ TeV, $\mu = 350$ GeV and $v \approx 246$ GeV, one has $\delta m_h^2 \approx 2.36\alpha_{20} \times 10^{11} (\text{GeV})^2$. Assuming $\alpha_{20} \sim 1/M^2$ for $M = 10$ TeV and the classical MSSM value of m_h to be equal to m_Z (reached for large $\tan\beta$), we obtain an increase of m_h from $d = 6$ operators alone of about $\Delta m_h = 12.15$ GeV to $m_h \approx 103$ GeV. An increase of α_{20} by a factor of 2.5 to $\alpha_{20} \sim 2.5/M^2$ would give $\Delta m_h \approx 28$ GeV and $m_h \approx 119.2$ GeV, which is above the LEP II bound.

The discussion above indicates that if we persist on using the loop correction to increase the Higgs mass, the effect of these operators will be to relax the strain of the little hierarchy. Indeed, the relative increase of Δm_h due to $d = 6$ operators

alone is mildly reduced, however, the effective quartic coupling of the Higgs is increased. This amounts to a reduction of the fine tuning for the electroweak scale [76]. The above choice of $M = 10$ TeV was partly motivated by the fine-tuning results of [71] and on convergence grounds: The expansion parameter of our effective analysis is m_q/M where m_q is any scale of the theory, in particular it can be the susy breaking scale m_0 . For $m_0 \simeq 3$ TeV and $c_{1,2} \simeq 2.5$ (of eq. (5.1)), one finds for $M = 10$ TeV that $c_{1,2} m_0/M \simeq 0.75$ which is already at the limit of validity of the expansion in the effective approach considered.

These simple estimates demonstrate that mass dimension six operators can indeed bring a significant increase of m_h to values compatible with the LEP II bound. However, the amount of increase depends on implicit assumptions like the type and number of operators present and whether their overall sign, as generated by the UV physics, is consistent with an increase of m_h . Take for example the case of the leading contribution to m_h in the large $\tan\beta$ case. One would prefer to generate the leading operators with supersymmetric coefficients satisfying

$$\alpha_{20} > 0, \alpha_{30} < 0, \alpha_{40} < 0 \tag{5.60}$$

in order to increase m_h . We have already mentioned that $\mathcal{O}_{1,2,3}$ can be generated by integrating out a massive gauge boson $U(1)'$ or $SU(2)$ triplets while \mathcal{O}_4 by a massive gauge singlet or $SU(2)$ triplets. Let us discuss the signs that these operators are generated with:

(a): Integrating out a massive vector superfield $U(1)'$ under which Higgs fields have opposite charges (to avoid a Fayet-Iliopoulos term), one finds $\alpha_{20} < 0$ and $\alpha_{30} > 0$ (also $\alpha_{10} < 0$), which is opposite to condition (5.60). However, this can change if for example there are additional pairs of massive Higgs doublets also

charged under the new $U(1)'$ since then O_3 could be generated with $\alpha_{30} < 0$. **(b):** Integrating out massive $SU(2)$ triplets that couple to the MSSM Higgs sector would bring $\alpha_{20} > 0$, $\alpha_{40} < 0$, $\alpha_{30} > 0$; the first two of these satisfy (5.60). **(c):** Integrating out a massive gauge singlet would bring $\alpha_{40} > 0$ which would actually decrease m_h . Finally, if we take into account further constraints coming from the ρ parameter [69], it turns out that it is α_{40} and α_{30} that can have the largest correction to m_h^2 . For generating them, the case of a massive gauge singlet or additional $U(1)'$ vector superfield would have the advantage of preserving gauge couplings unification at one-loop.

For smaller $\tan\beta$, operators $O_{5,6,7}$ could bring significant corrections to m_h but it is more difficult to generate these in a renormalisable setup. For example, $O_{5,6}$ can be generated by integrating out a pair of massive Higgs doublets and a massive gauge singlet but the overall sign of $\alpha_{50,60}$ would depend on the details of the model. This discussion shows that while effective operators can in principle increase m_h , deriving a renormalisable model that would generate them with appropriate signs for their (supersymmetric) coefficients is not a simple issue. However this does not exclude the possibility, since the examples given are rather simplistic. Other generating mechanisms for O_i could be in place⁶ with appropriate signs to increase m_h .

⁶For some models with extended MSSM Higgs sector see [128, 129, 130, 131].

CHAPTER 6

NONLINEAR MSSM

In the previous chapters we used EFT to study in a model independent way the effects of new physics beyond MSSM in the multiTeV scale. Nevertheless, MSSM itself contains new physics at scale \sqrt{f} , the SUSY breaking scale. If we take this scale to be around multiTeV, new effects appear by the presence of a goldstino, which is the dominant component of the gravitino. Goldstino couplings are best described in terms of nonlinear supersymmetry, as briefly presented in section 2.3. One way to realize symmetries in a nonlinear fashion is by using appropriate constraints. In supersymmetry, these are constraints on superfields. In the following we apply the method of constrained superfields in order to construct the most general couplings of a goldstino to full MSSM.

6.1 The Model

We couple the constrained superfield X_{nl} of eq. (2.44) to the SUSY part of the MSSM, to find the “nonlinear” supersymmetry version of MSSM. At energy scales below m_{soft} , similar constraints can be applied to the MSSM superfields themselves, corresponding to integrating out the superpartners. Here, the only difference from ordinary MSSM is in the supersymmetry breaking sector. Supersymmetry is broken spontaneously via a vacuum expectation value (v.e.v.)

of F_X , fixed by its equation of motion. The Lagrangian of nonlinear MSSM is:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_X + \mathcal{L}_H + \mathcal{L}_m + \mathcal{L}_{AB} + \mathcal{L}_g \quad (6.1)$$

Let us detail these terms. \mathcal{L}_0 is the usual MSSM SUSY Lagrangian

$$\begin{aligned} \mathcal{L}_0 = & \sum_{\Phi, H_{1,2}} \int d^4\theta \Phi^\dagger e^{V_i} \Phi + \left\{ \int d^2\theta \left[\mu H_1 H_2 + H_2 Q U^c + Q D^c H_1 + L E^c H_1 \right] + h.c. \right\} \\ & + \sum_{\text{SM groups}} \frac{1}{16 g^2 \kappa} \int d^2\theta \text{Tr} [W^\alpha W_\alpha] + h.c., \quad \Phi : Q, D^c, U^c, E^c, L, \end{aligned} \quad (6.2)$$

where κ is a constant canceling the trace factor and the gauge coupling g is shown explicitly. The family matrices in the superpotential are implicit to lighten the notation.

The SUSY breaking couplings originate from the MSSM fields couplings to the goldstino superfield; this is done by the replacement $S \rightarrow m_{soft} X_{nl}/f$ [54], where S is the usual spurion also used in the previous chapters, with $S = \theta\theta m_{soft}$ and m_{soft} a generic notation for the soft terms (denoted below $m_{1,2}, m_0$). One has for the Higgs sector

$$\begin{aligned} \mathcal{L}_H = & \sum_{i=1,2} c_i \int d^4\theta X_{nl}^\dagger X_{nl} H_i^\dagger e^{V_i} H_i \\ = & \sum_{i=1,2} c_i \left\{ |\phi_X|^2 \left[|\mathcal{D}_\mu h_i|^2 + F_{h_i}^\dagger F_{h_i} + h_i^\dagger \frac{D_i}{2} h_i + \left(\frac{i}{2} \bar{\psi}_{h_i} \bar{\sigma}^\mu \mathcal{D}_\mu \psi_{h_i} - \frac{1}{\sqrt{2}} h_i^\dagger \lambda_i \psi_{h_i} + h.c. \right) \right] \right. \\ & + \frac{1}{2} h_i^\dagger (\mathcal{D}_\mu + \overleftarrow{\mathcal{D}}_\mu) h_i \partial^\mu |\phi_X|^2 + \bar{\psi}_X \bar{\psi}_{h_i} \psi_X \psi_{h_i} - \frac{1}{2} [\phi_X^\dagger (\partial^\mu - \overleftarrow{\partial}^\mu) \phi_X] [h_i^\dagger (\mathcal{D}_\mu - \overleftarrow{\mathcal{D}}_\mu) h_i] \\ & + \left[-\frac{i}{2} \phi_X^\dagger \psi_X \sigma^\mu \bar{\psi}_{h_i} (\mathcal{D}_\mu - \overleftarrow{\mathcal{D}}_\mu) h_i - \frac{1}{\sqrt{2}} \phi_X^\dagger \psi_X h_i^\dagger \lambda_i h_i - \phi_X^\dagger \psi_X F_{h_i}^\dagger \psi_{h_i} + \phi_X^\dagger F_X F_{h_i}^\dagger h_i \right. \\ & + \frac{i}{2} (\bar{\psi}_X \bar{\sigma}^\mu \psi_X) (h_i^\dagger \mathcal{D}_\mu h_i) + \frac{i}{2} (\phi_X^\dagger \partial_\mu \phi_X) (\bar{\psi}_{h_i} \bar{\sigma}^\mu \psi_{h_i}) + \frac{i}{2} \bar{\psi}_X \bar{\sigma}^\mu (\partial_\mu - \overleftarrow{\partial}_\mu) \phi_X (h_i^\dagger \psi_{h_i}) \\ & \left. - \bar{\psi}_X F_X \bar{\psi}_{h_i} h_i + h.c. \right] + \left[\partial_\mu \phi_X^\dagger \partial^\mu \phi_X + F_X^\dagger F_X + \left(\frac{i}{2} \bar{\psi}_X \bar{\sigma}^\mu \partial_\mu \psi_X + h.c. \right) |h_i|^2 \right], \end{aligned} \quad (6.3)$$

Here $\mathcal{D}, \partial, (\overleftarrow{\mathcal{D}}, \overleftarrow{\partial})$ act only on the first field to their right (left) respectively and h_i, ψ_{h_i}, F_{h_i} denote SU(2) doublets. Also

$$c_1 = -m_1^2/f^2, \quad c_2 = -m_2^2/f^2. \quad (6.4)$$

Similar terms exist for all matter fields

$$\mathcal{L}_m = \sum_{\Phi} c_{\Phi} \int d^4\theta X_{nl}^{\dagger} X_{nl} \Phi^{\dagger} e^V \Phi, \quad c_{\Phi} = -\frac{m_{\Phi}^2}{f^2}, \quad \Phi : Q, U^c, D^c, L, E^c, \quad (6.5)$$

One can eventually set $m_{\Phi} = m_0$ (all Φ). The bi- and trilinear SUSY breaking couplings are

$$\begin{aligned} \mathcal{L}_{AB} &= \frac{B'}{f} \int d^2\theta X_{nl} H_1 H_2 \quad (6.6) \\ &+ \frac{A_u}{f} \int d^2\theta X_{nl} H_2 Q U^c + \frac{A_d}{f} \int d^2\theta X_{nl} Q D^c H_1 + \frac{A_e}{f} \int d^2\theta X_{nl} L E^c H_1 + h.c. \\ &= \frac{B'}{f} \left\{ \phi_X \left[h_1 \cdot F_{h_2} + F_{h_1} \cdot h_2 - \psi_{h_1} \cdot \psi_{h_2} \right] - h_1 \cdot (\psi_X \psi_{h_2}) - (\psi_X \psi_{h_1}) \cdot h_2 + F_X h_1 \cdot h_2 \right\} \\ &+ \left\{ \frac{A_u}{f} \left[\phi_X h_2 \cdot (\phi_Q F_U - \psi_Q \psi_U + F_Q \phi_U) - \phi_X (\psi_{h_2} \cdot \phi_Q \psi_U + \psi_{h_2} \cdot \psi_Q \phi_U - F_{h_2} \cdot \phi_Q \phi_U) \right. \right. \\ &- \psi_X (h_2 \cdot \phi_Q \psi_U + h_2 \cdot \psi_Q \phi_U + \psi_{h_2} \cdot \phi_Q \phi_U) + F_X h_2 \cdot \phi_Q \phi_U \left. \right] - [U \rightarrow D, H_2 \rightarrow H_1] \\ &- [U \rightarrow E, H_2 \rightarrow H_1, Q \rightarrow L] \left. \right\} + h.c. \quad (6.7) \end{aligned}$$

where $B' \equiv B m_0 \mu$. Finally, the supersymmetry breaking couplings in the gauge sector are

$$\begin{aligned} \mathcal{L}_g &= \sum_{i=1}^3 \frac{1}{16 g_i^2 \kappa} \frac{2 m_{\lambda_i}}{f} \int d^2\theta X_{nl} \text{Tr} [W^{\alpha} W_{\alpha}]_i + h.c. \\ &= \sum_{i=1}^3 \frac{m_{\lambda_i}}{2f} \left\{ \phi_X \left[2i \lambda^a \sigma^{\mu} \Delta_{\mu} \bar{\lambda}^a - \frac{1}{2} F^{\alpha\mu\nu} F_{\mu\nu}^a + D^a D^a - \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \right] \right. \\ &\quad \left. - \sqrt{2} \psi_X \sigma^{\mu\nu} \lambda^a F_{\mu\nu}^a - \sqrt{2} \psi_X \lambda^a D^a + F_X \lambda^a \lambda^a \right\}_i + h.c. \quad (6.8) \end{aligned}$$

with $m_{\lambda_{1,2,3}}$ the masses of the three gauginos and gauge group index i for $U(1)$, $SU(2)$, $SU(3)$ respectively. Above we introduced the notation $\Delta_{\mu} \bar{\lambda}^a = \partial_{\mu} \bar{\lambda}^a - g t^{abc} V_{\mu}^b \bar{\lambda}^c$. Equations (6.1) to (6.8), along with (2.45), define the model, with spontaneous supersymmetry breaking ensured by non-zero $\langle F_X \rangle$.

Since $\phi_X \sim 1/f$, the Lagrangian contains terms of order higher than $1/f^2$. In the calculation of the onshell Lagrangian we shall restrict the calculations to up

to and including $1/f^2$ terms. This requires solving for F_ϕ of matter fields up to and including $1/f^2$ terms and for F_X up to and including $1/f^3$ terms (due to its leading contribution which is $-f$). Doing so, in the final Lagrangian no kinetic mixing is present at this order. Using the expressions of the auxiliary fields, one then computes the F -part of the scalar potential of the Higgs sector, to find:

$$V_F = |\mu|^2 \left[|h_1|^2 + |h_2|^2 \right] + \frac{|f + (B'/f) h_1 \cdot h_2|^2}{1 + c_1 |h_1|^2 + c_2 |h_2|^2} + \mathcal{O}(1/f^3) \quad (6.9)$$

with $h_1 \cdot h_2 \equiv h_1^0 h_2^0 - h_1^- h_2^+$ and $|h_i|^2 \equiv h_i^\dagger h_i = h_i^{0*} h_i^0 + h_i^{-*} h_i^-$. One can work with this potential, however, for convenience, if $|c_{1,2}| |h_{1,2}|^2 \ll 1$, we can approximate V_F by expanding the denominator in a series of powers of these coefficients. Our analysis below is then valid for $|c_{1,2}| |h_{1,2}|^2 \ll 1$. After adding the gauge contribution, we find the following result for the scalar potential of the Higgs sector:

$$\begin{aligned} V = & f^2 + (|\mu|^2 + m_1^2) |h_1|^2 + (|\mu|^2 + m_2^2) |h_2|^2 + (B' h_1 \cdot h_2 + h.c.) \quad (6.10) \\ & + \frac{1}{f^2} \left[m_1^2 |h_1|^2 + m_2^2 |h_2|^2 + B' h_1 \cdot h_2 \right]^2 + \frac{g_1^2 + g_2^2}{8} \left[|h_1|^2 - |h_2|^2 \right]^2 + \frac{g_2^2}{2} |h_1^\dagger h_2|^2 + \mathcal{O}(1/f^3) \end{aligned}$$

This is the full Higgs potential. The first term in the last line is a new term, absent in MSSM (generated by eliminating F_X of X_n). Its effects for phenomenology will be analyzed later. The ignored higher order terms in $1/f$ involve non-renormalizable $h_{1,2}^6$ interactions in V .

6.2 New Couplings in the Lagrangian

In this section we compute the new interactions induced by Lagrangian (6.1), which are not present in the MSSM. Many of the new couplings are actually

dimension-four in fields, with a (dimensionless) f -dependent coupling. The couplings are important in the case of a low SUSY breaking scale in the hidden sector and a light gravitino scenario. Some of the new couplings also involve the goldstino field and are relevant for phenomenology.

As mentioned earlier, from the SUSY breaking part of the Lagrangian only terms up to $1/f^2$ were kept in the total Lagrangian. After eliminating all terms proportional to F -auxiliary fields of X, H_i, Q, D^c, U^c, E^c and L , one obtains new couplings \mathcal{L}^{new} beyond those of the usual on shell, supersymmetric part of MSSM, which are unchanged and not shown. One finds the on shell Lagrangian

$$\mathcal{L}^{new} \equiv \mathcal{L}_F^{aux} + \mathcal{L}_D^{aux} + \mathcal{L}_m^{extra} + \mathcal{L}_g^{extra} \quad (6.11)$$

Let us detail these terms. Firstly,

$$\mathcal{L}_F^{aux} = \mathcal{L}_{F(1)}^{aux} + \mathcal{L}_{F(2)}^{aux} \quad (6.12)$$

with

$$\begin{aligned} \mathcal{L}_{F(1)}^{aux} = & -\left[f^2 + (m_1^2|h_1|^2 + m_2^2|h_2|^2 + m_\Phi^2|\phi_\Phi|^2) \right] \\ & - \left[B' h_1 \cdot h_2 + A_u h_2 \cdot \phi_Q \phi_U + A_d \phi_Q \phi_D \cdot h_1 + A_e \phi_L \phi_E \cdot h_1 + \frac{1}{2} m_{\lambda_i} \lambda_i \lambda_i + h.c. \right] \end{aligned} \quad (6.13)$$

recovering the usual MSSM soft terms and the additional contributions:

$$\begin{aligned} \mathcal{L}_{F(2)}^{aux} = & \left\{ \frac{\bar{\psi}_X \bar{\psi}_X}{2 f^2} \left[\mu(m_1^2 + m_2^2) h_1 \cdot h_2 - (m_1^2 + m_Q^2 + m_D^2) h_1 \cdot \phi_Q \phi_D - (m_1^2 + m_L^2 + m_E^2) h_1 \cdot \phi_L \phi_E \right. \right. \\ & - (m_2^2 + m_Q^2 + m_U^2) \phi_Q \phi_U \cdot h_2 + (B' h_2 - A_d \phi_Q \phi_D - A_e \phi_L \phi_E)^\dagger (\mu h_2 - \phi_Q \phi_D - \phi_L \phi_E) \\ & + (B' h_1 - A_u \phi_Q \phi_U)^\dagger (\mu h_1 - \phi_Q \phi_U) + (A_d \phi_D h_1 - A_u h_2 \phi_U)^\dagger (\phi_D h_1 - h_2 \phi_U) \\ & + A_d (|\phi_Q \cdot h_1|^2 + |\phi_E h_1|^2) + A_u |h_2 \cdot \phi_Q|^2 + A_e |\phi_L \cdot h_1|^2 \left. \right] + h.c. \left. \right\} - \frac{1}{f^2} \left[B' h_1 \cdot h_2 \right. \\ & + A_u h_2 \cdot \phi_Q \phi_U + A_d \phi_Q \phi_D \cdot h_1 + A_e \phi_L \phi_E \cdot h_1 + \frac{m_{\lambda_i}}{2} \lambda_i \lambda_i + (m_1^2|h_1|^2 + m_2^2|h_2|^2 + m_\Phi^2|\phi_\Phi|^2) \left. \right]^2 \\ & - \frac{1}{f} \left[m_1^2 \bar{\psi}_X \bar{\psi}_{h_1} h_1 + m_2^2 \bar{\psi}_X \bar{\psi}_{h_2} h_2 + m_\Phi^2 \bar{\psi}_X \bar{\psi}_\Phi \phi_\Phi + h.c. \right] + \mathcal{O}(1/f^3) \end{aligned} \quad (6.14)$$

A summation is understood over the SM group indices $i = 1, 2, 3$ in the gaugino term and over $\Phi = Q, U^c, D^c, L, E^c$ in the mass terms; appropriate contractions among $SU(2)_L$ doublets are understood for holomorphic products, when the order displayed is relevant. The leading interactions $\mathcal{O}(1/f)$ are those in the last line and are dimension-four in fields. Similar couplings exist at $\mathcal{O}(1/f^2)$ and involve scalar and gaugino fields. Yukawa matrices are restored in (6.14) by replacing $\phi_Q\phi_D \rightarrow \phi_Q\gamma_d\phi_D$, $\phi_Q\phi_U \rightarrow \phi_Q\gamma_u\phi_U$, $\phi_L\phi_E \rightarrow \phi_L\gamma_e\phi_E$, as already explained.

There are also new couplings from terms involving the auxiliary components of the vector superfields of the SM. Integrating them out one finds:

$$\begin{aligned} \mathcal{L}_D^{aux} = & \frac{-1}{2} \left[\tilde{D}_1 + \frac{1}{4f^2} (m_{\lambda_1} \psi_X \psi_X + h.c.) \tilde{D}_1 + \frac{1}{\sqrt{2}f} (m_{\lambda_1} \psi_X \lambda_1 + h.c.) \right]^2 \\ & + \frac{-1}{2} \left[\tilde{D}_2^a + \frac{1}{4f^2} (m_{\lambda_2} \psi_X \psi_X + h.c.) \tilde{D}_2^a + \frac{1}{\sqrt{2}f} (m_{\lambda_2} \psi_X \lambda_2^a + h.c.) \right]^2 \\ & + \frac{-1}{2} \left[\tilde{D}_3^a + \frac{1}{4f^2} (m_{\lambda_3} \psi_X \psi_X + h.c.) \tilde{D}_3^a + \frac{1}{\sqrt{2}f} (m_{\lambda_3} \psi_X \lambda_3^a + h.c.) \right]^2 + \mathcal{O}(f^{-3}) \end{aligned} \quad (6.15)$$

with notation:

$$\begin{aligned} \tilde{D}_1 &= -\frac{1}{2} g_1 (-h_1^\dagger h_1 + h_2^\dagger h_2 + 1/3 \phi_Q^\dagger \phi_Q - 4/3 \phi_U^\dagger \phi_U + 2/3 \phi_D^\dagger \phi_D - \phi_L^\dagger \phi_L + 2 \phi_E^\dagger \phi_E) \\ \tilde{D}_2^a &= -\frac{1}{2} g_2 (h_1^\dagger \sigma^a h_1 + h_2^\dagger \sigma^a h_2 + \phi_Q^\dagger \sigma^a \phi_Q + \phi_L^\dagger \sigma^a \phi_L) \\ \tilde{D}_3^a &= -\frac{1}{2} g_3 (\phi_Q^\dagger t^a \phi_Q - \phi_U^\dagger t^a \phi_U - \phi_D^\dagger t^a \phi_D) \end{aligned} \quad (6.16)$$

for the corresponding MSSM expressions; here $t^a/2$ are the $SU(3)$ generators. From (6.15) one can easily read the new, f -dependent couplings in the gauge sector, absent in the MSSM.

The total Lagrangian also contains extra terms, not proportional to the auxiliary fields, and *not* present in the MSSM. In the matter sector these are:

$$\begin{aligned}
\mathcal{L}_m^{extra} &= \frac{1}{4f^2} |\partial_\mu(\psi_X \psi_X)|^2 + \left(\frac{i}{2} \bar{\psi}_X \bar{\sigma}^\mu \partial_\mu \psi_X + h.c. \right) \tag{6.17} \\
&- \sum_{i=1}^2 \frac{m_i^2}{f^2} \left\{ \bar{\psi}_X \bar{\psi}_{h_i} \psi_X \psi_{h_i} + \left[\frac{i}{2} (\bar{\psi}_X \bar{\sigma}^\mu \psi_X) (h_i^\dagger \mathcal{D}_\mu h_i) + \frac{i}{2} |h_i|^2 \bar{\psi}_X \bar{\sigma}^\mu \partial_\mu \psi_X + h.c. \right] \right\} \\
&- \left[m_i^2 \rightarrow m_\Phi^2, H_i \rightarrow \Phi \right] + \left\{ \frac{B'}{f} \left[\frac{1}{2f} \psi_X \psi_X \psi_{h_1} \cdot \psi_{h_2} - h_1 \cdot (\psi_X \psi_{h_2}) - (\psi_X \psi_{h_1}) \cdot h_2 \right] \right. \\
&+ \frac{A_u}{f} \left[\frac{1}{2f} \psi_X \psi_X (h_2 \cdot \psi_Q \psi_U + \psi_{h_2} \cdot \phi_Q \psi_U + \psi_{h_2} \cdot \psi_Q \phi_U) - \psi_X (h_2 \cdot \phi_Q \psi_U + h_2 \cdot \psi_Q \phi_U \right. \\
&+ \left. \left. \psi_{h_2} \cdot \phi_Q \phi_U \right) \right] + \left[\frac{A_d}{f} \left(\frac{1}{2f} \psi_X \psi_X (\psi_Q \psi_D \cdot h_1 + \phi_Q \psi_D \cdot \psi_{h_1} + \psi_Q \phi_D \cdot \psi_{h_1}) \right. \right. \\
&- \left. \left. \psi_X (\phi_Q \psi_D \cdot h_1 + \psi_Q \phi_D \cdot h_1 + \phi_Q \phi_D \cdot \psi_{h_1}) \right) + (D \rightarrow E, L \rightarrow Q) \right] + h.c. \left. \right\} + O(1/f^3) \tag{6.18}
\end{aligned}$$

Note the presence of interactions that are dimension-four in fields ($B'/f h_1 \psi_X \psi_{h_2}$, etc) that can be relevant for phenomenology at low f . There are also new couplings in the gauge sector

$$\begin{aligned}
\mathcal{L}_g^{extra} &= \sum_{i=1}^3 \frac{m_{\lambda_i}}{2f} \left[\frac{\psi_X \psi_X}{-2f} \left(2i \lambda^a \sigma^\mu \Delta_\mu \bar{\lambda}^a - \frac{1}{2} F_{\mu\nu}^a F^{a\mu\nu} - \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a \right) \right. \\
&- \left. \sqrt{2} \psi_X \sigma^{\mu\nu} \lambda^a F_{\mu\nu}^a \right] + h.c. + O(1/f^3), \tag{6.19}
\end{aligned}$$

with $i = 1, 2, 3$ is the gauge group index and $\sigma^{\mu\nu} = i/4(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$. The new couplings of \mathcal{L}^{new} together with the on shell part of the purely supersymmetric part of the MSSM Lagrangian (on shell \mathcal{L}_0 of (6.2)) gives the final effective Lagrangian of the model. From this, the full scalar potential is identified.

6.3 Implications for the Higgs Masses

Let us consider the Higgs scalar potential found in (6.10) and analyze its implications for the Higgs masses. From the neutral Higgs part of the potential one finds the masses of the CP even and CP odd Higgs fields. Since eq. (6.10) is valid up to $1/f^3$ terms, it is sufficient to restrict the expressions up to this order.

Firstly, at the minimum of the scalar potential one has:

$$\begin{aligned}
m_1^2 - m_2^2 &= \cot 2\beta \left[B' + \frac{f^2 (-1 + \sqrt{w_0})(-B' + m_Z^2 \sin 2\beta)}{v^2 (2\mu^2 + m_Z^2 \cos^2 2\beta + B' \sin 2\beta)} \right] \\
m_1^2 + m_2^2 &= \frac{1}{\sin 2\beta} \left[-B' + \frac{f^2 (-1 + \sqrt{w_0})(B' + 2\mu^2 \sin 2\beta)}{v^2 (2\mu^2 + m_Z^2 \cos^2 2\beta + B' \sin 2\beta)} \right]
\end{aligned} \tag{6.20}$$

where

$$w_0 \equiv 1 - \frac{v^2}{f^2} (4\mu^2 + 2m_Z^2 \cos^2 2\beta + 2B' \sin 2\beta) \tag{6.21}$$

One finds the following results (upper sign for m_h^2):

$$\begin{aligned}
m_{h,H}^2 &= \frac{1}{2} \left[m_Z^2 + \frac{-2B'}{\sin 2\beta} \mp \sqrt{w_1} \right] + \frac{v^2}{32f^2} \left\{ 4B' \left[2B' + (4\mu^2 + 2m_Z^2 \cos^2 2\beta) / \sin 2\beta \right] \right. \\
&+ 4 \left[2B'^2 + 8\mu^4 + 2m_Z^2(4\mu^2 + m_Z^2) \cos^2 2\beta + 8B'\mu^2 \sin 2\beta \right] \\
&\mp \frac{\csc^2 2\beta}{\sqrt{w_1}} \left[-2(B'^2 + 4\mu^4)m_Z^2 + 4\mu^2 m_Z^4 + m_Z^6 + 8(2\mu^4 m_Z^2 - B'^2(4\mu^2 + m_Z^2)) \cos 4\beta \right. \\
&- m_Z^2(6B'^2 + 8\mu^4 + 4\mu^2 m_Z^2 + m_Z^4) \cos 8\beta - 8B'(B'^2 - 8\mu^4) \sin 2\beta \\
&\left. \left. + B'(-8B'^2 + 16\mu^2 m_Z^2 + m_Z^4) \sin 6\beta + B' m_Z^4 \sin 10\beta \right] \right\} + \mathcal{O}(1/f^3)
\end{aligned} \tag{6.22}$$

with

$$w_1 = \left(m_Z^2 + \frac{-2B'}{\sin 2\beta} \right)^2 - 4m_Z^2 \left(\frac{-2B'}{\sin 2\beta} \right) \cos^2 2\beta \tag{6.23}$$

Further, the mass m_A of the pseudoscalar Higgs has a simple form (no expansion):

$$m_A^2 = \frac{-2B'}{\sin 2\beta} \left\{ \frac{3}{4} + \frac{1}{4} \sqrt{w_0} - \frac{v^2}{4f^2} B' \sin 2\beta \right\} \tag{6.24}$$

and, as usual, the Goldstone mode has mass $m_G = 0$.

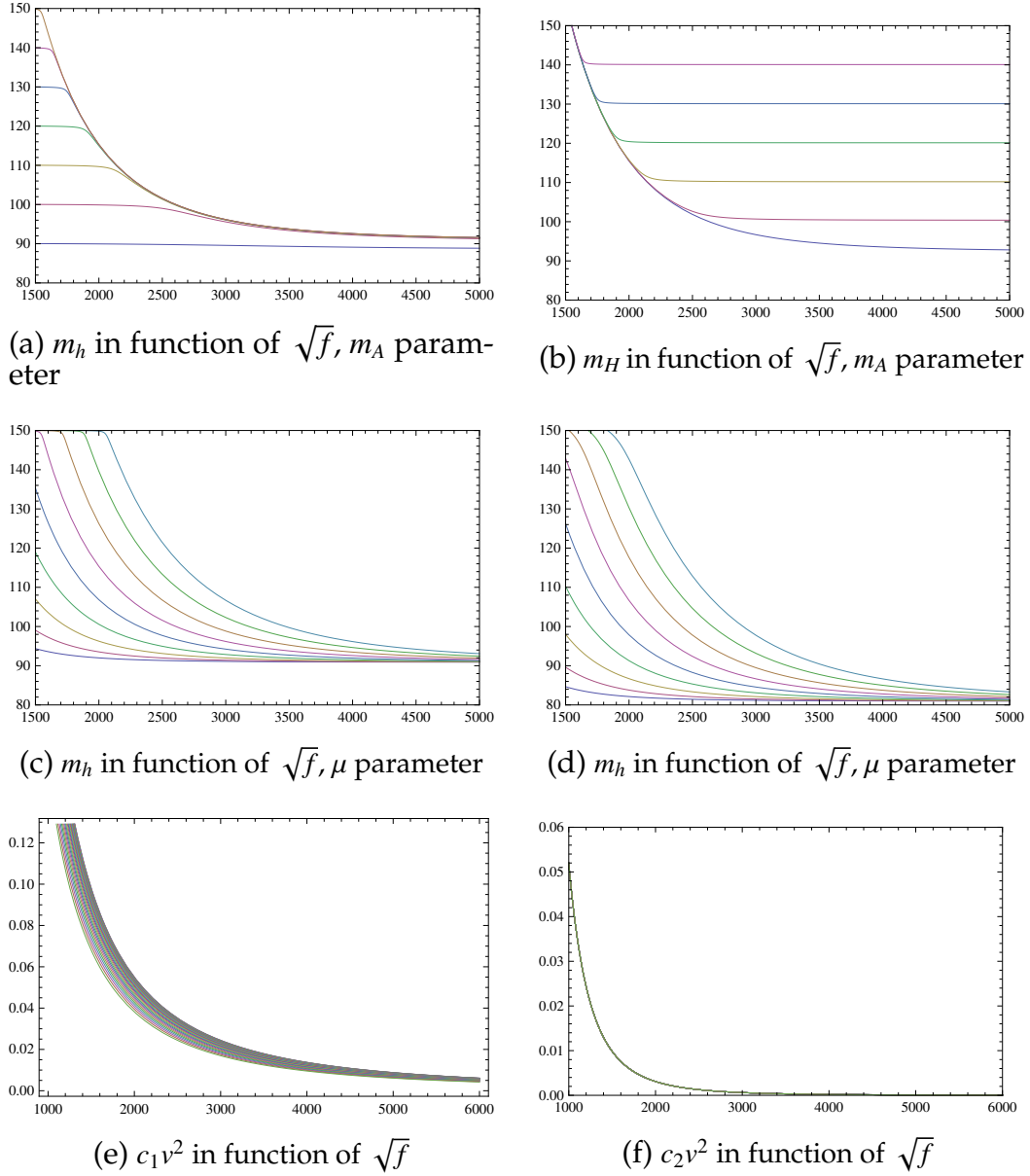


Figure 6.1: The tree-level Higgs masses (in GeV) and expansion coefficients as functions of \sqrt{f} (in GeV). In (a), (b) $\mu = 900$ GeV, $\tan\beta = 50$, m_A increases upwards from 90 to 150 GeV in steps of 10 GeV. Larger m_A has little impact on m_h for relevant \sqrt{f} . In (c), (d), $m_A = 150$ GeV, and m_h increases as μ varies from 400 to 1200 GeV, in steps of 100 GeV. In (c) $\tan\beta = 50$ while in (d) $\tan\beta = 5$, showing a milder dependence on $\tan\beta$ than in MSSM. For $\tan\beta \geq 10$ there is little difference from (c). In (e), (f) the expansion coefficients are shown, for $m_A = [90, 650]$ GeV with steps of 10 GeV, $\mu = 900$ GeV, $\tan\beta = 50$; they are less than unity (even at larger μ), as required for a convergent expansion.

It is instructive to consider the limit of large $u \equiv \tan\beta$, with $B' < 0$ fixed, when

$$m_h^2 = \left[m_Z^2 + \mathcal{O}(1/u) \right] + \frac{v^2}{2f^2} \left[(2\mu^2 + m_Z^2)^2 + \frac{4}{u} B' (2\mu^2 + m_Z^2) + \mathcal{O}(1/u^2) \right] + \mathcal{O}(f^{-3}) \quad (6.25)$$

$$m_H^2 = \left[\frac{-2B'}{\sin 2\beta} + \mathcal{O}(1/u) \right] + \frac{v^2 B'}{4f^2} \left[(2\mu^2 + m_Z^2)u + 4B' + \frac{1}{u} (2\mu^2 - 11m_Z^2) + \mathcal{O}(1/u^2) \right] + \mathcal{O}(f^{-3})$$

which shows that a large μ can increase m_h (decrease m_H). However, for phenomenology it is customary to use m_A as an input instead of B' , in which case the masses $m_{h,H}$ take the form

$$m_{h,H}^2 = \frac{1}{2} \left[m_A^2 + m_Z^2 \mp \sqrt{w} \right] \pm \frac{v^2}{16f^2} \frac{1}{\sqrt{w}} \left[16m_A^2\mu^4 + 4m_A^2\mu^2 m_Z^2 + (m_A^2 - 8\mu^2) m_Z^4 \right. \\ - 2m_Z^6 \pm 2(-2m_A^2\mu^2 + 8\mu^4 + 4\mu^2 m_Z^2 + m_A^4) \sqrt{w} + m_A^2 m_Z^4 \cos 8\beta \\ + m_A^4 (m_A^2 - 8\mu^2 - 3m_Z^2) \sin^2 2\beta + \cos 4\beta \left[-2m_Z^2 (8\mu^4 + 4\mu^2 m_Z^2 + m_Z^4) - m_A^2 (6\mu^2 + m_Z^2) \right. \\ \left. \left. \pm 2(2m_A^2\mu^2 + 4\mu^2 m_Z^2 + m_A^4) \sqrt{w} - m_A^2 (m_A^2 + 5m_Z^2) \sin^2 2\beta \right] \right] + \mathcal{O}(1/f^3) \quad (6.26)$$

where the first term (bracket) is just the MSSM contribution. The upper (lower) signs correspond to m_h (m_H) and $w = (m_A^2 + m_Z^2)^2 - 4m_A^2 m_Z^2 \cos^2 2\beta$. At large $\tan\beta$ with m_A fixed one finds¹ (with $u \equiv \tan\beta$)

$$m_h^2 = \left[m_Z^2 + \mathcal{O}(1/u^2) \right] + \frac{v^2}{2f^2} \left[(2\mu^2 + m_Z^2)^2 + \mathcal{O}(1/u^2) \right] + \mathcal{O}(1/f^3)$$

$$m_H^2 = \left[m_A^2 + \mathcal{O}(1/u^2) \right] + \frac{1}{f^2} \mathcal{O}(1/u^2) + \mathcal{O}(1/f^3) \quad (6.27)$$

In this limit the increase of m_h is driven by a large μ and is apparently of SUSY origin, but the quartic Higgs couplings giving this effect involved combinations of soft masses (see (6.10)). These soft masses combine to give, at the EW minimum, the μ -dependent increase in (6.27).

¹In (6.27) $m_A > m_Z$ is assumed, otherwise just exchange m_h^2 with m_H^2 .

Some simple numerical examples are relevant for the size of the corrections to the Higgs masses, relative to their MSSM values. The largest correction to m_h for large $\tan\beta$ is dominated by μ and f . For example, if $(\mu/\sqrt{f})^2 = (1/2.25)^2 \approx 1/5$, $v = 246$ GeV, with $\mu = 900$ GeV then $\sqrt{f} = 2$ TeV, giving $m_h = 114.4$ GeV. Another example is with $\mu = 1.2$ TeV, $\sqrt{f} = 2.7$ TeV, $((\mu/\sqrt{f})^2 \approx 1/5)$, giving again $m_h = 114.4$ GeV. Smaller $\mu \approx 600$ GeV can still allow m_h just above the LEP bound if $\sqrt{f} = 1.35$ TeV, for similar value for $(\mu/\sqrt{f})^2 = 1/5$ and for the rest of the parameters. This shows that one can have a *classical* value of m_h near or marginally above the LEP bound and larger than the classical MSSM value ($= m_Z$). The plots in Figure 6.1 illustrate better this change of m_h and m_H for various values of \sqrt{f} . For a low value of \sqrt{f} near or above 1.35 TeV, the LEP bound is still satisfied for m_h , while at large \sqrt{f} the MSSM case is recovered. By varying \sqrt{f} our results can interpolate between low and high scale (in the hidden sector) SUSY breaking. Quantum corrections increase m_h further, just as in the MSSM.

Regarding the usual MSSM tree-level flat direction $|h_1^0| = |h_2^0|$ one can show that the potential in this direction can have a minimum for the case (not considered in MSSM) of $m_1^2 + m_2^2 + 2|\mu|^2 < 2|B'|$, equal to $V_m = f^2 - (1/4)f^2(m_1^2 + m_2^2 + 2|\mu|^2 + 2B')^2 / (m_1^2 + m_2^2 + B')^2$. Compared to the usual MSSM minimum, the former can be situated above it only for values of f which do not comply with the original assumptions of $m_{1,2}^2, |B'| < f$. On the other hand, the case with V_m situated below the MSSM minimum does not allow one to recover the MSSM ground state in the decoupling limit of large f , and in conclusion the “flat” direction is not of physical interest here.

6.4 Other Phenomenological Implications

6.4.1 Fine Tuning of the Electroweak Scale

The increase of m_h beyond the MSSM tree level bound and the presence of new quartic Higgs couplings have implications in the fine tuning. In MSSM the smallness of the effective quartic coupling λ (fixed by the gauge sector) is at the origin of an increased amount of fine tuning of the electroweak scale for large soft masses. For soft masses significantly larger than the electroweak (EW) scale, (also needed to increase the MSSM value for m_h above LEP bound via quantum corrections), fine tuning increases rapidly and may become a potential problem (sometimes referred to as the “little hierarchy” problem). Let us see why in the present model this problem is alleviated. One can write $v^2 = -m^2/\lambda$ where

$$\begin{aligned}\lambda &\equiv \frac{g_1^2 + g_2^2}{8} [\cos^2 2\beta + \delta \sin^4 \beta] + \frac{1}{f^2} \left| m_1^2 \cos^2 \beta + m_2^2 \sin^2 \beta + (1/2) B' \sin 2\beta \right|^2 \\ m^2 &\equiv (|\mu|^2 + m_1^2) \cos^2 \beta + (|\mu|^2 + m_2^2) \sin^2 \beta + B' \sin 2\beta\end{aligned}\quad (6.28)$$

The first term in λ is due to MSSM only, while the second one, which is positive, is due to the new quartic Higgs terms in (6.10). Here δ accounts for the top/stop quantum effects to $|h_2|^4$ term in the potential, which becomes $(1+\delta)(g_1^2+g_2^2)/8|h_2|^4$; usually $\delta \sim O(1)$ (ignoring couplings other than top Yukawa). This quantum effect is only included for a comparison to the new quartic Higgs term. The important point to note is that a larger λ gives a suppression in the fine tuning measure Δ :

$$\Delta = \frac{\partial \ln v^2}{\partial \ln p} = \frac{\partial \ln(-m^2/\lambda)}{\partial \ln p}, \quad p = A, B', m_0^2, \mu^2, m_{\lambda_i}^2. \quad (6.29)$$

Here p is an MSSM parameter with respect to which fine tuning is evaluated.

In the large $\tan\beta$ limit, the fine tuning of the electroweak scale becomes (see the Appendix in [71]):

$$\Delta = -\frac{(|\mu|^2 + m_2^2)'}{v^2 m_2^4/f^2 + (1 + \delta) m_2^2/2} + O(1/\tan\beta), \quad (|\mu|^2 + m_2^2)' \equiv \frac{\partial(|\mu|^2 + m_2^2)}{\partial \ln p} \quad (6.30)$$

For small $\tan\beta$ a similar result is obtained in which one replaces m_2 by m_1 . The first term in denominator comes from the new correction to the effective quartic coupling λ . Larger soft masses $m_{1,2}$ increase λ and this can actually reduce fine tuning, see the denominator in Δ . Therefore, in this case heavier superpartners do not necessarily bring an increased fine tuning amount (as it usually happens in the MSSM). The only limitation here is the size of the ratio $m_{1,2}^2/f \leq 1$ for convergence of the nonlinear formalism. In the limit this coefficient approaches its upper bound (say $\sim 1/3$), the two contributions in the denominator have comparable size (for $\delta \sim 1$ and $v = 246$ GeV) and fine tuning is reduced by a factor ≈ 2 from that in the absence of the new term in the denominator (*i.e.* the MSSM case).

6.4.2 Limiting Cases and Loop Corrections

Some interesting limits of our “nonlinear” MSSM model are worth considering. Firstly, in the limit of large f (*i.e.* large SUSY breaking scale in the hidden sector) and with $m_{1,2}, B'$ fixed, the new quartic term in (6.10) vanishes, while the usual explicit soft SUSY breaking terms specific to the Higgs sector remain. This is just the MSSM case. All other couplings suppressed by inverse powers of f are negligible in this limit. Another limiting case is that of very small f . For our analysis to be valid, one needs to satisfy the condition $B', m_{1,2}^2 \leq f$. When

f reaches this minimal bound, the new quartic couplings in (6.10), not present in the MSSM, increase and eventually become closer to unity. The analysis is then less reliable and additional effective contributions in the Lagrangian, suppressed by higher powers like $1/f^4$ and beyond, may become relevant for SUSY breaking effects.

Finally, one remark regarding the calculation of radiative corrections using (6.10) and the electroweak symmetry breaking (EWSB). In our case EWSB was assumed to take place by appropriate values of $m_{1,2}^2, B'$. However, the same EWSB mechanism as in the MSSM is at work here, via quantum corrections to these masses, which near the EW scale turn $m_2^2 + \mu^2$ negative and trigger radiative EWSB. Indeed, if the loops of the MSSM states are cut off as usual at the high GUT scale (well above \sqrt{f}) and with the new Higgs quartic couplings regarded as an effective, classical operator, radiative EWSB can take place as in the MSSM. A similar example is the case of a MSSM Higgs sector extended with additional effective operators of dimension $d = 5$ such as $(1/M) \int d^2\theta (H_1 H_2)^2$ giving a dimension-four (in fields) contribution to the scalar potential $V \supset h_1 h_2 (|h_1|^2 + |h_2|^2)$; this is regarded as an effective operator and radiative EWSB is implemented as in the MSSM, see for example [65, 71].

It is interesting to remark that that the loop corrections induced by the (effective) quartic couplings proportional to $1/f^2$ in eq. (6.10), can be under control at large f . Indeed, the loop integrals this coupling induces can be quadratically divergent and are then cut-off at momentum $p^2 \leq f$; but the loop effects come with a coupling factor that behaves like $1/f^2$, so overall they will be suppressed like $1/f$ and can then be under control even at large f . It would be interesting to check if for a large enough f , radiative EW breaking is still achievable if the

usual MSSM effects are also cut at this scale (with less an energy range to trigger EWSB).

6.4.3 Invisible Decays of Higgs and Z Bosons

Let us analyze some implications of the interactions involving the goldstino field, described by the Lagrangian found above. An interesting possibility, for a light enough neutralino, is the decay of the neutral higgses into a goldstino and the lightest neutralino χ_1^0 (this is the NLSP, while the goldstino is the LSP). The coupling Higgs-goldstino-neutralino is only suppressed by $1/f$. It arises from the following terms in \mathcal{L}^{new} and from the terms in the on shell, supersymmetric part of usual MSSM Lagrangian (6.2), hereafter denoted $\mathcal{L}_0^{onshell}$:

$$\begin{aligned} \mathcal{L}^{new} + \mathcal{L}_0^{onshell} \supset & -\frac{1}{f} \left[m_1^2 \psi_X \psi_{h_1^0} h_1^{0*} + m_2^2 \psi_X \psi_{h_2^0} h_2^{0*} \right] - \frac{B'}{f} \left[\psi_X \psi_{h_2^0} h_1^0 + \psi_X \psi_{h_1^0} h_2^0 \right] \\ & - \frac{1}{f} \sum_{i=1,2} \frac{m_{\lambda_i}}{\sqrt{2}} \tilde{D}_i^a \psi_X \lambda_i^a - \frac{1}{\sqrt{2}} \left[g_2 \lambda_2^3 - g_1 \lambda_1 \right] \left[h_1^{0*} \psi_{h_1^0} - h_2^{0*} \psi_{h_2^0} \right] + h.c. \end{aligned} \quad (6.31)$$

The last term (present in the MSSM) also brings a goldstino interaction. This is possible through the goldstino components of the higgsinos $\psi_{h_{1,2}^0}$ and EW gauginos $\lambda_{1,2}$. The goldstino components are found via the equations of motion, after EWSB, to give (see also [54]):

$$\begin{aligned} \mu \psi_{h_1^0} &= \frac{1}{f \sqrt{2}} \left(-m_2^2 v_2 - B' v_1 - \frac{1}{2} v_2 \langle g_2 D_2^3 - g_1 D_1 \rangle \right) \psi_X + \dots \\ \mu \psi_{h_2^0} &= \frac{1}{f \sqrt{2}} \left(-m_1^2 v_1 - B' v_2 + \frac{1}{2} v_1 \langle g_2 D_2^3 - g_1 D_1 \rangle \right) \psi_X + \dots \\ \lambda_1 &= \frac{-1}{f \sqrt{2}} \langle D_1 \rangle \psi_X + \dots, \quad \lambda_2^3 = \frac{-1}{f \sqrt{2}} \langle D_2^3 \rangle \psi_X + \dots \end{aligned} \quad (6.32)$$

which can be further simplified by using the MSSM minimum conditions in

the terms multiplied by $1/f$ (allowed in this approximation). As a consistency check we also showed that the determinant of the neutralino mass matrix (now a 5×5 matrix, to include the goldstino) vanishes up to corrections of order $O(f^{-4})$. This is consistent with our approximation for the Lagrangian, and verifies the existence of a massless goldstino (ultimately “eaten” by the gravitino). Using (6.31) and (6.32), one finds after some calculations (for previous calculations of this decay see [77, 78, 79]):

$$\mathcal{L}^{new} + \mathcal{L}_0^{onshell} \supset -\frac{1}{f\sqrt{2}} \sum_{j,k=1}^4 \left[\psi_X \chi_j^0 H^0 \delta_k \chi_{jk}^* + \psi_X \chi_j^0 h^0 \delta'_k \chi_{jk}^* \right] + h.c. \quad (6.33)$$

where

$$\begin{aligned} \delta_1 &= m_Z \sin \theta_w [m_{\lambda_1} \cos(\alpha + \beta) + \mu \sin(\alpha - \beta)], \\ \delta_2 &= -m_Z \cos \theta_w [m_{\lambda_2} \cos(\alpha + \beta) + \mu \sin(\alpha - \beta)], \\ \delta_3 &= -m_A^2 \sin \beta \sin(\alpha - \beta) - \mu^2 \cos \alpha \\ \delta_4 &= m_A^2 \cos \beta \sin(\alpha - \beta) - \mu^2 \sin \alpha, \quad \delta'_i = \delta_i \Big|_{\alpha \rightarrow \alpha + \pi/2} \end{aligned} \quad (6.34)$$

\mathcal{X} is the matrix that diagonalizes the MSSM neutralino mass matrix²: $M_d^2 = \mathcal{X} M M^\dagger \mathcal{X}^\dagger$, and can be easily evaluated numerically (see [80] for its analytical expression). Further H^0, h^0 are Higgs mass eigenstates (of mass $m_{h,H}$ computed earlier) and $h_i^0 = 1/\sqrt{2} (v_i + h_i^{0'} + i\sigma_i)$ with $\langle h_i^{0'} \rangle = 0, \langle \sigma_i \rangle = 0$; the relation of H^0, h^0 to $h_{1,2}^{0'}$ is a rotation, which in this case can be just that of the MSSM (due to extra $1/f$ suppression in the coupling³). The angle α is

$$\tan 2\alpha = \tan 2\beta \frac{m_A^2 + m_Z^2}{m_A^2 - m_Z^2}, \quad -\pi/2 \leq \alpha \leq 0 \quad (6.35)$$

²The exact form of M is: $M_{11} = m_{\lambda_1}, M_{12} = 0, M_{13} = -m_Z \cos \beta \sin \theta_w, M_{14} = m_Z \sin \beta \sin \theta_w, M_{21} = 0, M_{22} = m_{\lambda_2}, M_{23} = m_Z \cos \beta \cos \theta_w, M_{24} = -m_Z \sin \beta \cos \theta_w, M_{33} = 0, M_{34} = \mu, M_{44} = 0$, also $M_{ij} = M_{ji}$. Note the sign of μ related to our definition of the holomorphic product of SU(2) doublets. With this notation, in the text $\chi_j^0 = \chi_{jk} \xi_k$, with $\xi_k^T \equiv (\lambda_1, \lambda_2^3, \psi_{h_1^0}, \psi_{h_2^0})$.

³The relation is $h_1^{0'} = H^0 \cos \alpha - h^0 \sin \alpha$, and $h_2^{0'} = H^0 \sin \alpha + h^0 \cos \alpha$.

If the lightest neutralino is light enough, $m_{\chi_1^0} < m_h$, then h^0, H^0 can decay into it and a goldstino which has a mass of order $f/M_{Planck} \sim 10^{-3}$ eV; if this is not the case, the decay of neutralino into h^0 and goldstino takes place, examined in [79].

In the former case, the partial decay rate is

$$\Gamma_{h^0 \rightarrow \chi_1^0 \psi_X} = \frac{m_h}{16\pi f^2} \left| \sum_{k=1}^4 \delta'_k \mathcal{X}_{1k} \right|^2 \left(1 - \frac{m_{\chi_1^0}^2}{m_{h^0}^2} \right)^2 \quad (6.36)$$

The partial decay rate has corrections coming from both higgsino ($\mathcal{X}_{13}, \mathcal{X}_{14}$) and gaugino fields ($\mathcal{X}_{11}, \mathcal{X}_{12}$), since they both acquire a goldstino component, see eqs. (6.32). The gaugino correction arises after gaugino-goldstino mixing, SUSY and EW symmetry breaking, (as shown by m_{λ_i}, m_Z dependence in δ'_k) and was not included in previous similar studies [77, 78, 79].

The partial decay rate is presented in Figure 6.2 for various values of μ, m_A and $m_{\lambda_{1,2}}$ which are parameters of the model. A larger decay rate requires a light $\mu \sim O(100)$ GeV, when the neutralino χ_1^0 has a larger higgsino component. At the same time an increase of m_h above the LEP bound requires a larger value for μ , close to $\mu \approx 700$ GeV if $\sqrt{f} \approx 1.5$ TeV, and $\mu \approx 850$ GeV if $\sqrt{f} \approx 2$ TeV, see Figure 6.1 (c). The results in Figure 6.2 show that the partial decay rate can be significant ($\sim 3 \times 10^{-6}$ GeV), if we recall that the total SM Higgs decay rate (for $m_h \approx 114$ GeV) is about 3×10^{-3} GeV, with a branching ratio of $h^0 \rightarrow \gamma\gamma$ of 2×10^{-3} , (Figure 2 in [81]). Thus the branching ratio of the process can be close to that of SM $h^0 \rightarrow \gamma\gamma$. The decay is not very sensitive to $\tan\beta$ (Figure 6.2 (b)), due to the extra contribution (beyond MSSM) from the quartic Higgs coupling.

An interesting coupling that is also present in the $1/f$ order is that of goldstino to Z_μ boson and to a neutralino. Depending on the relative mass relations, it can bring about a decay of $Z_\mu (\chi_j^0)$ into $\chi_j^0 (Z_\mu)$ and a goldstino, respectively.

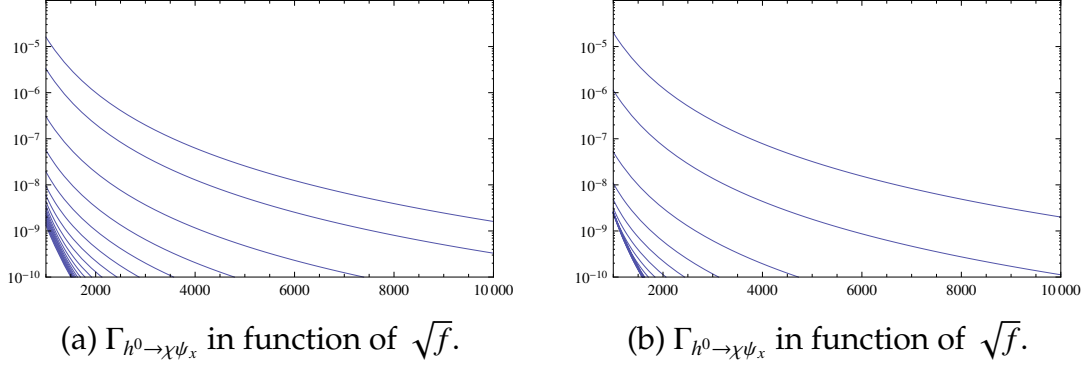


Figure 6.2: The partial decay rate of $h^0 \rightarrow \psi_X \chi_1^0$ for (a): $\tan\beta = 50$, $m_{\lambda_1} = 70$ GeV, $m_{\lambda_2} = 150$ GeV, μ increases from 50 GeV (top curve) by a step 50 GeV, $m_A = 150$ GeV. Compare against Figure 6.1 (c) corresponding to a similar range for the parameters. At larger μ , m_h increases, but the partial decay rate decreases. Similar picture is obtained at low $\tan\beta \sim 5$. (b): As for (a) but with $\tan\beta = 5$. Compare against Figure 6.1 (d). Note that the total SM decay rate, for $m_h \sim 114$ GeV, is of order 10^{-3} , thus the branching ratio in the above cases becomes comparable to that of SM Higgs going into $\gamma\gamma$ (see Figure 2 in [81]).

The relevant terms are

$$\begin{aligned}
\mathcal{L}^{new} + \mathcal{L}_0^{onshell} &\supset -\frac{1}{4} \bar{\psi}_{h_1^0} \bar{\sigma}^\mu \psi_{h_1^0} (g_2 V_2^3 - g_1 V_1)_\mu + \frac{1}{4} \bar{\psi}_{h_2^0} \bar{\sigma}^\mu \psi_{h_2^0} (g_2 V_2^3 - g_1 V_1)_\mu \} \\
&- \sum_{i=1}^2 \frac{m_{\lambda_i}}{\sqrt{2} f} \psi_X \sigma^{\mu\nu} \lambda_i^a F_{\mu\nu,i}^a + h.c.
\end{aligned} \tag{6.37}$$

where the last term was generated in (6.19) (i labels the gauge group). Since the higgsinos acquired a goldstino component ($\propto \psi_X/f$) via mass mixing, the first line above induces additional $\mathcal{O}(1/f)$ couplings of the higgsino to goldstino and to $Z_\mu = (1/g)(g_2 V_2^3 - g_1 V_1)_\mu$ with $g^2 = g_1^2 + g_2^2$. After some calculations one finds the coupling $Z_\mu \chi_j^0 \psi_X$:

$$\begin{aligned}
\mathcal{L}^{new} + \mathcal{L}_0^{onshell} &= \frac{1}{f \sqrt{2}} \sum_{j=1}^4 \left[\bar{\psi}_X \bar{\sigma}^\mu \chi_j^0 Z_\mu (\mu m_Z w_j - m_Z^2 v_j) - \bar{\psi}_X (\bar{\sigma}^\mu \partial^\nu - \bar{\sigma}^\nu \partial^\mu) \chi_j^0 Z_{\mu\nu} v_j \right] \\
&+ h.c.
\end{aligned} \tag{6.38}$$

where

$$w_j = \cos\beta \mathcal{X}_{j4}^* - \sin\beta \mathcal{X}_{j3}^*, \quad v_j = -\sin\theta_w \mathcal{X}_{j1}^* + \cos\theta_w \mathcal{X}_{j2}^*, \quad Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \quad (6.39)$$

If $m_{\chi_1^0}$ is lighter than Z_μ then a decay of the latter into $\chi_1^0 + \psi_X$ is possible. The decay rate of this process is (with $j = 1$):

$$\Gamma_{Z \rightarrow \psi_X \chi_1^0} = \frac{m_Z^5}{32\pi f^2} \left[\zeta_1 |w_j|^2 + \zeta_2 |v_j|^2 + \zeta_3 (w_j v_j^* + w_j^* v_j) \right] \left(1 - \frac{m_{\chi_1^0}^2}{m_Z^2} \right)^2 \quad (6.40)$$

with $\zeta_1 = 2(2 + r^2)\mu^2/m_Z^2$, $\zeta_2 = 2(8 + r^2)(1 + 2r^2)$, $\zeta_3 = -2(4 + 5r^2)\mu/m_Z$ where $r = m_{\chi_j}/m_Z$ (in (6.38) and subsequent one can actually replace μ by m_{χ_j} and $w_j \rightarrow w_j^*$, with $\mathcal{X}_{j4} \leftrightarrow \mathcal{X}_{j3}$).

The decay rate should be within the LEP error for Γ_Z , which is 2.3 MeV [82] (ignoring theoretical uncertainties which are small). From this, one finds a lower bound for \sqrt{f} , which can be as high as $\sqrt{f} \approx 700$ GeV for the parameter space considered previously in Figure 6.1, while generic values are $\sqrt{f} \sim \mathcal{O}(400)$ GeV. Therefore the results for the increase of m_h , that needed a value for \sqrt{f} in the TeV region, escape this constraint. This constraint does not apply if the lightest neutralino has a mass larger than m_Z , when the opposite decay ($\chi_j \rightarrow Z \psi_X$) takes place (this can be arranged for example by a larger m_{λ_1}).

There also exists the interesting possibility of an invisible decay of Z_μ gauge boson into a pair of goldstino fields, that we review here [54, 56, 69]. This is induced by the following terms in the Lagrangian, after the Higgs field acquires a VEV:

$$\begin{aligned} \mathcal{L}^{new} + \mathcal{L}_0^{onshell} \supset & \left\{ \frac{1}{4f^2} \bar{\psi}_X \bar{\sigma}^\mu \psi_X (g_2 V_2^3 - g_1 V_1)_\mu (m_1^2 v_1^2/2 - m_2^2 v_2^2/2) \right. \\ & \left. - \frac{1}{4} \bar{\psi}_{h_1^0} \bar{\sigma}^\mu \psi_{h_1^0} (g_2 V_2^3 - g_1 V_1)_\mu + \frac{1}{4} \bar{\psi}_{h_2^0} \bar{\sigma}^\mu \psi_{h_2^0} (g_2 V_2^3 - g_1 V_1)_\mu \right\} + h.c. \quad (6.41) \end{aligned}$$

With (6.32) and (6.41) one finds the coupling of Z boson to a pair of goldstinos:

$$\mathcal{L}^{new} + \mathcal{L}_0^{onshell} \supset \frac{m_Z^2}{4f^2} \bar{\psi}_X \bar{\sigma}^\mu \psi_X Z_\mu \langle D_Z \rangle + h.c. \quad (6.42)$$

where $\langle D_Z \rangle \equiv \cos \theta_W \langle D_2^3 \rangle - \sin \theta_W \langle D_1 \rangle = -(m_Z^2/g) \cos 2\beta + \mathcal{O}(1/f)$. The decay rate is then

$$\Gamma_{Z \rightarrow \psi_X \psi_X} = \frac{m_Z}{24\pi g^2} \left[\frac{m_Z^4}{2f^2} \right]^2 \cos^2 2\beta \quad (6.43)$$

in agreement with previous results obtained for $B' = 0$ [56, 54, 69]. The decay rate is independent of m_A and should be within the LEP error for Γ_Z (2.3 MeV [82]). One can then easily see that the increase of the Higgs mass above the LEP bound (114.4 GeV) seen earlier in Figure 6.1 is consistent with the current bounds for this decay rate, which thus places only mild constraints on f , below the TeV scale (≈ 200 GeV) [56, 69].

Similarly, \mathcal{L}^{new} can also induce Higgs decays into goldstino pairs. The terms in \mathcal{L}^{new} that contribute to Higgs decays are $\mathcal{L}_{F(2)}^{aux}$, \mathcal{L}_D^{aux} , \mathcal{L}_m^{extra} together with the MSSM higgsino-Higgs-gaugino coupling (last term in (6.31)). After using (6.32), expanding the Higgs fields about their v.e.v., one finds:

$$\mathcal{L}^{new} + \mathcal{L}_0^{onshell} \supset \frac{\mu v}{4f^2} m_A^2 \cos 2\beta \bar{\psi}_X \bar{\psi}_X \left[h_1^{0'} \sin \beta - h_2^{0'} \cos \beta \right] + h.c. + \mathcal{O}(1/f^3) \quad (6.44)$$

which, similarly to Z couplings, is independent of gaugino masses. Here $v = 246$ GeV and $h_i^0 = 1/\sqrt{2} (v_i + h_i^{0'} + i\sigma_i)$, $\langle h_i^{0'} \rangle = 0$, $\langle \sigma_i \rangle = 0$. In the mass eigenstates basis one simply replaces the square bracket in (6.44) by $[H^0 \sin(\beta - \alpha) - h^0 \cos(\beta - \alpha)]$. One can also replace m_A by $m_A^2 = m_h^2 + m_H^2 - m_Z^2 + \mathcal{O}(1/f^2)$, where the Higgs masses

can be taken to be the MSSM values (up to higher order corrections in $1/f$). The decay rate of h^0 into a pair of goldstinos is then

$$\Gamma_{h^0 \rightarrow \psi_X \psi_X} = \frac{m_h}{8\pi f^4} g_{h^0 \psi_X \psi_X}^2 \quad (6.45)$$

where $g_{h^0 \psi_X \psi_X}$ is the coupling of $h^0 \psi_X \psi_X$ of the above Lagrangian. For relevant values of f above ~ 1 TeV it turns out that this decay rate is very small relative to other partial decay rates of the Higgs in the MSSM/SM. For example, for a total decay rate near 10^{-3} GeV (valid near a Higgs mass of order $\mathcal{O}(100)$ GeV), the branching ratio of this decay mode is well below the usual ones and below that of SM Higgs going into $\gamma\gamma$, by a factor $\approx 10^{-3} - 10^{-2}$.

CHAPTER 7

SUMMARY OF RESULTS

This part of the thesis consists of two different effective analyses in the context of MSSM.

In the first one, covered in chapters 3, 4 and 5, we considered an extension by the complete set of R-parity conserving, mass dimension 5 operators for the MSSM and by dimension 5 and 6 for its Higgs sector. This set included all supersymmetric and supersymmetry breaking terms, the latter being incorporated by the use of spurions. Some of these operators are not physical since they can be related to each other by field redefinitions. We performed the appropriate, spurion dependent, redefinitions that allowed us to write down the full irreducible set of dimension 5 and dimension 6 operators. We further restricted the parameter space by applying phenomenological constraints, in particular from flavor changing neutral currents. We then studied the phenomenological consequences of the model both in the production of new couplings and in the mass of the Higgs.

The new couplings include “wrong” Higgs Yukawa terms which are also generated at one loop in pure MSSM. One significant effect of these terms is the $\tan\beta$ enhancement of the mass of the bottom quark. If the scale of the effective operators is at the multiTeV scale, the effective contribution is compa-

rable or even bigger than the loop contribution. We also found couplings of type 2 quarks - 2 squarks and 2 quarks - 2 sleptons. These are also relevant for LHC since they contribute to processes of squark production. The corresponding pure MSSM channels become weaker for higher collision energy, contrary to the effective contribution which is simply suppressed by $1/M$. The two can become comparable for energy of the TeV scale as in LHC.

The effective analysis presented offers a solution to the little hierarchy problem of MSSM. This problem is related to the fact that the tree level calculation for m_h in pure MSSM reveals an upper bound, equal to $m_Z = 91.2$ GeV, which is in complete disagreement with the lower bound of 114 GeV from the LEP II experiment. The only way to overpass this discrepancy in pure MSSM is to suppose significant loop corrections implying very heavy stops or large stop mixing. In any case fine tuning is reintroduced and this is what we call the little hierarchy problem. However, we showed that effective operators can significantly raise the mass of the Higgs thus reducing the fine tuning. This result suggests an alternative interpretation of the little hierarchy. Instead of viewing it as a deficiency of MSSM, it can be viewed as an indication for new physics at the multiTeV range.

The second effective analysis, presented in chapter 6, is not related to some “new physics” but to the SUSY breaking sector. Models of low energy SUSY breaking predict a very light gravitino. In the low energy regime, the dynamics of the gravitino can be accurately described by the dynamics of its goldstino component. So if the breaking scale is around TeV, apart from the pure MSSM spectrum we need to include the goldstino mode.

The effective description of the goldstino mode is done via nonlinear real-

ization of supersymmetry. There are various ways to study such systems. We chose the language of “constrained superfields” as the most general and easy to reproduce the couplings of goldstinos to MSSM fields. We wrote the full set of couplings and studied their phenomenological significance. One important effect is again related to the mass of the Higgs. It is shown that the presence of a goldstino can also increase m_h providing us with yet another way to alleviate the little hierarchy, even without the hypothesis of new physics. Furthermore, we found that invisible decays of Higgs to goldstinos and other neutralinos can be of comparable size with the standard decay to two photons. Finally, assuming that the lightest neutralino is lighter than the Z gauge boson, we got a bound on the SUSY breaking scale of around 700 GeV from invisible Z boson decays.

Part II

BRANE - BULK INTERACTIONS IN N = 2 GLOBAL AND LOCAL SUPERSYMMETRY

CHAPTER 8

PRELIMINARIES

8.1 The Dirac Born Infeld Action as the Effective Action of a D-brane

In 1934, a few years before the development of Quantum Electrodynamics, M. Born and L. Infeld proposed a generalization of Maxwell's electrodynamics that was free of the notorious divergence in the self-energy of the electron [83]. Their inspiration derived from how Special Relativity (SR) accommodated what they called "the principle of finiteness", that consistent theories should not allow physical quantities to become infinite.

In SR, the Newtonian kinetic energy of a particle is replaced by a function that imposes an upper limit in the velocity.

$$\frac{1}{2}mv^2 \rightarrow mc^2 \left(1 - \sqrt{1 - \frac{v^2}{c^2}} \right). \quad (8.1)$$

The deeper reason behind this replacement is the *principle of relativity*, that the kinetic action be invariant under Lorentz transformations. Born and Infeld

suggested a similar replacement for electrodynamics

$$\frac{1}{2}(H^2 - E^2) \rightarrow b^2 \left(\sqrt{1 + \frac{1}{b^2}(H^2 - E^2)} - 1 \right), \quad (8.2)$$

where b is a constant with the same dimension as the fields. They supported it by constructing a general expression for the Lorentz invariant action of a tensor field $A_{\mu\nu}$. In a few lines, this is what they did: Under a coordinate transformation, the measure d^4x becomes Jd^4x and the determinant $|A|$ becomes $J^{-2}|A|$, where J is the Jacobian of the transformation. It is obvious then that $\sqrt{|A|}d^4x$ forms an invariant piece. As for any arbitrary tensor, we can split $A_{\mu\nu}$ into a sum of its symmetric and antisymmetric part. The symmetric part was identified with the metric $g_{\mu\nu}$ and the antisymmetric with the field strength $F_{\mu\nu}$. A general expression for an invariant Lagrangian is then:

$$\mathcal{L} = \sqrt{-|g + F|} + \alpha \sqrt{-|g|} + \beta \sqrt{-|F|}. \quad (8.3)$$

However, the last piece is a total derivative and can be ignored. Also, $\alpha = -1$ by the requirement that we reduce to Maxwell's electrodynamics in the limit of small fields. After restoring dimensions we find that in flat space the Lagrangian takes the form:

$$\mathcal{L} = b^2 \left(1 - \sqrt{-|\eta + \frac{F}{b}|} \right) = b^2 \left(1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2} - \frac{(F^{\mu\nu}\tilde{F}_{\mu\nu})^2}{16b^4}} \right), \quad (8.4)$$

where $\tilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}/2$ is the dual field strength. We see that their derivation led to the suggested action (8.2) up to the piece $F\tilde{F}$ that does not affect the resolution of the electron self energy problem. In fact, we see that the electric field E has a maximum value b , in direct correspondence to the maximum velocity c of a particle in SR. As a consequence, the electric potential at zero distance doesn't diverge as $1/r$ but rather takes a maximum value proportional to $\sqrt{b/e}$, with e the electron charge.

The Born Infeld (BI) action offered an ingenious solution to the apparent divergence of the electric field at short distances. However, it was a classical solution to a problem that is purely quantum mechanical. The advent of Quantum Electrodynamics and renormalizable quantum field theories in the following years resolved, beyond many other things, the self energy problem.

Little attention was paid to the BI action until 50 years later. In a paper by E. Fradkin and A. Tseytlin in 1984, it was shown that the low energy effective action for open bosonic strings propagating in a background of constant field strength is given precisely by the BI action [84]. The same action is obtained in the superstrings case, too [85, 86]. In this framework, the maximal value 'b' of the field strength is interpreted as the string tension $T = 1/2\pi\alpha'$. At such extreme values, higher harmonics of the string can be excited and thus the energy of the field is transferred into these modes. In a way, the extended nature of strings *smears* the singularity. This was a remarkable discovery as it provided a closed expression where α' corrections are summed up to all orders.

The connection with D-branes, which were discovered some years later, didn't take long to reveal. It was soon demonstrated that the effective action for the coupling of a D-brane with NSNS bulk fields is given by the Dirac-Born-Infeld (DBI) action [87]. The DBI action is merely a dimensional reduction of a generalization of BI action to include the coupling to the dilaton and the anti-symmetric tensor. The effective action of a D-brane was extended after it was discovered that these non perturbative objects break half of the bulk supersymmetries and act as sources for the RR fields of the closed string spectrum [88]. This introduced a second piece in the effective action given by Wess-Zumino terms [89]. All in all, the bosonic part of the world-volume effective action of a

Dp-brane at the string tree level is given by:

$$S_{Dp} = -T_p \int d^{p+1}x e^{-\phi} \left(\sqrt{-|g|} - \sqrt{-|g + 2\pi\alpha' F + B|} \right) + \mu_p \int \sum_l e^{2\pi\alpha' F+B} \wedge C_l \quad (8.5)$$

at string frame. T_p is the brane tension, μ_p is the brane's charge for the various RR fields denoted by C_l (so l is even in type IIB and odd in type IIA) while B is the NSNS 2-form.

In the previous paragraph we mentioned that D-branes are objects that break half of the bulk supersymmetries and that their low energy effective action is described by the DBI action. To be more precise, it has been shown that the broken half of the supersymmetry is realized nonlinearly on the worldvolume of the D-brane. These facts lead us to the following question: Is it possible to apply the tools of nonlinear realizations developed in the previous part of the thesis, in order to “reproduce” the low energy effective action of a D-brane? In the following chapters we show that for the general case of $\mathcal{N} = 2$ bulk supersymmetry, it is. We do this by defining appropriate $\mathcal{N} = 2$ superfields and then upgrading the constrained superfields technique to $\mathcal{N} = 2$ superspace. The constraint breaks one supersymmetry leaving one linear and one nonlinear in the effective theory. The result comes out to be precisely the DBI action plus Wess - Zumino terms.

8.2 Quaternion-Kähler and Hyper-Kähler Manifolds

Supersymmetric Lagrangians of interacting matter typically contain complicated, field dependent terms in their kinetic part forming a nonlinear σ model. An efficient way to study the structure of the allowed couplings is to view the

fields as coordinates of a Riemannian manifold. Restrictions that supersymmetry imposes on the couplings are translated into restrictions on the corresponding manifold of the σ model.

$\mathcal{N} = 1$ global supersymmetry requires that the manifold of hypermultiplet scalars is Kähler while for $\mathcal{N} = 1$ supergravity it is further restricted to be Hodge. Adding one more supersymmetry brings further conditions: The scalar manifold in global $\mathcal{N} = 2$ is restricted to be hyperKähler while in local $\mathcal{N} = 2$ it is quaternion-Kähler. Since we will focus on $\mathcal{N} = 2$ supersymmetric models, we briefly present some basic facts about these two manifolds.

A quaternion-Kähler manifold is a $4n$ real dimensional Kähler manifold with holonomy contained in $Sp(2) \times Sp(2n)$. It has three complex structures

$$J^i J^k = -\delta^{ik} \mathbb{I} + \epsilon^{ikl} J^l \quad (8.6)$$

with $i, k, l = 1, 2, 3$ and a hermitian metric such that, for each i

$$g_{\alpha\beta} J_{\kappa}^{i\alpha} J_{\lambda}^{i\beta} = g_{\kappa\lambda} .$$

It is also Einstein, which means that its Ricci tensor is proportional to the metric:

$$R_{\alpha\beta} = 2\rho(2 + n)g_{\alpha\beta} .$$

and is strictly non-vanishing. In addition, it has a self-dual Weyl curvature (Weyl tensor is the traceless component of Riemann tensor). In 4D ($n=1$) the holonomy is $Sp(2) \times Sp(2) \sim SO(4)$ so the holonomy condition is empty. In this case the proper condition is self-duality of the Weyl tensor. In $\mathcal{N} = 2$ nonlinear σ models coupled to SUGRA the Einstein parameter is identified as $\rho = -k^2$ where $k^2 = 8\pi G_N$ (G_N is Newton's constant). In the zero curvature limit ($k \rightarrow 0$) we obtain global supersymmetry and a manifold which is Ricci-flat (Ricci tensor is zero).

By properly taking the global supersymmetry limit in a SUGRA theory with matter couplings, we should reduce to some global matter coupling theory which, as we mentioned, is described by a hyper-Kähler manifold. Hyper-Kähler manifolds are defined as the $4n$ real dimensional, connected, Riemannian manifolds whose holonomy group is contained in $Sp(2n)$. All hyper-Kähler manifolds are also Kähler and Ricci-flat, that is $R_{\mu\nu} = 0$. This matches with the zero curvature limit of the quaternion-Kähler. However, hyper-Kähler are not a subclass of quaternion-Kähler whose Ricci scalar and $Sp(2)$ connection are strictly non-zero.

8.3 Superspace Conventions

The notation used henceforth is somewhat different from the one of the previous part, being more suitable for the work done here. We present the notation as well as some ingredients that will be proven useful in the following chapters.

The $\mathcal{N} = 1$ supersymmetry variation of a superfield V_1 is $\delta V_1 = (\epsilon Q + \bar{\epsilon} \bar{Q})V_1$, with supercharges verifying the algebra

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu. \quad (8.7)$$

On V_1 , the supersymmetry algebra is

$$[\delta_1, \delta_2]V_1 = -2i(\epsilon_1 \sigma^\mu \bar{\epsilon}_2 - \epsilon_2 \sigma^\mu \bar{\epsilon}_1) \partial_\mu V_1. \quad (8.8)$$

The covariant derivatives

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} - i(\sigma^\mu \bar{\theta})_\alpha \partial_\mu, \quad \bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i(\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu \quad (8.9)$$

anticommute with supercharges and verify

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \quad (8.10)$$

as well.

The second supersymmetry will transform V_1 into another superfield V_2 and these two will form an $\mathcal{N} = 2$ supermultiplet. It is known that the covariant derivatives themselves offer a good differential realization of the supersymmetry algebra; this is easily seen here by comparing (8.7) and (8.10). So we choose to realize the second supersymmetry algebra on the covariant derivatives by postulating the following transformations:

$$\delta^* V_1 = -\frac{i}{\sqrt{2}}(\eta D + \bar{\eta} \bar{D}) V_2, \quad \delta^* V_2 = i\sqrt{2}(\eta D + \bar{\eta} \bar{D}) V_1. \quad (8.11)$$

where η_α is the spinorial parameter of the second supersymmetry. What we have presented here is the realization of $\mathcal{N} = 2$ supersymmetry in terms of $\mathcal{N} = 1$ superfields. We will see later that for our purposes, we can also define $\mathcal{N} = 2$ chiral superfields, which will be very useful in simplifying various expressions.

The $\mathcal{N} = 1$ supersymmetry variations of the components (z, ψ, f) of a chiral superfield Φ , $\bar{D}_{\dot{\alpha}}\Phi = 0$, are

$$\begin{aligned} \delta z &= \sqrt{2} \epsilon \psi, \\ \delta \psi_\alpha &= -\sqrt{2} [f \epsilon_\alpha + i(\sigma^\mu \bar{\epsilon})_\alpha \partial_\mu z], \\ \delta f &= -\sqrt{2} i \partial_\mu \psi \sigma^\mu \bar{\epsilon}. \end{aligned} \quad (8.12)$$

The bosonic expansions of the chiral superfields that will appear later are:

$$\begin{aligned} W_\alpha(y, \theta) &= \theta_\alpha d(y) + \frac{i}{2}(\theta \sigma^\mu \bar{\sigma}^\nu)_\alpha F_{\mu\nu}(y), \\ \chi_\alpha(y, \theta) &= -\frac{1}{4}\theta_\alpha C(y) + \frac{1}{4}(\theta \sigma^\mu \bar{\sigma}^\nu)_\alpha b_{\mu\nu}(y), \\ \Phi(y, \theta) &= \phi(y) - \theta \theta f_\phi(y), \end{aligned} \quad (8.13)$$

and any other chiral superfield has an expansion similar to Φ . In this notation $\bar{\chi}_{\dot{\alpha}} = (\chi_{\alpha})^*$ but $\bar{W}_{\dot{\alpha}} = -(W_{\alpha})^*$. Since $L = D^{\alpha}\chi_{\alpha} - \bar{D}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}$, the linear superfield has bosonic expansion

$$L(x, \theta, \bar{\theta}) = C + \theta\sigma^{\mu}\bar{\theta}v_{\mu} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square C, \quad (8.14)$$

$$v_{\mu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^{\nu}b^{\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^{[\nu}b^{\rho\sigma]} = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}H^{\nu\rho\sigma}.$$

With these expansions,

$$\int d^2\theta d^2\bar{\theta} \left[-L^2 + \frac{1}{2}(\Phi + \bar{\Phi})^2 \right]$$

is the Lagrangian of a free, canonically-normalized, single-tensor $\mathcal{N} = 2$ multiplet. Its bosonic content is

$$\frac{1}{2}(\partial_{\mu}C)(\partial^{\mu}C) + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}, \quad H_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}.$$

For more details on the single tensor multiplet see section 9.1.

The identities

$$DD\theta\theta = \bar{D}\bar{D}\bar{\theta}\bar{\theta} = -4, \quad \int d^2\theta d^2\bar{\theta} = -\frac{1}{4}\int d^2\theta\bar{D}\bar{D} = -\frac{1}{4}\int d^2\bar{\theta}DD, \quad (8.15)$$

only valid under a space-time integral $\int d^4x$, are commonly used. Also,

$$\begin{aligned} D_{\alpha}D_{\beta} &= \frac{1}{2}\epsilon_{\alpha\beta}DD, & \bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}} &= -\frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{D}\bar{D}, \\ [D_{\alpha}, \bar{D}\bar{D}] &= -4i(\sigma^{\mu}\bar{D})_{\alpha}\partial_{\mu}, & [\bar{D}_{\dot{\alpha}}, DD] &= +4i(D\sigma^{\mu})_{\dot{\alpha}}\partial_{\mu}, \\ DDW_{\alpha} &= 4i(\sigma^{\mu}\partial_{\mu}\bar{W})_{\alpha}, & \bar{D}\bar{D}\bar{W}_{\dot{\alpha}} &= -4i(\partial_{\mu}W\sigma^{\mu})_{\dot{\alpha}}. \end{aligned}$$

CHAPTER 9

THE LINEAR $\mathcal{N} = 2$ MAXWELL-DILATON SYSTEM

Our first objective is to describe, in the context of linear $\mathcal{N} = 2$ supersymmetry, the coupling of the single-tensor multiplet to $\mathcal{N} = 2$ super-Maxwell theory. Since these two supermultiplets admit off-shell realizations, they can be described in superspace without reference to a particular Lagrangian. Gauge transformations of the Maxwell multiplet use a single-tensor multiplet, we then begin with the latter.

9.1 The Single-Tensor Multiplet

In global $\mathcal{N} = 1$ supersymmetry, a real antisymmetric tensor field $b_{\mu\nu}$ is described by a chiral, spinorial superfield χ_α with $8_B + 8_F$ fields [90]¹:

$$\chi_\alpha = -\frac{1}{4}\theta_\alpha(C + iC') + \frac{1}{4}(\theta\sigma^\mu\bar{\sigma}^\nu)_\alpha b_{\mu\nu} + \dots \quad (\bar{D}_{\dot{\alpha}}\chi_\alpha = 0), \quad (9.1)$$

C and C' being the real scalar partners of $b_{\mu\nu}$. The curl $h_{\mu\nu\rho} = 3\partial_{[\mu}b_{\nu\rho]}$ is described by the real superfield

$$L = D^\alpha\chi_\alpha - \bar{D}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \quad (9.2)$$

¹The notation $m_B + n_F$ stands for ‘ m bosonic and n fermionic fields’.

Chirality of χ_α implies linearity of L : $DDL = \overline{DDL} = 0$. The linear superfield L is invariant under the supersymmetric gauge transformation²

$$\chi_\alpha \longrightarrow \chi_\alpha + \frac{i}{4} \overline{DD} D_\alpha \Delta, \quad \bar{\chi}_{\dot{\alpha}} \longrightarrow \bar{\chi}_{\dot{\alpha}} + \frac{i}{4} DD \bar{D}_{\dot{\alpha}} \Delta, \quad (9.3)$$

of χ_α : this is the supersymmetric extension of the invariance of $h_{\mu\nu\rho}$ under $\delta b_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]}$. Considering bosons only, the gauge transformation (9.3) eliminates three of the six components of $b_{\mu\nu}$ and the scalar field C' . Accordingly, L only depends on the invariant curl $h_{\mu\nu\rho}$ and on the invariant real scalar C . The linear L describes then $4_B + 4_F$ fields. Using either χ_α or L , we will find two descriptions of the single-tensor multiplet of global $\mathcal{N} = 2$ supersymmetry [17, 18, 19].

In the gauge-invariant description using L , the $\mathcal{N} = 2$ multiplet is completed with a chiral superfield Φ ($8_B + 8_F$ fields in total). The second supersymmetry transformations (with parameter η_α) are

$$\begin{aligned} \delta^* L &= -\frac{i}{\sqrt{2}}(\eta D\Phi + \overline{\eta D\Phi}), \\ \delta^* \Phi &= i\sqrt{2}\overline{\eta DL}, \quad \delta^* \bar{\Phi} = i\sqrt{2}\eta DL, \end{aligned} \quad (9.4)$$

where D_α and $\bar{D}_{\dot{\alpha}}$ are the usual $\mathcal{N} = 1$ supersymmetry derivatives verifying $\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i(\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu$. It is easily verified that the $\mathcal{N} = 2$ supersymmetry algebra closes on L and Φ .

We may try to replace L by χ_α with second supersymmetry transformation $\delta^* \chi_\alpha = -\frac{i}{\sqrt{2}}\Phi \eta_\alpha$, as suggested when comparing eqs. (9.2) and (9.4). However, with superfields χ_α and Φ only, the $\mathcal{N} = 2$ algebra only closes up to a gauge transformation (9.3). This fact, and the unusual number $12_B + 12_F$ of fields, indicate that (χ_α, Φ) is a gauge-fixed version of the off-shell $\mathcal{N} = 2$ multiplet.

² Δ is an arbitrary real superfield.

We actually need another chiral $\mathcal{N} = 1$ superfield Y to close the supersymmetry algebra. The second supersymmetry variations are

$$\begin{aligned}\delta^* Y &= \sqrt{2} \eta \chi, \\ \delta^* \chi_\alpha &= -\frac{i}{\sqrt{2}} \Phi \eta_\alpha - \frac{\sqrt{2}}{4} \eta_\alpha \overline{DD} \bar{Y} - \sqrt{2} i (\sigma^\mu \bar{\eta})_\alpha \partial_\mu Y, \\ \delta^* \Phi &= 2 \sqrt{2} i \left[\frac{1}{4} \overline{DD} \eta \chi + i \partial_\mu \chi \sigma^\mu \bar{\eta} \right].\end{aligned}\tag{9.5}$$

One easily verifies that the Y -dependent terms in $\delta^* \chi_\alpha$ induce a gauge transformation (9.3). Hence, the linear L and its variation $\delta^* L$ do not feel Y . The superfields χ_α , Φ and Y have $16_B + 16_F$ field components. Gauge transformation (9.3) eliminates $4_B + 4_F$ fields. To further eliminate $4_B + 4_F$ fields, a new gauge variation

$$Y \quad \longrightarrow \quad Y - \frac{1}{2} \overline{DD} \Delta',\tag{9.6}$$

with Δ' real, is then postulated. We will see below that this variation is actually dictated by $\mathcal{N} = 2$ supersymmetry. There exists then a gauge in which $Y = 0$ but in this gauge the supersymmetry algebra closes on χ_α only up to a transformation (9.3). This is analogous to the Wess-Zumino gauge of $\mathcal{N} = 1$ supersymmetry, but in our case, this particular gauge respects $\mathcal{N} = 1$ supersymmetry and gauge symmetry (9.3).

Two remarks should be made at this point. Firstly, the superfield Y will play an important role in the construction of the Dirac-Born-Infeld interaction with nonlinear $\mathcal{N} = 2$ supersymmetry. As we will see later on³, it includes a four-index antisymmetric tensor field in its highest component. Secondly, a constant (θ -independent) background value $\langle \Phi \rangle$ breaks the second supersymmetry only, $\delta^* \chi_\alpha = -\frac{i}{\sqrt{2}} \langle \Phi \rangle \eta_\alpha + \dots$. It is a natural source of partial supersymmetry breaking in the single-tensor multiplet. Notice that the condition $\delta^* \langle \Phi \rangle = 0$ is equivalent

³See section 9.4.

to $\overline{D}_\alpha(D\chi - \overline{D}\overline{\chi}) = 0$.

An invariant kinetic action for the gauge invariant single-tensor multiplet involves an arbitrary function solution of the three-dimensional Laplace equation (for the variables L , Φ and $\overline{\Phi}$) [18]:

$$\mathcal{L}_{ST} = \int d^2\theta d^2\overline{\theta} \mathcal{H}(L, \Phi, \overline{\Phi}), \quad \frac{\partial^2 \mathcal{H}}{\partial L^2} + 2 \frac{\partial^2 \mathcal{H}}{\partial \Phi \partial \overline{\Phi}} = 0. \quad (9.7)$$

In the dual hypermultiplet formulation the Laplace equation is replaced by a Monge-Ampère equation. We will often insist on theories with axionic shift symmetry $\delta\Phi = ic$ (c real), dual to a double-tensor theory. In this case, \mathcal{H} is a function of L and $\Phi + \overline{\Phi}$ so that the general solution of Laplace equation is

$$\mathcal{L}_{ST} = \int d^2\theta d^2\overline{\theta} H(\mathcal{V}) + \text{h.c.}, \quad \mathcal{V} = L + \frac{i}{\sqrt{2}}(\Phi + \overline{\Phi}), \quad (9.8)$$

with an arbitrary analytic function $H(\mathcal{V})$.

The single-tensor multiplet as well as its Poincaré duals will play a central role in what follows. For this reason in Appendix C we give a detailed presentation of these multiplets and the duality transformations that switch from one to the other.

9.2 The Maxwell Multiplet, Fayet-Iliopoulos Terms

Take two real vector superfields V_1 and V_2 . Variations

$$\delta^* V_1 = -\frac{i}{\sqrt{2}}[\eta D + \overline{\eta \overline{D}}]V_2, \quad \delta^* V_2 = \sqrt{2}i[\eta D + \overline{\eta \overline{D}}]V_1 \quad (9.9)$$

provide a representation of $\mathcal{N} = 2$ supersymmetry with $16_B + 16_F$ fields. We may reduce the supermultiplet by imposing on V_1 and V_2 constraints consistent with

the second supersymmetry variations: for instance, the single-tensor multiplet is obtained by requiring $V_1 = L$ and $V_2 = \Phi + \bar{\Phi}$. Another option is to impose a gauge invariance: we may impose that the theory is invariant under⁴

$$\delta_{U(1)} V_1 = \Lambda_\ell, \quad \delta_{U(1)} V_2 = \Lambda_c + \bar{\Lambda}_c, \quad (9.10)$$

where Λ_ℓ and Λ_c form a single-tensor multiplet,

$$\Lambda_\ell = \bar{\Lambda}_\ell, \quad DD\Lambda_\ell = 0, \quad \bar{D}_{\dot{\alpha}}\Lambda_c = 0, \quad (9.11)$$

with transformations (9.4). Defining the gauge invariant superfields⁵

$$\begin{aligned} W_\alpha &= -\frac{1}{4} \overline{DD} D_\alpha V_2, & \bar{W}_{\dot{\alpha}} &= -\frac{1}{4} DD \bar{D}_{\dot{\alpha}} V_2, \\ X &= \frac{1}{2} \overline{DD} V_1, & \bar{X} &= \frac{1}{2} DD V_1, \end{aligned} \quad (9.12)$$

the variations (9.9) imply⁶

$$\begin{aligned} \delta^* X &= \sqrt{2} i \eta^\alpha W_\alpha, & \delta^* \bar{X} &= \sqrt{2} i \bar{\eta}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}, \\ \delta^* W_\alpha &= \sqrt{2} i \left[\frac{1}{4} \eta_\alpha \overline{DD} \bar{X} + i(\sigma^\mu \bar{\eta})_\alpha \partial_\mu X \right], & (9.13) \\ \delta^* \bar{W}_{\dot{\alpha}} &= \sqrt{2} i \left[\frac{1}{4} \bar{\eta}_{\dot{\alpha}} DD X - i(\eta \sigma^\mu)_{\dot{\alpha}} \partial_\mu \bar{X} \right]. \end{aligned}$$

While (V_1, V_2) describes the $\mathcal{N} = 2$ supersymmetric extension of the gauge potential $A_{\mu'}$, (W_α, X) is the multiplet of the gauge curvature $F_{\mu\nu} = 2 \partial_{[\mu} A_{\nu]}$ [91].

The $\mathcal{N} = 2$ gauge invariant Lagrangian depends on the derivatives of a holomorphic prepotential $\mathcal{F}(X)$:

$$\begin{aligned} \mathcal{L}_{Max.} &= \frac{1}{4} \int d^2\theta \left[\mathcal{F}''(X) WW - \frac{1}{2} \mathcal{F}'(X) \overline{DD} \bar{X} \right] + \text{c.c.} \\ &= \frac{1}{4} \int d^2\theta \mathcal{F}''(X) WW + \text{c.c.} + \frac{1}{2} \int d^2\theta d^2\bar{\theta} \left[\mathcal{F}'(X) \bar{X} + \bar{\mathcal{F}}'(\bar{X}) X \right] + \partial_\mu(\dots). \end{aligned} \quad (9.14)$$

⁴For clarity, we use the following convention for field variations: δ^* refers to the second ($\mathcal{N} = 2$) supersymmetry variations of the superfields and component fields; $\delta_{U(1)}$ indicates the Maxwell gauge variations; δ appears for gauge variations of superfields or field components related (by supersymmetry) to $\delta b_{\mu\nu} = 2 \partial_{[\mu} \Lambda_{\nu]}$.

⁵Remember that with this (standard) convention, $\bar{W}_{\dot{\alpha}}$ is *minus* the complex conjugate of W_α .

⁶There is a phase choice in the definition of X : a phase rotation of X can be absorbed in a phase choice of η .

In the construction of the Maxwell multiplet in terms of X and W_α , one expects a triplet of Fayet-Iliopoulos terms,

$$\mathcal{L}_{F.I.} = -\frac{1}{4}(\xi_1 + ia) \int d^2\theta X - \frac{1}{4}(\xi_1 - ia) \int d^2\bar{\theta} \bar{X} + \xi_2 \int d^2\theta d^2\bar{\theta} V_2, \quad (9.15)$$

with real parameters ξ_1 , ξ_2 and a . They may generate background values of the auxiliary components f_X and d_2 of X and V_2 which in general break both supersymmetries:

$$\delta^* X = \sqrt{2}i \eta \theta \langle d_2 \rangle + \dots, \quad \delta^* W_\alpha = \sqrt{2}i \eta_\alpha \langle \bar{f}_X \rangle + \dots \quad (9.16)$$

In terms of V_1 and V_2 however, the relation $X = \frac{1}{2} \overline{DD} V_1$ implies that $\text{Im } f_X$ is the curl of a three-index antisymmetric tensor (see section 9.4) and that its expectation value is turned into an integration constant of the tensor field equation [92, 93]. As a consequence,

$$-\frac{1}{4}(\xi_1 + ia) \int d^2\theta X - \frac{1}{4}(\xi_1 - ia) \int d^2\bar{\theta} \bar{X} = \xi_1 \int d^2\theta d^2\bar{\theta} V_1 + \text{derivative}$$

and the Fayet-Iliopoulos Lagrangian becomes

$$\mathcal{L}_{F.I.} = \int d^2\theta d^2\bar{\theta} [\xi_1 V_1 + \xi_2 V_2], \quad (9.17)$$

with two real parameters only.

The Maxwell multiplet with superfields (X, W_α) and the single-tensor multiplet (Y, χ_α, Φ) have a simple interpretation in terms of chiral superfields on $\mathcal{N} = 2$ superspace. We will use this formalism to construct their interacting Lagrangians in section 9.5.

9.3 The Chern-Simons Interaction

With a Maxwell field $F_{\mu\nu} = 2 \partial_{[\mu} A_{\nu]}$ (in W_α) and an antisymmetric tensor $b_{\mu\nu}$ (in χ_α or L), one may expect the presence of a $b \wedge F$ interaction

$$\epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} F_{\rho\sigma} = 2 \epsilon^{\mu\nu\rho\sigma} A_\mu \partial_\nu b_{\rho\sigma} + \text{derivative}.$$

This equality suggests that its $\mathcal{N} = 2$ supersymmetric extension also exists in two forms: either as an integral over chiral superspace of an expression depending on χ_α , W_α , X , Φ and Y , or as a real expression using L , $\Phi + \bar{\Phi}$, V_1 and V_2 .

In the ‘real’ formulation, the $\mathcal{N} = 2$ Chern-Simons term is⁷

$$\mathcal{L}_{CS} = -g \int d^2\theta d^2\bar{\theta} [LV_2 + (\Phi + \bar{\Phi})V_1], \quad (9.18)$$

with a real coupling constant g . It is invariant (up to a derivative) under the gauge transformations (9.10) of V_1 and V_2 with L and Φ left inert. Notice that the introduction of Fayet-Iliopoulos terms for V_1 and V_2 corresponds respectively to the shifts $\Phi + \bar{\Phi} \rightarrow \Phi + \bar{\Phi} - \xi_1/g$ and $L \rightarrow L - \xi_2/g$ in the Chern-Simons term.

The ‘chiral’ version uses the spinorial prepotential χ_α instead of L . Turning expression (9.18) into a chiral integral and using $X = \frac{1}{2} \bar{D}\bar{D} V_1$ leads to

$$\mathcal{L}_{CS,\chi} = g \int d^2\theta [\chi^\alpha W_\alpha + \frac{1}{2} \Phi X] + g \int d^2\bar{\theta} [-\bar{\chi}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + \frac{1}{2} \bar{\Phi} \bar{X}], \quad (9.19)$$

which differs from \mathcal{L}_{CS} by a derivative. The chiral version of the Chern-Simons term $\mathcal{L}_{CS,\chi}$ transforms as a derivative under the gauge variation (9.3) of χ_α . Its invariance under constant shift symmetry of $\text{Im } \Phi$ follows from $X = \frac{1}{2} \bar{D}\bar{D} V_1$. It does not depend on Y .

⁷The dimensions in mass unit of our superfields are as follows: $V_1, V_2 : 0$, $X, Y : 1$, $W_\alpha, \chi_\alpha : 3/2$, $\Phi, L : 2$. The coupling constant g is then dimensionless.

The consistent Lagrangian for the Maxwell – single-tensor system with Chern-Simons interaction is then

$$\mathcal{L}_{ST} + \mathcal{L}_{Max.} + \mathcal{L}_{CS} \quad \text{or} \quad \mathcal{L}_{ST} + \mathcal{L}_{Max.} + \mathcal{L}_{CS,\chi}. \quad (9.20)$$

The first two contributions include the kinetic terms and self-interactions of the multiplets while the third describes how they interact. Each of the three terms is separately $\mathcal{N} = 2$ supersymmetric.

Using a $\mathcal{N} = 1$ duality, a linear multiplet can be transformed into a chiral superfield with constant shift symmetry and the opposite transformation of course exists. Hence, performing both transformations, a single-tensor multiplet Lagrangian (L, Φ) with constant shift symmetry of the chiral Φ has a ‘double-dual’ second version. Suppose that we start with a Lagrangian where Maxwell gauge symmetry acts as a Stückelberg gauging of the single-tensor multiplet:⁸

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \mathcal{H}(L - gV_1, \Phi + \bar{\Phi} - gV_2). \quad (9.21)$$

The shift symmetry of $\text{Im } \Phi$ has been gauged and \mathcal{L} is invariant under gauge transformations (9.10) combined with

$$\delta_{U(1)}L = g\Lambda_\ell, \quad \delta_{U(1)}\Phi = g\Lambda_c, \quad (9.22)$$

and under $\mathcal{N} = 2$ supersymmetry if \mathcal{H} verifies Laplace equation (9.7). If we perform a double dualization $(L, \Phi + \bar{\Phi}) \rightarrow (\tilde{\Phi} + \bar{\tilde{\Phi}}, \tilde{L})$, we obtain the dual theory

$$\begin{aligned} \tilde{\mathcal{L}} &= \int d^2\theta d^2\bar{\theta} \tilde{\mathcal{H}}(\tilde{L}, \tilde{\Phi} + \bar{\tilde{\Phi}}) + g \int d^2\theta \left[\tilde{\chi}^\alpha W_\alpha + \frac{1}{2} \tilde{\Phi} X \right] + \text{c.c.} \\ &= \int d^2\theta d^2\bar{\theta} \left[\tilde{\mathcal{H}}(\tilde{L}, \tilde{\Phi} + \bar{\tilde{\Phi}}) - g\tilde{L}V_2 \right] + \frac{g}{2} \int d^2\theta \tilde{\Phi} X + \text{c.c.} \end{aligned} \quad (9.23)$$

⁸Strictly speaking, the coupling constant g in this theory has dimension $(\text{energy})^2$. There is an irrelevant energy scale involved in the duality transformation of a dimension two L into a dimension two chiral superfield. Hence, g in eq. (9.23) is again dimensionless.

where $\tilde{\mathcal{H}}$ is the result of the double Legendre transformation

$$\tilde{\mathcal{H}}(\tilde{y}, \tilde{x}) = \mathcal{H}(x, y) - \tilde{x}x - \tilde{y}y. \quad (9.24)$$

The dual theory is then the sum of the ungauged Lagrangian (9.7) and of the Chern-Simons coupling (9.18). This *single-tensor – single-tensor* duality is actually $\mathcal{N} = 2$ covariant: if \mathcal{H} solves Laplace equation, so does $\tilde{\mathcal{H}}$, and every intermediate step of the duality transformation can be formulated with explicit $\mathcal{N} = 2$ off-shell supersymmetry.

We have then found two classes of couplings of Maxwell theory to the single-tensor multiplet. Firstly, using the supersymmetric extension of the $b \wedge F$ coupling, as in eqs. (9.20). Secondly, using a Stückelberg gauging (9.21) of the single-tensor kinetic terms. The first version only is directly appropriate to perform an electric-magnetic duality transformation. However, since the second version can always be turned into the first one by a single-tensor – single-tensor duality, electric-magnetic duality of the second version requires this preliminary step: both theories have the same ‘magnetic’ dual.

9.4 The Significance of V_1 , X and Y

In the description of the $\mathcal{N} = 2$ Maxwell multiplet in terms of two $\mathcal{N} = 1$ real superfields, V_2 describes as usual the gauge potential A_μ , a gaugino λ_α and a real auxiliary field d_2 (in Wess-Zumino gauge). We wish to clarify the significance and the field content of the superfields V_1 and $X = \frac{1}{2}\overline{DD}V_1$, as well as the related content of the chiral superfield Y used in the description in terms of the spinorial potential χ_α of the single-tensor multiplet (Y, χ_α, Φ) .

The vector superfield V_1 has the $\mathcal{N} = 2$ Maxwell gauge variation $\delta_{U(1)} V_1 = \Lambda_\ell$, with a real linear parameter superfield Λ_ℓ . In analogy with the Wess-Zumino gauge commonly applied to V_2 , there exists then a gauge where

$$V_1(x, \theta, \bar{\theta}) = \theta\sigma^\mu\bar{\theta} v_{1\mu} - \frac{1}{2}\theta\theta\bar{x} - \frac{1}{2}\bar{\theta}\theta x - \frac{1}{\sqrt{2}}\theta\theta\bar{\theta}\psi_X - \frac{1}{\sqrt{2}}\bar{\theta}\theta\theta\psi_X + \frac{1}{2}\theta\theta\bar{\theta}\theta d_1. \quad (9.25)$$

This gauge leaves a residual invariance acting on the vector field $v_{1\mu}$ only:

$$\delta_{U(1)} v_1^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu\Lambda_{\rho\sigma}. \quad (9.26)$$

This indicates that the vector v_1^μ is actually a three-index antisymmetric tensor,

$$v_1^\mu = \frac{1}{6}\epsilon^{\mu\nu\rho\sigma}A_{\nu\rho\sigma}, \quad (9.27)$$

with Maxwell gauge invariance

$$\delta_{U(1)}A_{\mu\nu\rho} = 3\partial_{[\mu}\Lambda_{\nu\rho]}. \quad (9.28)$$

By construction, $X = \frac{1}{2}\overline{DD}V_1$ is gauge invariant. In chiral variables,

$$X(y, \theta) = x + \sqrt{2}\theta\psi_X - \theta\theta(d_1 + i\partial_\mu v_1^\mu). \quad (9.29)$$

Hence, while $\text{Re } f_X = d_1$,

$$\text{Im } f_X = \partial_\mu v_1^\mu = \frac{1}{24}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma}, \quad F_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]} \quad (9.30)$$

is the gauge-invariant curl of $A_{\mu\nu\rho}$. It follows that the field content (in Wess-Zumino gauge) of V_1 is the second gaugino ψ_X , the complex scalar of the Maxwell multiplet x , a real auxiliary field d_1 and the three-form field $A_{\mu\nu\rho}$, which corresponds to a single, non-propagating component field. The gauge-invariant chiral X includes the four-form curvature $F_{\mu\nu\rho\sigma}$.

At the Lagrangian level, the implication of relations (9.30) is as follows. Suppose that we compare two theories with the same Lagrangian $\mathcal{L}(u)$ but either

with $u = \phi$, a real scalar, or with $u = \partial_\mu V^\mu$, as in eq. (9.30). Since $\mathcal{L}(\phi)$ does not depend on $\partial_\mu \phi$, the scalar ϕ is auxiliary. The field equations for both theories are

$$\frac{\partial}{\partial \phi} \mathcal{L}(\phi) = 0, \quad \partial_\nu \frac{\partial}{\partial u} \mathcal{L}(u) \Big|_{u=\partial_\mu V^\mu} = 0$$

The second case allows a supplementary integration constant k related to the possible addition of a ‘topological’ term proportional to $\partial_\mu V^\mu$ to the Lagrangian [92, 93]:

$$\frac{\partial}{\partial u} \mathcal{L}(u) \Big|_{u=\partial_\mu V^\mu} = k.$$

In the first case, the same integration constant appears if one considers the following modified theory and field equation:

$$\mathcal{L}(\phi) - k \phi \quad \longrightarrow \quad \frac{\partial}{\partial \phi} \mathcal{L}(\phi) = k.$$

Returning to our super-Maxwell case, the relation is $\phi = \text{Im } f_X$ and the modification of the Lagrangian is then

$$-k \text{Im } f_X = -\frac{ik}{2} \int d^2\theta X + \text{c.c.} \quad (9.31)$$

This is the third Fayet-Iliopoulos term, which becomes a ‘hidden parameter’ [92] when using V_1 instead of X .

Consider finally the single-tensor multiplet (Y, χ_α, Φ) and the supersymmetric extension of the antisymmetric-tensor gauge symmetry, as given in Eqs. (9.3) and (9.6):

$$\delta Y = -\frac{1}{2} \overline{DD} \Delta', \quad \delta \chi_\alpha = \frac{i}{4} \overline{DD} D_\alpha \Delta, \quad \delta \Phi = 0.$$

Using expansion (9.29), there is a gauge in which Y reduces simply to

$$Y = -i\theta\theta \text{Im } f_Y \quad (9.32)$$

and one should identify $\text{Im } f_Y$ as a four-index antisymmetric tensor field,

$$\text{Im } f_Y = \frac{1}{24} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}, \quad (9.33)$$

with residual gauge invariance

$$\delta C_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} \Lambda_{\nu\rho\sigma]}. \quad (9.34)$$

The antisymmetric tensor $C_{\mu\nu\rho\sigma}$ describes a single field component which can be gauged away using $\Lambda_{\nu\rho\sigma}$. Applying this extended Wess-Zumino gauge to the $\mathcal{N} = 2$ multiplet (Y, χ_α, Φ) , the fields described by these $\mathcal{N} = 1$ superfields are as given in the following table.

| $\mathcal{N} = 1$ superfield | Field | Gauge invariance | Number of fields |
|------------------------------|---------------|--|-------------------|
| χ_α | $b_{\mu\nu}$ | $\delta b_{\mu\nu} = 2 \partial_{[\mu} \Lambda_{\nu]}$ | $6_B - 3_B = 3_B$ |
| | C | | 1_B |
| | χ_α | | 4_F |
| Φ | Φ | $\delta C_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} \Lambda_{\nu\rho\sigma]}$ | 2_B |
| | f_Φ | | 2_B (auxiliary) |
| | ψ_Φ | | 4_F |
| | Y | | $1_B - 1_B = 0_B$ |

The propagating bosonic fields $b_{\mu\nu}$, C and Φ (four bosonic degrees of freedom) have kinetic terms defined by Lagrangian \mathcal{L}_{ST} , eq. (9.7).

9.5 Chiral $\mathcal{N} = 2$ Superspace

Many results of the previous section can be reformulated in terms of chiral superfields on $\mathcal{N} = 2$ superspace. We now turn to a discussion of this framework, including an explicitly $\mathcal{N} = 2$ covariant formulation of electric-magnetic duality.

9.5.1 Chiral $\mathcal{N} = 2$ Superfields

A chiral superfield on $\mathcal{N} = 2$ superspace can be written as a function of $y^\mu, \theta, \tilde{\theta}$:

$$\bar{D}_{\dot{\alpha}} \mathcal{Z} = \bar{\bar{D}}_{\dot{\alpha}} \mathcal{Z} = 0 \quad \longrightarrow \quad \mathcal{Z} = \mathcal{Z}(y, \theta, \tilde{\theta}) \quad (9.35)$$

with $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta} - i\tilde{\theta}\sigma^\mu\bar{\tilde{\theta}}$ and $\bar{D}_{\dot{\alpha}} y^\mu = \bar{\bar{D}}_{\dot{\alpha}} y^\mu = 0$. Its second supersymmetry variations are

$$\delta^* \mathcal{Z} = i(\eta\tilde{Q} + \bar{\eta}\bar{\tilde{Q}})\mathcal{Z}, \quad (9.36)$$

with supercharge differential operators \tilde{Q}_α and $\bar{\tilde{Q}}_{\dot{\alpha}}$ which we do not need to explicitly write. It includes four $\mathcal{N} = 1$ chiral superfields and $16_B + 16_F$ component fields and we may use the expansions

$$\begin{aligned} \mathcal{Z}(y, \theta, \tilde{\theta}) &= Z(y, \theta) + \sqrt{2}\tilde{\theta}^\alpha\omega_\alpha(y, \theta) - \tilde{\theta}\tilde{\theta}F(y, \theta) \\ &= Z(y, \theta) + \sqrt{2}\tilde{\theta}^\alpha\omega_\alpha(y, \theta) - \tilde{\theta}\tilde{\theta}\left[\frac{i}{2}\Phi_Z(y, \theta) + \frac{1}{4}\overline{DD}Z(y, \theta)\right], \end{aligned} \quad (9.37)$$

where $\tilde{\theta}$ and $\bar{\tilde{D}}_{\dot{\alpha}}$ are the Grassmann coordinates and the super-derivatives associated with the second supersymmetry. The second supersymmetry variations (9.36) are easily obtained by analogy with the $\mathcal{N} = 1$ chiral supermultiplet:

$$\begin{aligned} \delta^* Z &= \sqrt{2}\eta\omega, \\ \delta^* \omega_\alpha &= -\sqrt{2}[F\eta_\alpha + i(\sigma^\mu\bar{\eta})_\alpha\partial_\mu Z] = -\frac{i}{\sqrt{2}}\Phi_Z\eta_\alpha - \frac{\sqrt{2}}{4}\eta_\alpha\overline{DD}Z - \sqrt{2}i(\sigma^\mu\bar{\eta})_\alpha\partial_\mu Z, \\ \delta^* F &= -\sqrt{2}i\partial_\mu\omega\sigma^\mu\bar{\eta}, \\ \delta^* \Phi_Z &= 2\sqrt{2}i\left[\frac{1}{4}\overline{DD}\eta\omega + i\partial_\mu\omega\sigma^\mu\bar{\eta}\right]. \end{aligned} \quad (9.38)$$

We immediately observe that the second expansion (9.37) leads to the second supersymmetry variations (9.5) of a single-tensor multiplet ($Y = Z, \chi = \omega, \Phi = \Phi_Z$). Similarly, the expansion

$$\mathcal{W}(y, \theta, \tilde{\theta}) = X(y, \theta) + \sqrt{2}i\tilde{\theta}W(y, \theta) - \tilde{\theta}\tilde{\theta}\frac{1}{4}\overline{DD}X(y, \theta), \quad (9.39)$$

which is obtained by imposing $\Phi_Z = 0$ in expansion (9.37), leads to the Maxwell supermultiplet (9.13) [94]. The Bianchi identity $D^\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}}$ is required by $\delta^* \Phi_Z = 0$. The $\mathcal{N} = 2$ Maxwell Lagrangian (9.14) rewrites then as an integral over chiral $\mathcal{N} = 2$ superspace,

$$\mathcal{L}_{Max.} = \frac{1}{2} \int d^2\theta \int d^2\tilde{\theta} \mathcal{F}(\mathcal{W}) + \text{c.c.}, \quad (9.40)$$

and the Fayet-Iliopoulos terms (9.17) can be written [95]

$$\mathcal{L}_{F.I.} = \int d^2\theta d^2\bar{\theta} [\xi_1 V_1 + \xi_2 V_2] = -\frac{1}{4} \int d^2\theta \int d^2\tilde{\theta} [\tilde{\theta}\tilde{\theta} \xi_1 - \sqrt{2}i \theta\tilde{\theta} \xi_2] \mathcal{W} + \text{c.c.} \quad (9.41)$$

Considering the unconstrained chiral superfield (9.37) with $16_B + 16_F$ fields, the reduction to the $8_B + 8_F$ components of the single-tensor multiplet is done by imposing gauge invariance (9.3) and (9.6). In terms of $\mathcal{N} = 2$ chiral superfields, this gauge symmetry is simply

$$\delta\mathcal{Y} = -\widehat{\mathcal{W}}, \quad (9.42)$$

where $\widehat{\mathcal{W}}$ is a Maxwell $\mathcal{N} = 2$ superfield parameter (9.39). In terms of $\mathcal{N} = 1$ superfields, this is

$$\delta Y = -\widehat{X}, \quad \delta\chi_\alpha = -i\widehat{W}_\alpha, \quad \delta\Phi = 0, \quad (9.43)$$

as in eqs. (9.3) and (9.6). Hence, a single-tensor superfield \mathcal{Y} is a chiral superfield \mathcal{Z} with the second expansion (9.37) and with gauge symmetry (9.42).

The chiral version of the Chern-Simons interaction (9.19) can be easily written on $\mathcal{N} = 2$ superspace. Using \mathcal{Y} with gauge invariance (9.42) and \mathcal{W} to respectively describe the single-tensor and the Maxwell multiplets. Then

$$\mathcal{L}_{CS,\mathcal{X}} = ig \int d^2\theta \int d^2\tilde{\theta} \mathcal{Y} \mathcal{W} + \text{c.c.} \quad (9.44)$$

It is gauge-invariant since for any pair of Maxwell superfields

$$i \int d^2\theta \int d^2\tilde{\theta} \mathcal{W} \widehat{\mathcal{W}} + \text{c.c.} = \text{derivative.} \quad (9.45)$$

Notice that the lowest component superfield Y of \mathcal{Y} does not contribute to the field equations derived from $\mathcal{L}_{CS,\mathcal{X}}$: it only contributes to this Lagrangian with a derivative.

Finally, a second method to obtain an interactive Lagrangian for the Maxwell–single-tensor system is then obvious. Firstly, a generic $\mathcal{N} = 2$ chiral superfield \mathcal{Z} can always be written as

$$\mathcal{Z} = \mathcal{W} + 2g\mathcal{Y}. \quad (9.46)$$

It is invariant under the single-tensor gauge variation (9.42) if one also postulates that

$$\delta\mathcal{W} = 2g\widehat{\mathcal{W}}, \quad (9.47)$$

which amounts to a $\mathcal{N} = 2$ Stückelberg gauging of the symmetry of the anti-symmetric tensor. With this decomposition, $F_{\mu\nu}$ and $b_{\mu\nu}$ only appear in the $\theta_\alpha\tilde{\theta}_\beta$ component of \mathcal{Z} through the gauge-invariant combination $F_{\mu\nu} - gb_{\mu\nu}$. The chiral integral

$$\mathcal{L} = \frac{1}{2} \int d^2\theta \int d^2\tilde{\theta} \mathcal{F}(\mathcal{W} + 2g\mathcal{Y}) + \text{c.c.} + \mathcal{L}_{ST} \quad (9.48)$$

provides a $\mathcal{N} = 2$ invariant Lagrangian describing $16_B + 16_F$ (off-shell) interacting fields. There exists a gauge in which $\mathcal{W} = 0$, in which case theory (9.48) describes a massive chiral $\mathcal{N} = 2$ superfield.

Theory (9.48) is actually related to the Chern-Simons Lagrangian (9.20) by electric-magnetic duality, as will be shown below.

9.5.2 Electric-Magnetic Duality

The description in chiral $\mathcal{N} = 2$ superspace of the Maxwell multiplet allows to derive a $\mathcal{N} = 2$ covariant version of electric-magnetic duality. The Maxwell Lagrangian (9.14) supplemented by the Chern-Simons coupling (9.19) can be written

$$\mathcal{L}_{electric} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{2} \mathcal{F}(\mathcal{W}) + ig \mathcal{Y} \mathcal{W} \right] + \text{c.c.}, \quad (9.49)$$

adding eqs. (9.40) and (9.44). Replace then \mathcal{W} by an unconstrained chiral superfield $\hat{\mathcal{Z}}$ (with $\mathcal{N} = 1$ superfields \hat{Z} , $\hat{\omega}_\alpha$ and $\hat{\Phi}$) and introduce a new Maxwell multiplet $\tilde{\mathcal{W}}$ (with $\mathcal{N} = 1$ superfields \tilde{X} and \tilde{W}_α). Using

$$\tilde{X} = \frac{1}{2} \overline{DD} \tilde{V}_1, \quad \tilde{W}_\alpha = -\frac{1}{4} \overline{DD} D_\alpha \tilde{V}_2,$$

we have

$$\begin{aligned} i \int d^2\theta \int d^2\tilde{\theta} \tilde{\mathcal{W}} \hat{\mathcal{Z}} + \text{c.c.} &= \int d^2\theta \left[\frac{1}{2} \hat{\Phi} \tilde{X} + \hat{\omega} \tilde{W} \right] + \text{c.c.} \\ &= - \int d^2\theta d^2\tilde{\theta} \left[\tilde{V}_1 (\hat{\Phi} + \overline{\hat{\Phi}}) + \tilde{V}_2 (D^\alpha \hat{\omega}_\alpha - \overline{D}_{\dot{\alpha}} \overline{\hat{\omega}}^{\dot{\alpha}}) \right]. \end{aligned} \quad (9.50)$$

Consider now the Lagrangian

$$\mathcal{L} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{2} \mathcal{F}(\hat{\mathcal{Z}}) + \frac{i}{2} \hat{\mathcal{Z}} (\tilde{\mathcal{W}} + 2g\mathcal{Y}) \right] + \text{c.c.} \quad (9.51)$$

Invariance under the gauge transformation of the single-tensor superfield, eq. (9.42), requires a compensating gauge variation of $\tilde{\mathcal{W}}$, as in eq. (9.47). Eliminating $\tilde{\mathcal{W}}$ leads back to theory (9.49) with $\hat{\mathcal{Z}} = \mathcal{W}$. This can be seen in two ways. Firstly, the condition

$$i \int d^2\theta \int d^2\tilde{\theta} \tilde{\mathcal{W}} \hat{\mathcal{Z}} + \text{c.c.} = \text{derivative}$$

leads to $\hat{\mathcal{Z}} = \mathcal{W}$, a $\mathcal{N} = 2$ Maxwell superfield, up to a background value. Secondly, using eqs. (9.50), we see that \tilde{V}_2 imposes the Bianchi identity on $\hat{\omega}$ while

\widetilde{V}_1 cancels $\widehat{\Phi}$ up to an imaginary constant.⁹ We will come back to the (important) role of a nonzero background value in the next section. For the moment we disregard it.

On the other hand, we may prefer to eliminate $\widehat{\mathcal{Z}}$, using its field equation

$$\mathcal{F}'(\widehat{\mathcal{Z}}) = -i\mathcal{V}, \quad \mathcal{V} \equiv \widetilde{\mathcal{W}} + 2g\mathcal{Y}, \quad (9.52)$$

which corresponds to a Legendre transformation exchanging variables $\widehat{\mathcal{Z}}$ and \mathcal{V} . Defining

$$\widetilde{\mathcal{F}}(\mathcal{V}) = \mathcal{F}(\widehat{\mathcal{Z}}) + i\mathcal{V}\widehat{\mathcal{Z}}, \quad (9.53)$$

we have

$$\widetilde{\mathcal{F}}'(\mathcal{V}) = i\widehat{\mathcal{Z}}, \quad \mathcal{F}'(\widehat{\mathcal{Z}}) = -i\mathcal{V}, \quad \widetilde{\mathcal{F}}''(\mathcal{V})\mathcal{F}''(\widehat{\mathcal{Z}}) = 1. \quad (9.54)$$

The dual (Legendre-transformed) theory is then

$$\widetilde{\mathcal{L}}_{\text{magnetic}} = \frac{1}{2} \int d^2\theta \int d^2\tilde{\theta} \widetilde{\mathcal{F}}(\widetilde{\mathcal{W}} + 2g\mathcal{Y}) + \text{c.c.} \quad (9.55)$$

or, expressed in $\mathcal{N} = 1$ superspace,¹⁰

$$\begin{aligned} \widetilde{\mathcal{L}}_{\text{magnetic}} = & \frac{1}{4} \int d^2\theta \left[\widetilde{\mathcal{F}}''(\widetilde{X} + 2gY) (\widetilde{W} - 2ig\chi)^\alpha (\widetilde{W} - 2ig\chi)_\alpha \right. \\ & \left. - \frac{1}{2} \widetilde{\mathcal{F}}'(\widetilde{X} + 2gY) \overline{D\widetilde{D}}(\widetilde{X} + 2g\overline{Y}) - 2ig \widetilde{\mathcal{F}}'(\widetilde{X} + 2gY) \Phi \right] + \text{c.c.} \end{aligned} \quad (9.56)$$

We then conclude that the presence of the Chern-Simons term in the electric theory induces a Stückelberg gauging in the dual magnetic theory.

As explained in ref. [95], the situation changes when Fayet-Iliopoulos terms (9.41) are present in the electric theory. In the magnetic theory coupled to the single-tensor multiplet, with Lagrangian (9.56), the gauging $\delta\widetilde{\mathcal{W}} = 2g\widehat{\mathcal{W}}$ forbids

⁹An unconstrained \widetilde{X} would forbid this constant.

¹⁰The free, canonically-normalized theory corresponds to $\mathcal{F}(\mathcal{W}) = \frac{1}{2}\mathcal{W}^2$ and $\widetilde{\mathcal{F}}(\mathcal{V}) = \frac{1}{2}\mathcal{V}^2$.

Fayet-Iliopoulos terms for the magnetic Maxwell superfields \widetilde{V}_1 and \widetilde{V}_2 . Spontaneous supersymmetry breaking by Fayet-Iliopoulos terms in the electric theory finds then a different origin in the magnetic dual.

For our needs, we only consider the Fayet-Iliopoulos term induced by V_1 , *i.e.* we add

$$\mathcal{L}_{FI} = \xi_1 \int d^4\theta V_1 = -\frac{1}{4}\xi_1 \int d^2\theta \int d^2\tilde{\theta} \tilde{\theta}\tilde{\theta} \mathcal{W} + \text{c.c.} \quad (9.57)$$

to $\mathcal{L}_{electric}$, eq. (9.49). In turn, this amounts to add

$$-\frac{1}{4}\xi_1 \int d^2\theta \int d^2\tilde{\theta} \tilde{\theta}\tilde{\theta} \hat{\mathcal{Z}} + \text{c.c.}$$

to theory (9.51). But, in contrast to expression (9.57), this modification is not invariant under the second supersymmetry: according to the first eq. (9.38), its δ^* variation

$$-\frac{\sqrt{2}}{4}\xi_1 \int d^2\theta \eta\omega + \text{c.c.}$$

is not a derivative.¹¹ To restore $\mathcal{N} = 2$ supersymmetry, we must deform the δ^* variation of $\widetilde{W}_\alpha - 2ig\chi_\alpha$ into

$$\delta_{deformed}^*(\widetilde{W}_\alpha - 2ig\chi_\alpha) = \frac{1}{\sqrt{2}}\xi_1\eta_\alpha + \delta^*(\widetilde{W}_\alpha - 2ig\chi_\alpha), \quad (9.58)$$

the second term being the usual, undeformed, variations (9.13) and (9.5). Hence, the magnetic theory has a goldstino fermion and linear $\mathcal{N} = 2$ supersymmetry partially breaks to $\mathcal{N} = 1$, as a consequence of the electric Fayet-Iliopoulos term.

Concretely, the magnetic theory is now

$$\begin{aligned} \widetilde{\mathcal{L}}_{magnetic} &= \frac{1}{2} \int d^2\theta \int d^2\tilde{\theta} \widetilde{\mathcal{F}}(\widetilde{\mathcal{W}} + 2g\mathcal{Y} + \frac{i}{2}\xi_1\tilde{\theta}\tilde{\theta}) + \text{c.c.} \\ &= \frac{1}{2} \int d^2\theta \int d^2\tilde{\theta} \left[\widetilde{\mathcal{F}}(\widetilde{\mathcal{W}} + 2g\mathcal{Y}) + \frac{i}{2}\xi_1\tilde{\theta}\tilde{\theta} \widetilde{\mathcal{F}}'(\widetilde{\mathcal{W}} + 2g\mathcal{Y}) \right] + \text{c.c.} \\ &= \left[\frac{1}{2} \int d^2\theta \int d^2\tilde{\theta} \widetilde{\mathcal{F}}(\widetilde{\mathcal{W}} + 2g\mathcal{Y}) + \frac{i}{4}\xi_1 \int d^2\theta \widetilde{\mathcal{F}}'(\widetilde{X} + 2gY) \right] + \text{c.c.} \end{aligned} \quad (9.59)$$

¹¹It would be a derivative if ω_α would be replaced by the Maxwell superfield W_α , as in eq. (9.57).

One easily checks that $\mathcal{N} = 2$ supersymmetry holds, using the deformed variations (9.58).

CHAPTER 10

NONLINEAR $\mathcal{N} = 2$ SUPERSYMMETRY AND THE DBI ACTION

In the previous sections, we have developed various aspects of the coupling of a Maxwell multiplet to a single-tensor multiplet in linear $\mathcal{N} = 2$ supersymmetry. With these tools, we can now address our main subject: show how a Dirac-Born-Infeld Lagrangian (DBI) coupled to the single-tensor multiplet arises from nonlinearization of the second supersymmetry.

It has been observed that the DBI Lagrangian with nonlinear second supersymmetry can be derived by solving a constraint invariant under $\mathcal{N} = 2$ supersymmetry imposed on the super-Maxwell theory [20, 21]. We start with a summary of this result, following mostly Roček and Tseytlin [21], and we then generalize the method to incorporate the fields of the single-tensor multiplet.

10.1 The $\mathcal{N} = 2$ Super-Maxwell DBI Theory

The constraint imposed on the $\mathcal{N} = 2$ Maxwell chiral superfield \mathcal{W} is [21]¹

$$\mathcal{W}^2 - \frac{1}{\kappa} \tilde{\theta}\tilde{\theta} \mathcal{W} = \left(\mathcal{W} - \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta} \right)^2 = 0. \quad (10.1)$$

¹See also Ref. [96] and very recently Ref. [54] in the context of $\mathcal{N} = 1$ supersymmetry.

It imposes a relation between the super-Maxwell Lagrangian superfield \mathcal{W}^2 and the Fayet-Iliopoulos ‘superfield’ $\tilde{\theta}\tilde{\theta}\mathcal{W}$, eq. (9.57). The real scale parameter κ has dimension $(\text{energy})^{-2}$. In terms of $\mathcal{N} = 1$ superfields, the constraint is equivalent to

$$X^2 = 0, \quad XW_\alpha = 0, \quad WW - \frac{1}{2}X\overline{DDX} = \frac{1}{\kappa}X. \quad (10.2)$$

The third equality leads to

$$X = \frac{2WW}{\frac{2}{\kappa} + \overline{DDX}} \quad (10.3)$$

which, since $W_\alpha W_\beta W_\gamma = 0$, implies the first two conditions. Solving the third constraint amounts to express X as a function of WW [20]². The DBI theory is then obtained using as Lagrangian the Fayet-Iliopoulos term (9.57) properly normalized:

$$\mathcal{L}_{DBI} = \frac{1}{4\kappa} \int d^2\theta X + \text{c.c.} = \frac{1}{8\kappa^2} \left[1 - \sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})} \right] + \dots \quad (10.4)$$

The constraints (10.1) and (10.2) are not invariant under the second linear supersymmetry, with variations δ^* . However, one easily verifies that the three constraints (10.2) are invariant under the deformed, nonlinear variation

$$\delta_{deformed}^* W_\alpha = \sqrt{2}i \left[\frac{1}{2\kappa}\eta_\alpha + \frac{1}{4}\eta_\alpha \overline{DDX} + i(\sigma^\mu \bar{\eta})_\alpha \partial_\mu X \right], \quad (10.5)$$

with $\delta^* X$ unchanged. The deformation preserves the $\mathcal{N} = 2$ supersymmetry algebra. It indicates that the gaugino spinor in $W_\alpha = -i\lambda_\alpha + \dots$ transforms inhomogeneously, $\delta^* \lambda_\alpha = -\frac{1}{\sqrt{2}\kappa}\eta_\alpha + \dots$, like a goldstino for the breaking of the second supersymmetry. In other words, at the level of the $\mathcal{N} = 2$ chiral superfield \mathcal{W} ,

$$\delta_{deformed}^* \mathcal{W} = -\frac{1}{\kappa}\tilde{\theta}\eta + i\left(\eta\tilde{Q} + \bar{\eta}\bar{\tilde{Q}}\right)\mathcal{W} = i\left(\eta\tilde{Q} + \bar{\eta}\bar{\tilde{Q}}\right)\left(\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}\right).$$

The deformed second supersymmetry variations $\delta_{deformed}^*$ act on \mathcal{W} as the usual variations δ^* act on the shifted superfield $\mathcal{W} - \frac{1}{2\kappa}\tilde{\theta}\tilde{\theta}$. In fact, this superfield transforms like a chiral $\mathcal{N} = 2$ superfield (9.37) with $Z = X$, $\omega_\alpha = iW_\alpha$ verifying the

²See Appendix B.

Bianchi identity and with $\Phi_Z = -i/\kappa$. The latter background value of Φ_Z may be viewed as the source of the partial breaking of linear supersymmetry.

Hence, the scale parameter κ introduced in the nonlinear constraint (10.1) appears as the scale parameter of the DBI Lagrangian and also as the order parameter of partial supersymmetry breaking. The Fayet-Iliopoulos term (10.4) has in principle an arbitrary coefficient $-\xi_1/4$, as in eq. (9.17). We have chosen $\xi_1 = -\kappa^{-1}$ to canonically normalize gauge kinetic terms.

The DBI Lagrangian is invariant under electric-magnetic duality.³ In our $\mathcal{N} = 2$ case, the invariance is easily established in the language of $\mathcal{N} = 2$ superspace. We first include the constraint as a field equation of the Lagrangian:

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\kappa} \tilde{\theta}\tilde{\theta} \mathcal{W} + \frac{1}{4} \Lambda \left(\mathcal{W} - \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta} \right)^2 \right] + \text{c.c.} \quad (10.6)$$

The field equation of the $\mathcal{N} = 2$ superfield Λ enforces (10.1). We then introduce two unconstrained $\mathcal{N} = 2$ chiral superfields U and Υ and the modified Lagrangian

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\kappa} \tilde{\theta}\tilde{\theta} \mathcal{W} + \frac{1}{4} \Lambda U^2 - \frac{1}{2} \Upsilon \left(U - \mathcal{W} + \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta} \right) \right] + \text{c.c.}$$

Since the Lagrange multiplier Υ imposes $U = \mathcal{W} - \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta}$, the equivalence with (10.6) is manifest. But we may also eliminate \mathcal{W} which only appears linearly in the last version of the theory. The result is

$$\Upsilon = -i\widetilde{\mathcal{W}} - \frac{1}{2} \left(\frac{1}{\kappa} - i\zeta \right) \tilde{\theta}\tilde{\theta}$$

where $\widetilde{\mathcal{W}}$ is a Maxwell $\mathcal{N} = 2$ superfield dual to \mathcal{W} and ζ an arbitrary real constant. As in subsection 9.5.2, $\mathcal{N} = 2$ supersymmetry of the theory with a Fayet-Iliopoulos term requires a nonlinear deformation of the δ^* variation of

³For instance, in the context of D3-branes of IIB superstrings, see Ref. [97]. Our procedure is inspired by Ref. [21].

$\widetilde{\mathcal{W}}$: $\widetilde{\mathcal{W}} - \frac{i}{2} \left(\frac{1}{\kappa} - i\zeta \right) \tilde{\theta}\tilde{\theta}$ should be a ‘good’ $\mathcal{N} = 2$ chiral superfield. Replacing Υ in the Lagrangian and taking $\zeta = 0$ leads to

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4} \Lambda U^2 + \frac{i}{2} U \left[\widetilde{\mathcal{W}} - \frac{i}{2\kappa} \tilde{\theta}\tilde{\theta} \right] + \frac{i}{4\kappa} \widetilde{\mathcal{W}} \tilde{\theta}\tilde{\theta} \right] + \text{c.c.}$$

Finally, eliminating U gives the magnetic dual

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\Lambda} \left(\widetilde{\mathcal{W}} - \frac{i}{2\kappa} \tilde{\theta}\tilde{\theta} \right)^2 + \frac{i}{4\kappa} \widetilde{\mathcal{W}} \tilde{\theta}\tilde{\theta} \right] + \text{c.c.} \quad (10.7)$$

One easily verifies that the resulting theory has the same expression as the initial ‘electric’ theory (10.4). The Lagrange multiplier Λ^{-1} imposes constraint (10.1) to $-i\widetilde{\mathcal{W}}$, which reduces to eq. (10.3) applied to $-i\widetilde{\mathcal{X}}$. The Lagrangian is then given by the Fayet-Iliopoulos term for this superfield.

10.2 Coupling the DBI Theory to a Single-Tensor Multiplet: a Super-Higgs Mechanism without Gravity

The $\mathcal{N} = 2$ super-Maxwell DBI theory is given by a Fayet-Iliopoulos term for a Maxwell superfield submitted to the quadratic constraint (10.1), which also provides the source of partial supersymmetry breaking. The second supersymmetry is deformed by the constraint: it is $\mathcal{W} - \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta}$ which transforms as a regular $\mathcal{N} = 2$ chiral superfield. Instead of expression (9.44), we are thus led to consider the following Chern-Simons interaction with the single-tensor multiplet:

$$\begin{aligned} \mathcal{L}_{CS,def.} &= ig \int d^2\theta \int d^2\tilde{\theta} \mathcal{Y} \left(\mathcal{W} - \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta} \right) + \text{c.c.} \\ &= g \int d^2\theta \left[\frac{1}{2} \Phi X + \chi^\alpha W_\alpha - \frac{i}{2\kappa} Y \right] + \text{c.c.} + \text{derivative.} \end{aligned} \quad (10.8)$$

The new term induced by the deformation of $\delta^* W_\alpha$ is proportional to the four-form field described by the chiral superfield Y , as explained in section 9.4 [see

eq. (9.33)]. This modified Chern-Simons interaction, invariant under the deformed second supersymmetry variations, may be simply added to the Maxwell DBI theory (10.6). We then consider the Lagrangian

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[ig\mathcal{Y} \left(\mathcal{W} - \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta} \right) - \frac{1}{4} \xi_1 \tilde{\theta}\tilde{\theta} \mathcal{W} + \frac{1}{2} \Lambda \left(\mathcal{W} - \frac{1}{2\kappa} \tilde{\theta}\tilde{\theta} \right)^2 \right] + \text{c.c.}, \quad (10.9)$$

for the constrained Maxwell and single-tensor multiplets, keeping the Fayet-Iliopoulos coefficient ξ_1 arbitrary. For a coherent theory with a propagating single-tensor multiplet, a kinetic Lagrangian \mathcal{L}_{ST} [eq. (9.7)] should also be added. Since

$$\int d^2\theta \int d^2\tilde{\theta} \left[ig\mathcal{Y} \mathcal{W} - \frac{1}{4} \xi_1 \tilde{\theta}\tilde{\theta} \mathcal{W} \right] + \text{c.c.} = \int d^2\theta \left[g\chi W + \frac{g}{2} \Phi X - \frac{1}{4} \xi_1 X \right] + \text{c.c.} + \text{deriv.},$$

we see that the Fayet-Iliopoulos term is equivalent to a constant real shift of Φ which, according to variations (9.5), partially breaks supersymmetry. We will choose to expand Φ around $\langle \Phi \rangle = 0$ and keep $\xi_1 \neq 0$.

Again, the constraint (10.1) imposed by the Lagrange multiplier Λ can be solved to express X as a function of WW : $X = X(WW)$. The result is [20]

$$X(WW) = \kappa \bar{W}W - \kappa^3 \bar{D}D \left[\frac{WW\bar{W}W}{1 + \kappa^2 A + \sqrt{1 + 2\kappa^2 A + \kappa^4 B^2}} \right], \quad (10.10)$$

where A and B are defined in Appendix B. The DBI Lagrangian coupled to the single-tensor multiplet reads then

$$\mathcal{L}_{DBI} = \int d^2\theta \left[\frac{1}{4} (2g\Phi - \xi_1) X(WW) + g\chi^\alpha W_\alpha - \frac{ig}{2\kappa} Y \right] + \text{c.c.} + \mathcal{L}_{ST}. \quad (10.11)$$

The bosonic Lagrangian depends on a single auxiliary field⁴, d_2 in W_α or V_2 :

$$\begin{aligned} \mathcal{L}_{DBI, bos.} &= \frac{1}{8\kappa} (2g \text{Re } \Phi - \xi_1) \left(1 - \sqrt{-8\kappa^2 d_2^2 - \det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})} \right) - \frac{g}{2} C d_2 \\ &+ g \epsilon^{\mu\nu\rho\sigma} \left(\frac{\kappa}{4} \text{Im } \Phi F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4} b_{\mu\nu} F_{\rho\sigma} + \frac{1}{24\kappa} C_{\mu\nu\rho\sigma} \right) + \mathcal{L}_{ST, bos.} \end{aligned} \quad (10.12)$$

⁴Since $X(WW)|_{\theta=0}$ is a function of fermion bilinears, the auxiliary f_Φ does not contribute to the bosonic Lagrangian and χ_α does not include any auxiliary field.

The real scalar field C is the lowest component of the linear superfield L . Contrary to $\langle \Phi \rangle$, its background value is allowed by $\mathcal{N} = 2$ supersymmetry. However, a non-zero $\langle C \rangle$ would induce a non-zero $\langle d_2 \rangle$ which would spontaneously break the residual $\mathcal{N} = 1$ linear supersymmetry. This is visible in the bosonic action which, after elimination of

$$d_{2,bos.} = \frac{gC}{2\kappa} \sqrt{\frac{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}{(2g \operatorname{Re} \Phi - \xi_1)^2 + 2g^2 C^2}}, \quad (10.13)$$

becomes

$$\begin{aligned} \mathcal{L}_{DBI,bos.} = & \frac{1}{8\kappa} (2g \operatorname{Re} \Phi - \xi_1) \left[1 - \sqrt{1 + \frac{2g^2 C^2}{(2g \operatorname{Re} \Phi - \xi_1)^2}} \sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})} \right] \\ & + g\epsilon^{\mu\nu\rho\sigma} \left(\frac{\kappa}{4} \operatorname{Im} \Phi F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4} b_{\mu\nu} F_{\rho\sigma} + \frac{1}{24\kappa} C_{\mu\nu\rho\sigma} \right) + \mathcal{L}_{ST,bos.} \end{aligned} \quad (10.14)$$

First of all, as expected, the theory includes a DBI Lagrangian for the Maxwell field strength $F_{\mu\nu}$, with scale $\sim \kappa$. With the Chern-Simons coupling to the single-tensor multiplet, the DBI term acquires a field-dependent coefficient,

$$-\frac{1}{8\kappa} \sqrt{(2g \operatorname{Re} \Phi - \xi_1)^2 + 2g^2 C^2} \sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}. \quad (10.15)$$

It also includes a $F \wedge F$ term which respects the axionic shift symmetry of $\operatorname{Im} \Phi$, a $b \wedge F$ coupling induced by (linear) $\mathcal{N} = 2$ supersymmetry and a ‘topological’ C_4 term induced by the nonlinear deformation. These terms are strongly reminiscent of those found when coupling a D-brane Lagrangian to IIB supergravity. The contribution of the four-form can be eliminated by a gauge choice of the single-tensor symmetry (9.34). We have however insisted on keeping off-shell (deformed) $\mathcal{N} = 2$ supersymmetry, hence the presence of this term.

The theory also includes a semi-positive scalar potential⁵

$$V(C, \operatorname{Re} \Phi) = \frac{2g \operatorname{Re} \Phi - \xi_1}{8\kappa} \left[\sqrt{1 + \frac{2g^2 C^2}{(2g \operatorname{Re} \Phi - \xi_1)^2}} - 1 \right] \quad (10.16)$$

⁵We only consider $2g \operatorname{Re} \Phi - \xi_1 > 0$, in order to have well-defined positive gauge kinetic terms.

which vanishes only if C is zero.⁶ The scalar potential determines then $\langle C \rangle = 0$ but leaves $\text{Re } \Phi$ arbitrary. Since

$$\langle d_2 \rangle = \frac{g\langle C \rangle}{2\kappa} \left\langle (2g \text{Re } \Phi - \xi_1)^2 + 2g^2 C^2 \right\rangle^{-1/2},$$

the vacuum line $\langle C \rangle = 0$ is compatible with linear $\mathcal{N} = 1$ and deformed second supersymmetry. While Φ is clearly massless, C has a mass term

$$-\frac{1}{2}M_C^2 C^2 = -\frac{g^2}{4\kappa(2 \text{Re } \Phi - \xi_1)} C^2.$$

The same mass is acquired by the $U(1)$ gauge field coupled to the antisymmetric tensor $b_{\mu\nu}$, and by the goldstino (the $U(1)$ gaugino in W_α) that forms a Dirac spinor with the fermion of the linear multiplet χ_α . In other words, the Chern-Simons coupling χW pairs the Maxwell goldstino with the linear multiplet to form a massive vector, while the chiral multiplet Φ remains massless with no superpotential.

At $\langle C \rangle = \langle \text{Re } \Phi \rangle = 0$, gauge kinetic terms are canonically normalized if $\xi_1 = -\kappa^{-1}$. The Maxwell DBI theory (10.4) is of course recovered when the Chern-Simons interaction decouples with $g = 0$. Notice finally that the kinetic terms \mathcal{L}_{ST} of the single-tensor multiplet are given by eq. (9.7), as with linear $\mathcal{N} = 2$ supersymmetry. Since the nonlinear deformation of the second supersymmetry does not affect $\delta^* L$ or $\delta^* \Phi$ even if $\langle \text{Re } \Phi \rangle \neq 0$, the function \mathcal{H} remains completely arbitrary.

The phenomenon described above provides a first instance of a super-Higgs mechanism without gravity: the nonlinear goldstino multiplet is ‘absorbed’ by the linear multiplet to form a massive vector $\mathcal{N} = 1$ superfield. One may wonder how this can happen without gravity; normally one expects that the gold-

⁶With respect to $\text{Re } \Phi$, the potential is stationary, $\frac{\partial V}{\partial \text{Re } \Phi} = 0$, only if $C = 0$. All local minima are then characterized by $C = 0$ and $\text{Re } \Phi$ arbitrary and are then (supersymmetric) global minima.

stino can be absorbed only by the gravitino in local supersymmetry. The reason of this novel mechanism is that the goldstino sits in the same multiplet of the linear supersymmetry as a gauge field which has a Chern-Simons interaction with the tensor multiplet. This will become clearer in Section 10.6, where we will show by a change of variables that this coupling is equivalent to an ordinary gauge interaction with a charged hypermultiplet, providing non derivative gauge couplings to the goldstino. Actually, this particular super-Higgs mechanism is an explicit realization of a phenomenon known in string theory where the $U(1)$ field of the D-brane world-volume becomes in general massive due to a Chern-Simons interaction with the R–R antisymmetric tensor of a bulk hypermultiplet.⁷

We have chosen a description in terms of the single-tensor multiplet because it admits an off-shell formulation well adapted to our problem. Our DBI Lagrangian (10.9), supplemented with kinetic terms \mathcal{L}_{ST} , admits however several duality transformations. Firstly, since it only depends on \mathcal{W} , we may perform an electric-magnetic duality transformation, as described in section 10.4. Then, for any choice of \mathcal{L}_{ST} , we can transform the linear $\mathcal{N} = 1$ superfield L into a chiral Φ' . The resulting theory is a hypermultiplet formulation with superfields (Φ, Φ') and $\mathcal{N} = 2$ supersymmetry realized only on-shell. As already explained in section 9.3, the $b \wedge F$ interaction is replaced by a Stückelberg gauging of the axionic shift symmetry of the new chiral Φ' : the Kähler potential of the hypermultiplet formulation is a function of $\Phi' + \overline{\Phi}' - gV_2$. Explicit formulae are given in the next section and in section 10.6 we will use this mechanism in the case of nonlinear $\mathcal{N} = 2$ QED. Finally, if kinetic terms \mathcal{L}_{ST} also respect the shift symmetry of $\text{Im } \Phi$, the chiral Φ can be turned into a second linear superfield L' , leading

⁷This can be avoided in the orientifold case: the $\mathcal{N} = 2$ bulk supermultiplets are truncated by the orientifold projection.

to two formulations which are also briefly described below.

10.3 Hypermultiplet, Double-Tensor and Single-Tensor Dual Formulations

As already mentioned, using the single-tensor multiplet is justified by the existence of an off-shell $\mathcal{N} = 2$ formulation. The hypermultiplet formulation, with two $\mathcal{N} = 1$ chiral superfields, is however more familiar and the first purpose of this subsection is to translate our results into this formalism. In the DBI theory (10.11), the linear superfield L only appears in

$$\mathcal{L}_{ST} + g \int d^2\theta \chi^\alpha W_\alpha + \text{c.c.} = \int d^2\theta d^2\bar{\theta} [\mathcal{H}(L, \Phi, \bar{\Phi}) + gLV_2] + \text{derivative.}$$

These contributions are not invariant under δ^* variations: the nonlinear deformation acts on W_α and on V_2 . Nevertheless, the linear superfield can be transformed into a new chiral superfield Φ' . The resulting 'hypermultiplet formulation' has Lagrangian

$$\begin{aligned} \mathcal{L}_{DBI, hyper.} &= \int d^2\theta d^2\bar{\theta} \mathcal{K}(\Phi' + \bar{\Phi}' - gV_2, \Phi, \bar{\Phi}) \\ &+ \int d^2\theta \left[\frac{1}{4} (2g\Phi - \xi_1) X(WW) - \frac{ig}{2\kappa} Y \right] + \text{c.c.} \end{aligned} \quad (10.17)$$

The Kähler potential is given by the Legendre transformation

$$\mathcal{K}(\Phi' + \bar{\Phi}', \Phi, \bar{\Phi}) = \mathcal{H}(U, \Phi, \bar{\Phi}) - U(\Phi' + \bar{\Phi}'), \quad (10.18)$$

where U is the solution of

$$\frac{\partial}{\partial U} \mathcal{H}(U, \Phi, \bar{\Phi}) = \Phi' + \bar{\Phi}'. \quad (10.19)$$

In the single-tensor formulation, $\mathcal{N} = 2$ supersymmetry implies that \mathcal{H} solves Laplace equation. As a result of the Legendre transformation, the determinant

of \mathcal{K} is constant and the metric is hyperkähler [18]. It should be noted that the Legendre transformation defines the new auxiliary scalar $f_{\Phi'}$ of Φ' according to

$$f_{\Phi'} = \left(\frac{\partial^2 \mathcal{H}}{\partial U \partial \Phi} \right)_{\theta=0} f_{\Phi}. \quad (10.20)$$

Hence, the hypermultiplet formulation has the same number of independent auxiliary fields as the single-tensor version: d_2 and f_{Φ} .

The second supersymmetry variation δ^* of Φ' is also defined by transformation (10.19): in the hypermultiplet formulation, $\mathcal{N} = 2$ is realized on-shell only, using the Lagrangian function. The nonlinear deformation of variations δ^* acts on V_2 . Since $W_{\alpha} = -\frac{1}{4} \overline{D} \overline{D} D_{\alpha} V_2$, eq. (10.5) indicates that

$$\delta^* V_2 = \frac{i}{\sqrt{2\kappa}} (\overline{\theta\theta\theta\eta} - \theta\theta\theta\overline{\eta}) + \sqrt{2}i (\eta D + \overline{\eta} \overline{D}) V_1.$$

The κ -dependent term in the δ^* variation of the Kähler potential term in $\mathcal{L}_{DBI, hyper}$ is then the same as the κ -dependent part in $g \delta^* \int d^2\theta \chi^{\alpha} W_{\alpha} + \text{c.c.}$, which is compensated by the variation of the four-form field. This can again be shown using eqs. (10.18) and (10.19). This hypermultiplet formulation will be used in Section 10.6, on the example of nonlinear DBI QED with a charged hypermultiplet.

For completeness, let us briefly mention two further formulations of the same DBI theory, using either a double-tensor, or a dual single-tensor $\mathcal{N} = 2$ multiplet. These possibilities appear if Lagrangian (10.11) has a second shift symmetry of $\text{Im } \Phi$. This is the case if the single-tensor kinetic Lagrangian has this isometry:

$$\mathcal{L}_{ST} = \int d^2\theta d^2\overline{\theta} \mathcal{H}(L, \Phi + \overline{\Phi}).$$

We may then transform Φ into a linear superfield L' using an $\mathcal{N} = 1$ duality transformation. Keeping L and turning Φ into L' leads to a double-tensor for-

mulation with superfields (L, L') . The Lagrangian has the form

$$\mathcal{L}_{DT} = \int d^2\theta d^2\bar{\theta} \mathcal{G}(L, L' - gV_1(WW)) - \int d^2\theta \left[\frac{1}{4} \xi_1 X(WW) - g\chi^\alpha W_\alpha + \frac{ig}{2\kappa} Y \right] + \text{c.c.} \quad (10.21)$$

The function \mathcal{G} is the Legendre transform of \mathcal{H} with respect to its second variable $\Phi + \bar{\Phi}$ and the real superfield $V_1(WW)$ is defined by the equation

$$X(WW) = \frac{1}{2} \overline{DD} V_1(WW). \quad (10.22)$$

It includes the DBI gauge kinetic term in its d_1 component and the Lagrangian depends on the new tensor $b'_{\mu\nu}$ through the combination $3 \partial_{[\mu} b'_{\nu\rho]} - g \omega_{\mu\nu\rho}$, where $\omega_{\mu\nu\rho} = 3 A_{[\mu} F_{\nu\rho]}$ is the Maxwell Chern-Simons form.

Finally, turning Φ and L into L' and Φ' , leads to another single-tensor theory with a Stückelberg gauging of both Φ' and L' , as in theory (9.21). In this case, the Lagrangian is

$$\mathcal{L}_{ST'} = \int d^2\theta d^2\bar{\theta} \tilde{\mathcal{H}}(\Phi' + \bar{\Phi}' - gV_2, L' - gV_1(WW)) - \int d^2\theta \left[\frac{1}{4} \xi_1 X(WW) + \frac{ig}{2\kappa} Y \right] + \text{c.c.} \quad (10.23)$$

While in the double-tensor theory (10.21) the second nonlinear supersymmetry only holds on shell, it is valid off shell in theory (10.23). The function $\tilde{\mathcal{H}}$ verifies Laplace equation, as required by $\mathcal{N} = 2$ linear supersymmetry.⁸ Using the supersymmetric Legendre transformation, one can show that the nonlinear deformation of $\delta^* V_2$, which affects $\delta^* \tilde{\mathcal{H}}$, is again balanced by the variation of the four-form superfield Y .

⁸See eq. (9.7).

10.4 The Magnetic Dual

To perform electric-magnetic duality on theory (10.9), we first replace it with

$$\begin{aligned} \mathcal{L}_{DBI} = & \int d^2\theta \int d^2\tilde{\theta} \left[ig\mathcal{Y} \left(\mathcal{W} - \frac{1}{2k}\tilde{\theta}\tilde{\theta} \right) - \frac{1}{4}\xi_1\tilde{\theta}\tilde{\theta}\mathcal{W} \right. \\ & \left. + \frac{1}{4}\Lambda U^2 - \frac{1}{2}\Upsilon \left(U - \mathcal{W} + \frac{1}{2k}\tilde{\theta}\tilde{\theta} \right) \right] + \text{c.c.} + \mathcal{L}_{ST}. \end{aligned} \quad (10.24)$$

Both U and Υ are unconstrained chiral $\mathcal{N} = 2$ superfields. The Lagrange multiplier Υ imposes $U = \mathcal{W} - \frac{1}{2k}\tilde{\theta}\tilde{\theta}$, which leads again to theory (10.9). The first two terms, which have gauge and $\mathcal{N} = 2$ invariance properties related to the Maxwell character of \mathcal{W} are left unchanged. The term quadratic in \mathcal{W} has been turned into a linear one using the Lagrange multiplier. Hence, the Maxwell superfield \mathcal{W} , which contributes to Lagrangian (10.24) by

$$\int d^2\theta \int d^2\tilde{\theta} \mathcal{W} \left(ig\mathcal{Y} + \frac{1}{2}\Upsilon - \frac{1}{4}\xi_1\tilde{\theta}\tilde{\theta} \right) + \text{c.c.}, \quad (10.25)$$

can as well be eliminated: Υ should be such that this contribution is a derivative. In terms of $\mathcal{N} = 1$ chiral superfields, \mathcal{W} has components X and W_α and since there exists two real superfields V_1 and V_2 such that $X = \frac{1}{2}\overline{DD}V_1$ and $W_\alpha = -\frac{1}{4}\overline{DDD}_\alpha V_2$, we actually need to eliminate V_1 and V_2 with result

$$\Upsilon = -i\widetilde{\mathcal{W}} - 2ig\mathcal{Y} + \frac{1}{2}(\xi_1 + i\zeta)\tilde{\theta}\tilde{\theta}. \quad (10.26)$$

In this expression, $\widetilde{\mathcal{W}}$ is a Maxwell $\mathcal{N} = 2$ superfield, the ‘magnetic dual’ of the eliminated \mathcal{W} . There is a new arbitrary real deformation parameter ζ , allowed by the field equation of V_2 . Notice however that $\xi_1 + i\zeta$ can be eliminated by a constant complex shift of Φ . Invariance of Υ under the single-tensor gauge variation (9.42) implies that $\delta\widetilde{\mathcal{W}} = 2g\widehat{\mathcal{W}} = -2g\delta\mathcal{Y}$ and

$$\mathcal{Z} \equiv \widetilde{\mathcal{W}} + 2g\mathcal{Y} \quad (10.27)$$

is then a gauge-invariant chiral superfield. As already mentioned, any unconstrained chiral $\mathcal{N} = 2$ superfield can be decomposed in this way and our theory may as well be considered as a description of the chiral superfields \mathcal{Z} and \mathcal{Y} with Lagrangian

$$\mathcal{L}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4} \Lambda U^2 + iU \left(\frac{1}{2} \mathcal{Z} + \frac{i}{4} (\xi_1 + i\zeta) \tilde{\theta}\tilde{\theta} \right) + \frac{i}{4\kappa} \tilde{\theta}\tilde{\theta} (\mathcal{Z} - 2g\mathcal{Y}) \right] + \text{c.c.} + \mathcal{L}_{ST}. \quad (10.28)$$

Invariance under the second supersymmetry implies that $\mathcal{Z} + \frac{i}{2} (\xi_1 + i\zeta) \tilde{\theta}\tilde{\theta}$ transforms as a standard $\mathcal{N} = 2$ chiral superfield and then

$$\delta_{deformed}^* \mathcal{Z} = i(\xi_1 + i\zeta) \tilde{\theta}\eta + i(\eta\tilde{Q} + \bar{\eta}\bar{\tilde{Q}}) \mathcal{Z}. \quad (10.29)$$

Eliminating U leads finally to

$$\tilde{\mathcal{L}}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\Lambda} \left(\mathcal{Z} + \frac{i}{2} (\xi_1 + i\zeta) \tilde{\theta}\tilde{\theta} \right)^2 + \frac{i}{4\kappa} \tilde{\theta}\tilde{\theta} (\mathcal{Z} - 2g\mathcal{Y}) \right] + \text{c.c.} + \mathcal{L}_{ST}, \quad (10.30)$$

which is the electric-magnetic dual of theory (10.9).⁹ The Lagrange multiplier superfield Λ^{-1} implies now the constraint

$$0 = \left(\mathcal{Z} + \frac{i}{2} (\xi_1 + i\zeta) \tilde{\theta}\tilde{\theta} \right)^2 = \mathcal{Z}^2 + i(\xi_1 + i\zeta) \tilde{\theta}\tilde{\theta} \mathcal{Z}. \quad (10.31)$$

Using the expansion (9.37),

$$\mathcal{Z}(y, \theta, \tilde{\theta}) = Z(y, \theta) + \sqrt{2} \tilde{\theta} \omega(y, \theta) - \tilde{\theta}\tilde{\theta} \left[\frac{i}{2} \Phi_Z(y, \theta) + \frac{1}{4} \overline{DD\bar{Z}}(y, \theta) \right],$$

with $Z = \tilde{X} + 2gY$, $\omega_\alpha = i\tilde{W}_\alpha + 2g\chi_\alpha$ and $\Phi_Z = 2g\Phi$, this constraint corresponds to

$$Z^2 = 0, \quad Z\omega_\alpha = 0, \quad \frac{1}{2} Z \overline{DD\bar{Z}} + \omega\omega = -iZ[\Phi_Z - (\xi_1 + i\zeta)].$$

In this case, and in contrast to the electric case, the constraint leading to the DBI theory is due to the scale $\langle \Phi_Z \rangle = 2g\langle \Phi \rangle$: we will actually choose $\zeta = 0$, absorb ξ_1

⁹It reduces to eq. (10.7) if $g = 0$.

into Φ_Z and consider the constraint $Z^2 = 0$ with a non-zero background value $\langle \Phi_Z \rangle$ breaking the second supersymmetry. Our magnetic theory is then

$$\tilde{\mathcal{L}}_{DBI} = \int d^2\theta \int d^2\tilde{\theta} \left[\frac{1}{4\Lambda} Z^2 + \frac{i}{4\kappa} \tilde{\theta}\tilde{\theta} (Z - 2g\mathcal{Y}) \right] + \text{c.c.} + \mathcal{L}_{ST}, \quad (10.32)$$

with constraints

$$Z^2 = 0, \quad Z\omega_\alpha = 0, \quad \frac{1}{2} Z \overline{DDZ} + \omega\omega = -iZ\Phi_Z, \quad (10.33)$$

the DBI scale arising from $\Phi_Z = \phi_Z + \langle \Phi_Z \rangle$. As in the Maxwell case, the third equation, which also reads

$$Z = \frac{i\omega\omega}{\Phi_Z - \frac{i}{2} \overline{DDZ}}, \quad (10.34)$$

implies $Z\omega_\alpha = Z^2 = 0$ and allows to express Z as a function of $\omega\omega$ and Φ , $Z = Z(\omega\omega, \Phi)$, using $\Phi_Z = 2g\Phi - \xi_1$. The magnetic theory (10.32) is then simply

$$\tilde{\mathcal{L}}_{DBI} = -\frac{1}{2\kappa} \text{Im} \int d^2\theta \left[Z(\omega\omega, \Phi) - 2gY \right] + \mathcal{L}_{ST}. \quad (10.35)$$

It is the electric-magnetic dual of expression (10.11). At this point, it is important to recall that ω and Φ are actually $\mathcal{N} = 1$ superfields components of $Z = \tilde{\mathcal{W}} + 2g\mathcal{Y}$, *i.e.*

$$\omega_\alpha = i\tilde{W}_\alpha + 2g\chi_\alpha. \quad (10.36)$$

The kinetic terms for the single-tensor multiplet (L, Φ) , $L = D\chi - \overline{D}\bar{\chi}$, are included in \mathcal{L}_{ST} while $Z(\omega\omega, \Phi)$ includes the DBI kinetic terms for the Maxwell $\mathcal{N} = 1$ superfield \tilde{W}_α . As in the electric case, the magnetic theory has a contribution proportional to the four-form field included in Y .

The third constraint (10.33) is certainly invariant under the variations (9.38), using $Z\omega_\alpha = 0$. But with a non-zero background value $\Phi = \phi + \langle \Phi \rangle$, the spinor ω_α transforms nonlinearly, like a goldstino:¹⁰

$$\delta^* \omega_\alpha = -\frac{i}{\sqrt{2}} \langle \Phi \rangle \eta_\alpha - \frac{i}{\sqrt{2}} \phi \eta_\alpha - \frac{\sqrt{2}}{4} \eta_\alpha \overline{DDZ} - \sqrt{2} i (\sigma^\mu \bar{\eta})_\alpha \partial_\mu Z. \quad (10.37)$$

¹⁰See eq. (10.29).

10.4.1 The Bosonic Lagrangian

The bosonic Lagrangian included in the magnetic theory (10.35) is

$$\begin{aligned}
\widetilde{\mathcal{L}}_{DBI,bos.} = & \frac{\text{Re } \Phi_Z}{8\kappa} - \frac{\text{Re } \Phi_Z}{8\kappa|\Phi_Z|^2} \left\{ -|\Phi_Z|^4 \det \left[\eta_{\mu\nu} - 2\sqrt{2}|\Phi_Z|^{-1}(\widetilde{F}_{\mu\nu} - gb_{\mu\nu}) \right] \right. \\
& - 8\widetilde{d}_2^2 (|\Phi_Z|^2 + 2g^2C^2) + 2g^2C^2|\Phi_Z|^2 \\
& \left. + 8gC\widetilde{d}_2 \epsilon^{\mu\nu\rho\sigma} (\widetilde{F}_{\mu\nu} - gb_{\mu\nu})(\widetilde{F}_{\rho\sigma} - gb_{\rho\sigma}) \right\}^{1/2} \\
& - \frac{\text{Im } \Phi_Z}{8\kappa|\Phi_Z|^2} \left[\epsilon^{\mu\nu\rho\sigma} (\widetilde{F}_{\mu\nu} - gb_{\mu\nu})(\widetilde{F}_{\rho\sigma} - gb_{\rho\sigma}) - 4gC\widetilde{d}_2 \right] \\
& + \frac{g}{24\kappa} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \mathcal{L}_{ST,bos.} .
\end{aligned} \tag{10.38}$$

It depends on a single auxiliary field, the Maxwell real scalar \widetilde{d}_2 , with field equation

$$\begin{aligned}
\widetilde{d}_{2,bos.} = & -\frac{gC}{2(|\Phi_Z|^2 + 2g^2C^2)} \epsilon^{\mu\nu\rho\sigma} (\widetilde{F}_{\mu\nu} - gb_{\mu\nu})(\widetilde{F}_{\rho\sigma} - gb_{\rho\sigma}) \\
& - \frac{gC \text{Im } \Phi_Z}{2|\Phi_Z|^2} \frac{\sqrt{-\det\left(\eta_{\mu\nu} + \frac{2\sqrt{2}}{\sqrt{2g^2C^2 + |\Phi_Z|^2}}(\widetilde{F}_{\mu\nu} - gb_{\mu\nu})\right)}}{\sqrt{(\text{Re } \Phi_Z)^2 + 2g^2C^2}} .
\end{aligned} \tag{10.39}$$

Eliminating \widetilde{d}_2 and using $\Phi_Z = 2g\Phi - \xi_1$ to reintroduce the superfield Φ of the single-tensor multiplet and the ‘original’ Fayet-Iliopoulos term ξ_1 , we finally obtain the magnetic, bosonic Lagrangian

$$\begin{aligned}
\widetilde{\mathcal{L}}_{DBI,bos.} = & \frac{2g \text{Re } \Phi - \xi_1}{8\kappa} - \frac{1}{8\kappa} \sqrt{(2g \text{Re } \Phi - \xi_1)^2 + 2g^2C^2} \\
& \times \sqrt{-\det\left(\eta_{\mu\nu} - \frac{2\sqrt{2}}{\sqrt{2g^2C^2 + |2g\Phi - \xi_1|^2}}(\widetilde{F}_{\mu\nu} - gb_{\mu\nu})\right)} \\
& - \frac{g \text{Im } \Phi}{4\kappa(2g^2C^2 + |2g\Phi - \xi_1|^2)} \epsilon^{\mu\nu\rho\sigma} (\widetilde{F}_{\mu\nu} - gb_{\mu\nu})(\widetilde{F}_{\rho\sigma} - gb_{\rho\sigma}) \\
& + \frac{g}{24\kappa} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \mathcal{L}_{ST,bos.} .
\end{aligned} \tag{10.40}$$

As in the electric case, the DBI term has a field-dependent coefficient,

$$-\frac{1}{8\kappa} \sqrt{(2g \operatorname{Re} \Phi - \xi_1)^2 + 2g^2 C^2} \sqrt{-\det\left(\eta_{\mu\nu} - \frac{1}{\sqrt{2g^2 C^2 + |2g\Phi - \xi_1|^2}} (\tilde{F}_{\mu\nu} - gb_{\mu\nu})\right)}, \quad (10.41)$$

and, as expected, the scalar potentials of the magnetic and electric [eq. (10.16)] theories are identical.

Define the complex dimensionless field

$$S = \kappa \sqrt{(2g \operatorname{Re} \Phi - \xi_1)^2 + 2g^2 C^2} + 2i\kappa g \operatorname{Im} \Phi, \quad (10.42)$$

for which $\kappa^{-2}|S|^2 = |2g\Phi - \xi_1|^2 + 2g^2 C^2$. In terms of S , the magnetic theory (10.40) rewrites as

$$\begin{aligned} \tilde{\mathcal{L}}_{DBI,bos.} &= \frac{2g \operatorname{Re} \Phi - \xi_1}{8\kappa} - \frac{1}{8\kappa^2} \operatorname{Re} \frac{1}{S} \sqrt{-\det(|S|\eta_{\mu\nu} - 2\sqrt{2}\kappa(\tilde{F}_{\mu\nu} - gb_{\mu\nu}))} \\ &\quad + \frac{1}{8} \operatorname{Im} \frac{1}{S} \epsilon^{\mu\nu\rho\sigma} (\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) + \frac{g}{24\kappa} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \mathcal{L}_{ST,bos.} \\ &= \frac{2g \operatorname{Re} \Phi - \xi_1}{8\kappa} - \frac{1}{8\kappa^2} \operatorname{Re} S \sqrt{-\det(\eta_{\mu\nu} - 2\sqrt{2}\kappa|S|^{-1}(\tilde{F}_{\mu\nu} - gb_{\mu\nu}))} \\ &\quad + \frac{1}{8} \operatorname{Im} \frac{1}{S} \epsilon^{\mu\nu\rho\sigma} (\tilde{F}_{\mu\nu} - gb_{\mu\nu})(\tilde{F}_{\rho\sigma} - gb_{\rho\sigma}) + \frac{g}{24\kappa} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \mathcal{L}_{ST,bos.}. \end{aligned} \quad (10.43)$$

This is to be compared with the electric theory (10.14):

$$\begin{aligned} \mathcal{L}_{DBI,bos.} &= \frac{2g \operatorname{Re} \Phi - \xi_1}{8\kappa} - \frac{1}{8\kappa^2} \operatorname{Re} S \sqrt{-\det(\eta_{\mu\nu} - 2\sqrt{2}\kappa F_{\mu\nu})} \\ &\quad + \frac{1}{8} \operatorname{Im} S \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} - \frac{g}{4} \epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} F_{\rho\sigma} + \frac{g}{24\kappa} \epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} + \mathcal{L}_{ST,bos.}. \end{aligned} \quad (10.44)$$

Hence, the duality from the electric to the magnetic theory corresponds to the transformations

$$b_{\mu\nu} \rightarrow 0, \quad F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} - gb_{\mu\nu}, \quad S \rightarrow S^{-1}, \quad \eta_{\mu\nu} \rightarrow |S|\eta_{\mu\nu}, \quad (10.45)$$

which can be also derived from electric-magnetic duality applied on the bosonic DBI theory only.

10.5 Double-Tensor Formulation and Connection with the String Fields

In IIB superstrings compactified to four dimensions with eight residual supercharges, the dilaton belongs to a double-tensor supermultiplet. This representation of $\mathcal{N} = 2$ supersymmetry includes two Majorana spinors, two antisymmetric tensors $B_{\mu\nu}$ (NS–NS) and $C_{\mu\nu}$ (R–R) with gauge symmetries

$$\delta_{gauge} B_{\mu\nu} = 2 \partial_{[\mu} \Lambda_{\nu]}, \quad \delta'_{gauge} C_{\mu\nu} = 2 \partial_{[\mu} \Lambda'_{\nu]} \quad (10.46)$$

and two (real) scalar fields, the NS–NS dilaton and the R–R scalar, for a total of $4_B + 4_F$ physical states. In principle, both antisymmetric tensors can be dualized to pseudoscalar fields with axionic shift symmetry, in a version of the effective field theory where the dilaton belongs to a hypermultiplet with four scalars in a quaternion-Kähler manifold possessing three perturbative shift isometries, since the R–R scalar has its own shift symmetry. It is easy to see that only two shift isometries, related to the two antisymmetric tensors, commute, while all three together form the Heisenberg algebra. Indeed, in the double-tensor basis, the R–R field strength is modified [98] due to its anomalous Bianchi identity to $3 \partial_{[\lambda} C_{\mu\nu]} - 3 C^{(0)} \partial_{[\lambda} B_{\mu\nu]}$. Thus, a shift of the R–R scalar $C^{(0)}$ by a constant λ is accompanied by an appropriate transformation of $C_{\mu\nu}$ to leave its modified field-strength invariant:

$$\delta_H C^{(0)} = \lambda, \quad \delta_H C_{\mu\nu} = \lambda B_{\mu\nu}. \quad (10.47)$$

It follows that δ_{gauge} , δ'_{gauge} and δ_H verify the Heisenberg algebra, with a single non-vanishing commutator

$$[\delta_{gauge}, \delta_H] = \delta'_{gauge}. \quad (10.48)$$

More details about the Heisenberg algebra in local and global supersymmetry are given in chapter 11 where we obtain the global supersymmetry limit of the universal hypermultiplet. Our aim is to use the Heisenberg algebra in order to establish the connection between the general formalism developed so far and string theory. This formalism would then describe the coupling of a D-brane with bulk fields in the limit of global supersymmetry.

To this end, we transform the $\mathcal{N} = 2$ double-tensor into a single-tensor representation by dualizing one of its two $\mathcal{N} = 1$ linear multiplet components L' , containing the R–R fields $C_{\mu\nu}$ and $C^{(0)}$, into a chiral basis $\Phi + \bar{\Phi}$. In this basis, the two R–R isometries correspond to constant complex shifts of the $\mathcal{N} = 1$ superfield Φ . Imposing this symmetry to the kinetic function of eqs. (9.7)–(9.8), one obtains (up to total derivatives, after superspace integration):

$$\mathcal{H}(L, \Phi, \bar{\Phi}) = \alpha \left(-\frac{1}{3}L^3 + \frac{1}{2}L(\Phi + \bar{\Phi})^2 \right) + \beta \left(-L^2 + \frac{1}{2}(\Phi + \bar{\Phi})^2 \right), \quad (10.49)$$

where α and β are constants. Note that the second term proportional to β can be obtained from the first by shifting $L + \beta/\alpha$. For $\alpha = 0$ however, it corresponds to the free case of quadratic kinetic terms for all fields of the single-tensor multiplet. The coupling to the Maxwell goldstino multiplet is easily obtained using eqs. (10.12), (10.22) and (9.18). Up to total derivatives, the action is:

$$\begin{aligned} \mathcal{L} = & \int d^2\theta d^2\bar{\theta} \left[\alpha \left(-\frac{1}{3}L^3 + \frac{1}{2}L(\Phi + \bar{\Phi})^2 \right) + \beta \left(-L^2 + \frac{1}{2}(\Phi + \bar{\Phi})^2 \right) \right. \\ & \left. - g(\Phi + \bar{\Phi})V_1(WW) \right] + g \int d^2\theta \left[\chi^\alpha W_\alpha - \frac{i}{2k}Y - \frac{\xi_1}{4g}X(WW) \right] + \text{c.c.} \end{aligned} \quad (10.50)$$

In general, the four-form field is not inert under the variation δ_H of eq. (10.47) [99]. In our single-tensor formalism, $\delta_H L = 0$ and $\delta_H \Phi = c$ where c is complex when combined with the axionic shift δ'_{gauge} of $\text{Im}\Phi$ dual to $C_{\mu\nu}$ of eq. (10.46); in addition

$$\delta_H Y = -ic\kappa X(WW). \quad (10.51)$$

With this variation, the Lagrangian, including the Chern-Simons interaction, is invariant under the Heisenberg symmetry.

We can now dualize back $\Phi + \bar{\Phi}$ to a second linear multiplet L' by first replacing it with a real superfield U :

$$\begin{aligned} \mathcal{L} = & \int d^2\theta d^2\bar{\theta} \left[\alpha \left(-\frac{1}{3}L^3 + \frac{1}{2}LU^2 \right) + \beta \left(-L^2 + \frac{1}{2}U^2 \right) - U(mL' + gV_1) \right] \\ & + g \int d^2\theta \left[\chi^\alpha W_\alpha - \frac{i}{2\kappa}Y - \frac{\xi_1}{4g}X \right] + \text{c.c.}, \end{aligned} \quad (10.52)$$

where the constant m corresponds to a rescaling of L' . Solving for U ,

$$U = \frac{mL' + gV_1}{\alpha L + \beta}, \quad (10.53)$$

delivers the double-tensor Lagrangian

$$\tilde{\mathcal{L}} = \int d^2\theta d^2\bar{\theta} \left[-\frac{\alpha}{3}L^3 - \beta L^2 - \frac{1}{2} \frac{(mL' + gV_1)^2}{\alpha L + \beta} \right] + g \int d^2\theta \left[\chi^\alpha W_\alpha - \frac{i}{2\kappa}Y - \frac{\xi_1}{4g}X \right] + \text{c.c.}, \quad (10.54)$$

where as before $V_1 = V_1(WW)$ and $X = X(WW) = \frac{1}{2}\overline{DD}V_1(WW)$. It is invariant under variation (10.51) of the four-form superfield combined with $\delta_H L' = 2c(\alpha L + \beta)/m$.

After elimination of the Maxwell auxiliary field (choosing $m = \sqrt{2}$)

$$d_{2,bos.} = \frac{gC}{2\kappa} \sqrt{\frac{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})}{\left(\frac{\sqrt{2}gC'}{\alpha C + \beta} - \xi_1\right)^2 + 2g^2C^2}}, \quad (10.55)$$

the component expansion of the bosonic Lagrangian is

$$\begin{aligned} \tilde{\mathcal{L}}_{bos.} = & (\alpha C + \beta) \left[\frac{1}{2}(\partial_\mu C)^2 + \frac{1}{2}\partial_\mu \left(\frac{C'}{\alpha C + \beta} \right)^2 + \frac{1}{12}(3\partial_{[\mu} b_{\nu\rho]})^2 \right] \\ & + \frac{1}{12(\alpha C + \beta)} \left(3\partial_{[\mu} b'_{\nu\rho]} + \frac{g\kappa}{\sqrt{2}}\omega_{\mu\nu\rho} - \frac{C'}{\alpha C + \beta} 3\partial_{[\mu} b_{\nu\rho]} \right)^2 \\ & - \frac{g}{4\kappa\sqrt{2}} \left(\frac{C'}{\alpha C + \beta} + \frac{\xi_1}{\sqrt{2}g} \right) + \frac{g}{4\kappa\sqrt{2}} \sqrt{\left(\frac{C'}{\alpha C + \beta} + \frac{\xi_1}{\sqrt{2}g} \right)^2 + C^2} \sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\kappa F_{\mu\nu})} \\ & - \frac{g}{4}\epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} F_{\rho\sigma} + \frac{g}{24\kappa}\epsilon^{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma}. \end{aligned} \quad (10.56)$$

in terms of the Maxwell Chern-Simons form $\omega_{\nu\rho\sigma} = 3 A_{[\nu} F_{\rho\sigma]}$.

This is the explicit expression of the interacting action (10.21) and the kinetic part for the double-tensor multiplet. It describes the global supersymmetry limit of the effective four dimensional action of a D-brane coupled to the universal dilaton hypermultiplet of the perturbative type II string. The precise identification of the fields will be done in section 11.3 in the dual single-tensor basis but we can already see the similarities here: As mentioned previously, its general form in the local case depends also on two constant parameters, upon imposing the perturbative Heisenberg isometries, that correspond to the tree and one-loop contributions [28]. We expect that these two parameters are related to α and β of our action. Moreover, by identifying the two antisymmetric tensors $b_{\mu\nu}$ and $b'_{\mu\nu}$ with the respective NS–NS $B_{\mu\nu}$ and R–R $C_{\mu\nu}$ and the combination $C' / (\alpha C + \beta)$ with the R–R scalar $C^{(0)}$, as the Heisenberg transformations indicate, one finds that the two actions match up to normalization factors depending on the NS–NS dilaton that should correspond to the scalar C .

10.6 Nonlinear $\mathcal{N} = 2$ QED

We will now show that the effective theory presented above describing a super-Higgs phenomenon of partial (global) supersymmetry breaking can be identified with the Higgs phase of nonlinear $\mathcal{N} = 2$ QED, up to an appropriate choice of the single-tensor multiplet kinetic terms. We will then analyze its vacuum structure in the generally allowed parameter space.

In linear $\mathcal{N} = 2$ quantum electrodynamics (QED), the Lagrangian couples a hypermultiplet with two chiral superfields (Q_1, Q_2) to the vector multiplet

(V_1, V_2) or (X, W_α) . The $U(1)$ gauge transformations of the hypermultiplet are linear, and Q_1 and Q_2 have opposite $U(1)$ charges:

$$\mathcal{L}_{QED} = \int d^2\theta d^2\bar{\theta} [\bar{Q}_1 Q_1 e^{V_2} + \bar{Q}_2 Q_2 e^{-V_2}] + \int d^2\theta \frac{i}{\sqrt{2}} X Q_1 Q_2 + \text{c.c.} + \mathcal{L}_{Max.} + \Delta\mathcal{L}, \quad (10.57)$$

where $\mathcal{L}_{Max.}$ includes (canonical) gauge kinetic terms and $\Delta\mathcal{L}$ contains three parameters:

$$\Delta\mathcal{L} = m \int d^2\theta Q_1 Q_2 + \text{c.c.} + \int d^2\theta d^2\bar{\theta} [\xi_1 V_1 + \xi_2 V_2]. \quad (10.58)$$

The hypermultiplet mass term with coefficient m can be eliminated by a shift of X and $\xi_{1,2}$ are the two Fayet-Iliopoulos coefficients. Since $\xi_1 \int d^2\theta d^2\bar{\theta} V_1 = -\frac{1}{4} \int d^2\theta \xi_1 X + \text{c.c.}$, the complete superpotential w is

$$w = \left(\frac{i}{\sqrt{2}} X + m \right) Q_1 Q_2 - \frac{1}{4} \xi_1 X.$$

There are six real auxiliary fields, f_{Q_1} , f_{Q_2} , d_1 and d_2 but only four are actually independent:¹¹ $Q_1 \bar{f}_{Q_1} = Q_2 \bar{f}_{Q_2}$. Since the metric is canonical, $\det K_{i\bar{j}} = 1$ and trivially hyperkähler. If $\xi_1 = \xi_2 = 0$, the gauge symmetry is not broken and the hypermultiplet mass $m + i\langle X \rangle / \sqrt{2}$ is arbitrary. Any nonzero ξ_1 or ξ_2 induces $U(1)$ symmetry breaking with all fields having the same mass. In any case, $\mathcal{N} = 2$ supersymmetry remains unbroken at the global minimum.

In order to first bring the theory to a form allowing dualization to our single-tensor formulation, we use the holomorphic field redefinition¹²

$$\begin{aligned} Q_1 &= a \sqrt{\Phi} e^{\Phi'}, & Q_2 &= ia \sqrt{\Phi} e^{-\Phi'}, \\ Q_1 Q_2 &= ia^2 \Phi, & Q_1 / Q_2 &= -ie^{2\Phi'}, \end{aligned} \quad (10.59)$$

¹¹We use the same notation for a chiral superfield Φ , Q_1 , Q_2 , ... and for its lowest complex scalar component field.

¹²This field redefinition has constant Jacobian.

with $a^2 = 1/\sqrt{2}$. The QED Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{QED} &= \frac{1}{\sqrt{2}} \int d^2\theta d^2\bar{\theta} \sqrt{\Phi\bar{\Phi}} \left[e^{\Phi'+\bar{\Phi}'+V_2} + e^{-\Phi'-\bar{\Phi}'-V_2} \right] + \mathcal{L}_{Max}. \\ &+ \int d^2\theta \left[-\frac{1}{2}\Phi(X - \sqrt{2}im) - \frac{1}{4}\xi_1 X \right] + \text{c.c.} + \xi_2 \int d^2\theta d^2\bar{\theta} V_2. \end{aligned} \quad (10.60)$$

While the gauge transformation of Φ' is $\delta_{U(1)}\Phi' = \Lambda_c$, Φ is gauge invariant. Since the Kähler potential is now a function of $\Phi' + \bar{\Phi}'$, with a Stückelberg gauging of the axionic shift of Φ' , the chiral Φ' can be dualized to a linear L using a $N = 1$ Legendre transformation. The result is

$$\begin{aligned} \mathcal{L}_{QED} &= \int d^2\theta d^2\bar{\theta} \left[\sqrt{2\Phi\bar{\Phi} + L^2} - L \ln \left(\sqrt{2\Phi\bar{\Phi} + L^2} + L \right) \right] + \mathcal{L}_{Max}. \\ &- \int d^2\theta \left[\frac{1}{2}X\Phi + \chi^\alpha W_\alpha - \frac{i}{\sqrt{2}}m\Phi + \frac{1}{4}\xi_1 X \right] + \text{c.c.} + \xi_2 \int d^2\theta d^2\bar{\theta} V_2. \end{aligned} \quad (10.61)$$

The dual single-tensor QED theory has off-shell $\mathcal{N} = 2$ invariance (the Laplace equation (9.7) is verified) and the two multiplets are now coupled by a $\mathcal{N} = 2$ Chern-Simons interaction (9.19). Notice that the free quadratic kinetic terms of the charged hypermultiplet lead to a highly non-trivial kinetic function in the single-tensor representation. Moreover, there are only four auxiliary fields, f_Φ , d_1 and d_2 . The Legendre transformation defines the scalar field C in L as

$$e^{2\text{Re}\Phi'} = \frac{1}{\sqrt{2\Phi\bar{\Phi}}} \left(\sqrt{2\Phi\bar{\Phi} + C^2} + C \right), \quad e^{-2\text{Re}\Phi'} = \frac{1}{\sqrt{2\Phi\bar{\Phi}}} \left(\sqrt{2\Phi\bar{\Phi} + C^2} - C \right) \quad (10.62)$$

and eqs. (10.59) relate then C and Φ with Q_1 and Q_2 :

$$C = |Q_1|^2 - |Q_2|^2, \quad \Phi = -\sqrt{2}i Q_1 Q_2. \quad (10.63)$$

According to eq. (10.11), the nonlinear DBI version of $\mathcal{N} = 2$ QED is obtained by replacing in Lagrangian (10.61) X by $X(WW)$, which includes DBI gauge kinetic terms, by omitting \mathcal{L}_{Max} . which is removed by the third constraint (10.2)

and by adding the four-form term $\frac{i}{2\kappa} \int d^2\theta Y + \text{c.c.}$:

$$\begin{aligned} \mathcal{L}_{QED,DBI} = & \int d^2\theta d^2\bar{\theta} \left[\sqrt{2\Phi\bar{\Phi} + L^2} - L \ln \left(\sqrt{2\Phi\bar{\Phi} + L^2} + L \right) + \xi_2 V_2 \right] \\ & - \int d^2\theta \left[\left(\frac{1}{2}\Phi + \frac{1}{4}\xi_1 \right) X(WW) - \frac{i}{\sqrt{2}}m\Phi + \chi^\alpha W_\alpha - \frac{i}{2\kappa}Y \right] + \text{c.c.} \end{aligned} \quad (10.64)$$

Notice that two additional terms appear compared to the action studied in Section 10: a Fayet-Iliopoulos term proportional to ξ_2 and a term linear in Φ which is also invariant under the second (nonlinear) supersymmetry (9.4); they generate, together with ξ_1 the general parameter space of nonlinear QED coupled to a charged hypermultiplet. Without loss of generality, we choose m to be real, while the choice $\xi_1 = -1/\kappa$ would canonically normalize gauge kinetic terms for a background where Φ vanishes. We may return to chiral superfields (Φ, Φ') or (Q_1, Q_2) to write the DBI theory as¹³

$$\begin{aligned} \mathcal{L}_{QED} = & \int d^2\theta d^2\bar{\theta} \left[\bar{Q}_1 Q_1 e^{V_2} + \bar{Q}_2 Q_2 e^{-V_2} + \xi_2 V_2 \right] \\ & + \int d^2\theta \left[\left(\frac{i}{\sqrt{2}}Q_1 Q_2 - \frac{1}{4}\xi_1 \right) X(WW) + mQ_1 Q_2 + \frac{i}{2\kappa}Y \right] + \text{c.c.} \end{aligned} \quad (10.65)$$

Since $X(WW)|_{\theta=0}$ only depends on fermion fields, the auxiliary fields f_1 and f_2 only contribute to the bosonic Lagrangian by a hypermultiplet mass term

$$\left(|f_1|^2 + |f_2|^2 \right)_{bos.} = m^2 \left(|Q_1|^2 + |Q_2|^2 \right)$$

to be added to the scalar potential obtained from eq. (10.16) with the substitutions

$$2g \operatorname{Re} \Phi - \xi_1 \longrightarrow 2\sqrt{2} \operatorname{Im}(Q_1 Q_2) - \xi_1, \quad gC \longrightarrow C + \xi_2 = \xi_2 + |Q_1|^2 - |Q_2|^2$$

(since we have chosen $g = 1$). The complete potential is then¹⁴

$$\begin{aligned} V_{QED,DBI} = & \frac{1}{8\kappa} \left(2\sqrt{2} \operatorname{Im}(Q_1 Q_2) - \xi_1 \right) \left[\sqrt{1 + \frac{2[\xi_2 + |Q_1|^2 - |Q_2|^2]^2}{[2\sqrt{2} \operatorname{Im}(Q_1 Q_2) - \xi_1]^2}} - 1 \right] \\ & + m^2 \left(|Q_1|^2 + |Q_2|^2 \right). \end{aligned} \quad (10.66)$$

¹³See eq. (10.17).

¹⁴The auxiliary d_2 is given in eq. (10.13).

The analysis is then very simple. The first line vanishes only for

$$\langle \xi_2 + |Q_1|^2 - |Q_2|^2 \rangle = 0, \quad \langle 2\sqrt{2} \operatorname{Im}(Q_1 Q_2) - \xi_1 \rangle > 0. \quad (10.67)$$

The first condition is the usual D -term equation $\langle d_2 \rangle = 0$ for the Maxwell superfield. The second condition is necessary to have a well-defined DBI gauge kinetic term at the minimum. Hence, if $m = 0$, conditions (10.67), which can always be solved, define the vacuum of the theory. Choosing $\langle Q_1 \rangle = v$ and $\langle Q_2 \rangle = \sqrt{v^2 + \xi_2}$, with v real (and arbitrary), we find a massive vector boson which, along with a real scalar and the two Majorana fermions

$$\frac{1}{\sqrt{2v^2 + \xi_2}} \left[v\psi_{Q_1} - \sqrt{v^2 + \xi_2} \psi_{Q_2} \right] \pm i\lambda,$$

makes a massive $\mathcal{N} = 1$ vector multiplet of mass $\sqrt{v^2 + \xi_2}/2$. Hence the potentially massless gaugino λ , with its goldstino-like second supersymmetry variation $\delta^* \lambda_\alpha = -\frac{1}{\sqrt{2}k} \eta_\alpha + \dots$, has been absorbed in the massive $U(1)$ gauge boson multiplet. This is possible only because the second supersymmetry transformation of the four-form field compensates the gaugino nonlinear variation. The fermion

$$\sqrt{v^2 + \xi_2} \psi_{Q_1} + v \psi_{Q_2}$$

is massless and corresponds to the fermion of the chiral superfield Φ in the single-tensor formalism, in agreement with our analysis in Section 10.2 [see below eq. (10.16)]. With two real scalars, it belongs to a massless $\mathcal{N} = 1$ chiral multiplet.

If $m \neq 0$, a supersymmetric vacuum has $\langle Q_1 \rangle = \langle Q_2 \rangle = 0$. It only exists if $\xi_2 = 0$ and $\xi_1 \neq 0$. The second condition is again to have DBI gauge kinetic terms on this vacuum. In this case, the $U(1)$ gauge symmetry is not broken, the goldstino vector multiplet remains massless and the hypermultiplet has mass

m. If $m \neq 0$, a nonzero Fayet-Iliopoulos coefficient ξ_2 breaks then $\mathcal{N} = 1$ linear supersymmetry. Note that the single-tensor formalism is appropriate for the description of the Higgs phase of nonlinear QED in a manifest $\mathcal{N} = 1$ superfield basis (with respect to the linear supersymmetry), while the charged hypermultiplet representation is obviously convenient for describing the Coulomb phase.

One can finally expand the action (10.65) in powers of κ in order to find the lowest dimensional operators that couple the goldstino multiplet of partial supersymmetry breaking to the $\mathcal{N} = 2$ hypermultiplet. Besides the dimension-four operators corresponding to the gauge factors $e^{\pm V_2}$, one obtains a dimension-six superpotential interaction $\sim \kappa Q_1 Q_2 W^2$ coming from the solution of the nonlinear constraint $X = \kappa W^2 + \mathcal{O}(\kappa^3)$; it amounts to a field-dependent correction to the $U(1)$ gauge coupling.

CHAPTER 11

THE UNIVERSAL HYPERMULTIPLY IN LOCAL AND GLOBAL SUPERSYMMETRY

11.1 On the Heisenberg Algebra and Global Supersymmetry

In the context of IIB superstrings, the Heisenberg algebra is generated by a combination of the gauge symmetries of the two antisymmetric tensors $B_{\mu\nu}$ (NS-NS) and $C_{\mu\nu}$ (R-R) and of the shift symmetry of the R-R scalar C_0 :

$$\delta B_{\mu\nu} = 2 \partial_{[\mu} \Lambda_{\nu]}, \quad \delta C_{\mu\nu} = 2 \partial_{[\mu} \tilde{\Lambda}_{\nu]} + \lambda B_{\mu\nu}, \quad \delta C_0 = \lambda. \quad (11.1)$$

As a consequence, the theory depends on the invariant three-forms

$$H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]}, \quad F_{\mu\nu\rho} = 3 \partial_{[\mu} C_{\nu\rho]} - C_0 H_{\mu\nu\rho} \quad (11.2)$$

and on $\partial_\mu C_0$. The Heisenberg algebra follows from

$$[\delta_1, \delta_2] C_{\mu\nu} = 2 \partial_{[\mu} \lambda_2 \Lambda_{1\nu]} - 2 \partial_{[\mu} \lambda_1 \Lambda_{2\nu]}. \quad (11.3)$$

After reduction to four dimensions, the gauge symmetries imply that each tensor can be dualized into a scalar field with axionic shift symmetry. The third global symmetry (with parameter λ) combines then with the axionic shifts to realize again the Heisenberg algebra on three scalar fields.

Indeed, one obtains three scalar fields φ , τ and $\eta = C_0$, with Heisenberg variations

$$\delta\eta = c_X, \quad \delta\varphi = c_Y, \quad \delta\tau = c_Z - c_X\varphi. \quad (11.4)$$

The scalars φ and τ are Poincaré dual to $C_{\mu\nu}$ and $B_{\mu\nu}$, respectively. The duality relations are, schematically,

$$\partial_\mu\varphi \sim \epsilon_{\mu\nu\lambda\rho}F^{\nu\lambda\rho}, \quad \partial_\mu\tau + \eta\partial_\mu\varphi \sim \epsilon_{\mu\nu\lambda\rho}H^{\nu\lambda\rho}.$$

The algebra is $[X, Y] \sim Z$, with Y and Z generating the axionic shifts (with parameters c_Y and c_Z), while X generates the shift of the R-R scalar (with parameter c_X). Notice that the central charge of the algebra is (depending on the representation) the gauge symmetry of the R-R tensor and the axionic symmetry of τ , dual to the NS-NS tensor.

The Heisenberg algebra is extended by a fourth perturbative generator M that rotates X, Y and commutes also with the central charge Z :

$$\delta_M\eta = c_M\varphi, \quad \delta_M\varphi = -c_M\eta, \quad \delta_M\tau = \frac{c_M}{2}(\eta^2 - \varphi^2). \quad (11.5)$$

Equivalently, M rotates the phase of the complex R-R scalar $\eta + i\varphi$. As a result, the perturbative symmetry becomes the two-dimensional Euclidean group E_2 with central extension Z .

11.1.1 Lagrangians

Consider a $\mathcal{N} = 1$ globally supersymmetric theory with two superfields, a chiral Φ and a real linear L . It contains three real scalars, $\text{Re}\phi = \text{Re}\Phi|_{\theta=0}$, $\text{Im}\phi = \text{Im}\Phi|_{\theta=0}$, and $C = L|_{\theta=0}$, and L also depends on the curl of an antisym-

metric tensor $H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]}$. The Lagrangian (up to two derivatives) is

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \mathcal{H}(L, \Phi, \bar{\Phi}) + \int d^2\theta W(\Phi) + \int d^2\bar{\theta} \bar{W}(\bar{\Phi}). \quad (11.6)$$

Besides the gauge invariance of $B_{\mu\nu}$ which does not act on the superfields, we also impose a two-parameter global symmetry acting on Φ with variations

$$\delta\Phi = \alpha - i\beta. \quad (11.7)$$

In this formulation, all three symmetries trivially commute. Nevertheless, in the version where $B_{\mu\nu}$ is dualized to a scalar, or in the version where $\text{Im } \phi$ (for instance) is transformed into a second antisymmetric tensor, the three-parameter symmetry realizes a Heisenberg algebra acting either on three scalars according to eq. (11.4), as in the hypermultiplet formulation of IIB strings compactified to four dimensions, or on two tensors and one scalar according to eqs. (11.1) and (11.3). The Lagrangian compatible with the required symmetry (11.7) has

$$\mathcal{H}(L, \Phi, \bar{\Phi}) = \mathcal{F}(L) + [AL + B]\Phi\bar{\Phi}, \quad W(\Phi) = k\Phi, \quad (11.8)$$

with an arbitrary function $\mathcal{F}(L)$ and real constants A and B .¹ The constant k generates a C -dependent potential $V = |k|^2/(AC + B)$ which does not admit a vacuum if $A \neq 0$. We take then $k = 0$.

The superfields Φ and L provide an off-shell representation of the $\mathcal{N} = 2$ single-tensor multiplet. On the $\mathcal{N} = 1$ Lagrangian, the condition for a second supersymmetry is [18]

$$\frac{\partial^2 \mathcal{H}}{\partial L^2} + 2 \frac{\partial^2 \mathcal{H}}{\partial \Phi \partial \bar{\Phi}} = 0, \quad (11.9)$$

which in turn indicates that

$$\mathcal{F}_{\mathcal{N}=2}(L) = -\frac{A}{3} L^3 - BL^2. \quad (11.10)$$

¹Of course, B can be eliminated by a constant shift of L .

The same theory is given by

$$\widehat{\mathcal{F}}_{N=2}(L) = -\frac{1}{3A^2} (AL + B)^3. \quad (11.11)$$

Hence, the $\mathcal{N} = 2$ theory compatible with complex shift symmetry of Φ is the sum

$$\mathcal{L}_{N=2} = \int d^2\theta d^2\bar{\theta} \left[A \left(-\frac{1}{3}L^3 + L\Phi\bar{\Phi} \right) + B(-L^2 + \Phi\bar{\Phi}) \right] \quad (11.12)$$

of a trilinear interacting term and of a free term where the symmetry is trivial. If canonical dimensions are assigned to L and Φ , A has dimension $(\text{mass})^{-1}$ and B is dimensionless.

For further use, we need the bosonic component expansion of this superfield theory. Using (8.14) and the expansion of Φ

$$\Phi(x, \theta, \bar{\theta}) = \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \theta\theta f - \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square\phi,$$

we obtain²

$$\begin{aligned} \mathcal{L}_{N=2, bos.} = & (AC + B) \left[\frac{1}{2}(\partial_\mu C)^2 + (\partial_\mu\phi)(\partial^\mu\bar{\phi}) + \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} \right] \\ & - \frac{i}{12}A \epsilon^{\mu\nu\rho\sigma}(\bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi})H_{\nu\rho\sigma}. \end{aligned} \quad (11.13)$$

Since, $\partial_{[\mu}H_{\nu\rho\sigma]} = 0$, the variation (11.7) of ϕ induces a total derivative. Kinetic terms are positive if $AC + B > 0$. If $A \neq 0$, B can be eliminated by shifting C . The (shifted) field C will be assumed strictly positive and the two options are an interacting, cubic theory with $A > 0$ and $B = 0$, or the free theory $A = 0$, $B > 0$.

We may then perform two supersymmetric duality transformations [90] on theory (11.8), either turning the linear L into a chiral S or turning the chiral Φ into a second linear multiplet L' . The first transformation leads to

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left[\widetilde{\mathcal{F}}(\mathcal{Y}) + B\Phi\bar{\Phi} \right], \quad (11.14)$$

²The auxiliary field f vanishes.

where $\widetilde{\mathcal{F}}(\mathcal{Y})$ is the Legendre transform of $\mathcal{F}(L)$ and the variable is³ $\mathcal{Y} = S + \bar{S} + A\Phi\bar{\Phi}$. Invariance of \mathcal{Y} under shift symmetries (11.7) requires a compensating variation of S :

$$\delta_H S = (\alpha\delta_X + \beta\delta_Y + \gamma\delta_Z)S = -A(\alpha + i\beta)\Phi + 2i\gamma, \quad (11.15)$$

where the axionic shift symmetry of $\text{Im } S$ is dual to the gauge symmetry of $B_{\mu\nu}$, and the subscripts X, Y, Z make clear the correspondence with the transformations (11.4). Indeed, since

$$[\delta'_H, \delta_H]S \equiv -A(\alpha' + i\beta')\delta_H\Phi + A(\alpha + i\beta)\delta'_H\Phi = 2iA(\alpha'\beta - \alpha\beta'), \quad [\delta_H, \delta'_H]\Phi = 0, \quad (11.16)$$

the chiral theory has Heisenberg symmetry. Moreover, the theory (11.14) has another symmetry M rotating the chiral superfield Φ , as already mentioned in the Introduction (see eq. (11.5)).

For the $\mathcal{N} = 2$ single-tensor theory (11.12), the dual hypermultiplet theory⁴ is

$$\mathcal{L}_{\mathcal{N}=2} = \int d^2\theta d^2\bar{\theta} \mathcal{K}(\mathcal{Y}) = \frac{2}{3A^2} \int d^2\theta d^2\bar{\theta} (A\mathcal{Y} + B^2)^{3/2}. \quad (11.17)$$

Eliminating some derivatives, the limiting case $A = 0$ is a free theory. As required for a hyper-Kähler sigma-model, the determinant of the Kähler metric is constant (and positive).

A useful change of variable is

$$\hat{S} = S - \frac{A}{2}\Phi^2, \quad \mathcal{Y} = \hat{S} + \bar{\hat{S}} + \frac{A}{2}(\Phi + \bar{\Phi})^2. \quad (11.18)$$

and transformation (11.15) becomes $\delta_H \hat{S} = -2A\alpha\Phi + 2i\gamma$. With these variables, the transformations with parameters β and γ only act as shift symmetries of

³Notice that $\int d^2\theta d^2\bar{\theta} \Phi\bar{\Phi} = \frac{1}{A} \int d^2\theta d^2\bar{\theta} \mathcal{Y} + \text{derivative}$.

⁴With positive Kähler metric.

$\text{Im } \Phi$ and $\text{Im } \hat{S}$ respectively. In terms of variables \mathcal{Y} , $\text{Im } \hat{S}$, $\text{Re } \Phi$ and $\text{Im } \Phi$, one immediately deduces that the most general Heisenberg-invariant supersymmetric theory is of the form (11.14).

Performing the second duality transformation of the chiral Φ into a linear L' , always leads to the dual theory

$$\mathcal{L} = \int d^2\theta d^2\bar{\theta} \left[\mathcal{F}(L) - \frac{1}{2} \frac{L'^2}{AL + B} \right], \quad (11.19)$$

with \mathcal{F} given in eq. (11.10). Expression (11.19) is actually the most general $\mathcal{N} = 1$ Lagrangian for L and L' with symmetry

$$\delta L' = \alpha(AL + B). \quad (11.20)$$

This transformation, which links the two antisymmetric tensors in L and L' as in variation (11.1), forms with their respective gauge symmetries a Heisenberg algebra realized as in type IIB strings.

Instead of $\text{Im } \Phi$, we could have chosen to dualize $e^{ia}\Phi$ for any phase a , since

$$\int d^2\theta d^2\bar{\theta} (AL + B)\Phi\bar{\Phi} = \frac{1}{2} \int d^2\theta d^2\bar{\theta} (AL + B)(e^{ia}\Phi + e^{-ia}\bar{\Phi})^2 + \text{derivative}.$$

The result would be again theory (11.19). This is a consequence of symmetry M , which is however fixed by the choice of dualization and does not act on L' .

11.1.2 Hyper-Kähler Metrics with Heisenberg Symmetry

The Kähler coordinates defined by $\mathcal{N} = 1$ chiral superfields S and Φ are not necessarily the most appropriate to describe a hyper-Kähler manifold. There is a 'standard' set of coordinates used to describe hyper-Kähler metrics with

shift isometries in the literature. For comparison purposes, we define in this subsection these coordinates in terms of our superfield components.

For any hyper-Kähler manifold with a shift symmetry, one can find coordinates in which the metric has the Gibbons-Hawking form [100]

$$ds^2 = f(\vec{x}) dx_i dx_i + f(\vec{x})^{-1} (d\tau + \omega_i dx_i)^2, \quad (11.21)$$

with condition $\vec{\nabla} \times \vec{\omega} = \vec{\nabla} f$. Imposing the requirement of a Heisenberg symmetry acting according to

$$\delta_H x_1 = \sqrt{2} \alpha, \quad \delta_H x_2 = -\sqrt{2} \beta, \quad \delta_H x_3 = 0, \quad \delta_H \tau = -\sqrt{2} \alpha x_2 + \gamma \quad (11.22)$$

also defines $d\tau + x_1 dx_2$ as the invariant derivative of τ and indicates that $\vec{\omega} = (0, x_1, 0)$. The value of $f(\vec{x})$ follows then from $\vec{\nabla} \times \vec{\omega} = \vec{\nabla} f$. This last condition is invariant under $\vec{\omega} \rightarrow \vec{\omega} + \vec{\nabla} \lambda(\vec{x})$, for any gauge function $\lambda(\vec{x})$. In turn, invariance of the metric requires the compensating transformation $\tau \rightarrow \tau - \lambda(\vec{x})$.

From the $\mathcal{N} = 2$ Kähler potential (11.17), the Kähler metric can be written⁵

$$ds^2 = \frac{1}{2}(A\mathcal{Y} + B^2)^{-1/2} \left[\frac{1}{4} d\mathcal{Y}^2 + \left(d \operatorname{Im} S + i \frac{A}{2} (\Phi d\bar{\Phi} - \bar{\Phi} d\Phi) \right)^2 \right] + (A\mathcal{Y} + B^2)^{1/2} d\Phi d\bar{\Phi}, \quad (11.23)$$

using coordinates $(\mathcal{Y}, \operatorname{Im} S, \operatorname{Re} \Phi, \operatorname{Im} \Phi)$. The supersymmetric duality transformation from L to S exchanges a real scalar $C = L|_{\theta=0}$, invariant under Heisenberg variations, and $\operatorname{Re} S$ with variation (11.15). The Legendre transformation defines the change of variable from \mathcal{Y} to C :

$$AC + B = \sqrt{A\mathcal{Y} + B^2}. \quad (11.24)$$

Then, in terms of coordinates $(C, \operatorname{Im} S, \operatorname{Re} \Phi, \operatorname{Im} \Phi)$, the metric becomes

$$ds^2 = \frac{AC + B}{2} [dC^2 + 2 d\Phi d\bar{\Phi}] + \frac{2}{(AC + B)} (d\tau + A \operatorname{Re} \Phi d \operatorname{Im} \Phi)^2. \quad (11.25)$$

⁵From here on, we do not distinguish chiral superfields S and Φ and their lowest complex scalar components.

This is the Gibbons-Hawking metric (11.21) with $\vec{x} = (\sqrt{2} \operatorname{Re} \Phi, \sqrt{2} \operatorname{Im} \Phi, C)$ and

$$\tau = \frac{1}{2} (\operatorname{Im} S - A \operatorname{Re} \Phi \operatorname{Im} \Phi) = \frac{1}{2} \operatorname{Im} \hat{S}.$$

The function

$$f(\vec{x}) = \frac{AC + B}{2} \quad (11.26)$$

solves the hyper-Kähler condition $\vec{\nabla} \times \vec{\omega} = \vec{\nabla} f$ with $\vec{\omega} = (0, \frac{A}{2} x_1, 0)$. Choosing for instance $\lambda = -\frac{A}{2} x_1 x_2$ turns then $\vec{\omega}$ into $(-\frac{A}{2} x_2, 0, 0)$ and $d\tau + \frac{A}{2} x_1 dx_2$ into $d\tau - \frac{A}{2} x_2 dx_1$. Similarly, a rotation of Φ

$$\delta_M x_1 = m x_2, \quad \delta_M x_2 = -m x_1,$$

which is compatible with the shift symmetry (11.7), corresponds to $\lambda(\vec{x}) = \frac{Am}{4} (x_2^2 - x_1^2)$. It is the isometry M of metric (11.25).

The conclusion is that the Gibbons-Hawking ansatz for the hyper-Kähler metric corresponds to coordinates where $\operatorname{Re} S$ is replaced by its Legendre dual C , which is also the lowest scalar component of the linear superfield dual to S .

11.2 The Universal Hypermultiplet in $\mathbf{N} = 2$ Supergravity

Hypermultiplet scalars of $\mathcal{N} = 2$ supergravity live on $4n$ -dimensional quaternion-Kähler manifolds with holonomy included in $Sp(2n) \times Sp(2)$. Supergravity requires that the curvature of these Einstein spaces is proportional to the gravitational coupling κ^2 [14]. Hence, the decoupling limit $\kappa \rightarrow 0$ turns the hypermultiplet manifold into a Ricci-flat hyper-Kähler space, as required by global $\mathcal{N} = 2$ supersymmetry [16]. For a single hypermultiplet, or a four-dimensional quaternion-Kähler manifold, the defining condition on the holonomy is not per-

inent since $Sp(2) \times Sp(2) \sim SO(4)$. The relevant condition is then self-duality of the Weyl tensor.

11.2.1 The Calderbank-Pedersen Metric with Heisenberg Symmetry

Calderbank and Pedersen [101] have classified all four-dimensional Einstein metrics with self-dual Weyl curvature and two commuting isometries. Using coordinates $(\rho, \eta, \varphi, \tau)$ with the isometries acting as shifts of φ and τ , their metrics are written in terms of any single function $F(\rho, \eta)$ verifying

$$\frac{\partial^2 F}{\partial \rho^2} + \frac{\partial^2 F}{\partial \eta^2} = \frac{3F}{4\rho^2}. \quad (11.27)$$

It is simple to see [28] that metrics with Heisenberg symmetry are then obtained if F does not depend on η , *i.e.* if⁶

$$\sqrt{\rho} F(\rho) = \frac{1}{2}[\rho^2 - \chi], \quad (11.28)$$

with an arbitrary real parameter χ . The Calderbank-Pedersen metric with Heisenberg symmetry (the CPH metric) reads then

$$ds_{CPH}^2 = \frac{\rho^2 + \chi}{(\rho^2 - \chi)^2} (d\rho^2 + d\eta^2 + d\varphi^2) + \frac{4\rho^2}{(\rho^2 - \chi)^2 (\rho^2 + \chi)} (d\tau + \eta d\varphi)^2. \quad (11.29)$$

The coordinate ρ is positive, $\rho > 0$, and positivity of the metric requires $\rho^2 + \chi > 0$, a stronger condition if χ is negative. It is an Einstein metric with negative curvature, and is Kähler only if $\chi = 0$. Notice that if $\chi \neq 0$, the rescaling $(\rho, \eta, \varphi, \tau) \rightarrow (|\chi|^{1/2}\rho, |\chi|^{1/2}\eta, |\chi|^{1/2}\varphi, |\chi|\tau)$ turns χ in metric (11.29) into ± 1 . This is

⁶The metric does not make sense without the $\rho^{3/2}$ contribution to F and the overall normalization of F is a choice of coordinates. Our χ is $\hat{\chi}$ in Ref. [28].

not true if we turn on string interactions, such as in the presence of D-branes where the dilaton, or equivalently the field ρ , couples to the Dirac-Born-Infeld (DBI) action in a non-trivial way (see section 11.3). For this reason, we keep explicitly χ throughout the paper. We may use a new coordinate $V = \rho^2$ with metric

$$ds_{CPH}^2 = \frac{V + \chi}{(V - \chi)^2} \left(\frac{dV^2}{4V} + d\eta^2 + d\varphi^2 \right) + \frac{4V}{(V - \chi)^2(V + \chi)} (d\tau + \eta d\varphi)^2. \quad (11.30)$$

The particular case $\chi = 0$ has extended symmetry: it is the $SU(2, 1)/SU(2) \times U(1)$ metric with Kähler potential

$$K(\hat{S}, \bar{\hat{S}}, \Phi, \bar{\Phi}) = -\ln V, \quad V = \hat{S} + \bar{\hat{S}} - (\Phi + \bar{\Phi})^2, \quad (11.31)$$

and with $\Phi = \frac{1}{\sqrt{2}}(\eta + i\varphi)$, $\tau = -\frac{1}{2} \text{Im } \hat{S}$.

The CPH metric is invariant under four isometry variations acting on coordinates (η, φ, τ) :

$$\begin{aligned} \delta_X \eta &= \sqrt{2}, & \delta_Y \eta &= 0, & \delta_Z \eta &= 0, & \delta_M \eta &= \varphi, \\ \delta_X \varphi &= 0, & \delta_Y \varphi &= -\sqrt{2}, & \delta_Z \varphi &= 0, & \delta_M \varphi &= -\eta, \\ \delta_X \tau &= -\sqrt{2} \varphi, & \delta_Y \tau &= 0, & \delta_Z \tau &= 1, & \delta_M \tau &= \frac{1}{2}(\eta^2 - \varphi^2). \end{aligned} \quad (11.32)$$

The non-zero commutators are

$$[X, Y] = 2Z, \quad [M, X] = Y, \quad [M, Y] = -X. \quad (11.33)$$

Hence, X, Y and Z generate the Heisenberg algebra and Z is a central extension of a two-dimensional euclidean algebra generated by M (which rotates φ and η), X and Y (which translate φ and η). With these conventions,

$$\delta_H \Phi = (\alpha X + \beta Y + \gamma Z)\Phi = \alpha - i\beta, \quad \delta_H \hat{S} = 4\alpha \Phi - 2i\gamma \quad (11.34)$$

and V is invariant.

The metric (11.30) appears in the one-loop-corrected Lagrangian of the universal hypermultiplet of type II strings, reduced to four dimensions, with the NS-NS and R-R tensors dualized to scalars with shift symmetry [28]. At one-loop order, the four-dimensional dilaton field is related to coordinate V and parameter χ by

$$e^{-2\phi_4} = V - \chi, \quad \chi = -\chi_1, \quad \chi_1 = \frac{\chi_E}{12\pi}, \quad (11.35)$$

where χ_E is the Euler number of the internal CY_3 manifold. The real number χ_1 encodes the one-loop correction [28]. Notice that this relation also indicates that $V - \chi = V + \chi_1 > 0$, which is stronger than $V = \rho^2 > 0$ if the Euler number is negative ($\chi > 0$). Since positivity of the CPH metric also requires $V + \chi > 0$ if $\chi < 0$, the domain of V is naturally restricted to $V > |\chi|$.

The R-R scalar is

$$C_0 \equiv \eta, \quad (11.36)$$

and is shifted by symmetry X . Finally, Poincaré duality gives the following equivalences

$$d\varphi \quad \sim \quad F_3 = dC_2 - \eta dB_2,$$

$$d\tau + \eta d\varphi \quad \sim \quad H_3 = dB_2.$$

In the scalar version, the central charge is the shift Z of τ (related to the NS-NS tensor B_2) while in the two-tensor version, it is the gauge variation of the (R-R) tensor C_2 . Writing η and φ in a complex Φ is conventional: we always use

$$\Phi = \frac{1}{\sqrt{2}}(\eta + i\varphi).$$

In the previous section, we found a unique four-dimensional hyper-Kähler manifold with Heisenberg symmetry. It also admits the fourth isometry M ro-

tating Φ . In the quaternion-Kähler case, the theorem of Calderbank-Pedersen [101] leads then to a very similar uniqueness conclusion. We will see how these two results are connected when taking an appropriate zero-curvature limit. But we first want to obtain the $\mathcal{N} = 2$ supergravity coupling of the universal hypermultiplet on the CPH manifold.

11.2.2 Coupling to $\mathcal{N} = 2$ Supergravity

There are different methods to construct hypermultiplet couplings to $\mathcal{N} = 2$ supergravity. The simplest procedure, which is however not the most general, is to use hypermultiplets coupled to local $\mathcal{N} = 2$ superconformal symmetry [102] and to perform a quaternionic quotient [29, 30] using supplementary hypermultiplet(s) and non-propagating vector multiplet(s). In this section, we use this procedure to obtain the supergravity theory of the one-loop-corrected dilaton hypermultiplet.

Related constructions, using more general but also more complicated methods, can be found in ref. [108], in the language of projective superspace or in ref. [109], using harmonic superspace.

Conformal $\mathcal{N} = 2$ supergravity is the gauge theory of $SU(2, 2|2)$, which has a $SU(2)_R \times U(1)_R$ R -symmetry with non-propagating gauge fields. Pure Poincaré $\mathcal{N} = 2$ supergravity is obtained from the superconformal coupling of one propagating vector multiplet⁷ (which may be charged under $U(1)_R$) and one hypermultiplet (charged under $SU(2)_R$) by gauge-fixing of the extraneous symmetries. These two multiplets include in particular the compensating fields used

⁷Its gauge field is the graviphoton.

in the gauge-fixing to the Poincaré theory.

For the superconformal construction of our particular hypermultiplet sigma-model, we also need a physical hypermultiplet, with positive kinetic metric, to describe the dilaton multiplet. In addition, for the quaternionic quotient, we need a non-propagating vector multiplet with gauge field W_μ , gauging a specific generator T to be discussed below, and, since the elimination of the algebraic vector multiplet involves three constraints and one gauge choice on scalar fields, we also need a third non-physical hypermultiplet. Its kinetic metric can have a positive or negative sign, depending on the constraints induced by the choice of T . Hence, we need to consider the $\mathcal{N} = 2$ superconformal theory of two vector multiplets and three hypermultiplets. The superconformal hypermultiplet scalar sector has then an ‘automatic’ $Sp(2, 4)$ global symmetry in which the gauge generator T of the quaternionic quotient is chosen.

11.2.3 $Sp(2, 4)$

In the following, we consider three hypermultiplets coupled to (superconformal) $\mathcal{N} = 2$ supergravity. One (compensating) hypermultiplet has negative signature, the physical hypermultiplet has positive signature, the third hypermultiplet, associated to the non-propagating vector multiplet, may have a positive or negative signature, depending on the constraints applied to the scalar fields. In any case, we are considering $Sp(2, 4)$ -invariant supergravity couplings of $\mathcal{N} = 2$ hypermultiplets.

The hypermultiplet scalars are A_i^α , with $SU(2)_R$ index $i = 1, 2$ and $Sp(2, 4)$ index $\alpha = 1, \dots, 6$. They transform in representation $(\mathbf{6}, \mathbf{2})$ of $Sp(2, 4) \times SU(2)_R$.

Their conjugates are⁸

$$A_\alpha^i = (A_i^\alpha)^* = \epsilon^{ij} \rho_{\alpha\beta} A_j^\beta \quad (11.37)$$

with $\rho^{\alpha\beta} \rho_{\beta\gamma} = -\delta_\gamma^\alpha$ and $\epsilon^{ij} \epsilon_{jk} = -\delta_k^i$. We choose the $Sp(2, 4)$ -invariant metric as

$$\rho = I_3 \otimes i\sigma_2 = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix} \quad (11.38)$$

and we use

$$d = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}, \quad \eta = \text{diag}(-1, 1, -1), \quad \rho d \rho = -d. \quad (11.39)$$

In our choice of η , direction 1 corresponds to the superconformal compensator, direction 2 to the physical hypermultiplet and our choice of quaternionic quotient will require a negative metric in direction 3; otherwise, our construction does not work. On scalar fields, $Sp(2, 4)$ acts according to

$$\delta A_i^\alpha = g t^\alpha{}_\beta A_i^\beta, \quad \delta A_\alpha^i = g t_\alpha{}^\beta A_\beta^i, \quad t_\alpha{}^\beta = -\rho_{\alpha\gamma} t^\gamma{}_\delta \rho^{\delta\beta}. \quad (11.40)$$

Since relation (11.37) also implies $t_\alpha{}^\beta = (t^\alpha{}_\beta)^*$, the choice (11.38) and the invariance of $d_\beta^\alpha A_\alpha^i A_i^\beta$ lead to

$$t = \begin{pmatrix} U & \eta Q \\ -\eta Q^* & U^* \end{pmatrix}, \quad U^\dagger = -\eta U \eta, \quad Q = Q^T, \quad t^\dagger = -d t d. \quad (11.41)$$

This is an element of $Sp(2, 4)$: U generates the $U(1, 2)$ subgroup (9 generators) and Q (12 generators) generates $Sp(2, 4)/U(1, 2)$. The (2×2) matrix $A^\dagger d t A$, with matrix elements $A_\alpha^i d_\beta^\alpha t^\beta{}_\gamma A_j^\gamma$, is antihermitian, as required by gauge invariance of $A^\dagger d A$, and traceless.

⁸We follow the conventions of the second paper of ref. [102].

11.2.4 The Heisenberg Subalgebra of $SU(1, 2)$ and $Sp(2, 4)$

At string tree-level, the universal hypermultiplet of the dilaton in type II strings lives, when formulated in terms of four real scalars, on the quaternion-Kähler and Kähler manifold $SU(1, 2)/SU(2) \times U(1) = U(1, 2)/U(2) \times U(1)$. Since $U(1, 2) = SU(1, 2) \times U(1)_0$ is maximal in $Sp(2, 4)$, $Sp(2, 4)$ has a unique generator commuting with $SU(1, 2)$: the generator of $U(1)_0$. At one-loop however, the isometry is reduced and includes the Heisenberg algebra which is known to be a subalgebra of $SU(1, 2)$. We need to find the most general generator T of $Sp(2, 4)$ which commutes with a Heisenberg subalgebra. In the following subsections, we will perform the quaternionic quotient construction induced by the gauging of T .

Since elements U of the $U(1, 2)$ algebra verify $U^\dagger = -\eta U \eta$ and we have chosen $\eta = \text{diag}(-1, 1, -1)$, a generic U is

$$U = \begin{pmatrix} ia & A & B \\ \bar{A} & ib & C \\ -\bar{B} & \bar{C} & ic \end{pmatrix}, \quad (11.42)$$

with a, b, c real, A, B, C complex and elements of $SU(1, 2)$ are traceless. On a three-dimensional complex vector, $U(1, 2)$ variations are $\delta A = UA$.

We may define the Heisenberg subalgebra as the $U(1, 2)$ transformations leaving $A_1 - A_2$ invariant: $(\delta_H A)_1 - (\delta_H A)_2 = (UA)_1 - (UA)_2 = 0$. The transformations acting on A_1 and A_2 are generated by the following three elements

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & i \\ i & -i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} i & -i & 0 \\ i & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (11.43)$$

which verify

$$0 = XZ = ZX = YZ = ZY = Z^2, \quad XY = -YX = Z, \quad X^2 = Y^2 = iZ. \quad (11.44)$$

The Heisenberg algebra

$$[X, Y] = 2Z, \quad [X, Z] = [Y, Z] = 0 \quad (11.45)$$

is then realized as a subalgebra of $SU(1, 2)$, with variations

$$\delta_H A = (\alpha X + \beta Y + \gamma Z) A = \begin{pmatrix} i\gamma & -i\gamma & \alpha + i\beta \\ i\gamma & -i\gamma & \alpha + i\beta \\ -\alpha + i\beta & \alpha - i\beta & 0 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} \quad (11.46)$$

in the fundamental representation. Since Z is a central charge of the Heisenberg algebra, we are interested in the elements of $U(1, 2)$ which commute with Z .

They form an algebra generated by five elements, U_0, M, X, Y and Z , with

$$U_0 = iI_3, \quad M = i \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (11.47)$$

(U_0 generates the abelian factor of $U(1, 2) = SU(1, 2) \times U(1)_0$). Besides the Heisenberg algebra generated by X, Y, Z , we also have

$$[M, X] = 3Y, \quad [M, Y] = -3X \quad (11.48)$$

and M generates a rotation of (X, Y) leaving $X^2 + Y^2 = 2iZ$ invariant: $[M, X^2 + Y^2] = 2i[M, Z] = 0$.

One then easily checks that the most general $U(1, 2)$ generator which commutes with the Heisenberg algebra generated by X, Y, Z is proportional to

$$\widehat{T} = U_0 + \chi Z = i \begin{pmatrix} 1 + \chi & -\chi & 0 \\ \chi & 1 - \chi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_0 = iI_3, \quad (11.49)$$

where χ is an arbitrary real number. If $\chi = 0$, $\widehat{T} = U_0$ commutes with the whole $U(1, 2)$. If $\chi \neq 0$, \widehat{T} commutes with the Heisenberg algebra supplemented by U_0 and M . The extension to $Sp(2, 4)$ is straightforward. Requiring that

$$T = \begin{pmatrix} \hat{T} & 0 \\ 0 & \hat{T}^* \end{pmatrix} \quad (11.50)$$

in $Sp(2, 4)$ commutes with an element of $Sp(2, 4)/U(1, 2)$ corresponds to find a (nonzero) symmetric matrix Q in eq. (11.41) such that $\hat{T}^\dagger Q$ is also antisymmetric, which is impossible.⁹ Hence, T is also the most general generator in $Sp(2, 4)$ which commutes with the Heisenberg algebra generated by X, Y and Z in $SU(1, 2)$. It actually commutes with X, Y, Z, M and U_0 .

11.2.5 $\mathcal{N} = 2$ Supergravity Scalar Lagrangian

To construct the scalar kinetic metric, the relevant terms of the $\mathcal{N} = 2$ conformal supergravity Lagrangian are [102, 29, 30]

$$\begin{aligned} e^{-1} \mathcal{L} = & d_\beta^\alpha (D_\mu A_i^\beta) (D^\mu A_\alpha^i) + (g d_\beta^\alpha A_\alpha^i T^\beta{}_\gamma A_k^\gamma Y_i^k + \text{c.c.}) \\ & + \frac{1}{6} R (-X_0 \bar{X}_0 + d_\beta^\alpha A_\alpha^i A_i^\beta) + d(X_0 \bar{X}_0 + \frac{1}{2} d_\beta^\alpha A_\alpha^i A_i^\beta). \end{aligned} \quad (11.51)$$

The complex scalar X_0 is the partner of the graviphoton, $Y_j^i, Y_i^i = 0$, is the triplet of real auxiliary scalars in the non-propagating vector multiplet with gauge field W_μ used in the quaternionic quotient. The covariant derivatives are

$$\begin{aligned} D_\mu A_i^\alpha &= \partial_\mu A_i^\alpha - g' W_\mu T^\alpha{}_\beta A_i^\beta - g V_{\mu i}{}^j A_j^\alpha, \\ D_\mu A_\alpha^i &= \partial_\mu A_\alpha^i - g' W_\mu T_\alpha{}^\beta A_\beta^i - g V_\mu{}^i{}_j A_\alpha^j, \end{aligned} \quad (11.52)$$

⁹This would not be true for $\hat{T} = Z$, which commutes with a larger subalgebra of $Sp(2, 4)$. The U_0 component is necessary.

where g and g' are $SU(2)_R$ and $U(1)_T$ coupling constant. The (anti-hermitian) $SU(2)$ gauge fields $V_{\mu i}^j$, $V_{\mu i}^i = 0$, and the real auxiliary scalar d belong to the multiplet of superconformal gauge fields:

$$V_{\mu i}^j = \frac{i}{2} V_{\mu}^x (\sigma^x)_i^j, \quad V_{\mu i}^j = \epsilon^{ik} \epsilon_{jl} V_{\mu k}^l = (V_{\mu i}^j)^*.$$

We will commonly use a matrix notation, with a 6×2 complex matrix A and its 2×6 conjugate A^\dagger replacing A_i^α and A_α^i . Condition (11.37) implies that A contains six complex components only. It also implies, in particular, that $A^\dagger dA = \frac{1}{2} \text{Tr}(A^\dagger dA) I_2$. Since $V_\mu = -V_\mu^\dagger$, the Lagrangian and the derivatives read

$$\begin{aligned} e^{-1} \mathcal{L} &= \text{Tr}(D_\mu A^\dagger) d(D^\mu A) + g \text{Tr} Y A^\dagger dTA + \text{c.c.} \\ &\quad + \frac{1}{6} R(-X_0 \bar{X}_0 + \text{Tr} A^\dagger dA) + d(X_0 \bar{X}_0 + \frac{1}{2} \text{Tr} A^\dagger dA); \end{aligned} \quad (11.53)$$

$$D_\mu A = \partial_\mu A - g' W_\mu TA - g A V_\mu,$$

$$D_\mu A^\dagger = \partial_\mu A^\dagger - g' W_\mu A^\dagger T^\dagger + g V_\mu A^\dagger.$$

Constraints are obtained from the elimination of the auxiliary fields and from the gauge-fixing of dilatation symmetry in the Poincaré theory:

- Einstein frame gauge-fixing condition and d auxiliary field equation:

$$X_0 \bar{X}_0 = \frac{1}{\kappa^2}, \quad \text{Tr} A^\dagger dA = -\frac{2}{\kappa^2}. \quad (11.54)$$

The second condition is invariant under $SU(2)_R$ and $Sp(4, 2)$. With an $SU(2)$ gauge choice, it allows to eliminate four scalar fields and would lead to the $Sp(4, 2)/Sp(4) \times Sp(2)$ sigma-model.

- Auxiliary fields Y_j^i :

$$A^\dagger dTA = 0. \quad (11.55)$$

Since this 2×2 matrix is traceless and antihermitian, these conditions eliminate three scalars and the associated abelian gauge invariance removes a fourth field.

The $SU(2)_R$ gauge fields $V_{\mu i}^j$ and the abelian W_μ have then algebraic field equations:

- Gauge field W_μ , associated with generator T :

$$W_\mu = \frac{\text{Tr}(\partial_\mu A^\dagger dTA - A^\dagger dT \partial_\mu A)}{2g' \text{Tr}(A^\dagger T^\dagger dTA)}. \quad (11.56)$$

- $SU(2)_R$ gauge fields $V_{\mu i}^j$:

$$V_\mu = -\frac{\partial_\mu A^\dagger dA - A^\dagger d \partial_\mu A}{g \text{Tr}(A^\dagger dA)}. \quad (11.57)$$

According to the second eq. (11.54), the denominator is $-2g/\kappa^2$.

At this point, the scalar kinetic Lagrangian in theory (11.51) reduces to

$$\begin{aligned} e^{-1} \mathcal{L} &= e^{-1} (\mathcal{L}_{kin.} + \mathcal{L}_T + \mathcal{L}_{SU(2)}) \\ &= \text{Tr}(\partial_\mu A^\dagger) d(\partial^\mu A) - g'^2 \text{Tr}(A^\dagger T^\dagger dTA) W^\mu W_\mu - \frac{g^2}{\kappa^2} \text{Tr}(V^\mu V_\mu). \end{aligned} \quad (11.58)$$

The scalar fields are submitted to constraints (11.54) and (11.55) and the gauge fields W_μ and $V_{\mu i}^j$ are defined by their field equations (11.56) and (11.57).

To study the constraints (11.54) and (11.55) for our specific choice (11.49) and (11.50) of gauged generator T , we introduce two three-component complex vectors:

$$A_i^\alpha = \begin{pmatrix} \vec{A}_+ & \vec{A}_- \\ -\vec{A}_-^* & \vec{A}_+^* \end{pmatrix}, \quad A_\alpha^i = \begin{pmatrix} \vec{A}_+^* & \vec{A}_-^* \\ -\vec{A}_- & \vec{A}_+ \end{pmatrix}, \quad (11.59)$$

verifying the reality condition (11.37). On each doublet A_{+a}, A_{-a} , $a = 1, 2, 3$, act two different $SU(2)$ groups. Firstly, the superconformal $SU(2)_R$ acts on \pm indices. Secondly, $Sp(2, 4) \supset Sp(2)_1 \times Sp(2)_2 \times Sp(2)_3 \sim SU(2)_1 \times SU(2)_2 \times SU(2)_3$ and $(A_{+a}, -A_{-a}^*)$ is a doublet of $SU(2)_a$. One could define three quaternions

$$Q_a = \begin{pmatrix} A_{+a} & A_{-a} \\ -A_{-a}^* & A_{+a}^* \end{pmatrix} \quad a = 1, 2, 3 \quad (11.60)$$

with a left action of $SU(2)_a$ and a right action of the superconformal $SU(2)_R$. They verify (for each a)

$$Q_a Q_a^\dagger = Q_a^\dagger Q_a = \det Q_a I_2, \quad \det Q_a = |A_{+a}|^2 + |A_{-a}|^2. \quad (11.61)$$

The second condition (11.54) from $\mathcal{N} = 2$ supergravity becomes:

$$\vec{A}_+^* \cdot \vec{A}_+ + \vec{A}_-^* \cdot \vec{A}_- = -\frac{1}{\kappa^2}, \quad \vec{A}^* \cdot \vec{A} = \vec{A}^\dagger \eta \vec{A} = -|A_1|^2 + |A_2|^2 - |A_3|^2. \quad (11.62)$$

With eq. (11.50), condition (11.55) leads to three (real) equations:

$$\vec{A}_+^\dagger i\eta \hat{T} \vec{A}_+ = \vec{A}_-^\dagger i\eta \hat{T} \vec{A}_-, \quad (11.63)$$

$$\vec{A}_-^\dagger i\eta \hat{T} \vec{A}_+ = 0$$

($[i\eta \hat{T}]^\dagger = i\eta \hat{T}$). With the explicit form of \hat{T} , eq. (11.49), and defining dimensionless fields $a_{\pm i} = \sqrt{2}\kappa A_{\pm i}$, the four constraints (11.62) and (11.63) read finally

$$\begin{aligned} I : \quad & |a_{+1}|^2 + |a_{-1}|^2 - |a_{+2}|^2 - |a_{-2}|^2 + |a_{+3}|^2 + |a_{-3}|^2 = 2, \\ II : \quad & -|a_{+1}|^2 + |a_{+2}|^2 - |a_{+3}|^2 - \chi |a_{+1} - a_{+2}|^2 \\ & = -|a_{-1}|^2 + |a_{-2}|^2 - |a_{-3}|^2 - \chi |a_{-1} - a_{-2}|^2, \end{aligned} \quad (11.64)$$

$$III : \quad 0 = -a_{+1}\bar{a}_{-1} + a_{+2}\bar{a}_{-2} - a_{+3}\bar{a}_{-3} - \chi(a_{+1} - a_{+2})(\bar{a}_{-1} - \bar{a}_{-2}).$$

They are invariant under Heisenberg variations (11.46) of \vec{a}_+ and \vec{a}_- . The case $\chi = 0$ has been considered by Galicki [29]. Since it leads to $SU(1,2)/SU(2) \times U(1)$, coordinates more appropriate for this larger isometry have been used.

11.2.6 Solving the Constraints

To solve the constraints (11.64), we insist on keeping in \vec{a}_- a field Φ which transforms under the Heisenberg variations¹⁰ $\delta_H \vec{a}_- = (\alpha X + \beta Y + \gamma Z) \vec{a}_-$ with a complex

¹⁰See eq. (11.46).

shift:

$$\delta_H \Phi = \alpha - i\beta. \quad (11.65)$$

This is the case if $a_{-1} = a_{-2}$, and a_{-3} is then invariant. We may define $\bar{\Phi} = a_{-1}/a_{-3}$ and constraint *III* reduces to $a_{+3} = (a_{+2} - a_{+1})\Phi$. Since

$$\delta_H \left(\frac{a_{+2} + a_{+1}}{a_{+2} - a_{+1}} \right) = -2i\gamma + 2(\alpha + i\beta) \frac{a_{+3}}{a_{+2} - a_{+1}} = -2i\gamma + 2\Phi \delta_H \bar{\Phi},$$

we finally define

$$S = \frac{a_{+2} + a_{+1}}{a_{+2} - a_{+1}} + Y, \quad \delta_H S = -2i\gamma + 2(\alpha + i\beta)\Phi \quad (11.66)$$

and the quantity

$$Y = S + \bar{S} - 2\Phi\bar{\Phi} \quad (11.67)$$

is invariant under Heisenberg variations. The algebra follows from $[\delta'_H, \delta_H] = (\alpha'\beta - \alpha\beta')[X, Y] = 2(\alpha'\beta - \alpha\beta')Z$:

$$[\delta'_H, \delta_H]S = 2(\alpha' + i\beta')\delta_H\Phi - 2(\alpha + i\beta)\delta'_H\Phi = -4i(\alpha'\beta - \alpha\beta') = 2(\alpha'\beta - \alpha\beta')Z.$$

These definitions are summarized in the choice

$$\vec{a}_- = \frac{K}{\Delta} \begin{pmatrix} \bar{\Phi} \\ \Phi \\ 1 \end{pmatrix}, \quad \vec{a}_+ = \frac{1}{\Delta} \begin{pmatrix} S - Y - 1 \\ S - Y + 1 \\ a \end{pmatrix}, \quad (11.68)$$

with complex fields S , Φ and a . The four available gauge choices have been used to take $\Delta = |\Delta|$, $K = |K|$ and $a_{-1} = a_{-2}$. Under Heisenberg variations, Δ and K are invariant. Hence, we are left with eight real scalar fields submitted to the four constraints (11.64) which drastically simplify:

$$\begin{aligned} I : \quad & \Delta^2(2 - |a_{+1}|^2 + |a_{+2}|^2 - |a_{+3}|^2) = K^2, \\ II : \quad & 2(S + \bar{S}) - |a|^2 - 4Y = 4\chi - K^2, \\ III : \quad & a = 2\Phi. \end{aligned} \quad (11.69)$$

Hence, the solution is

$$\vec{d}_- = \sqrt{\frac{Y+2\chi}{Y+\chi}} \begin{pmatrix} \bar{\Phi} \\ \Phi \\ 1 \end{pmatrix}, \quad \vec{d}_+ = \frac{1}{\sqrt{2(Y+\chi)}} \begin{pmatrix} S-Y-1 \\ S-Y+1 \\ 2\Phi \end{pmatrix}. \quad (11.70)$$

The solution implies $Y+\chi > 0$ if $\chi > 0$ or $Y+2\chi > 0$ if $\chi < 0$. The scalar kinetic Lagrangian (11.58) obtained from this solution is¹¹

$$\begin{aligned} \kappa^2 \mathcal{L} &= \frac{(Y+3\chi)}{4(Y+2\chi)(Y+\chi)^2} (\partial_\mu Y)^2 - \frac{2}{Y+\chi} \partial_\mu \Phi \partial^\mu \bar{\Phi} \\ &+ \frac{1}{2(Y+\chi)(Y+3\chi)} \left[\text{Im}(\partial_\mu S - 2\bar{\Phi} \partial_\mu \Phi) \right]^2 \\ &+ \frac{1}{2(Y+\chi)^2} \left[\text{Im}(\partial_\mu S - 2\bar{\Phi} \partial_\mu \Phi) \right]^2 + \frac{4(Y+2\chi)}{(Y+\chi)^2} \partial_\mu \Phi \partial^\mu \bar{\Phi}. \end{aligned} \quad (11.71)$$

The first line comes from the basic scalar kinetic terms $\mathcal{L}_{kin.}$ in Lagrangian (11.58). The second line is the contribution \mathcal{L}_T of the gauge field of T , the third line arises from the supergravity $SU(2)_R$ gauge fields. Each term is separately invariant under Heisenberg variations. Collecting terms, the final form of the theory is

$$\begin{aligned} \kappa^2 \mathcal{L} &= \frac{Y+3\chi}{(Y+\chi)^2} \left[\frac{1}{4} \frac{(\partial_\mu Y)^2}{Y+2\chi} + 2\partial_\mu \Phi \partial^\mu \bar{\Phi} \right] \\ &+ \frac{Y+2\chi}{(Y+3\chi)(Y+\chi)^2} \left(\partial_\mu \text{Im} \hat{S} - 4 \text{Re} \Phi \partial_\mu \text{Im} \Phi \right)^2, \end{aligned} \quad (11.72)$$

where

$$\hat{S} = S + \Phi^2, \quad (11.73)$$

for which $Y = \hat{S} + \bar{\hat{S}} - (\Phi + \bar{\Phi})^2$ and $\text{Im}(dS - 2\bar{\Phi} d\Phi) = d \text{Im} \hat{S} - 4 \text{Re} \Phi d \text{Im} \Phi$. From the existence of solutions (11.70) and positivity of the Lagrangian, the range of Y is $Y+\chi > 0$ if $\chi > 0$ and $Y+3\chi > 0$ if $\chi < 0$ Writing as usual

$$\mathcal{L} = \frac{1}{\kappa^2} g_{ab}(\partial_\mu q^a)(\partial^\mu q^b) = G_{ab}(\partial_\mu q^a)(\partial^\mu q^b), \quad (11.74)$$

¹¹All fields and parameter χ are dimensionless.

$q^a = (Y, \text{Re } \Phi, \text{Im } \Phi, \text{Im } \hat{S})$, and comparing $ds^2 = g_{ab} dq^a dq^b$ with expression (11.30), we see that the hypermultiplet kinetic metric g_{ab} is the CPH metric with

$$Y = V - 2\chi = \rho^2 - 2\chi, \quad (11.75)$$

and with¹²

$$\Phi = \frac{1}{\sqrt{2}}(\eta + i\varphi), \quad \text{Im } \hat{S} = -2\tau. \quad (11.76)$$

Positivity of kinetic terms is obtained if $V = \rho^2 > |\chi|$ which is, as explained at the end of subsection 11.2.1, the natural domain of V .

As already observed, the case $\chi = 0$ corresponds to the $SU(2, 1)/SU(2) \times U(1)$ metric

$$ds^2 = \frac{1}{Y^2} \left[\frac{1}{4} dY^2 + (d \text{Im } \hat{S} - 4 \text{Re } \Phi d \text{Im } \Phi)^2 \right] + \frac{2}{Y} d\Phi d\bar{\Phi}. \quad (11.77)$$

With Kähler coordinates \hat{S} and Φ , the Kähler potential is $K = -\ln Y$, with $Y = V = \hat{S} + \bar{\hat{S}} - (\Phi + \bar{\Phi})^2$.

This relatively simple construction of the one-loop-corrected dilaton hypermultiplet metric allows easily to derive the full $\mathcal{N} = 2$ supergravity Lagrangian, using $\mathcal{N} = 2$ superconformal tensor calculus [102, 29, 30].

11.3 Zero-Curvature Hyper-Kähler Limit

All quaternion-Kähler metrics are Einstein spaces with nonzero curvature. With one hypermultiplet, the scalar kinetic Lagrangian (11.74) verifies [14]

$$R_{ab} = -6 g_{ab} = -6\kappa^2 G_{ab}. \quad (11.78)$$

¹²This choice is not unique. We may for instance rotate Φ using isometry M .

The link with global $\mathcal{N} = 2$ supersymmetry is realized by defining a $\kappa \rightarrow 0$ hyper-Kähler limit of the CPH metric (11.30) or (11.72) in which, if feasible, the Heisenberg algebra does not contract to an abelian symmetry. As observed in Subsection 11.2.1, the magnitude of χ can be eliminated by rescaling of the coordinates (in the absence of D-branes). We then have three $|\chi|$ -independent cases to examine: firstly, positive χ , with $V > 0$; secondly, $\chi = 0$ ($V > 0$) which is $SU(1,2)/SU(2) \times U(1)$; thirdly, a negative χ , with $V > |\chi|$. In each case, we should seek to find a parameter-free zero-curvature limit. The most interesting case turns out to be χ negative, which we first study.

With χ negative, we are interested in the CPH metric in the region $V + \chi \sim 0$. We then apply to metric (11.30) the following change of variables:

$$\begin{aligned} V &= 2|\chi|\kappa^{2/3}\mu^{-1/3}C - \chi, & \varphi &= \sqrt{|\chi|}\kappa^{2/3}\mu^{-1/3}\hat{\varphi}, \\ \eta &= \sqrt{|\chi|}\kappa^{2/3}\mu^{-1/3}\hat{\eta}, & \tau &= |\chi|\kappa^{4/3}\mu^{1/3}\hat{\tau}, \end{aligned} \quad (11.79)$$

where μ is an arbitrary mass scale. Positivity of the metric, $V + \chi > 0$ implies $C > 0$. While the original fields are dimensionless, the new, hatted, fields ($C, \hat{\varphi}, \hat{\eta}, \hat{\tau}$) have canonical dimension. With this choice of dependence in κ , the resulting metric is

$$\begin{aligned} ds^2 = g_{ab}dq^a dq^b &= \frac{\kappa^2}{2} \frac{\mu C}{[(\kappa\mu)^{2/3}C + \mu]^2} \left[\frac{dC^2}{2\kappa^{2/3}\mu^{-1/3}C + 1} + d\hat{\eta}^2 + d\hat{\varphi}^2 \right] \\ &+ \frac{\kappa^2\mu^2}{2C} \frac{2(\kappa\mu)^{2/3}C + \mu}{[(\kappa\mu)^{2/3}C + \mu]^2} \left[d\hat{\tau} + \frac{1}{\mu}\hat{\eta}d\hat{\varphi} \right]^2, \end{aligned} \quad (11.80)$$

since $\chi = -|\chi|$. Using this metric in Lagrangian (11.74), the overall factor κ^2 cancels and we can take the limit $\kappa \rightarrow 0$, with result

$$\mathcal{L}_{\kappa \rightarrow 0} = \frac{C}{2\mu} \left[(\partial_\mu C)^2 + (\partial_\mu \hat{\eta})^2 + (\partial_\mu \hat{\varphi})^2 \right] + \frac{\mu}{2C} \left[\partial_\mu \hat{\tau} + \frac{1}{\mu} \hat{\eta} \partial_\mu \hat{\varphi} \right]^2. \quad (11.81)$$

This scalar Lagrangian has the hyper-Kähler metric with Heisenberg symmetry (11.25) with $A = 1/\mu$ and $B = 0$ and with relations $\Phi = \frac{1}{\sqrt{2}}(\hat{\eta} + i\hat{\varphi})$, $\hat{\tau} = 2\tau$. As

noticed earlier, parameter B can always be absorbed in a shift of C , as long as $A \neq 0$.

Notice that to obtain limit (11.81), we only need the change of variables (11.79) up to higher orders in κ . In particular, according to eq. (11.35), we may write the four-dimensional string dilaton as

$$\begin{aligned}
e^{-2\phi_4} &= 2|\chi|\kappa^{2/3}\mu^{-1/3}C - 2\chi, \\
\phi_4 &= \langle\phi_4\rangle - \kappa^{2/3}\mu^{-1/3}\hat{\phi}_4, \\
e^{-2\langle\phi_4\rangle} &= -2\chi = 2|\chi|, & C &= 2\hat{\phi}_4,
\end{aligned}
\tag{11.82}$$

in terms of the fluctuation $\hat{\phi}_4$ and of the background value $\langle\phi_4\rangle$. Since $|\chi| = \chi_1 = \chi_E/(12\pi)$, we are considering the case of a positive Euler number $\chi_E = 2(h_{11} - h_{21})$, with h_{11}, h_{12} the corresponding Betti numbers of the CY_3 manifold. A typical example with a single hypermultiplet would be IIA strings on a CY_3 manifold with $h_{21} = 0$. Positivity-related questions with several hypermultiplets, as is in particular the case with a negative Euler number, should be reanalyzed.

Comparing the scalings (11.79) and the identification of the string coupling in the last eq. (11.82), we see that the R-R fields η and φ carry as expected a supplementary factor g_{string} .

We could also consider the single-tensor version of the theory. Dualizing $\hat{\tau}$ into $H_{\mu\nu\rho}$, we find

$$\begin{aligned}
\mathcal{L}_{\kappa \rightarrow 0, ST} &= \frac{C}{\mu} \left[\frac{1}{2}(\partial_\mu C)^2 + \frac{1}{12}H^{\mu\nu\rho}H_{\mu\nu\rho} + (\partial_\mu \bar{\Phi})(\partial^\mu \Phi) \right] \\
&\quad - \frac{i}{12\mu}\epsilon^{\mu\nu\rho\sigma}(\bar{\Phi}\partial_\mu\Phi - \Phi\partial_\mu\bar{\Phi})H_{\nu\rho\sigma}.
\end{aligned}
\tag{11.83}$$

This is the bosonic sector (11.13) of the single-tensor theory (11.12) with again $A = 1/\mu$ and $B = 0$. Then, for negative χ , the $\mathcal{N} = 2$ supergravity hypermultiplet

with Heisenberg symmetry is described in the global supersymmetry limit by the unique nontrivial theory with the same symmetry.

For completeness, we may also consider the case of the CPH metric with positive χ . The interesting limiting regions are $V \sim 0$ and $V - \chi \sim 0$. If $V = \rho^2 \ll \chi$,

$$ds_{CPH}^2 = \frac{1}{\chi}(d\rho^2 + d\eta^2 + d\varphi^2) + \frac{4\rho^2}{\chi^3}(d\tau + \eta d\varphi)^2. \quad (11.84)$$

The appropriate rescalings are $(\rho, \eta, \varphi, \tau) = (\sqrt{\chi}\kappa\hat{\rho}, \sqrt{\chi}\kappa\hat{\eta}, \sqrt{\chi}\kappa\hat{\varphi}, \chi\hat{\tau})$ to obtain

$$ds_{CPH}^2 = \kappa^2 \left[d\hat{\rho}^2 + d\hat{\eta}^2 + d\hat{\varphi}^2 + 4\hat{\rho}^2(d\hat{\tau} + \kappa^2 \hat{\eta}d\hat{\varphi})^2 \right]. \quad (11.85)$$

The Heisenberg symmetry acting on the rescaled fields has algebra $[X, Y] = 2\kappa^2 Z$.

In the limit $\kappa \rightarrow 0$, it contracts to $[X, Y] = 0$ and we find

$$\lim_{\kappa \rightarrow 0} \frac{1}{\kappa^2} ds_{CPH}^2 = d\hat{\rho}^2 + 4\hat{\rho}^2 d\hat{\tau}^2 + d\hat{\eta}^2 + d\hat{\varphi}^2, \quad (11.86)$$

which is the trivial four-dimensional euclidean space. The second region of interest if $\chi > 0$ is $V - \chi \sim 0$. First, we change coordinates to

$$V = 2\lambda C + \chi, \quad \eta = \lambda\hat{\eta}/\sqrt{\chi}, \quad \varphi = \lambda\hat{\varphi}/\sqrt{\chi}, \quad \tau = \lambda\hat{\tau} \quad (11.87)$$

and the metric for $\lambda \rightarrow 0$ and χ finite reads

$$ds_{CPH}^2 = \frac{1}{2C^2} \left[dC^2 + d\hat{\eta}^2 + d\hat{\varphi}^2 + d\hat{\tau}^2 \right]. \quad (11.88)$$

This limiting metric is $SO(1,4)/SO(4)$, again with $R_{ij} = -6g_{ij}$ and with radius $\sim \langle C \rangle$. In the large radius, zero-curvature limit, the metric is trivial. Finally, in the $SU(1,2)/SU(2) \times U(1)$ case $\chi = 0$, the zero-curvature limit is again trivial.

The conclusion is that in the zero-curvature limit, the CPH one-loop Lagrangian for the dilaton hypermultiplet is the hyper-Kähler $\mathcal{N} = 2$ sigma-model with Heisenberg symmetry (11.12). If the one-loop parameter χ is negative, then

$A \neq 0$ and the Heisenberg algebra has a non-trivial realization in this limit. If $\chi \geq 0$ however, $A = 0$ and the limit of $\mathcal{N} = 2$ global supersymmetry is the free hypermultiplet. In the string context, the above non-trivial limit can be taken if the string coupling is tuned at a fixed value, according to the third line of eq. (11.82), which applies with positive Euler number.

In chapter 10 we constructed the interaction of a hypermultiplet with the Dirac-Born-Infeld Maxwell Lagrangian. The hypermultiplet sector has a full linear $\mathcal{N} = 2$ supersymmetry while the second supersymmetry is nonlinearly realized on the Maxwell superfield W_α . As an application of our results, we can easily use our identification of the string universal hypermultiplet. The bosonic DBI action, after elimination of the Maxwell auxiliary field and using the single-tensor formulation, is¹³

$$\begin{aligned} \mathcal{L}_{DBI} = & \frac{1}{8\mathcal{F}}(2g\text{Re}\Phi - \frac{1}{\mathcal{F}}) \left[1 - \sqrt{1 + \frac{2g^2C^2}{(2g\text{Re}\Phi - \frac{1}{\mathcal{F}})^2} \sqrt{-\det(\eta_{\mu\nu} + 2\sqrt{2}\mathcal{F}F_{\mu\nu})}} \right] \\ & + g\epsilon^{\mu\nu\rho\sigma} \left(\frac{\mathcal{F}}{4}\text{Im}\Phi F_{\mu\nu}F_{\rho\sigma} - \frac{1}{4}B_{\mu\nu}F_{\rho\sigma} + \frac{1}{24\mathcal{F}}C_{\mu\nu\rho\sigma} \right). \end{aligned} \quad (11.89)$$

In this expression, \mathcal{F} is the breaking scale of the second, nonlinearly realized supersymmetry (with dimension $(\text{energy})^{-2}$) and g is the Chern-Simons coupling¹⁴ (equal to the string coupling for a D3-brane). The four-form field $C_{\mu\nu\rho\sigma}$ is a component of the single-tensor multiplet required by supersymmetry of the nonlinear theory [see section 9.4].

Since we have control of the kinetic Lagrangian of the universal string hypermultiplet in the global supersymmetry limit, we can then identify the single-

¹³In chapter 10, this is the electric version of the theory, induced by a $\mathcal{N} = 2$ Chern-Simons coupling $gB \wedge F$.

¹⁴In contrast to chapter 10, we have defined single-tensor fields with canonical dimension so that g has dimension (energy) . We also chose the Fayet-Iliopoulos term to be $1/\mathcal{F}$ so that gauge kinetic terms are canonically normalized at $\text{Re}\Phi = 0$.

tensor fields in terms of string fields. First, C is the global dilaton and $B_{\mu\nu}$ is the NS-NS tensor. Then, the complex scalar Φ includes the R-R fields. The supersymmetric minimum of the scalar potential included in theory (11.89) implies $\langle C \rangle = 0$ and Φ corresponds to flat directions of this vacuum.

CHAPTER 12

SUMMARY OF RESULTS

This part of the thesis constitutes a detailed study, in the context of global supersymmetry, of the D-brane effective action of $\mathcal{N} = 2$ compactifications in type II string theory, including both the gauge part as well as the couplings of the brane to bulk fields. From a field theoretic point of view, this is the interaction of the Maxwell goldstino multiplet of $\mathcal{N} = 2$ nonlinear supersymmetry to a hypermultiplet with at least one isometry. The hypermultiplet is described by its Poincaré dual single tensor multiplet where $\mathcal{N} = 2$ supersymmetry can be realized off shell. The nonlinear breaking of the second SUSY is realized with a supersymmetric constraint while the coupling of the single-tensor to the goldstino multiplet is realized with a supersymmetric generalization of the usual Chern-Simons term $B \wedge F$. This system has equivalent descriptions in terms of different chiral and tensor multiplets. We proved the equivalence of these descriptions by performing $\mathcal{N} = 1$ and $\mathcal{N} = 2$ Poincaré type dualities which led us to a net of theories summarized in the figure below.

Up to appropriate field redefinitions, this system is also equivalent to the Higgs phase of $\mathcal{N} = 2$ nonlinear QED coupled to a charged hypermultiplet. The system also explores a phase with all supersymmetries broken and a phase with the $U(1)$ gauge symmetry unbroken. In the Higgs phase an interesting phenomenon appears. The goldstino multiplet combines with the hypermultiplet to

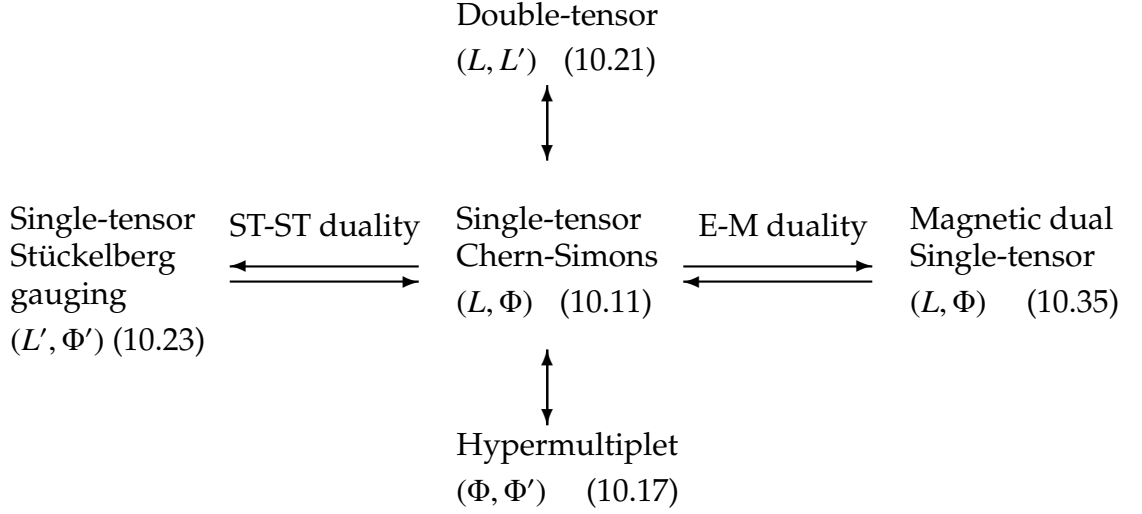


Figure 12.1: Web of dualities: double arrows indicate duality transformations preserving off-shell $\mathcal{N} = 2$ supersymmetry, simple arrows are $\mathcal{N} = 1$ off-shell dualities only, leading to theories with on-shell $\mathcal{N} = 2$ supersymmetry. The $\mathcal{N} = 1$ superfields and the related equations are indicated.

form a massive vector multiplet and a massless chiral multiplet. In the massive multiplet, the goldstino combines with a hypermultiplet fermion and becomes massive, thus realizing a new type of super-Higgs mechanism that doesn't involve a gravitino. This is possible because the hypermultiplet is charged under the $U(1)$ partner of the goldstino.

The next step is to find how the Lagrangian of our system eq. (10.14) relates with the global limit of the low energy effective D-brane action in $\mathcal{N} = 2$ compactifications. In other words, we have to relate the field basis used in our construction with the string basis of the universal hypermultiplet. To do that we need to specify the correct global limit of the universal hypermultiplet. At string tree level, the universal hypermultiplet is described by the symmetric coset

$SU(2, 1)/SU(2) \times U(1)$. At the quantum level this isometry structure reduces to the centrally extended Euclidean algebra E_2 which contains a Heisenberg subalgebra. Requiring that the same isometry structure survive in the global limit we found that apart from the trivial global limit of canonical kinetic terms (which destroys this isometry), there is also a limit leading to a hyperKähler manifold. An independent derivation of the most general hyperKähler manifold that satisfies the Heisenberg isometries had as a result precisely the same manifold that we obtained from this global limit. We could then identify the string basis of the system.

APPENDIX A

COEFFICIENTS FOR THE HIGGS MASSES

For completeness, we present the expressions of the coefficients in eq. (5.47):

$$\begin{aligned} \gamma_1^\pm &= \frac{\pm v^2}{2u^2(1+u^2)^3 w^{1/2}} \\ &\times \left[(B_0 m_0 \mu_0)^2 (1+u^2)^4 - 2m_Z^2 u^2 [m_Z^2(1-u^2)^2 + (1+u^2)(8\mu_0^2 u^2 \pm (u^2-1)w^{1/2})] \right. \\ &\left. + (B_0 m_0 \mu_0) u(1+u^2)^2 [m_Z^2(1+u^2) - (\pm w^{1/2}(1+u^2) + 16\mu_0^2 u^2)] \right] \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \gamma_2^\pm &= \frac{\pm v^2}{2(1+u^2)^3 w^{1/2}} \\ &\times \left[(B_0 m_0 \mu_0)^2 (1+u^2)^4 - 2m_Z^2 u^2 [8\mu_0^2(1+u^2) + m_Z^2(1-u^2)^2 \pm w^{1/2}(1-u^4)] \right. \\ &\left. - (B_0 m_0 \mu_0) u(1+u^2)^2 [16\mu_0^2 - m_Z^2(1+u^2) \pm (1+u^2)w^{1/2}] \right] \end{aligned} \quad (\text{A.2})$$

$$\begin{aligned} \gamma_3^\pm &= \gamma_4^\pm = \frac{\pm v^2}{u(1+u^2)^2 w^{1/2}} \{ \mu_0^2 [-B_0 m_0 \mu_0 (1+u^2)^3 + m_Z^2 u(1-6u^2+u^4) \mp u(1+u^2)^2 w^{1/2}] \\ &+ B_0 m_0 \mu_0 u^2 (1+u^2) m_Z^2 + m_Z^2 u^3 (m_Z^2 \mp w^{1/2}) \} \end{aligned} \quad (\text{A.3})$$

$$\begin{aligned} \gamma_5^\pm &= \frac{\mp v^2}{8u^3(1+u^2)^3 w^{1/2}} \left[(B_0 m_0 \mu_0)^2 (1+u^2)^4 (-1+3u^2) - (B_0 m_0 \mu_0) u(1+u^2)^2 \right. \\ &\times \left[-2m_Z^2(1+5u^2) + 2\mu_0^2(1+8u^2+25u^4+2u^6) \pm (1+u^2)(3u^2-1)w^{1/2} \right] \\ &- u^2 m_Z^2 [m_Z^2(1-19u^2-u^4+3u^6) - 2\mu_0^2(1+u^2)(1-16u^2-23u^4+2u^6) \\ &\pm (1+u^2)^2(1+3u^2)w^{1/2}] + 2\mu_0^2 u^2 [\pm (1+u^2)^2(1-9u^2+2u^4)w^{1/2}] \left. \right] \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \gamma_6^\pm &= \frac{\pm v^2}{8u^2(1+u^2)^3 w^{1/2}} \left[(B_0 m_0 \mu_0)^2 u(1+u^2)^4 (-3+u^2) - (B_0 m_0 \mu_0) (1+u^2)^2 \right. \\ &\times \left[2m_Z^2(5+u^2)u^4 - 2\mu_0^2(2+25u^2+8u^4+u^6) \pm (1+u^2)(u^2-3)u^2 w^{1/2} \right] \end{aligned}$$

$$\begin{aligned}
& + u m_Z^2 [m_Z^2 (3 - u^2 - 19u^4 + u^6) u^2 - 2\mu_0^2 (1 + u^2)(2 - 23u^2 - 16u^4 + u^6) \\
& \pm u^2 (1 + u^2)^2 (3 + u^2) w^{1/2}] - 2\mu_0^2 u [\pm (1 + u^2)^2 (2 - 9u^2 + u^4) w^{1/2}] \quad (A.5)
\end{aligned}$$

$$\begin{aligned}
\gamma_7^\pm & = \frac{\mp v^2 m_Z^2}{16u^2(1 + u^2)^3 w^{1/2}} \left[-B_0 m_0 \mu_0 (1 + u^2)(1 + 40u^2 - 114u^4 + 40u^6 + u^8) \right. \\
& \left. + m_Z^2 (u + 30u^5 + u^9) \pm u(1 + u^2)^2 (1 - 10u^2 + u^4) w^{1/2} \right] \quad (A.6)
\end{aligned}$$

$$\gamma_x^\pm = \frac{\pm 8 (u^2 - 1)^2 v^4}{u (1 + u^2)^3 w^{3/2}} [m_Z^2 u - B_0 m_0 \mu_0 (1 + u^2)][2m_Z^2 u - B_0 m_0 \mu_0 (1 + u^2)] m_0 \mu_0 \quad (A.7)$$

$$\gamma_y^\pm = \mp \frac{(-1 + u^2)^2 v^4}{(1 + u^2)^4 w^{3/2}} [m_Z^2 u - B_0 m_0 \mu_0 (1 + u^2)]^2 (4m_0^2) \quad (A.8)$$

$$\begin{aligned}
\gamma_z^\pm & = \frac{\mp v^4}{\mu_0^2 u^2 (1 + u^2)^3 w^{3/2}} \quad (A.9) \\
& \times \left[-2 (B_0 m_0 \mu_0)^3 u (1 + u^2)^4 + m_Z^4 u^2 (1 + u^2)(4\mu_0^2 (-1 + u^2)^2 - u^2 (2m_Z^2 \pm w^{1/2})) \right. \\
& + 2 B_0 m_0 \mu_0 m_Z^2 u [-2\mu_0^2 (u^4 - 1)^2 + u^2 (m_Z^2 (1 - 14u^2 + u^4) \pm (u^4 - 6u^2 + 1) w^{1/2})] \\
& \left. + (B_0 m_0 \mu_0)^2 (1 + u^2) [\mu_0^2 (u^4 - 1)^2 + u^2 (2m_Z^2 (1 - 14u^2 + u^4) \mp (1 + u^2)^2 w^{1/2})] \right] (4\mu_0^2)
\end{aligned}$$

APPENDIX B

THE SOLUTION OF THE QUADRATIC CONSTRAINT

In sec. 10.4, the quadratic constraint $\mathcal{Z}^2 = 0$ must be solved to obtain the magnetic DBI theory coupled to a single-tensor multiplet. Using the expansion

$$\mathcal{Z}(y, \theta, \tilde{\theta}) = Z(y, \theta) + \sqrt{2} \tilde{\theta} \omega(y, \theta) - \tilde{\theta} \tilde{\theta} \left[\frac{i}{2} \Phi_Z + \frac{1}{4} \overline{DDZ}(y, \theta) \right],$$

in terms of the $\mathcal{N} = 1$ chiral superfields Z , ω_α and Φ_Z , the constraint is equivalent to the single equation

$$Z = -\frac{\omega\omega}{i\Phi_Z + \frac{1}{2}\overline{DDZ}}. \quad (\text{B.1})$$

The electric constraint equation (10.3), which was solved by Bagger and Galperin [20] using a method which applies to eq. (B.1) as well, corresponds to the particular case $\omega_\alpha = iW_\alpha$, $\Phi_Z = -i/\kappa$ and $Z = X$. Following then Ref. [20], the solution of eq. (B.1) is

$$Z(\omega\omega, \Phi_Z) = \frac{i}{\Phi_Z} \left(\omega\omega + \overline{DD} \left[\frac{\omega\omega\overline{\omega\omega}}{|\Phi_Z|^2 + A + \sqrt{|\Phi_Z|^4 + 2A|\Phi_Z|^2 + B^2}} \right] \right), \quad (\text{B.2})$$

where

$$\begin{aligned} A &= -\frac{1}{2}(DD\omega\omega + \overline{DD}\overline{\omega\omega}) = A^*, \\ B &= -\frac{1}{2}(DD\omega\omega - \overline{DD}\overline{\omega\omega}) = -B^*. \end{aligned}$$

Another useful expression is

$$Z(\omega\omega, \Phi_Z) = \frac{i}{\Phi_Z} \left(\omega\omega + \overline{DD} \left[\frac{\omega\omega\overline{\omega\omega}}{(DD\omega\omega)(\overline{DD}\overline{\omega\omega})} \left\{ |\Phi_Z|^2 + A - \sqrt{|\Phi_Z|^4 + 2A|\Phi_Z|^2 + B^2} \right\} \right] \right). \quad (\text{B.3})$$

In the text, we need the bosonic content of $Z(\omega\omega, \Phi_Z)$. We write:

$$\omega_\alpha(y, \theta) = \theta_\alpha \rho + \frac{1}{2} (\theta\sigma^\mu\overline{\sigma}^\nu)_\alpha P_{\mu\nu} + \dots, \quad (\text{B.4})$$

where ρ is a complex scalar (2 bosons), $P_{\mu\nu}$ a real antisymmetric tensor (6 bosons) and dots indicate omitted fermionic terms. Hence,

$$\begin{aligned} \omega\omega &= \theta\theta \left[\rho^2 + \frac{1}{2} P^{\mu\nu} P_{\mu\nu} + \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} P_{\mu\nu} P_{\rho\sigma} \right] + \dots, \\ A &= 2(\rho^2 + \overline{\rho}^2) + 2P^{\mu\nu} P_{\mu\nu} + \dots, \\ B &= 2(\rho^2 - \overline{\rho}^2) + i\epsilon^{\mu\nu\rho\sigma} P_{\mu\nu} P_{\rho\sigma} + \dots \end{aligned}$$

Since the bosonic expansion of ω_α carries one θ_α , it follows from solution (B.2) that the bosonic $Z(\omega\omega, \Phi_Z)$ has a $\theta\theta$ component only, and that this component only depends on ρ , $P_{\mu\nu}$ and the lowest scalar component of Φ_Z (which we also denote by Φ_Z). As a consequence, the bosonic $Z(\omega\omega, \Phi_Z)$ does not depend on the auxiliary scalar f_{Φ_Z} of Φ_Z . We then find:

$$Z(\Phi_Z, \omega\omega)_{bos.} = \frac{i\overline{\Phi}_Z}{|\Phi_Z|^2} \omega\omega - \frac{i\overline{\Phi}_Z}{4|\Phi_Z|^2} \theta\theta \left(|\Phi_Z|^2 + A - \sqrt{|\Phi_Z|^4 + 2A|\Phi_Z|^2 + B^2} \right)_{\theta=0}. \quad (\text{B.5})$$

The parenthesis is real. In terms of component fields:

$$\begin{aligned} Z &= -\frac{i\overline{\Phi}_Z}{4|\Phi_Z|^2} \theta\theta \left[|\Phi_Z|^2 - i\epsilon^{\mu\nu\rho\sigma} P_{\mu\nu} P_{\rho\sigma} - 2(\rho^2 - \overline{\rho}^2) \right] \\ &+ \frac{i\overline{\Phi}_Z}{4|\Phi_Z|^2} \theta\theta \left[\left(|\Phi_Z|^2 + 2(\rho^2 + \overline{\rho}^2) \right)^2 - 16\rho^2\overline{\rho}^2 + 4(\rho^2 - \overline{\rho}^2) i\epsilon^{\mu\nu\rho\sigma} P_{\mu\nu} P_{\rho\sigma} \right. \\ &\quad \left. + 4|\Phi_Z|^2 P^{\mu\nu} P_{\mu\nu} - \left(\epsilon^{\mu\nu\rho\sigma} P_{\mu\nu} P_{\rho\sigma} \right)^2 \right]^{1/2} + \dots \end{aligned} \quad (\text{B.6})$$

The decomposition (10.27), $\mathcal{Z} = \widetilde{\mathcal{W}} + 2g\mathcal{Y}$, indicates that

$$\rho = -\frac{g}{2} C + i\widetilde{d}_2, \quad P_{\mu\nu} = gb_{\mu\nu} - \widetilde{F}_{\mu\nu}, \quad \Phi_Z = 2g\Phi. \quad (\text{B.7})$$

In Lagrangian (10.35), we need the imaginary part of the $\theta\theta$ component of $Z(\omega\omega, \Phi_Z)$:

$$\begin{aligned}
\text{Im } Z(\omega\omega, \Phi_Z)|_{\theta\theta} &= -\frac{g \text{Re } \Phi}{2} + \frac{\text{Re } \Phi}{8g|\Phi|^2} \left\{ 16g^4|\Phi|^4 + 8g^2|\Phi|^2(g^2C^2 - 4\tilde{d}_2^2) - 16g^2C^2\tilde{d}_2^2 \right. \\
&\quad + 16g^2|\Phi|^2(\tilde{F}_{\mu\nu} - g b_{\mu\nu})(\tilde{F}^{\mu\nu} - g b^{\mu\nu}) \\
&\quad + 8gC\tilde{d}_2 \epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - g b_{\mu\nu})(\tilde{F}_{\rho\sigma} - g b_{\rho\sigma}) \\
&\quad \left. - [\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - g b_{\mu\nu})(\tilde{F}_{\rho\sigma} - g b_{\rho\sigma})]^2 \right\}^{1/2} \\
&\quad + \frac{\text{Im } \Phi}{8g|\Phi|^2} \left[\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - g b_{\mu\nu})(\tilde{F}_{\rho\sigma} - g b_{\rho\sigma}) - 4gC\tilde{d}_2 \right].
\end{aligned} \tag{B.8}$$

We now use

$$\begin{aligned}
-\det(|\Phi|\eta_{\mu\nu} + \frac{\sqrt{2}}{g} P_{\mu\nu}) &= -|\Phi|^4 \det(\eta_{\mu\nu} + \frac{\sqrt{2}}{g|\Phi|} P_{\mu\nu}) \\
&= |\Phi|^4 + \frac{|\Phi|^2}{g^2} P^{\mu\nu} P_{\mu\nu} - \frac{1}{16g^4} (\epsilon^{\mu\nu\rho\sigma} P_{\mu\nu} P_{\rho\sigma})^2
\end{aligned} \tag{B.9}$$

to rewrite

$$\begin{aligned}
\text{Im } Z(\omega\omega, \Phi_Z)|_{\theta\theta} &= -\frac{g \text{Re } \Phi}{2} + \frac{\text{Re } \Phi}{4g|\Phi|^2} \left\{ -4g^4|\Phi|^4 \det \left[\eta_{\mu\nu} - \frac{\sqrt{2}}{g|\Phi|} (\tilde{F}_{\mu\nu} - g b_{\mu\nu}) \right] \right. \\
&\quad - 4g^2\tilde{d}_2^2(2|\Phi|^2 + C^2) + 2g^4C^2|\Phi|^2 \\
&\quad \left. + 2gC\tilde{d}_2 \epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - g b_{\mu\nu})(\tilde{F}_{\rho\sigma} - g b_{\rho\sigma}) \right\}^{1/2} \\
&\quad + \frac{\text{Im } \Phi}{8g|\Phi|^2} \left[\epsilon^{\mu\nu\rho\sigma}(\tilde{F}_{\mu\nu} - g b_{\mu\nu})(\tilde{F}_{\rho\sigma} - g b_{\rho\sigma}) - 4gC\tilde{d}_2 \right].
\end{aligned} \tag{B.10}$$

As a check, choosing $\Phi = -1/(2g\kappa)$ and $g = 0$ to decouple the single-tensor multiplet leads back to theory (10.4) since in that case $\tilde{d}_2 = 0$.

APPENDIX C

EQUIVALENT DESCRIPTIONS OF THE DILATON MULTIPLLET

We present in detail three dual descriptions of the dilaton multiplet as well as the duality transformations that take us from one to another. We start by repeating the analysis of section 9.1 on the single-tensor multiplet, this time with more details, and then we go on to the hyper- and the two-tensor multiplets.

C.1 The Single Tensor Formulation

The single-tensor multiplet [18, 103, 104] is the $\mathcal{N} = 2$ extension of the antisymmetric tensor field $b_{\mu\nu}$ with gauge symmetry $\delta_{gauge} b_{\mu\nu} = 2\partial_{[\mu}\Lambda_{\nu]}$. It admits two descriptions, either in terms of the gauge-invariant curl $\partial_{[\mu}b_{\nu\rho]}$ or in terms of the antisymmetric tensor field submitted to its gauge transformation.

In the case of $\mathcal{N} = 1$ supersymmetry, a real linear superfield L , $DDL = 0$, $L = \bar{L}$, describes the curl of the antisymmetric tensor. It can be expressed in terms of a chiral spinor potential including the antisymmetric tensor:

$$L = D^\alpha \chi_\alpha - \bar{D}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}, \quad (\text{C.1})$$

with $\bar{D}_{\dot{\alpha}} \chi_\alpha = 0$. The gauge invariance of the two-form field acts on the potential

χ_α according to

$$\chi_\alpha \longrightarrow \chi_\alpha + i\overline{D}D D_\alpha \Delta, \quad \bar{\chi}_{\dot{\alpha}} \longrightarrow \bar{\chi}_{\dot{\alpha}} + iD\overline{D}\bar{D}_{\dot{\alpha}}\Delta, \quad (\text{C.2})$$

which, since $D^\alpha \overline{D}D D_\alpha = \overline{D}_{\dot{\alpha}} D D \overline{D}^{\dot{\alpha}}$, leaves invariant the linear superfield L for any real Δ . The potential χ_α includes the antisymmetric tensor in its θ component:

$$\chi_\alpha = \dots - \frac{1}{4}\theta_\alpha C + \frac{1}{2}(\theta\sigma^\mu\bar{\sigma}^\nu)_\alpha b_{\mu\nu} + \dots, \quad (\text{C.3})$$

C being the real scalar partner of $b_{\mu\nu}$. The two descriptions of the $\mathcal{N} = 2$ single-tensor multiplet use either L or χ_α , completed with one or two chiral $\mathcal{N} = 1$ superfields.

In the gauge-invariant description using L , the $\mathcal{N} = 2$ multiplet is completed with a chiral superfield Φ ($8_B + 8_F$ fields in total). The second supersymmetry transformations are

$$\begin{aligned} \delta^* L &= -\frac{i}{\sqrt{2}}(\eta D\Phi + \overline{\eta D\Phi}), \\ \delta^* \Phi &= i\sqrt{2}\overline{\eta}DL, \quad \delta^* \overline{\Phi} = i\sqrt{2}\eta DL, \end{aligned} \quad (\text{C.4})$$

The supersymmetry algebra closes (off-shell) on L and Φ .

Alternatively, in terms of χ_α and Φ , eqs. (C.3) suggest the variations

$$\begin{aligned} \delta^* \chi_\alpha &= -\frac{i}{\sqrt{2}}\Phi\eta_\alpha, \quad \delta^* \bar{\chi}_{\dot{\alpha}} = \frac{i}{\sqrt{2}}\overline{\Phi}\bar{\eta}_{\dot{\alpha}}, \\ \delta^* \Phi &= 2\sqrt{2}i\left[\frac{1}{4}\overline{D}D\eta\chi + i\partial_\mu\chi\sigma^\mu\bar{\eta}\right], \\ \delta^* \overline{\Phi} &= -2\sqrt{2}i\left[\frac{1}{4}D\overline{D}\eta\chi - i\eta\sigma^\mu\partial_\mu\bar{\chi}\right]. \end{aligned} \quad (\text{C.5})$$

On χ_α however, the supersymmetry algebra closes up to a gauge transformation (C.2):

$$\begin{aligned} [\delta_1^*, \delta_2^*]\chi_\alpha &= -2i(\eta_2\sigma^\mu\bar{\eta}_1 - \eta_1\sigma^\mu\bar{\eta}_2)\partial_\mu\chi_\alpha \\ &+ \frac{i}{2}\overline{D}D D_\alpha \left[i\eta_1\theta\bar{\eta}_2\bar{\chi} - i\bar{\eta}_1\theta\eta_2\chi - i\eta_2\theta\bar{\eta}_1\bar{\chi} + i\bar{\eta}_2\theta\eta_1\chi \right]. \end{aligned} \quad (\text{C.6})$$

This result suggests that the $\mathcal{N} = 1$ superfields Φ and χ_α do not complete a true off-shell supermultiplet of $\mathcal{N} = 2$ supersymmetry. Another hint is given by the degrees of freedom: Φ and χ_α contain $12_B + 12_F$ fields and gauge invariance (C.2), which is only compatible with $\mathcal{N} = 1$, removes $4_B + 4_F$ fields, to give the expected $8_B + 8_F$ degrees of freedom in L and Φ . We should then expect that the $\mathcal{N} = 2$ supermultiplet of the potential χ_α (including the antisymmetric tensor among its component fields) has $16_B + 16_F$ fields, with an extended gauge transformation using a Maxwell $\mathcal{N} = 2$ multiplet and removing $8_B + 8_F$ components.

From the structure of relation (C.6), one may guess that the introduction of another chiral superfield Y (with $4_B + 4_F$ fields) with $\delta^* Y \sim \eta\chi$ would be appropriate if we also add to $\delta^* \chi_\alpha$ a gauge transformation proportional to

$$i \overline{DD} D_\alpha [i\eta\theta\bar{Y} - i\bar{\eta}\theta Y] = -\eta_\alpha \overline{DD} \bar{Y} - 4i(\sigma^\mu \bar{\eta})_\alpha \partial_\mu Y.$$

This modification, being a gauge transformation of χ_α , does not affect $\delta^* L$. One then easily verifies that the second supersymmetry variations

$$\begin{aligned} \delta^* Y &= \sqrt{2} \eta\chi, \\ \delta^* \chi_\alpha &= -\frac{i}{\sqrt{2}} \Phi \eta_\alpha - \frac{\sqrt{2}}{4} \eta_\alpha \overline{DD} \bar{Y} - \sqrt{2} i (\sigma^\mu \bar{\eta})_\alpha \partial_\mu Y, \end{aligned} \tag{C.7}$$

with $\delta^* \Phi$ as in (C.5), close the $\mathcal{N} = 2$ superalgebra.

It is then natural to generalize gauge transformation (C.2) to $\mathcal{N} = 2$, using a Maxwell supermultiplet with $\mathcal{N} = 1$ superfields \widehat{W}_α and \widehat{X} :

$$\delta_{gauge} \chi_\alpha = i \widehat{W}_\alpha, \quad \delta_{gauge} Y = \widehat{X}, \quad \delta_{gauge} \Phi = 0. \tag{C.8}$$

Since $L = D\chi - \overline{D}\bar{\chi}$, the Bianchi identity verified by \widehat{W} implies the gauge invariance of L . The second variation, which is the same as transformation (C.2), contains in particular $\delta_{gauge} b_{\mu\nu} = \widehat{F}_{\mu\nu}$. This $\mathcal{N} = 2$ gauge transformation removes $8_B + 8_F$ component fields, leaving as expected $8_B + 8_F$ fields.

It may be useful to remark that giving a constant background value to the chiral $\mathcal{N} = 1$ superfield Φ seems to break $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$. According to the second variation (C.7), χ_α transforms like a Goldstino if Φ acquires a background value. The lowest component of χ_α does however transform under gauge symmetry (C.2) and a Goldstino is generated only if a gauge-invariant quantity is created in a theory where the single-tensor multiplet interacts with other fields. In a theory depending only on the gauge-invariant L and Φ , a background value of Φ does not break the second supersymmetry: it is invariant under transformations (C.4).¹

The chiral superfield Y does not contain any physical state: neither L nor ϕ do depend on Y . There is a gauge similar to the Wess-Zumino gauge of $\mathcal{N} = 1$ supersymmetry in which $Y = 0$. This gauge choice respects $\mathcal{N} = 1$ supersymmetry and gauge symmetry (C.2).

An invariant kinetic action for the single-tensor multiplet involves an arbitrary function solution of the three-dimensional Laplace equation (for the variables L , Φ and $\bar{\Phi}$) [18]:

$$\mathcal{L}_{ST} = \int d^2\theta d^2\bar{\theta} \mathcal{H}(L, \Phi, \bar{\Phi}), \quad \frac{\partial^2 \mathcal{H}}{\partial L^2} + 2 \frac{\partial^2 \mathcal{H}}{\partial \Phi \partial \bar{\Phi}} = 0. \quad (\text{C.9})$$

It is in particular straightforward to show that

$$\mathcal{L}_{ST} = \int d^2\theta d^2\bar{\theta} H(\mathcal{V}) + \text{h.c.}, \quad (\text{C.10})$$

with

$$\mathcal{V} = L + \frac{i}{\sqrt{2}}(\Phi + \bar{\Phi})$$

transforms with a derivative under the second supersymmetry for any function $H(\mathcal{V})$. It is also invariant under a constant shift of $\text{Im } \Phi$, the symmetry which

¹A background value of the scalar C in χ_α [see expansion (C.3)] does not break supersymmetry. It corresponds to a constant background value of L .

allows dualization of Φ into the second linear superfield of the double-tensor multiplet.

C.2 Hypermultiplet Formulation

In terms of $\mathcal{N} = 1$ superfields, a hypermultiplet has two chiral superfields Φ and T . The linear L of the single-tensor multiplet has been dualized to a chiral T with axionic shift symmetry. Since the duality involves a Legendre transformation using the Lagrangian function, the second supersymmetry transformations will not any longer hold off-shell when acting on Φ and T : the hypermultiplet does not admit an off-shell formulation.

We start with the single-tensor Lagrangian

$$\mathcal{L}_{ST} = \int d^2\theta d^2\bar{\theta} \mathcal{H}(L, \Phi, \bar{\Phi}). \quad (\text{C.11})$$

To dualize the theory, use a real vector superfield U and rewrite

$$\mathcal{L}_{ST} = \int d^2\theta d^2\bar{\theta} [\mathcal{H}(U, \Phi, \bar{\Phi}) - m(T + \bar{T})U], \quad (\text{C.12})$$

with an arbitrary real parameter m . Eliminating U with

$$\frac{\partial}{\partial U} \mathcal{H}(U, \Phi, \bar{\Phi}) = m(T + \bar{T}), \quad (\text{C.13})$$

one obtains the dual hypermultiplet theory

$$\tilde{\mathcal{L}}_{ST} = \int d^2\theta d^2\bar{\theta} K(T + \bar{T}, \Phi, \bar{\Phi}), \quad K(T + \bar{T}, \Phi, \bar{\Phi}) = \mathcal{H}(u, \Phi, \bar{\Phi}) - m(T + \bar{T})u, \quad (\text{C.14})$$

where $U = u(T + \bar{T}, \Phi, \bar{\Phi})$ is the solution of the Legendre transformation (C.13).

One can then derive various relations between derivatives of the Kähler potential K and derivatives of \mathcal{H} :

$$\begin{aligned} K_{T\bar{T}} &= -\frac{m^2}{\mathcal{H}_{UU}}, & K_{\Phi\bar{\Phi}} &= \mathcal{H}_{\Phi\bar{\Phi}} - \frac{\mathcal{H}_{U\Phi}\mathcal{H}_{U\bar{\Phi}}}{\mathcal{H}_{UU}}, \\ K_{T\bar{\Phi}} &= m\frac{\mathcal{H}_{U\bar{\Phi}}}{\mathcal{H}_{UU}}, & K_{\Phi\bar{T}} &= m\frac{\mathcal{H}_{U\Phi}}{\mathcal{H}_{UU}}, \end{aligned} \quad (\text{C.15})$$

using the notation

$$\mathcal{H}_{UU} = \frac{\partial^2 \mathcal{H}}{\partial U^2}, \quad \mathcal{H}_{\Phi\bar{\Phi}} = \frac{\partial^2 \mathcal{H}}{\partial \Phi \partial \bar{\Phi}}, \quad \dots$$

As a consequence, the determinant of the (2×2) Kähler metric is

$$K_{T\bar{T}}K_{\Phi\bar{\Phi}} - K_{T\bar{\Phi}}K_{\Phi\bar{T}} = -m^2 \frac{\mathcal{H}_{\Phi\bar{\Phi}}}{\mathcal{H}_{UU}}. \quad (\text{C.16})$$

In this $\mathcal{N} = 1$ Legendre transformation, the condition for $\mathcal{N} = 2$ supersymmetry has not been used. Hence for a single-tensor multiplet, the second eq. (C.9) implies [105, 106]

$$K_{T\bar{T}}K_{\Phi\bar{\Phi}} - K_{T\bar{\Phi}}K_{\Phi\bar{T}} = \frac{1}{2}m^2 \quad (\text{C.17})$$

(Monge-Ampère equation). This result implies Ricci-flatness which, for a two-dimensional complex manifold, indicates that the hypermultiplet scalar manifold is hyper-Kähler, as expected in general [16]. Hypermultiplet scalar kinetic terms are²

$$\begin{aligned} &K_{T\bar{T}} \left[\partial_\mu T + \frac{K_{\Phi\bar{T}}}{K_{T\bar{T}}} \partial_\mu \Phi \right] \left[\partial^\mu \bar{T} + \frac{K_{T\bar{\Phi}}}{K_{T\bar{T}}} \partial^\mu \bar{\Phi} \right] + \frac{m^2}{2K_{T\bar{T}}} \partial_\mu \Phi \partial^\mu \bar{\Phi} \\ &= -\frac{1}{\mathcal{H}_{UU}} \left| m \partial_\mu T - \mathcal{H}_{U\Phi} \partial_\mu \Phi \right|^2 - \frac{1}{2} \mathcal{H}_{UU} (\partial_\mu \Phi)(\partial^\mu \bar{\Phi}). \end{aligned} \quad (\text{C.18})$$

using the same notation T and Φ for the chiral superfields and for their lowest scalar components. The chiral superfields T and Φ are Kähler coordinates.

One should remark that adding to \mathcal{H} the quantity

$$\Delta \mathcal{H} = L[g(\Phi) + \bar{g}(\bar{\Phi})] \quad (\text{C.19})$$

²Positivity of kinetic terms requires that $\mathcal{H}_{UU} < 0$.

does not change the single-tensor theory:³ its superspace integral is a derivative. Since

$$\Delta\mathcal{H}_U = g(\Phi) + \bar{g}(\bar{\Phi}), \quad \Delta\mathcal{H}_{U\Phi} = g_\Phi(\Phi),$$

the Legendre transformation (C.13) and the kinetic terms (C.18) are affected by a modification of T :

$$T \quad \longrightarrow \quad T - \frac{g(\Phi)}{m}. \quad (\text{C.20})$$

Hence, for a given single-tensor theory defined by the function \mathcal{H} , we have a family of hypermultiplet theories generated by the arbitrary function $g(\Phi)$. In other words, the chiral superfield dual to L can be defined as $T - \frac{g(\Phi)}{m}$, for any function g .

The hyper-Kähler scalar metric is commonly expressed in “mixed” coordinates where u , the solution of the Legendre transformation (C.13), is used instead of $\text{Re } T$. Defining then coordinates

$$q^a = (\tau, x^i) = (\text{Im } T, \sqrt{2} \text{Re } \phi, \sqrt{2} \text{Im } \phi, u), \quad a = 0, i, \quad i = 1, 2, 3, \quad (\text{C.21})$$

the line-element can be written

$$\begin{aligned} ds^2 &= g_{ab} dq^a dq^b \\ &= -\frac{\mathcal{H}_{UU}}{4} du^2 + \mathcal{H}_{\Phi\bar{\Phi}} d\Phi d\bar{\Phi} - \frac{m^2}{\mathcal{H}_{UU}} \left[d \text{Im } T + \frac{i}{2m} (\mathcal{H}_{U\Phi} d\Phi - \mathcal{H}_{U\bar{\Phi}} d\bar{\Phi}) \right]^2. \end{aligned} \quad (\text{C.22})$$

With the condition for $\mathcal{N} = 2$ supersymmetry, $\mathcal{H}_{\Phi\bar{\Phi}} = -\frac{1}{2}\mathcal{H}_{UU}$, this is

$$\begin{aligned} ds^2 &= -\frac{\mathcal{H}_{UU}}{4} [du^2 + 2 d\Phi d\bar{\Phi}] - \frac{m^2}{\mathcal{H}_{UU}} \left[d \text{Im } t + \frac{i}{2m} (\mathcal{H}_{U\Phi} d\Phi - \mathcal{H}_{U\bar{\Phi}} d\bar{\Phi}) \right]^2 \\ &= \frac{m}{2} (V dx^i dx^i + V^{-1} [d\tau - \omega^i dx^i]^2), \end{aligned} \quad (\text{C.23})$$

with functions $V(x^i)$ and $\omega^i(x^j)$ given by

$$V = -\frac{\mathcal{H}_{UU}}{2m}, \quad \omega^1 = \frac{\text{Im } \mathcal{H}_{U\Phi}}{\sqrt{2m}}, \quad \omega^2 = \frac{\text{Re } \mathcal{H}_{U\Phi}}{\sqrt{2m}}, \quad \omega^3 = 0. \quad (\text{C.24})$$

³It is a trivial solution of Laplace equation.

Using again the condition for $\mathcal{N} = 2$ supersymmetry, which implies that the metric is hyper-Kähler, one finds that

$$\vec{\nabla} V = \vec{\nabla} \wedge \vec{\omega}. \quad (\text{C.25})$$

This indicates that V solves Laplace equation

$$\partial^i \partial^i V = (\partial_u^2 + 2 \partial_\Phi \partial_{\bar{\Phi}}) V = 0, \quad (\text{C.26})$$

in agreement with its definition (C.24). A (four-dimensional) hyper-Kähler metric with shift symmetry of $\tau = \text{Im } T$ is then defined by V and ω^i related by equations (C.25) [100]. Given a metric of this form, the single-tensor formulation of the $\mathcal{N} = 2$ supersymmetric theory is then obtained by integrating eqs. (C.24) to find \mathcal{H} . Notice that eq. (C.25) remains valid if

$$\vec{\omega} \longrightarrow \vec{\omega} + \vec{\nabla} \mathcal{F},$$

for an arbitrary real function \mathcal{F} . The metric is unchanged if coordinate τ is changed according to

$$\tau \longrightarrow \tau + \mathcal{F}.$$

Comparing with eqs. (C.19) and (C.20), one sees that $\mathcal{F} = \frac{1}{\sqrt{2m}} \text{Im } g(\Phi)$.

The Kähler formulation with complex coordinates T and Φ is defined by relations

$$K_{T\bar{T}} = \frac{m}{2V}, \quad K_{\Phi\bar{T}} = -\frac{m}{\sqrt{2}V}(\omega^2 + i\omega^1) \quad (\text{C.27})$$

($\omega^3 = 0$) and by the Legendre transformation $K_T = -mu$ [see eqs. (C.15) and (C.14)].

Notice that if the theory is also invariant under the shift of $\text{Im } \Phi$, then is \mathcal{H} a real function of L (or U) and $\Phi + \bar{\Phi}$ and $\omega^1 = 0$. Relation (C.25) implies then that V does not depend on x^2 : obviously, V does not depend on $\text{Im } \Phi$.

As an example, the Taub-NUT metric is considered in Appendix D.2.

C.3 Two-Tensor Formulation

Similarly, we can turn $\Phi + \bar{\Phi}$ into a second linear superfield L' to obtain the two-tensor formulation of the kinetic Lagrangian (C.11). Rewriting it as

$$\mathcal{L}_{ST} = \int d^2\theta d^2\bar{\theta} [\mathcal{H}(L, V) - mL' V], \quad (\text{C.28})$$

with an unconstrained real superfield V to impose $V = \Phi + \bar{\Phi}$ and an arbitrary parameter m . If we instead eliminate V by its field equation

$$\mathcal{H}_V = mL', \quad \mathcal{H}_V = \frac{\partial}{\partial V} \mathcal{H}(L, V), \quad (\text{C.29})$$

the resulting two-tensor theory is

$$\mathcal{L}_{2T} = \int d^2\theta d^2\bar{\theta} \mathcal{G}(L, L'), \quad \mathcal{G}(L, L') = \mathcal{H}(L, V) - mL' V, \quad (\text{C.30})$$

with V replaced by the solution $V(L, L')$ of eq. (C.29). Again the Legendre transformation generates relations between derivatives of \mathcal{G} and \mathcal{H} :

$$\mathcal{G}_{LL} = \mathcal{H}_{LL} - \frac{\mathcal{H}_{LV}^2}{\mathcal{H}_{VV}}, \quad \mathcal{G}_{LL'} = m \frac{\mathcal{H}_{LV}}{\mathcal{H}_{VV}}, \quad \mathcal{G}_{L'L'} = -\frac{m^2}{\mathcal{H}_{VV}}. \quad (\text{C.31})$$

As in the hypermultiplet formulation, we have a determinant relation

$$\mathcal{G}_{LL} \mathcal{G}_{L'L'} - \mathcal{G}_{LL'}^2 = -m^2 \frac{\mathcal{H}_{LL}}{\mathcal{H}_{VV}}. \quad (\text{C.32})$$

The bosonic kinetic terms of the two-tensor formulation can then be written

$$\begin{aligned}
\mathcal{L}_{2T,kin.} &= -\frac{1}{4}\mathcal{G}_{LL}\left[(\partial_\mu C)(\partial_\mu C) + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right] \\
&\quad -\frac{1}{4}\mathcal{G}_{L'L'}\left[(\partial_\mu C')(\partial_\mu C') + \frac{1}{12}H'_{\mu\nu\rho}H'^{\mu\nu\rho}\right] \\
&\quad -\frac{1}{2}\mathcal{G}_{LL'}\left[(\partial_\mu C)(\partial_\mu C') + \frac{1}{12}H_{\mu\nu\rho}H'^{\mu\nu\rho}\right] \\
&= -\frac{1}{4}\mathcal{H}_{LL}\left[(\partial_\mu C)(\partial_\mu C) + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right] \\
&\quad +\frac{m^2}{4\mathcal{H}_{VV}}\left[(\partial_\mu C' - \frac{1}{m}\mathcal{H}_{LV}\partial_\mu C)(\partial^\mu C' - \frac{1}{m}\mathcal{H}_{LV}\partial^\mu C) \right. \\
&\quad \quad \left. +\frac{1}{12}(H'_{\mu\nu\rho} - \frac{1}{m}\mathcal{H}_{LV}H_{\mu\nu\rho})(H'^{\mu\nu\rho} - \frac{1}{m}\mathcal{H}_{LV}H^{\mu\nu\rho})\right],
\end{aligned} \tag{C.33}$$

with $H_{\mu\nu\rho} = 3\partial_{[\mu}B_{\nu\rho]}$ and $H'_{\mu\nu\rho} = 3\partial_{[\mu}B'_{\nu\rho]}$ and, as before, V should be replaced by the solution $V(L, L')$.

The condition imposed by the second supersymmetry has not been imposed yet. In the single-tensor formulation, $\mathcal{N} = 2$ supersymmetry is obtained if $\mathcal{H}_{LL} = -2\mathcal{H}_{VV}$. The two-tensor version (C.30) has then $\mathcal{N} = 2$ supersymmetry if

$$\mathcal{G}_{LL}\mathcal{G}_{L'L'} - \mathcal{G}_{LL'}^2 = 2m^2, \tag{C.34}$$

i.e. if the determinant is a positive constant. Bosonic kinetic terms of the $\mathcal{N} = 2$ theory are then

$$\begin{aligned}
\mathcal{L}_{2T,kin.} &= -\frac{m^2}{2\mathcal{G}_{L'L'}}\left[(\partial_\mu C)(\partial_\mu C) + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right] \\
&\quad -\frac{1}{4}\mathcal{G}_{L'L'}\left[(\partial_\mu C' + \frac{\mathcal{G}_{LL'}}{\mathcal{G}_{L'L'}}\partial_\mu C)(\partial^\mu C' + \frac{\mathcal{G}_{LL'}}{\mathcal{G}_{L'L'}}\partial^\mu C) \right. \\
&\quad \quad \left. +\frac{1}{12}(H'_{\mu\nu\rho} + \frac{\mathcal{G}_{LL'}}{\mathcal{G}_{L'L'}}H_{\mu\nu\rho})(H'^{\mu\nu\rho} + \frac{\mathcal{G}_{LL'}}{\mathcal{G}_{L'L'}}H^{\mu\nu\rho})\right], \\
&= -\frac{1}{4}\mathcal{H}_{LL}\left[(\partial_\mu C)(\partial_\mu C) + \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}\right] \\
&\quad -\frac{m^2}{2\mathcal{H}_{LL}}\left[(\partial_\mu C' - \frac{1}{m}\mathcal{H}_{LV}\partial_\mu C)(\partial^\mu C' - \frac{1}{m}\mathcal{H}_{LV}\partial^\mu C) \right. \\
&\quad \quad \left. +\frac{1}{12}(H'_{\mu\nu\rho} - \frac{1}{m}\mathcal{H}_{LV}H_{\mu\nu\rho})(H'^{\mu\nu\rho} - \frac{1}{m}\mathcal{H}_{LV}H^{\mu\nu\rho})\right].
\end{aligned} \tag{C.35}$$

While the first supersymmetry imposes a relation between scalar and tensor kinetic terms, the second imposes a specific relation between the kinetic terms of the two linear superfields.

In comparing with the reduction of a IIB supergravity Lagrangian, one should then choose a gravity frame in which the relation between scalar and tensor kinetic terms is verified. The first supersymmetry and kinetic terms (C.33) are then sufficient for this choice.

APPENDIX D

OBTAINING THE TAUB-NUT METRIC FROM CONFORMAL SUPERGRAVITY

D.1 $SU(2, 1)/SU(2) \times U(1)$ and its Global Hyper-Kähler Limit

The superconformal construction of the $\mathcal{N} = 2$ $SU(2, 1)/SU(2) \times U(1)$ sigma-model coupled to $\mathcal{N} = 2$ supergravity starts with one vector multiplet (for the graviphoton) and three hypermultiplets. However, with these states only, eliminating auxiliary fields and imposing Poincaré gauge conditions would lead to the $Sp(4, 2)/Sp(4) \times Sp(2)$ theory. We need an additional non-propagating vector multiplet with gauge field W_μ to eliminate four more scalars and to reduce the theory to $SU(2, 1)/SU(2) \times U(1)$. The vector field will be used to gauge a $U(1)$ or $SO(1, 1)$ subgroup of $Sp(4, 2)$ with generator T . This is very much similar to what we do in section 11.2 where we obtain the universal hypermultiplet from conformal $\mathcal{N} = 2$ supergravity.

The basic difference here is that in order to reduce to a Taub-NUT metric, we need to start with a different signature for η :

$$\eta = \text{diag}(-1, 1, 1) \tag{D.1}$$

The first steps of writing down the supergravity scalar Lagrangian and imposing the proper constraints is exactly the same as in subsec. 11.2.5 until eq. (11.62) where the the different choice of signature appears explicitly:

$$\vec{A}_+^* \cdot \vec{A}_+ + \vec{A}_-^* \cdot \vec{A}_- = -\frac{1}{\kappa^2}, \quad \vec{A}^* \cdot \vec{A} = \vec{A}^\dagger \eta \vec{A} = -|A_1|^2 + |A_2|^2 + |A_3|^2. \quad (\text{D.2})$$

From that point on, in order to obtain the Taub-NUT metric we proceed as follows. We first define

$$q_a = \frac{1}{\kappa} Q_a Q_1^{-1}, \quad (\text{D.3})$$

and $q_1 = \frac{1}{\kappa} I_2$ will not be used herebelow. Defining the new coordinates q_a left invariant by the superconformal $SU(2)$ is equivalent to identify the superconformal $SU(2)$ with $SU(2)_1$ and choose a gauge for Q_1 . Explicitly,

$$q_a = \begin{pmatrix} q_{+a} & q_{-a} \\ -q_{-a}^* & q_{+a}^* \end{pmatrix} = \frac{1}{\kappa \det Q_1} \begin{pmatrix} A_{+a} A_{+1}^* + A_{-a} A_{-1}^* & -A_{+a} A_{-1} + A_{-a} A_{+1} \\ -A_{-a}^* A_{+1}^* + A_{+a}^* A_{-1}^* & A_{-a}^* A_{-1} + A_{+a}^* A_{+1} \end{pmatrix}.$$

Similarly,

$$Q_a = \kappa \begin{pmatrix} q_{+a} A_{+1} - q_{-a} A_{-1}^* & q_{+a} A_{-1} + q_{-a} A_{+1}^* \\ -q_{-a}^* A_{+1} - q_{+a}^* A_{-1}^* & -q_{-a}^* A_{-1} + q_{+a}^* A_{+1}^* \end{pmatrix}.$$

The second condition (D.2) is now written as

$$-\det Q_1 + \det Q_2 + \det Q_3 = -\frac{1}{\kappa^2}, \quad \det Q_1 = \frac{1}{\kappa^2(1 - \kappa^2 \det q_2 - \kappa^2 \det q_3)}. \quad (\text{D.4})$$

Both Q_a and q_a have dimension (mass)¹ and they verify $\det Q_a \leq \kappa^{-2}$, $\det q_a \leq \kappa^{-2}$.

We will use the $SU(2)$ symmetry to choose

$$A_{+1} = \sqrt{\det Q_1} = A_{+1}^*, \quad A_{-1} = 0, \quad q_a = \frac{1}{\kappa \sqrt{\det Q_1}} Q_a \quad (a = 2, 3). \quad (\text{D.5})$$

Notice that with this choice A_{+1} and q_a are respectively of order κ^{-1} and κ^0 . Actually, in the global supersymmetry limit $\kappa \rightarrow 0$, the constraint reduces to $A_{+1} = \kappa^{-1}$. The $SU(2)$ gauge fields and their contributions to the Lagrangian are of order κ^2 .

With the above choices, the sigma-model Lagrangian for the scalar fields becomes

$$\begin{aligned}
\mathcal{L}_{scalar} &= 2\kappa^2 A_{+1}^2 \left[(\partial_\mu q_{+2})(\partial_\mu q_{+2}^*) + (\partial_\mu q_{-2})(\partial_\mu q_{-2}^*) \right. \\
&\quad \left. + (\partial_\mu q_{+3})(\partial_\mu q_{+3}^*) + (\partial_\mu q_{-3})(\partial_\mu q_{-3}^*) \right] \\
&\quad + \frac{1}{2} \kappa^6 A_{+1}^4 [\partial_\mu (\det q_2 + \det q_3)]^2 \\
&\quad - \frac{g^2}{\kappa^2} V_{\mu i}{}^j V_{\mu j}{}^i \\
&\quad + g'^2 W_\mu W^\mu d_\beta^\alpha T^\gamma{}_\alpha T^\beta{}_\delta A_\gamma^i A_i^\delta \\
&= \mathcal{L}_0 + \mathcal{L}_{A_{+1}} + \mathcal{L}_{SU(2)} + \mathcal{L}_{O(1,1)},
\end{aligned} \tag{D.6}$$

with A_{+1} as in the first eq. (D.5). Notice that the term in the third line is

$$\mathcal{L}_{A_{+1}} = \frac{2}{\kappa^2} \partial_\mu \ln(\kappa A_{+1}) \partial^\mu \ln(\kappa A_{+1}).$$

It vanishes in the limit $\kappa \rightarrow 0$. The $SU(2)$ gauge fields do not depend on derivatives of A_{+1} :

$$\begin{aligned}
\mathcal{L}_{SU(2)} &= -\frac{g^2}{\kappa^2} V_{\mu i}{}^j V_{\mu j}{}^i = -\frac{1}{4} \kappa^6 A_{+1}^4 \text{Tr} \left[q_2^\dagger \overleftrightarrow{\partial}_\mu q_2 + q_3^\dagger \overleftrightarrow{\partial}_\mu q_3 \right]^2 \\
&= -\frac{1}{2} \kappa^6 A_{+1}^4 \left[(q_{+a}^* \overleftrightarrow{\partial}_\mu q_{+a} - q_{-a}^* \overleftrightarrow{\partial}_\mu q_{-a})^2 + 4(q_{+a}^* \overleftrightarrow{\partial}_\mu q_{-a})(q_{-a}^* \overleftrightarrow{\partial}_\mu q_{+a}) \right],
\end{aligned} \tag{D.7}$$

where a is summed over values $a = 2, 3$ only. This contribution also cancels in the limit $\kappa \rightarrow 0$ where $\mathcal{L}_{scalar, \kappa \rightarrow 0} = \mathcal{L}_0 + \mathcal{L}_{SO(1,1)}$, with $\kappa A_{+1} = 1$.

With $g' = 0$ and without the constraint (11.55), one obtains the sigma-model $HP^2 = Sp(4, 2)/Sp(4) \times Sp(2)$. Expressed in terms of the quaternion (2×2) ma-

trices q_2 and q_3 , it reads:

$$\begin{aligned} \mathcal{L}_{HP^2} = & \kappa^2 A_{+1}^2 \text{Tr}[(\partial_\mu q_2)^\dagger (\partial^\mu q_2) + (\partial_\mu q_3)^\dagger (\partial^\mu q_3)] \\ & + \kappa^6 A_{+1}^4 \text{Tr}[(q_2^\dagger \partial_\mu q_2 + q_3^\dagger \partial_\mu q_3)(\partial^\mu q_2^\dagger q_2 + \partial^\mu q_3^\dagger q_3)]. \end{aligned} \quad (\text{D.8})$$

In the limit $\kappa \rightarrow 0$, $\kappa A_{+1} \rightarrow 1$ and the sigma-model metric is trivial.

If we choose the $U(1)$ generator T as in eq. (11.50):

$$T = \begin{pmatrix} \hat{T} & 0 \\ 0 & \hat{T}^* \end{pmatrix}, \quad \hat{T}^\dagger = -\eta \hat{T} \eta, \quad (\text{D.9})$$

then constraint (D.2) leads to three (real) equations:

$$\begin{aligned} \vec{A}_+^\dagger i\eta \hat{T} \vec{A}_+ &= \vec{A}_-^\dagger i\eta \hat{T} \vec{A}_-, \\ \vec{A}_-^\dagger i\eta \hat{T} \vec{A}_+ &= 0 \end{aligned} \quad (\text{D.10})$$

($[i\eta \hat{T}]^\dagger = i\eta \hat{T}$). With the $SO(1,1)$ generator

$$\hat{T} = \begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \quad (\text{D.11})$$

(λ real) the three constraints are:

$$\begin{aligned} \lambda(A_{+2}^* A_{+1} - A_{+1}^* A_{+2}) + iA_{+3}^* A_{+3} &= \lambda(A_{-2}^* A_{-1} - A_{-1}^* A_{-2}) + iA_{-3}^* A_{-3}, \\ \lambda(A_{-2}^* A_{+1} - A_{-1}^* A_{+2}) + iA_{-3}^* A_{+3} &= 0. \end{aligned} \quad (\text{D.12})$$

These conditions survive in the global supersymmetry limit $\kappa \rightarrow 0$, where also $\det Q_1 \rightarrow \kappa^{-2}$, if λA_{+1} has a finite limit. Since $\kappa A_{+1} \rightarrow 1$, we then assume that¹

¹ ℓ has dimension (mass)¹.

$\lambda = \ell\kappa$. In terms of the coordinates q_a , the conditions are:

$$\begin{aligned}
& \ell \left[(|A_{+1}|^2 - |A_{-1}|^2)(q_{+2}^* - q_{+2}) - 2A_{+1}A_{-1}q_{-2}^* + 2A_{+1}^*A_{-1}^*q_{-2} \right] \\
& = i \left[(|A_{-1}|^2 - |A_{+1}|^2)(|q_{+3}|^2 - |q_{-3}|^2) + 2A_{+1}A_{-1}q_{+3}q_{-3}^* + 2A_{+1}^*A_{-1}^*q_{-3}q_{+3}^* \right], \\
& \ell \left[A_{+1}A_{-1}^*(q_{+2}^* - q_{+2}) + A_{+1}A_{+1}q_{-2}^* + A_{-1}^*A_{-1}^*q_{-2} \right] \\
& = i \left[A_{+1}A_{-1}^*(|q_{-3}|^2 - |q_{+3}|^2) - A_{+1}A_{+1}q_{+3}q_{-3}^* + A_{-1}^*A_{-1}^*q_{+3}q_{-3}^* \right].
\end{aligned}$$

Using $SU(2)$ symmetry to choose as earlier $A_{-1} = 0$, we obtain

$$i\ell (q_{+2}^* - q_{+2}) = |q_{+3}|^2 - |q_{-3}|^2, \quad (\text{D.13})$$

$$\ell q_{-2} = i q_{+3}^* q_{-3},$$

independent of κ . In the limiting case $\ell = 0$, $q_3 = 0$ and the resulting constraint (D.4) leads to the four-dimensional $Sp(2, 2)/Sp(2) \times Sp(2)$. As a $SO(1, 1)$ gauge choice, we may take $\text{Re } q_{+2} = 0$, which leads to

$$\begin{aligned}
q_{+2} &= \frac{i}{2\ell} (|q_{+3}|^2 - |q_{-3}|^2), \\
q_{-2} &= \frac{i}{\ell} q_{+3}^* q_{-3},
\end{aligned} \quad (\text{D.14})$$

$$\det q_2 = |q_{+2}|^2 + |q_{-2}|^2 = \frac{1}{4\ell^2} (\det q_3)^2.$$

With $A_{-1} = 0$ and A_{+1} real, the unconstrained fields are $q_{\pm 3}$, with $q_{\pm 2}$ given by eqs. (D.14) and with relations

$$q_3 = \frac{1}{\kappa \sqrt{\det Q_1}} \begin{pmatrix} A_{+3} & A_{-3} \\ -A_{-3}^* & A_{+3}^* \end{pmatrix}, \quad (\text{D.15})$$

$$A_{+1} = \sqrt{\det Q_1} = \frac{1}{\kappa} \left[1 - \kappa^2 \det q_3 - \frac{\kappa^2}{4\ell^2} (\det q_3)^2 \right]^{-1/2}.$$

In terms of quaternion matrices, conditions (D.14) correspond to

$$q_2 = \frac{i}{2\ell} q_3^\dagger J q_3, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{D.16})$$

With the gauge choices $A_{-1} = \text{Re } q_{+2} = 0$ and A_{+1} real, the $SO(1, 1)$ gauge field reads

$$W_\mu = \frac{i}{2g} \frac{q_{+3}^* \overleftrightarrow{\partial}_\mu q_{+3} + q_{-3}^* \overleftrightarrow{\partial}_\mu q_{-3}}{\ell^2 + \det q_3 - \frac{\kappa^2}{4} (\det q_3)^2} \quad (\text{D.17})$$

in terms of $q_{\pm 3}$. Its contribution to the scalar Lagrangian is

$$\mathcal{L}_{SO(1,1)} = \frac{1}{2} \kappa^2 A_{+1}^2 \frac{(q_{+3}^* \overleftrightarrow{\partial}_\mu q_{+3} + q_{-3}^* \overleftrightarrow{\partial}_\mu q_{-3})^2}{\ell^2 + \det q_3 - \frac{\kappa^2}{4} (\det q_3)^2}. \quad (\text{D.18})$$

To calculate the various contributions to the scalar Lagrangian (D.6), we introduce new (real) coordinates (r, θ, ϕ, τ) :

$$q_{+3} = r \cos \frac{\theta}{2} e^{i(\phi+\tau)/2}, \quad q_{-3} = r \sin \frac{\theta}{2} e^{-i(\phi-\tau)/2}. \quad (\text{D.19})$$

With these variables,

$$\begin{aligned} \det q_3 &= r^2, \\ |dq_{+3}|^2 + |dq_{-3}|^2 &= dr^2 + \frac{r^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r^2}{4} (d\tau + \cos \theta d\phi)^2, \\ q_{+2} &= \frac{ir^2}{2\ell} \cos \theta, \quad q_{-2} = \frac{ir^2}{2\ell} \sin \theta e^{-i\phi}, \\ |dq_{+2}|^2 + |dq_{-2}|^2 &= \frac{r^2}{\ell^2} \left[dr^2 + \frac{r^2}{4} (d\theta^2 + \sin^2 \theta d\phi^2) \right], \\ \kappa A_{+1} &= \left[1 - \kappa^2 r^2 - \frac{\kappa^2}{4\ell^2} r^4 \right]^{-1/2}. \end{aligned}$$

The basic scalar kinetic terms become

$$\begin{aligned} \mathcal{L}_0 &= 2\kappa^2 A_{+1}^2 \left[|\partial_\mu q_{+2}|^2 + |\partial_\mu q_{-2}|^2 + |\partial_\mu q_{+3}|^2 + |\partial_\mu q_{-3}|^2 \right] \\ &= 2\kappa^2 A_{+1}^2 \left[\left(1 + \frac{r^2}{\ell^2} \right) \left[(\partial_\mu r)^2 + \frac{r^2}{4} \{ (\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \phi)^2 \} \right] \right. \\ &\quad \left. + \frac{r^2}{4} (\partial_\mu \tau + \cos \theta \partial_\mu \phi)^2 \right]. \end{aligned} \quad (\text{D.20})$$

The contribution of the $SO(1, 1)$ gauge field is

$$\mathcal{L}_{SO(1,1)} = -\frac{r^4}{2} \kappa^2 A_{+1}^2 \frac{(\partial_\mu \tau + \cos \theta \partial_\mu \phi)^2}{\ell^2 + r^2 - \frac{\kappa^2}{4} r^4}. \quad (\text{D.21})$$

The contribution of the $SU(2)$ gauge fields is

$$\mathcal{L}_{SU(2)} = \frac{1}{2} \kappa^6 A_{+1}^4 r^4 \left[(\partial_\mu \tau + \cos \theta \partial_\mu \phi)^2 + \left(1 + \frac{r^2}{2\ell^2}\right)^2 \{(\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \phi)^2\} \right]. \quad (\text{D.22})$$

Finally

$$\mathcal{L}_{A_{+1}} = 2\kappa^6 A_{+1}^4 \left(1 + \frac{r^2}{2\ell^2}\right)^2 r^2 (\partial_\mu r)(\partial^\mu r). \quad (\text{D.23})$$

Both $\mathcal{L}_{SU(2)}$ and $\mathcal{L}_{A_{+1}}$ vanish (like κ^2) in the limit $\kappa \rightarrow 0$. Then, summing the four contributions leads to the scalar Lagrangian

$$\begin{aligned} \mathcal{L} = & \frac{1}{2\ell^2} \kappa^4 A_{+1}^4 \left(1 + \frac{\ell^2}{r^2} - \frac{\kappa^2 r^2}{4}\right) \left[4r^2 (\partial_\mu r)^2 + r^4 \{(\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \phi)^2\}\right] \\ & + \frac{1}{2\ell^2} \kappa^4 A_{+1}^4 \frac{(\ell^2 + \frac{\kappa^2 r^4}{4})^2}{1 + \frac{\ell^2}{r^2} - \frac{\kappa^2 r^2}{4}} (\partial_\mu \tau + \cos \theta \partial_\mu \phi)^2. \end{aligned} \quad (\text{D.24})$$

If we define a new variable $R = r^2/\ell$, choosing a positive ℓ , the theory becomes

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} \kappa^4 A_{+1}^4 \left(1 + \frac{\ell}{R} - \frac{\kappa^2 \ell R}{4}\right) \left[(\partial_\mu R)^2 + R^2 \{(\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \phi)^2\}\right] \\ & + \frac{1}{2} \kappa^4 A_{+1}^4 \frac{(1 + \frac{\kappa^2 R^2}{4})^2}{1 + \frac{\ell}{R} - \frac{\kappa^2 \ell R}{4}} \ell^2 (\partial_\mu \tau + \cos \theta \partial_\mu \phi)^2, \end{aligned} \quad (\text{D.25})$$

where²

$$\kappa^4 A_{+1}^4 = \left[1 - \kappa^2 \ell R - \frac{\kappa^2 R^2}{4}\right]^{-2}. \quad (\text{D.26})$$

The parameter ℓ defines the energy scale of the field R while the length κ defines the curvature of the quaternionic manifold. The metric defined by these kinetic terms is Einstein with

$$R_{ab} = -6\kappa^2 g_{ab}, \quad (\text{D.27})$$

as expected for a single hypermultiplet quaternionic space [14].

The limit $\kappa \rightarrow 0$ leads to

$$\begin{aligned} \mathcal{L}_{\kappa \rightarrow 0} = & \frac{1}{2} \left[\left(1 + \frac{\ell}{R}\right) \left[(\partial_\mu R)^2 + R^2 \{(\partial_\mu \theta)^2 + \sin^2 \theta (\partial_\mu \phi)^2\}\right] \right. \\ & \left. + \frac{\ell^2}{1 + \frac{\ell}{R}} (\partial_\mu \tau + \cos \theta \partial_\mu \phi)^2 \right]. \end{aligned} \quad (\text{D.28})$$

²Positivity implies $R \leq \frac{2}{\kappa} (\sqrt{1 + \kappa^2 \ell^2} - \kappa \ell)$.

We will see later [eq. (D.35)] that the metric of this scalar Lagrangian is the Taub-NUT metric with $2M = \ell$.

There are four isometries acting on θ , ϕ and τ . Three are the spherical symmetries of $(\partial_\mu\theta)^2 + \sin^2\theta(\partial_\mu\phi)^2$, the fourth isometry is the shift of τ . Explicitly, the metric is invariant under

$$\begin{aligned}\delta\theta &= \sin\phi c_2 + \cos\phi c_3, \\ \delta\phi &= c_1 + \cotg\theta(\cos\phi c_2 - \sin\phi c_3), \\ \delta\tau &= c_4 - \frac{1}{\sin\theta}(\cos\phi c_2 - \sin\phi c_3).\end{aligned}\tag{D.29}$$

where C_I , $I = 1, 2, 3, 4$ are the real parameters of the isometries. The $SU(2)$ algebra is verified by transformations with parameters c_1 , c_2 and c_3 .

We introduce cartesian coordinates x_i , $i = 1, 2, 3$ instead of the polar coordinates R, θ, ϕ :

$$x_1 = R \sin\theta \cos\phi, \quad x_2 = R \sin\theta \sin\phi, \quad x_3 = R \cos\theta.$$

Using

$$\frac{x_1 dx_2 - x_2 dx_1}{x_1^2 + x_2^2} = d\phi, \quad \frac{x_3}{R} = \cos\theta, \quad R = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

We can rewrite our Lagrangian in the following form:

$$\mathcal{L} = F(R) (\partial_\mu x_i)(\partial^\mu x_i) + G(R) (\partial_\mu \tau + \omega_i \partial_\mu x_i)^2.\tag{D.30}$$

We find

$$\begin{aligned}F(R) &= \frac{1}{2} k^4 A_{+1}^4 \left[1 + \frac{\ell}{R} - \frac{\kappa^2 \ell R}{4} \right], \\ G(R) &= \frac{1}{2} k^4 A_{+1}^4 \frac{(1 + \frac{\kappa^2 R^2}{4})^2}{1 + \frac{\ell}{R} - \frac{\kappa^2 \ell R}{4}} \ell^2, \\ \omega_1 &= -\frac{x_2 x_3}{R(x_1^2 + x_2^2)}, \quad \omega_2 = \frac{x_1 x_3}{R(x_1^2 + x_2^2)}, \quad \omega_3 = 0.\end{aligned}\tag{D.31}$$

In the limit $\kappa \rightarrow 0$, $F(R)G(R) = \ell^2/4$.

Notice that

$$\frac{d}{dR} \left[1 + \frac{\ell}{R} - \frac{\kappa^2 \ell R}{4} \right] = -\frac{\ell}{R^2} \left[1 + \frac{\kappa^2 R^2}{4} \right]$$

In a set of Kahler coordinates $z^i = (T, \Phi)$, one can in general write

$$ds^2 = K_{T\bar{T}} \left(dT + \frac{K_{\Phi\bar{T}}}{K_{T\bar{T}}} d\Phi \right) \left(d\bar{T} + \frac{K_{T\bar{\Phi}}}{K_{T\bar{T}}} d\bar{\Phi} \right) + \frac{\det K_{i\bar{j}}}{K_{T\bar{T}}} d\Phi d\bar{\Phi}.$$

For an Einstein space with $R_{i\bar{j}} = \partial_i \partial_{\bar{j}} \ln \det K_{k\bar{l}} = \Delta K_{i\bar{j}}$,

$$ds^2 = K_{T\bar{T}} \left(dT + \frac{K_{\Phi\bar{T}}}{K_{T\bar{T}}} d\Phi \right) \left(d\bar{T} + \frac{K_{T\bar{\Phi}}}{K_{T\bar{T}}} d\bar{\Phi} \right) + \frac{Ae^{\Delta K}}{K_{T\bar{T}}} d\Phi d\bar{\Phi},$$

where A is an arbitrary positive constant.³ Defining $K = -n \ln Y$, the line element is

$$ds^2 = K_{T\bar{T}} \left(dT + \frac{K_{\Phi\bar{T}}}{K_{T\bar{T}}} d\Phi \right) \left(d\bar{T} + \frac{K_{T\bar{\Phi}}}{K_{T\bar{T}}} d\bar{\Phi} \right) + \frac{AY^{-n\Delta}}{K_{T\bar{T}}} d\Phi d\bar{\Phi}.$$

If we further assume that the Kähler potential K is a function of $T + \bar{T}$, Φ and $\bar{\Phi}$, since T is dual to a linear superfield,

$$dK_T = K_{T\bar{T}} d(T + \bar{T}) + K_{\Phi\bar{T}} d\Phi + K_{T\bar{\Phi}} d\bar{\Phi},$$

and the line element becomes

$$\begin{aligned} ds^2 &= K_{T\bar{T}} \left(d \operatorname{Re} T + \frac{K_{\Phi\bar{T}}}{2K_{T\bar{T}}} d\Phi + \frac{K_{T\bar{\Phi}}}{2K_{T\bar{T}}} d\bar{\Phi} \right)^2 \\ &\quad + K_{T\bar{T}} \left(d \operatorname{Im} T + \frac{K_{\Phi\bar{T}}}{2iK_{T\bar{T}}} d\Phi - \frac{K_{T\bar{\Phi}}}{2iK_{T\bar{T}}} d\bar{\Phi} \right)^2 + \frac{AY^{-n\Delta}}{K_{T\bar{T}}} d\Phi d\bar{\Phi} \\ &= \frac{1}{4K_{T\bar{T}}} (dK_T)^2 + K_{T\bar{T}} \left(d \operatorname{Im} T + \frac{K_{\Phi\bar{T}}}{2iK_{T\bar{T}}} d\Phi - \frac{K_{T\bar{\Phi}}}{2iK_{T\bar{T}}} d\bar{\Phi} \right)^2 + \frac{AY^{-n\Delta}}{K_{T\bar{T}}} d\Phi d\bar{\Phi}. \end{aligned}$$

³ A could be in principle a harmonic function $f(T, \Phi) + \bar{f}(\bar{T}, \bar{\Phi})$ but this case is irrelevant for us.

D.2 Taub-NUT

The Taub-NUT (Taub-Newman-Unti-Tamburino) metric [107] describes a four-dimensional euclidean space with self-dual curvature. It is then Ricci-flat and a solution of the vacuum Einstein equations. Hence, it is also hyper-Kähler and appropriate to describe the scalar sector of a globally $\mathcal{N} = 2$ hypermultiplet theory.

The Taub-NUT metric is commonly expressed in coordinates where

$$ds_{TN}^2 = \frac{r+M}{r-M} dr^2 + (r^2 - M^2)(\sigma_1^2 + \sigma_2^2) + 4M^2 \frac{r-M}{r+M} \sigma_3^2. \quad (\text{D.32})$$

The one-forms

$$\begin{aligned} \sigma_1 &= \cos \tau d\theta + \sin \tau \sin \theta d\varphi, \\ \sigma_2 &= -\sin \tau d\theta + \cos \tau \sin \theta d\varphi, \\ \sigma_3 &= \cos \theta d\varphi + d\tau \end{aligned} \quad (\text{D.33})$$

verify

$$d\sigma_x = -\epsilon_{xyz} \sigma_y \wedge \sigma_z \quad (x, y, z = 1, 2, 3). \quad (\text{D.34})$$

The coordinates τ , θ and φ are angular variables ($0 \leq \theta \leq \pi$, $0 \leq \tau \leq 4\pi$, $0 \leq \varphi \leq 2\pi$), $r > M$ and M is a (real) parameter. A more convenient form is obtained by shifting the singularity from $r = M$ to $R = 0$ with the redefinition $R = r - M$. The metric becomes [100]

$$ds_{TN}^2 = V[dR^2 + R^2 d\Omega] + \frac{4M^2}{V}[d\tau + \cos \theta d\varphi]^2, \quad V = 1 + \frac{2M}{R}, \quad (\text{D.35})$$

where

$$d\Omega = \sigma_1^2 + \sigma_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2. \quad (\text{D.36})$$

This form is reminiscent of a (euclidean) Schwarzschild metric. Since

$$\frac{1}{4M^2} ds_{TN}^2 = \frac{\rho+1}{\rho}[d\rho^2 + \rho^2 d\Omega] + \frac{\rho}{\rho+1}[d\tau + \cos \theta d\varphi]^2, \quad \rho = \frac{R}{2M}, \quad (\text{D.37})$$

the constant $2M$ sets the scale of the radial coordinate R . Notice that the determinant of the metric is

$$\det g_{ab} = 4M^2 V^2 R^4 \sin^2 \theta = 4M^2 R^2 \sin^2 \theta (R + 2M)^2. \quad (\text{D.38})$$

It would be constant in Kähler coordinates.

In cartesian coordinates $q^a = (\tau, x^i)$, with $dx^i dx^i = dR^2 + R^2 d\Omega$, the metric is

$$ds_{TN}^2 = 2M \left(\frac{V}{2M} dx^i dx^i + \frac{2M}{V} [d\tau - \omega^i dx^i]^2 \right), \quad (\text{D.39})$$

with

$$\omega^1 = -\frac{x^2 x^3}{R(x^{1^2} + x^{2^2})}, \quad \omega^2 = \frac{x^1 x^3}{R(x^{1^2} + x^{2^2})}, \quad \omega^3 = 0. \quad (\text{D.40})$$

The relation [100]

$$\vec{\nabla} \frac{V}{2M} = \vec{\nabla} \wedge \vec{\omega}, \quad (\text{D.41})$$

which is required for four-dimensional hyperkähler manifold, is verified.

The Taub-NUT metric (D.35) is invariant under $SU(2) \times U(1)$ isometries

$$\begin{aligned} \delta\theta &= \sin\phi c_2 + \cos\phi c_3, \\ \delta\phi &= c_1 + \cotg\theta(\cos\phi c_2 - \sin\phi c_3), \\ \delta\tau &= c_4 - \frac{1}{\sin\theta}(\cos\phi c_2 - \sin\phi c_3), \end{aligned} \quad (\text{D.42})$$

where c_I , $I = 1, 2, 3, 4$ are constant real parameters. The $SU(2)$ algebra is generated by transformations with parameters c_1 , c_2 and c_3 . On cartesian coordinates x^i , the action of the $SU(2)$ isometries is

$$\delta x^1 = -c_1 x^2 + c_3 x^3, \quad \delta x^2 = c_1 x^1 + c_2 x^3, \quad \delta x^3 = -c_3 x^1 - c_2 x^2. \quad (\text{D.43})$$

On the Kähler coordinate $\Phi = (x^1 + ix^2)/\sqrt{2}$,

$$\delta\Phi = ic_1 \Phi + \frac{1}{\sqrt{2}}(c_3 + ic_2)U, \quad \delta U = -\frac{1}{\sqrt{2}}(c_3 - ic_2)\Phi - \frac{1}{\sqrt{2}}(c_3 + ic_2)\bar{\Phi}, \quad (\text{D.44})$$

where $U = x^3$, leaving $U^2 + 2\Phi\bar{\Phi}$ invariant.⁴

The single-tensor $\mathcal{N} = 2$ theory leading to the Taub-NUT scalar manifold in the hypermultiplet formulation is defined by the function

$$\mathcal{H}(L, \Phi, \bar{\Phi}) = -\frac{1}{2}[L^2 - \Phi\bar{\Phi}] + 2M \left[\sqrt{L^2 + 2\Phi\bar{\Phi}} - L \ln \left(L + \sqrt{L^2 + 2\Phi\bar{\Phi}} \right) \right], \quad (\text{D.45})$$

obtained by integrating eqs. (C.24). The real superfield $L^2 + 2\Phi\bar{\Phi}$ is R^2 . Since the action of isometries does not respect in general the chiral or linear nature of a superfield, we do not expect \mathcal{H} to be invariant, but the line element (the kinetic terms) should be invariant.

In the hypermultiplet formulation, the line element reads

$$ds_{TN}^2 = \frac{1}{2} \mathcal{V} d\Phi d\bar{\Phi} + \mathcal{V}^{-1} \left| m dT + M \mathcal{U} \frac{d\Phi}{\Phi} \right|^2, \quad (\text{D.46})$$

where

$$\mathcal{V} = 1 + \frac{2M}{\sqrt{U^2 + 2\Phi\bar{\Phi}}}, \quad \mathcal{U} = 1 - \frac{U}{\sqrt{U^2 + 2\Phi\bar{\Phi}}} \quad (\text{D.47})$$

and U is defined (as a function of $T + \bar{T}$ and $\Phi\bar{\Phi}$) by the Legendre transformation (C.13):

$$U + 2M \ln \left(U + \sqrt{U^2 + 2\Phi\bar{\Phi}} \right) = -m(T + \bar{T}). \quad (\text{D.48})$$

This equation cannot be analytically inverted. The determinant of the Kähler metric is constant, as in eq. (C.17), and the second eq. (D.44) indicates that the $SU(2)$ isometries act on $T + \bar{T}$ according to

$$\delta(T + \bar{T}) = \frac{1}{\sqrt{2}m} \left(1 + \frac{2M}{U + R} \right) [(c_3 - ic_2)\Phi + (c_3 + ic_2)\bar{\Phi}]. \quad (\text{D.49})$$

To compare eqs. (D.46) and (D.39), we need to rewrite ds_{TN}^2 in coordinates (τ, x^i) with $x^i = (\sqrt{2} \text{Re } \Phi, \sqrt{2} \text{Im } \Phi, U)$ and $R^2 = x^i x^i = U^2 + 2\Phi\bar{\Phi}$. Hence,

$$\mathcal{V} = 1 + \frac{2M}{R} = V(R), \quad \mathcal{U} = 1 - \frac{U}{R}$$

⁴The phase rotation of Φ has parameter c_1 .

and, according to eq. (D.48),

$$2m d \operatorname{Re} T = -dU - 2M \frac{dU + dR}{U + R} = -V dU - \frac{2M}{R(U + R)} d(\Phi \bar{\Phi}).$$

We first obtain

$$\begin{aligned} ds_{TN}^2 &= V^{-1} \left[m d \operatorname{Re} T + \frac{M U}{2 \Phi \bar{\Phi}} d(\Phi \bar{\Phi}) \right]^2 + \frac{V}{2} d\Phi d\bar{\Phi} \\ &\quad + V^{-1} \left[m d \operatorname{Im} T + \frac{M U}{\Phi \bar{\Phi}} (\operatorname{Re} \Phi d \operatorname{Im} \Phi - \operatorname{Im} \Phi d \operatorname{Re} \Phi) \right]^2. \end{aligned}$$

Since $\frac{M U}{2 \Phi \bar{\Phi}} = \frac{M}{R(U+R)}$, we have

$$\begin{aligned} ds_{TN}^2 &= \frac{1}{4} V \left[(dU)^2 + 2 d\Phi d\bar{\Phi} \right] \\ &\quad + V^{-1} \left[m d \operatorname{Im} T + \frac{2M}{R(U+R)} (\operatorname{Re} \Phi d \operatorname{Im} \Phi - \operatorname{Im} \Phi d \operatorname{Re} \Phi) \right]^2 \quad (\text{D.50}) \\ &= \frac{1}{4} \left(V dx^i dx^i + 4m^2 V^{-1} \left[d \operatorname{Im} T + \frac{M}{mR(U+R)} (x^1 dx^2 - x^2 dx^1) \right]^2 \right). \end{aligned}$$

Finally, we set $m = M$ and use

$$\begin{aligned} \frac{1}{R(U+R)} (x^1 dx^2 - x^2 dx^1) &= -\frac{U}{R(R^2 - U^2)} (x^1 dx^2 - x^2 dx^1) + \frac{i}{2} d \ln(\bar{\Phi}/\Phi) \\ &= -\omega_i dx^i + \frac{i}{2} d \ln(\bar{\Phi}/\Phi), \end{aligned}$$

with ω_i as in eq. (D.40). Finally,

$$ds_{TN}^2 = \frac{1}{4} \left(V dx^i dx^i + 4M^2 V^{-1} \left[d \operatorname{Im} T + \frac{i}{2} d \ln(\bar{\Phi}/\Phi) - \omega_i dx^i \right]^2 \right). \quad (\text{D.51})$$

Comparison with expression (D.39) indicates that the fourth coordinate is

$$\tau = \operatorname{Im} T + \frac{i}{2} \ln(\bar{\Phi}/\Phi). \quad (\text{D.52})$$

The action of $SU(2) \times U(1)$ isometries on τ is

$$\delta \tau = c_4 - \frac{R}{2\sqrt{2}} \frac{c_2 - ic_3}{\Phi} - \frac{R}{2\sqrt{2}} \frac{c_2 + ic_3}{\bar{\Phi}}. \quad (\text{D.53})$$

Hence,

$$\begin{aligned} \delta \operatorname{Im} T &= c_4 - c_1 - \frac{1}{\sqrt{2}} \frac{1}{R-U} [(c_2 + ic_3)\Phi + (c_2 - ic_3)\bar{\Phi}] \\ \delta T &= i(c_4 - c_1) + \frac{1}{\sqrt{2}} \left[\frac{R}{\Phi \bar{\Phi}} + \frac{1}{2M} \right] (c_3 - ic_2)\Phi - \frac{1}{\sqrt{2}} \left[\frac{U}{\Phi \bar{\Phi}} - \frac{1}{2M} \right] (c_3 + ic_2)\bar{\Phi} \quad (\text{D.54}) \end{aligned}$$

To summarize, Kähler coordinates T and Φ of the Taub-NUT metric are related to standard variables by $(\tau, x^i) = (\tau, \sqrt{2} \operatorname{Re} \Phi, \sqrt{2} \operatorname{Im} \Phi, U)$. Eq. (D.52) defines $\operatorname{Im} T$ while the Legendre transformation (D.48) gives implicitly $\operatorname{Re} T$.

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