PERTURBING WARPED THROATS: EXPLICIT PREDICTIONS FROM A CORNER OF THE LANDSCAPE

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PERTURBING WARPED THROATS:

EXPLICIT PREDICTIONS FROM A CORNER OF THE LANDSCAPE

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Obtaining robust predictions from string theory has proven to be very challenging, simultaneously due to the vast number of possible vacua in the landscape and to the overwhelming geometric complexity of Calabi-Yau manifolds. We will present a promising new approach, applicable in the subset of string theory vacua possessing a warped throat region, in the form of a systematic procedure for perturbing Calabi-Yau cones. Most of the complexity of the bulk geometry is filtered out by the warping and effective descriptions for models constructed in the warped regions can be explicitly obtained. We will demonstrate the application of our procedure by analyzing the potential for angular moduli of an anti-D3-brane sitting at the tip of a Klebanov-Strassler throat.

BIOGRAPHICAL SKETCH

Sohang Gandhi was born in Cleveland, OH on January 23, 1981, but was raised in Orlando, FL. He became interested in science at an early age and knew he wanted to become a physicist since he was in high school. It was upon reading "A Brief History of Time" by Stephen Hawking, and understanding virtually nothing, that he decided to devote all his energy to being able to comprehend what the author was talking about. He attended the University of Central Florida where he earned his Bachelors in computational physics with honors. He then joined the PhD program in theoretical physics at Cornell University. He began his research career in the area of particle phenomenology but upon taking a course in string theory, was so inspired by the beauty of the subject that he shifted his field and began doing research with the lecturer, Liam McAllister. He spent the next four years deeply immersed in the geometric subtleties of the extra dimensions of string theory.

This work is dedicated to my late mother, Lekha C. Gandhi. All of my accomplishments spring from the sacrifices she made for her children, and the strength and dignity with which she faced life.

ACKNOWLEDGEMENTS

I owe a great deal of thanks to my advisor Liam McAllister. The are few academic mentors that are as dedicated to their students as him. He not only taught me how to do research as a physicist but he also showed me the mindset that is critical for success in all of life's endeavors. My previous advisor, Maxim Perelstein, was a major source of support throughout my graduate career. I am indebted to him for his kindness during many difficult times. I have learned a great deal from other faculty members at Cornell and beyond and would like to thank Jim Alexander, Marcus Berg, Csaba Csaki, Yuval Grossman, Chris Henley, André LeClair, and Henry Tye for sharing their wisdom inside and outside of the classroom.

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I also owe a lot to my family, especially my mother who sacrificed so much so that I could pursue an education and succeed in my life. I would also like to thank my brother and father, my aunts and uncles, and all the friends that I have made in Ithaca during my graduate studies.

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CHAPTER 1

INTRODUCTION

String theory is our current best candidate for a "Theory of Everything," successfully combining quantum theory and gravity in a consistent framework. This is truly a monumental theoretical achievement and the predictions of string theory at the smallest scales, where the quantum nature of gravity becomes important, are highly distinct. Qualitatively, it predicts that the fundamental constituents of the universe are tiny loops of vibrating energy. Nevertheless, at longer length scales—from those at which current particle accelerators operate up to the scales of galaxies and the cosmos as a whole—the predictivity of string theory seems to break down. This is because string theory contains a vast number of possibilities for the low energy configuration of the universe. This so called string theory landscape [1] has been estimated to contain over 10^{500} possible universes [1].

Yet theorists have had great difficulty explicitly constructing realistic models, in string theory, of the principal physical aspects of the observed universe (namely of the particle interactions and the inflationary stage of cosmic expansion). This is due primarily to the overwhelming geometric complexity into which the six extra-dimensions tend to settle. If string theory is to have any hope of being observationally verified, techniques must be developed for constructing complete models of realistic four dimensional physics and for extracting any characteristics of these models that are robust within the landscape.

[1]The term landscape has many controversial connotations in the string theory community. In this work, we simply use the term to refer to the set of all possible string theory ground states.

In this work, we set the foundations for a promising approach to addressing these issues in a particular corner of the landscape. We will see that when the geometry of the six extra dimensions possesses a highly warped region, models constructed in this region are to a high degree isolated from the details of the remaining space. Furthermore, the form of any corrections to the models generated by the geometry outside this region can be obtained explicitly up to a finite and manageable set of undetermined parameters. In tuning the values of these parameters one is effectively varying over the multitude of possible configurations for the remaining geometry. In this way we obtain a high degree of control over variations in the landscape and are able to calculate all physical characteristics explicitly, perhaps with the aid of a computer.

1.1 The Need for a More Fundamental Theory

The so called "standard model" successfully accounts for all physical phenomena observed to date and is the most precisely tested scientific theory. It is composed first of a quantum theory of the elementary matter particles (the quarks and leptons) as well as the interactions between these particles (as mediated by the gauge bosons). To this one must tack on Einstein's classical theory of gravity. This asymmetry between the treatment of gravity and the other fundamental interactions causes no problems in the regimes of current observations; in particle accelerators, gravity has a negligible effect on particle dynamics and on cosmic scales, quantum fluctuations are not important in determining the overall space-time geometry of the universe.

However, if we were to probe vastly larger energies (and correspondingly smaller lengths) than those obtained by today's accelerators, eventually quantum fluctuations of gravity will be excited. But it is known that Einstein's theory of gravity is nonrenormalizable meaning that, for quantum process below some smallest length scale (called the Planck Length l_p), the theory yields infinities for physical observables. Thus the standard model cannot be the whole story and there must be a more fundamental physical theory that takes over at some scale above the Planck Length.

String theory avoids the above problem by effectively imposing a lower bound on physical length scales. It does this, not by some awkward discretization of space-time, but by positing that the fundamental building blocks of the universe are one-dimensional loops of energy (strings) rather than point-like particles. In a particle theory of nature, one can in principle cause particles in a collider to approach arbitrarily close to one another by launching them towards each other at high enough energies. For strings, however, some of the energy the accelerator provides will end up in the vibrational motion of the string. As the string vibrates more violently, it tends to "spaghettify," elongating and forming highly convoluted configurations (see figure 1.1 below). It then effectively fills up a volume whose characteristic length grows with energy and this prevents the centers of two converging strings from approaching arbitrarily close. If one were to optimize between the two competing effects, one would find a smallest separation distance that can be achieved called the string length $l_s > l_p$.

Let us summarize the salient features of the theory: beyond the closed strings (strings forming loops), the theory also contains open strings with ends terminating on extended objects called D-branes. D-branes are multidimensional objects that are dynamical elements of the theory in their own right. Also,

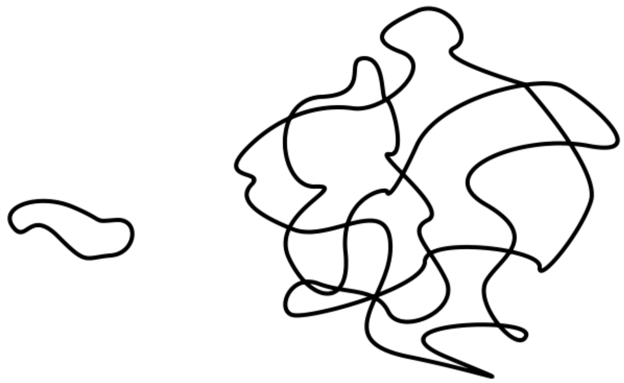

Figure 1.1: Low energy strings (left) have simple topology while high energy strings (right) become highly convoluted effectively occupying a volume which grows with energy

the theory is consistent only if the strings vibrate in ten dimensions (nine space and one time). In order for this to be consistent with the fact that we have not observed them, the extra dimensions must be very tiny and curled up on themselves.

1.2 Low Energy Physics from String Theory

At the string length and below, the predictions of string theory are more or less unique. Only a few consistent variants of the theory have been found and the discovery of various dualities linking them strongly suggests that they are merely different regimes of a single, even more fundamental theory called M-theory (see chapter 8 of [2]). However, there is a high degree of degeneracy in the theory for low energy physics. This is because there is a multitude of ground states that the universe can settle into.

The situation is analogous to that which arises for condensed matter systems. For instance there are two distinct crystal structures that carbon can form in its solid state — graphite and diamond (see figure 1.2). Although the fundamental building blocks (carbon atoms) and their interactions (interatomic forces) are the same for each case, the two "ground states" have very distinct physical characteristics; e.g. graphite is soft and opaque while diamond is hard and transparent. In string theory, the ground state is something much more grand than a crystal configuration—it is the state of the universe as a whole including its geometry, the types of matter and forces it contains and the values for the constants of nature. There aren't just two but a vast number of ground states that the universe can settle into, each one having distinct physical characteristics. Also, unlike the carbon example where external conditions (temperature and pressure) uniquely determine which crystal configuration is formed, all of these string theory vacua are at zero temperature and there are a large number of them that are sufficiently stable to be possible configurations for our universe. If it is even possible to detect strings directly at particle accelerators, it will most likely require vastly higher energies than can be obtained today. Thus if we are to have a reasonable hope of observationally verifying string theory we must find ways to get a handle on this landscape of vacua.

Let us be more specific about what characterizes a string theory vacuum. We want a four dimensional universe so the six extra dimensions must be compact. The non-compact four dimensions must reproduce the geometry of the observed universe. For late cosmological times they must form a (nearly) flat, Minkowski space. The ten dimensional metric must take the form,

$$\mathrm{d}s^2 = e^{2A(y)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{-2A(y)}g_{mn}\mathrm{d}y^m \mathrm{d}y^n \,, \tag{1.1}$$

in order to preserve the Lorentz symmetry of the four dimensional (external)

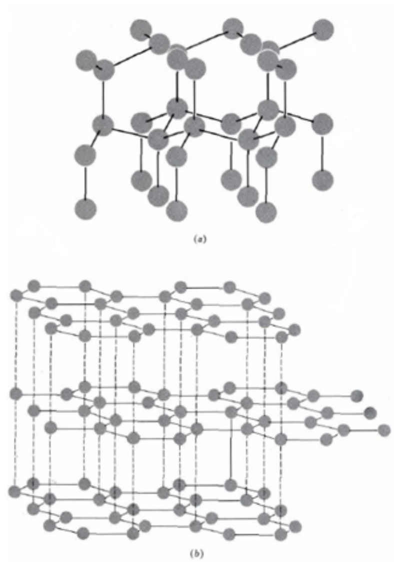

Figure 1.2: Crystal structure of graphite (top) and diamond (bottom). Taken from [3].

space. In the above, the metric is factorized into two pieces (see figure 1.3); the coordinates x^μ, $\mu = 1, \ldots, 4$, parameterize the external space and the coordinates y^m, $m = 5, \ldots, 10$, parameterize the internal six dimensions. The met-

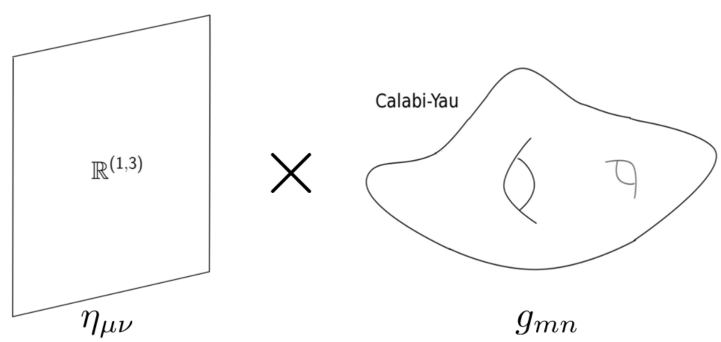

Figure 1.3: The geometry of the ten-dimensional space must take the form of a product of flat, four-dimensional Minkoski space and a compact, six-dimensional Calabi-Yau.

ric $\eta_{\mu\nu}$ of the external space is fixed by observation to be (approximately) the

Minkowski metric. It turns out that it is difficult to maintain mathematical control over the compactification unless the internal dimensions form the topology of a "Calabi-Yau" space (see [4] for details on these spaces). The metric g_{mn} on the Calabi-Yau is required to have vanishing Ricci tensor. The warp factor $e^{2A(y)}$ represents stretching and shrinking of the external space-time as one moves in the internal space. This is illustrated in figure 1.6 for the example where the internal space is a one-dimensional line segment. This provides a natural mechanism for generating hierarchies of scales between systems that are separated along the internal directions.

There are many other fields predicted by string theory besides the space-time metric. The additional bosonic fields (Lorentz invariance prohibits any of the fermonic fields from being turned on) are the p-form fluxes. These are generalizations of the Maxwell field strength tensor of electromagnetism—a totally antisymmetric two-tensor—to totally antisymmetric tensors with an arbitrary number of indices, p. Each solution for these p-form fields may be characterized by its integrals over p-dimensional closed surfaces (or cycles) in the Calabi-Yau that are topologically nontrivial (i.e. they cannot be smoothly contracted to a point due to some type of obstructing hole in the space). Specifying the magnitude of such integrals (the "flux quanta") for all topologically distinct cycles in the space uniquely fixes the solution. It turns out that these integrals may take only a discrete set of values with an upper bound on their magnitude.

There is also the "dilaton" scalar field whose magnitude at any point in the internal dimensions measures the strength of the interactions between strings at that location. Finally, there are various localized objects in the theory, e.g. p-dimensional branes ($p < 10$) which can be "wrapped" on nontrivial cylces. All this is illustrated in figure 1.4.

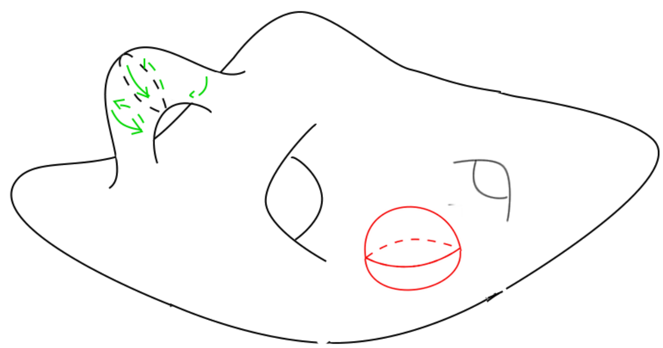

Figure 1.4: Nontrivial cycles in the Calabi-Yau can support fluxes (green) and branes (red). A p-form flux can thread a p-dimensional cycle. Above, a one-form flux threads the circumference of a "handle" which is a one-dimensional cycle. A p-dimensional brane must wrap a $(p-3)$-dimensional cycle, the remaining dimensions of the brane filling the external space.

Without fluxes turned on, there is typically a vast family of Ricci flat metrics on a given Calabi-Yau, parametrized by hundreds of scalar parameters. Each of these parameters is a light scalar field in the external space-time and having a large number of these produces inconsistencies with observation. In type IIB string theory, a good fraction of these "moduli" (the ones that parametrize the shape of the space) can be stabilized by turning on fluxes [5]. By stabilize, we mean that the values of these parameters are fixed to be one or a few discrete values by the equations of motion, thus effectively removing these degrees of freedom as moduli. There are also mechanisms to stabilize the remaining moduli (the ones parametrizing the size of the space) e.g. by adding nonperturbative elements to the solution [6] or by including loop corrections to the supergravity equations of motion [7].

Once we have specified the Calabi-Yau, the flux quanta and the types and positions of any localized objects in the space, the solutions for the remaining fields are uniquely determined. We will thus take these degrees of freedom as parameterizing the space of string vacua. It is not known whether or not the number of six dimensional Calabi-Yau manifolds is finite. It is known that the number is vast; tens of thousands of examples have been enumerated [4]. Even if we fix the choice of Calabi-Yau, there are typically hundreds of integers that need to be specified in order to fix the fluxes, and each of these integers can take on hundreds of possible values. This leads to an exponentially large set of vacua for the universe.

1.3 Local Model Building and Warped Throats

Although we can indirectly deduce a vast number of possible solutions for our universe in string theory, it has thus far proven very difficult to fully and explicitly construct an example. This is due to the geometric complexity of compact Calabi-Yau manifolds. While one can infer the existence of large numbers of such spaces, explicit construction of such a compact Calabi-Yau and its Ricci flat metric has only been achieved in very artificial examples. The key insight of the local model building approach is that there are many *non-compact* Calabi-Yau solutions (called "warped throats") which are explicitly known. Since both compact and non-compact Calabi-Yau manifolds locally satisfy the same equations of motion, it is possible to find compact Calabi-Yau manifolds that possess regions that locally approximate the geometry of finite patches of non-compact Calabi-Yau manifolds. If all relevant systems are constructed within this region, one can make the approximation of using the warped throat geometry in place of the region's local geometry and calculations can be performed explicitly.

Let now describe these warped throat geometries in detail. We begin with flat ten-dimensional Minkowski space,

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x_\mu + \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_5^2\right) , \tag{1.2}$$

where we have chosen to use Minkowski coordinates for the four-dimensional external space-time and spherical coordinates for the six extra dimensions. We have denoted the canonical metric on the five sphere by $\mathrm{d}\Omega_5^2$. If we place a stack of D3-branes (extend objects taking up three spatial dimensions) at $r = 0$, oriented so that they fill the external space but are only a point in the extra six dimensions, then we get a metric which resembles that of a black hole

$$\mathrm{d}s^2 = \left(1 + \frac{L^4}{r^4}\right)^{-1/2} \mathrm{d}x^\mu \mathrm{d}x_\mu + \left(1 + \frac{L^4}{r^4}\right)^{+1/2} \left(\mathrm{d}r^2 + r^2 \mathrm{d}\Omega_5^2\right) , \tag{1.3}$$

where $L^4 = 4\pi g_s \mathcal{N} l_s^4$, g_s is the string coupling constant and $\mathcal{N}$ is the number of stacked D3-branes which must be taken large for the above solution to be valid. If we ignore the three dimensions along which the D3-branes extend, the above solution is the Schwarzchild black hole in seven dimensions. With the directions along the branes included however, we see that the singularity is not a line (corresponding to a point source) but a four-dimensional hypersurface (corresponding to a planar source). Similarly, the horizon is not strictly spherical as in the Schwarzchild solution. Rather, it runs parallel to the singular surface in the external directions. Such a solution is called a "black brane."

For large r, the metric (1.3) asymptotes to flat Minkowski space as we would expect by locality. We are interested in the geometry near the horizon where $r \to 0$. Here the metric becomes

$$\mathrm{d}s^2 \approx \left(\left(\frac{r}{L}\right)^2 \mathrm{d}x^\mu \, \mathrm{d}x_\mu + \frac{\mathrm{d}r^2}{r^2} L^2\right) + L^2 \, \mathrm{d}\Omega_5^2 \, . \tag{1.4}$$

The second term is the metric of a five-sphere whose size becomes constant for small r as depicted in figure 1.5 which makes apparent why the geometry is called a "throat." The first term is the standard metric on five-dimensional Anti-deSitter space (AdS) so that the total geometry is $\mathrm{AdS}_5 \times S^5$.

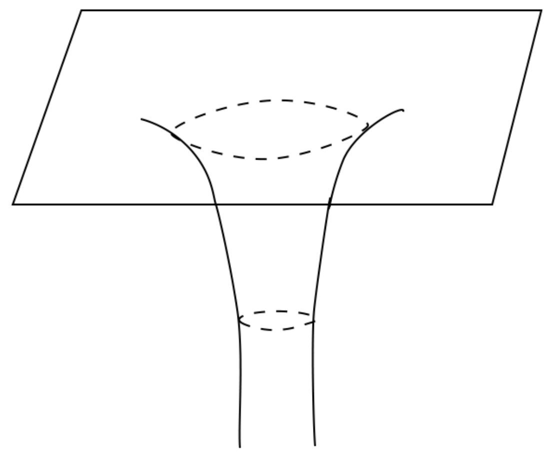

Figure 1.5: Far from the stack of branes the geometry becomes flat but for small r the geometry is tube-like.

Note that if we instead started with the space

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x_\mu + \left(\mathrm{d}r^2 + r^2 \mathrm{d}s^2_{\mathcal{B}_5}\right) , \tag{1.5}$$

where $\mathcal{B}_5$ is any five-dimensional compact Einstein space, then by placing a stack of D3-branes at $r = 0$ one would obtain the near horizon geometry

$$\mathrm{d}s^2 \approx \left(\left(\frac{r}{L}\right)^2 \mathrm{d}x^\mu\, \mathrm{d}x_\mu + \frac{\mathrm{d}r^2}{r^2} L^2\right) + L^2\, \mathrm{d}s^2_{\mathcal{B}_5} , \tag{1.6}$$

which is $AdS_5 \times \mathcal{B}_5$. In the case $\mathcal{B}_5 \neq S^5$, the metric (1.5) has a conical singularity at $r = 0$.

The occurrence of the AdS factor is of great significance, as we will see in the next sections, and we should therefore review some of its noteworthy properties. First off, the isometries of AdS include the four dimensional Poincare group on the coordinates x^μ plus a scaling symmetry which takes $x^\mu \to \lambda x^\mu$ and $r \to \lambda^{-1} r$. This scaling symmetry corresponds with a physical distortion of scales that occurs as one moves in the r direction; e.g. if one sends two light signals in the positive r direction, initially starting at position r_i and spaced a distance l alone the x^μ directions, then they will arrive at a position r_f spaced a distance $\frac{r_i}{r_f} \cdot l$. Indeed, we can think of AdS space as a family of copies of Minkowski space, parametrized by r, whose relative size varies with r (see figure 1.6).

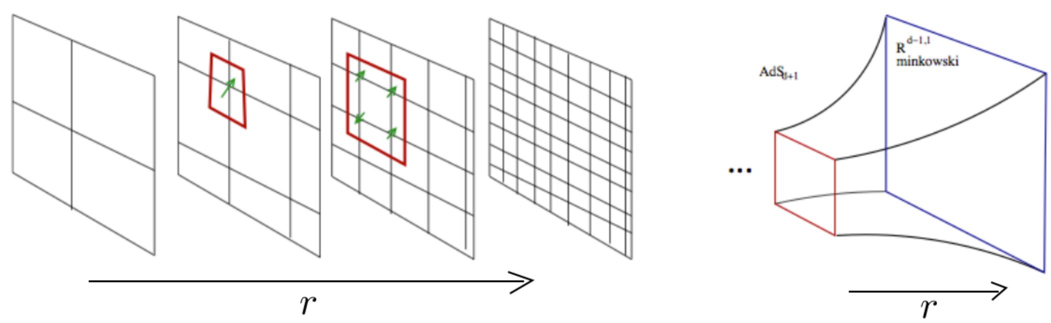

Figure 1.6: AdS is a family of copies of Minkowski space, parameterized by r, which become more and more stretched as we move to small r. Taken from [8].

Furthermore, the AdS space has a boundary at $r = \infty$. Although starting from a finite r and moving out to $r = \infty$ while keeping the other coordinates fixed corresponds to moving through an infinite proper distance, massless particles traveling on null geodesics can make it out to $r = \infty$ and back in a finite time. This means that, unlike for Minkowski space, initial data in the interior of the space (the bulk) is not sufficient to determine the future evolution; we must also specify boundary conditions.

The AdS metric is ill defined as $r \to \infty$. In order to define a metric on the boundary space, one must perform a conformal transformation $\mathrm{d}s^2 \to f^2(r)\,\mathrm{d}s^2$ such that the resulting metric is finite as $r \to \infty$. While this procedure produces a Minkowski metric on the boundary, any f(r) that scales asymptotically as $\frac{1}{r^2}$ is an equally valid choice. This means that any metric on the boundary space obtained by a rescaling is equally valid. Therefore the boundary space has a conformal structure. Any physics on the boundary that maps to dynamics in the bulk must be conformally invariant.

Actually, the throat solution we have obtained still requires some modification. The horizon at $r = 0$ is an infinite proper distance away from any point in the interior with $r \neq 0$. It would be preferable to have the throat terminate smoothly at a small but nonzero value of r, rather than allowing it to extend indefinitely downward. If we take $\mathcal{B}_5 \neq S^5$ in equation (1.6), the presence of the conical singularity allows for the existence of an exotic type of brane called a fractional D3-brane. Once a stack of these are added, the geometry is deformed as depicted in figure 1.7. For large r the geometry is asymptotically that of equation (1.6). However, AdS is smoothly cut off at a small finite value of r. The canonical example of this type of solution that we shall be using in this work is the Klebanov-Strassler (KS) throat [9]. This is the case where $\mathcal{B}_5$ is taken to be the homogeneous space $T^{1,1} = (SU(2) \times SU(2))/U(1)$.

It is easy to arrange for such a throat to be present in a compactification. Conical singularities are generic in compact Calabi-Yau manifolds. If we place the appropriate configuration of branes at such a singularity, the local geometry will be that of a warped throat. For large r, the throat will taper into the original compact space instead of extending indefinitely, as depicted in figure 1.8.

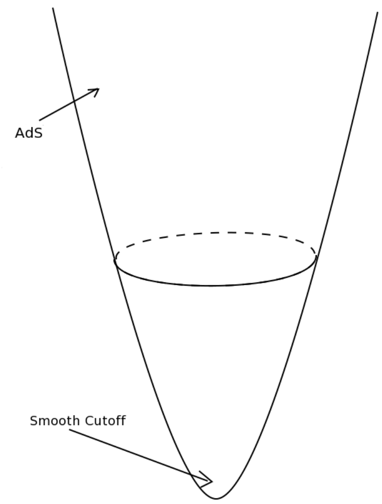

Figure 1.7: The KS solution is like a cone with a rounded tip. The large r regime approximates AdS but the space is smoothly cutoff at small but finite r. The circular dimension indicated above actually represents the five-dimensional angular space $\mathcal{B}_5$.

1.4 The Gauge / Gravity Duality

Besides providing an explicitly calculable local geometry for model building, warped throats have the appealing feature of having dual descriptions in terms of strongly coupled field theories. The simplest form of the so called gauge / gravity duality states that a theory of gravity on Anti-deSitter space is dual to a conformal field theory (CFT) on its boundary space. More precisely, for every field of the gravity theory ϕ there is a dual operator O_ϕ in the CFT. The dimension of the operator is determined by the AdS mass of ϕ; e.g. if ϕ is a scalar, then the dimension is given by $\Delta(\Delta - d) = m^2L^2$, where d is the number of dimensions of the AdS space. One can obtain correlation functions of the

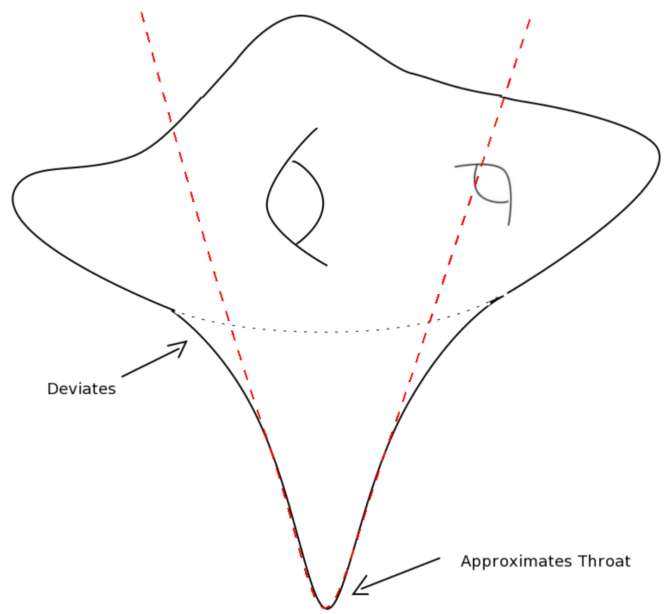

Figure 1.8: Deep in the warped region the geometry is coincident with the infinite throat. At larger distance the region tapers off into the compact bulk.

operator O_ϕ from the gravity side using the prescription

$$\left\langle e^{-\int \mathrm{d}x^4\, \phi_0(x)\, O_\phi} \right\rangle_{CFT} = Z_{\text{Quantum Gravity on AdS}}\left[\phi|_{r\to\infty} = r^{\Delta-4}\, \phi_0(x)\right] \approx e^{-S_{\text{Gravity On-Shell}}(\phi)}\Big|_{\phi|_{r\to\infty}=r^{\Delta-4}\,\phi_0(x)} . \tag{1.7}$$

In the above, $Z_{\text{Quantum Gravity on AdS}}\,[\phi|_{r\to\infty} = \phi_0(x)]$ is the partition function for the quantum gravity theory on AdS. It is to be evaluated with the boundary conditions that $\phi(r, x)$ has the asymptotic form $r^{\Delta-4}\,\phi_0(x)$ for $r \to \infty$. On the second line we have assumed that the CFT is strongly coupled in which case the dual gravity description becomes classical. A saddle point approximation can then be applied by evaluating the gravity action, on-shell, for the given boundary conditions. By taking functional derivatives with respect to $\phi_0(x)$ one generates the desired correlation functions.

The CFT has two dimensionless parameters: the Yang-Mills coupling g_{YM}

and the rank N of the gauge group (for simplicity we assume $SU(N)$). These are given in terms of gravity parameters as

$$g_{YM} N = \frac{1}{4\pi} \frac{L^4}{l_s^4}, \qquad N = \frac{L^4}{l_p^4}. \tag{1.8}$$

The isometry group of AdS is isomorphic to the conformal group. In fact the AdS transformation $x^\mu \to \lambda x^\mu$, $r \to \lambda^{-1} x^\mu$ maps to the scaling transformation of the conformal theory. This implies that the radial coordinate r of AdS is dual to the energy scale in the CFT; the radial dependence of a gravity solution for a field gives the RG flow of the corresponding operator in the CFT. Large r corresponds to the UV of the field theory and small r to the IR. This corresponds with our description in figure 1.6 of AdS as a family of copies of Minkowski spaces, related to each other by a rescaling.

The Klebanov-Strassler throat is also dual to a gauge theory. Since the KS throat is only asymptotically AdS, the dual theory is only approximately conformal in the UV. The deformation of the tip of the KS solution corresponds to the spontaneous breaking of the conformal symmetry by condensation and a mass gap in the IR. Attaching the throat to a compact bulk also has a gauge theory dual. This corresponds to cutting the field theory off at some UV scale (corresponding the the value of r where the throat is glued into the bulk) and coupling it to gravity and the bulk fields.

1.5 Corrections to the Local Throat Approximation

The central innovation of the local model building approach is to build physical models deep in a warped throat region of a compactification where the geometry looks like that of a noncompact throat as in figure 1.8. This amounts to a zeroth order approximation of replacing the warped throat region by a finite segment of the noncompact throat as in figure 1.9. Gluing into a compact bulk

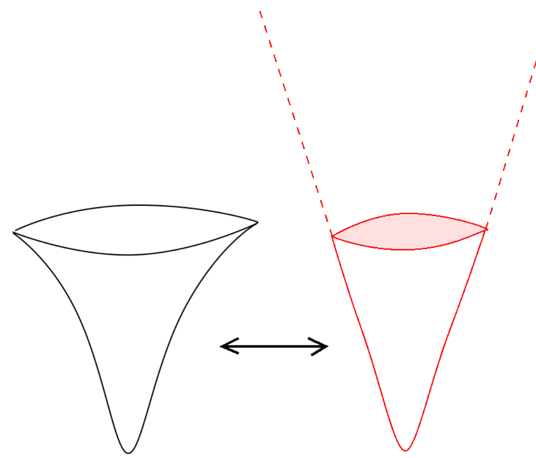

Figure 1.9: Our lowest order approximation is replacing the warped region of the compactification with a finite segment of the infinite throat solution.

will generate corrections to this picture—in order to do so smoothly, the geometry must be deformed in the UV and these deformations will propagate to the IR (figure 1.10). It turns out that while many features of models are insensitive to these corrections, there are key physical observables that are determined by them, solely. For example, the warped throat solutions preserve supersymmetry. Thus if a model of particle physics is being constructed, supersymmetry must be broken by elements in the bulk. Then important quantities such as soft breaking terms are determined solely by bulk corrections.

This might seem a big problem for the local model building program: the original purpose was to remove the need to contend with the uncontrollable bulk, but now we see that effects from the bulk are unavoidable. On the contrary, the remarkable geometric properties of warped throats make it so that only a manageable set of simple and explicitly calculable corrections from the

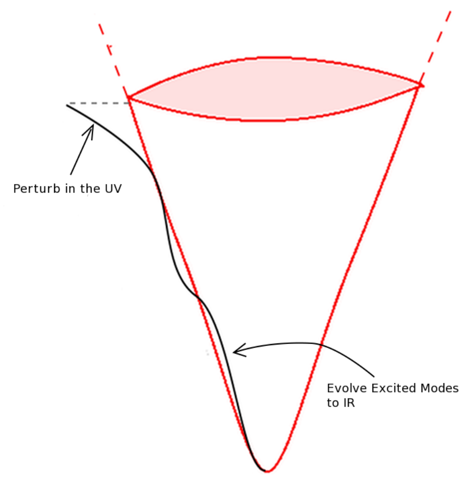

Figure 1.10: We can incorporate the effects of gluing into the bulk by considering arbitrary perturbations to the UV boundary conditions.

bulk survive in the deep IR—all of the unmanageable parts of the information from the bulk are still filtered out. The responsible "remarkable geometric properties" are the very same that allow the throat to have a gauge theory dual.

Actually the mechanism responsible for this in the gauge theory picture is quite familiar: it is just the Wilsonian Renormalization Group (RG) flow. Let us review Wilson's approach to obtaining an effective low energy description when the UV physics is unknown. We imagine that there is some energy cutoff Λ beyond which the UV physics becomes important. This is modeled by having a cutoff on our low energy CFT where it is coupled some new dynamics. If we integrate these degrees of freedom out of the fundamental Lagrangian, then we get some general deformations to the CFT at the scale Λ:

$$\mathcal{L}_{UV} = \mathcal{L}_{CFT} + \delta\mathcal{L}_{UV} \,. \tag{1.9}$$

There is a vast number of possible operators that could be included in $\delta\mathcal{L}_{UV}$. However, if we run this effective Lagrangian to the IR where we are interested in doing our physics, only the most relevant modes survive

$$\mathcal{L}_{IR} = \mathcal{L}_{CFT} + \sum_{\Delta \le 4} c_\Delta \, \mathcal{O}_\Delta \,. \tag{1.10}$$

The above sum runs over all relevant operators $\mathcal{O}$ and the c_Δ are undetermined Wilson coefficients. The set of possible operators that can appear in the sum is typically highly constrained by renormalizability and by symmetries. So all though we do not know the UV physics that determines the Wilson coefficients, we only have a finite and rather manageable set of unknown numbers to contend with. We can determine these by making a finite set of measurements. After this is done, we have full predictivity for all other physical processes. Said another way, this procedure has constrained the Lagrangian enough so that we can make correlative predictions that are experimentally observable.

It would be nice if we could apply this type of analysis directly on the gauge dual of our throat. Unfortunately, the gauge dual is strongly coupled and so we cannot compute the running of operators. However it is straightforward to perform the gravity dual of the Wilsonian procedure. The unknown UV physics corresponds to the bulk of the compactification. Integrating it out corresponds to replacing this bulk with arbitrary deformations of the UV boundary conditions of the throat. We then propagate to the IR by solving the equations of motion for the various supergravity fields. We will find that only a moderate number of modes survive in the IR for each of the supergravity fields:

$$\delta\phi(r, \Psi) \approx \sum_{\Delta \le 4} c_\Delta \, r^{\Delta - 4} \, Y_\Delta(\Psi) \,. \tag{1.11}$$

In the above Ψ stands for the set of angular coordinates on $\mathcal{B}_5$. There is no x^μ dependence since the resulting solution must be Poincaré invariant. The $Y_\Delta(\Psi)$ are angular harmonics on $\mathcal{B}_5$. Equation (1.11) then strongly constrains the form of the corrected supergravity fields in the IR of the throat. Moreover we can estimate the orders of magnitude of the c_Δ on physical grounds. Then, just as we can make predictions for colliders in the absence of data on UV physics with an effective field theory, we can sweepingly extract correlative predictions from a sizable portion of the landscape without having any direct control over the bulk Calabi-Yau manifolds in these vacua.

1.6 Outline

This program for treating corrections to the local geometry was initiated in [10, 11, 12]. The purpose of this work is to carry it out fully, so that we have a complete and systematic procedure for dealing with such corrections, and also to demonstrate how to apply the scheme effectively.

In chapter 2 we describe such an expansion procedure in full detail: in §2.2 we overview our scheme, and then expand the equations of motion of type IIB supergravity. We then present our method: we show that upon obtaining the homogeneous solutions for all supergravity fields, as well as all the associated Green's functions, it is straightforward to write down the inhomogeneous solution for any field of interest, to any desired order. In §2.3 we summarize the homogeneous solutions for each field, deferring details to Appendix B. In §2.4 we write down formal Green's function solutions for arbitrary fields. In §2.5 we obtain the radial scalings of the various contributions to the supergravity fields, allowing efficient identification of the most important fields in a given problem. We summarize in §2.6.

In chapter 3 we apply the scheme to the stabilization of the moduli associated with an anti-D3-brane sitting at the tip of a KS throat: in §3.1 we set down some preliminaries about the compactifications that we will consider and about the anti-D3-brane potential. In §3.2 we derive the set of appreciable corrections to the throat geometry that can be induced by compactification. In §3.3 we assemble all necessary information concerning the spectrum of Kaluza-Klein excitations of $T^{1,1}$. We use these results in §3.4 we determine the size of the mass term for the anti-D3-brane moduli that results from compactification effects. We conclude in §3.5.

Appendix A presents the structure of the source terms in the equations of motion, while Appendix B contains the details of the homogeneous solutions and Green's functions for the scalar, flux, and metric modes. In appendix C we provide the spectroscopy of metric modes on the Klebanov-Strassler throat which is used in the main text. The treatment in this appendix is pedagogical with the aim of enabling the reader to generalize the derivation for the conifold to an arbitrary cone over a homogeneous base space.

CHAPTER 2

SYSTEMATIC PERTURBATION OF FLUX COMPACTIFICATIONS

2.1 Introduction

Flux compactifications of type IIB string theory provide a promising framework for phenomenological and cosmological models in string theory, but the study of general compact spaces remains difficult. Warped throat regions, which arise naturally in this setting, are comparatively tractable: a throat region can be approximated by a portion of a noncompact warped cone, and explicit computations performed in the local model then serve to characterize the corresponding sector of the four-dimensional effective theory.

A significant challenge in this context is that the best-understood warped throat solutions, such as the Klebanov-Strassler throat [9], are noncompact and supersymmetric, while realistic model-building with dynamical four-dimensional gravity requires a finite throat region subject to supersymmetry breaking. It is therefore important to understand finite, non-supersymmetric warped throat regions of flux compactifications with stabilized moduli.

To first approximation, a finite warped throat can be replaced by a finite segment of a noncompact warped cone, terminating in the ultraviolet (UV) at some finite value of the radial coordinate, $r = r_{UV}$, where the throat is glued into a compact space. We seek here to understand corrections to this approximation generated by compactification. From the viewpoint of the supergravity fields in the throat, the properties of the bulk space determine boundary conditions on the gluing surface, or UV brane. For a given compact space, one could in principle pursue a solution for the throat fields with the corresponding boundary conditions, in a perturbation expansion around the solution obtained in the noncompact limit that decouples the bulk sources. A significant simplification is that the solution in a region at radial location $r_\star \ll r_{UV}$ is accurately described by the finite set of modes that diminish least rapidly towards the infrared (IR). In the dual field theory, this is just the statement that in the deep IR, a description in terms of the handful of most relevant operators is sufficient. However, even after making use of this radial expansion, the equations of motion are coupled in a complicated way, making an analytic solution impractical in general.

Our starting point is the observation that in an interesting class of compactifications, an additional expansion is available. In the scenarios [6, 7, 13] for Kähler moduli stabilization, the solution is nearly conformally Calabi-Yau, with fluxes that are nearly imaginary self-dual (ISD). We can therefore formulate a double perturbation expansion whose small parameters are $r_\star/r_{UV}$, and the size of the deviations on the UV brane from the ISD, conformally Calabi-Yau solution. For brevity we will refer to these as the radial expansion and the ISD expansion.

Upon expanding the equations of motion to any order n in the ISD expansion, we find a very convenient structure that allows us to disentangle and solve the equations for the various supergravity fields. To understand this structure, consider the much simpler model of k scalar functions φ_A, $A = 1, \dots k$, of a single variable r, obeying a general first-order system of equations. On general grounds the equations of motion for n-th order perturbations $\varphi_A^{(n)}$ around some chosen background $\varphi_A = \varphi_A^{(0)}(r)$ take the form

$$\frac{\mathrm{d}}{\mathrm{dr}}\varphi_A^{(n)} = N_A{}^B \varphi_B^{(n)} + \mathcal{S}_A\,, \tag{2.1}$$

where the matrix ${N_A}^B$ depends on the fields $\varphi_C^{(0)}(r)$, and the source term $\mathcal{S}_A$ depends on the fields $\varphi_C^{(m)}(r)$, $m < n$, $C = 1, \dots k$. If the coefficient matrix ${N_A}^B$ were constant, one could readily solve this system by standard techniques, whereas for ${N_A}^B = {N_A}^B(r)$ an analytic solution generally requires that ${N_A}^B$ has some special structure.

In particular, if ${N_A}^B$ is *triangular*, i.e. if ${N_A}^B = 0$ for A < B, then the equations of motion can be solved iteratively, as we shall explain at length. For constant ${N_A}^B$, finding a basis in which equation is in triangular form is an easy exercise in linear algebra, but the presence of the derivative operator makes this task highly nontrivial when ${N_A}^B$ is nonconstant. In fact, the problem of finding a basis in which a given ${N_A}^B(r)$ takes a triangular form involves solving a system of coupled differential equations that is no easier, in general, than the original system.

A key result in this chapter is a simple basis in which the supergravity equations of motion expanded to n-th order around an ISD background take a triangular[1] form, allowing us to construct an iterative Green's function solution. In contrast to the toy model above, the fields are not all scalars, and are governed by second-order partial differential equations (i.e., the fields have nontrivial dependence on the angular directions of the cone), but the nature of the simplification is identical. At each order n in perturbation theory, a privileged field $\varphi_1^{(n)}$ at the top of the triangle is sourced by no other fields at order n, so that a Green's function solution is straightforward. The next field $\varphi_2^{(n)}$ is sourced only by $\varphi_1^{(n)}$, while $\varphi_3^{(n)}$ is sourced by $\varphi_1^{(n)}$ and $\varphi_2^{(n)}$, etc. Thus, we can solve each successive equation by substituting the solutions from the preceding equations in the

[1]In fact, we find that the equations take a *strictly* triangular form, analogous to ${N_A}^B = 0$ for A ≤ B.

triangle. The same Green's functions apply at every order, so that one need only solve for a single set of Green's functions, one for each field, and then the solutions to the supergravity equations are readily obtained to any desired order in a purely algebraic way. We stress that the triangular structure that plays a central role in this work appears in the equations of motion expanded around *any* ISD background, which need not be a warped Calabi-Yau cone (and need not be supersymmetric). We focus on cones because the explicit metric and separable structure of the cone permit direct solution of the equations of motion.

In this chapter we explain this approach in detail, then determine all necessary Green's functions, so that the enterprising reader can obtain the supergravity solution for a general warped Calabi-Yau cone attached to a flux compactification, to any desired order. In practice, we give supergravity solutions as functions of the angular harmonics on the Sasaki-Einstein base of the cone, with radial scalings determined by the corresponding eigenvalues. For the case of $T^{1,1}$, the necessary eigenvalues and eigenfunctions are available in the literature; to use our method for a more general cone, one would need to compute the angular harmonics on the base.

A related approach was used in [11, 12] to study the inflationary model of [14], which involves the attraction of a D3-brane toward an anti-D3-brane in a warped throat. However, the works [11, 12] made extensive use of the facts that a D3-brane couples only to a particular scalar combination of the supergravity fields, denoted by Φ_-, and that the dominant source for Φ_- is imaginary anti-self-dual (IASD) flux G_-. Thus, it was possible to restrict attention to the fields Φ_- and G_-, and to truncate at quadratic order. In this work we fully complete this program for all supergravity fields, to all orders, permitting a much broader range of applications.

We remark that a similar structure in the equations of motion for global symmetry singlet perturbations linearized around the Klebanov-Strassler background was identified in [15] and played a role e.g. in [16, 17]. In contrast to those works, we establish and utilize a triangular structure to all orders, in expansion around a general ISD background. Our explicit results and separable solutions are not restricted to the singlet sector, but apply only in the approximately-conformal region above the tip of a warped Calabi-Yau cone, whereas the formulation of [15] applies throughout the deformed conifold.

Another useful result of this chapter is a simple formula for the radial scaling (i.e., parametric dependence on $r_\star/r_{UV}$) of a general n-th order correction. In a canonical basis, the n-th order corrections at some point in the throat have the same scalings as the n-th order products of the harmonic modes at that point. In particular, this implies that the 'running' sizes of the harmonic modes are faithful expansion parameters. We anticipate that our formula for the scaling of a general perturbation will be of use in determining the parametric sizes of physical effects mediated through warped geometries.

Although KKLT compactifications provide significant motivation for the geometries described herein, our approach applies more broadly, to type IIB compactifications subject to controllably small violations of the ISD conditions. In this connection, we remark that one might naively expect that all modes of the supergravity fields have coefficients of order unity at $r = r_{\rm UV}$, where the throat merges into the bulk. Then, for a sufficiently long throat, any relevant modes will grow exponentially large, and the throat geometry will be destroyed in the IR. We will find instead that, for a class of throats of broad interest, all relevant modes either violate the ISD conditions or violate the supersymmetry of the background throat geometry. In particular, we will show that in the concrete example of a Klebanov-Strassler throat in a KKLT compactification, all relevant modes remain perturbatively small all the way to the tip of the throat. Extending this result to more general throats in more general nearly-ISD compactifications is an interesting direction for the future.

Although we give detailed results for perturbations induced by boundary conditions on the UV brane, corresponding to sources such as D-branes, orientifold planes, fluxes, and quantum effects in the bulk, our methods apply equally well to the study of perturbations induced in the infrared.

2.2 Setup and Method

We begin by writing down the equations of motion and describing the ISD background around which we perturb. In §2.2.2 we expand the equations of motion, and in §2.2.3 we show that in our chosen basis, the equations of motion for the perturbations take on a triangular form at any order. Using this structure, we develop an iterative, purely algebraic method for solving the perturbed equations to all orders.

2.2.1 Equations of motion and background solution

We consider type IIB compactifications of the form

$$\mathrm{d}s^2 = e^{2A(y)} g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{-2A(y)} g_{mn} \mathrm{d}y^m \mathrm{d}y^n \,, \tag{2.2}$$

$$\tilde{F}_5 = (1 + \star_{10})\, \mathrm{d}\alpha(y) \wedge \sqrt{-\det g_{\mu\nu}}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, \tag{2.3}$$

$$G_{mnl} = G_{mnl}(y)\,, \quad m, n, l = 4, \ldots 9\,, \tag{2.4}$$

$$G_{\mu NL} = 0\,, \quad \mu = 0, \ldots 3\,, \quad N, L = 0, \ldots 9\,, \tag{2.5}$$

$$\tau = \tau(y)\,, \tag{2.6}$$

where we are using the conventions and notation of [5], with the modification that $g_{mn}^{\text{here}} = \tilde{g}_{mn}^{\text{there}}$. We generalize the setup of [5] slightly by allowing for a maximally symmetric spacetime $g_{\mu\nu}$. If we define the quantities

$$G_\pm \equiv (\star_6 \pm i) G_3\,, \tag{2.7}$$

$$\Phi_\pm \equiv e^{4A} \pm \alpha\,, \tag{2.8}$$

$$\Lambda \equiv \Phi_+ G_- + \Phi_- G_+\,, \tag{2.9}$$

then the equations of motion and Bianchi identities take the form

$$\nabla^2 \Phi_\pm = \frac{(\Phi_+ + \Phi_-)^2}{96\, \mathrm{Im}\, \tau} |G_\pm|^2 + \mathcal{R}_4 + \frac{2}{\Phi_+ + \Phi_-} |\nabla \Phi_\pm|^2\,, \tag{2.10}$$

$$\mathrm{d}\Lambda + \frac{i}{2\, \mathrm{Im}\, \tau} \mathrm{d}\tau \wedge (\Lambda + \bar{\Lambda}) = 0\,, \tag{2.11}$$

$$\mathrm{d}(G_3 + \tau H_3) = 0\,, \tag{2.12}$$

$$\nabla^2 \tau = \frac{\nabla\tau \cdot \nabla\tau}{i\, \mathrm{Im}(\tau)} + \frac{\Phi_+ + \Phi_-}{48 i}\, G_+ \cdot G_-\,, \tag{2.13}$$

$$\begin{aligned} R^6_{mn} = {} & \frac{\nabla_{(m}\tau \nabla_{n)}\bar{\tau}}{2\, (\mathrm{Im}\, \tau)^2} + \frac{2}{(\Phi_+ + \Phi_-)^2} \nabla_{(m}\Phi_+ \nabla_{n)}\Phi_- - g_{mn} \frac{\mathcal{R}_4}{2\, (\Phi_+ + \Phi_-)} \\ & - \frac{\Phi_+ + \Phi_-}{32\, \mathrm{Im}\, \tau} \Big(G_{+\,(m}{}^{pq}\, \bar{G}_{-\,n)\,pq} + G_{-\,(m}{}^{pq}\, \bar{G}_{+\,n)\,pq} \Big), \end{aligned} \tag{2.14}$$

where $\mathcal{R}_4$ is the four-dimensional Ricci scalar of $g_{\mu\nu}$, and covariant derivatives ∇_m and contractions are constructed and performed using g_{mn}. We have also dropped all contributions from localized sources. We will make use of an equivalent form for the Φ_+ equation of motion:

$$-\nabla^2 \Phi_+^{-1} = \frac{1}{96\, \mathrm{Im}\, \tau} \frac{(\Phi_+ + \Phi_-)^2}{\Phi_+^2} |G_+|^2 + \frac{\mathcal{R}_4}{\Phi_+^2} + \frac{2}{\Phi_+^2} \Big\{ \frac{1}{(\Phi_+ + \Phi_-)} - \frac{1}{\Phi_+} \Big\} (\nabla \Phi_+)^2\,. \tag{2.15}$$

In this work we will set $\mathcal{R}_4 \to 0$, $g_{\mu\nu} = \eta_{\mu\nu}$, which is appropriate for modeling late-time physics. For an example of incorporating curvature corrections in the context of inflation, see [12].

The background solution of equations (3.9)-(3.13) for our analysis will obey the conditions

$$G_- = 0\,, \tag{2.16}$$

$$\Phi_- = 0\,, \tag{2.17}$$

$$\nabla\tau = 0\,. \tag{2.18}$$

In a slight abuse of language, we will refer to (3.16)-(2.18) as the *ISD conditions*, and to the corresponding background as an ISD solution. (Properly speaking, (2.18) can be violated in solutions usually described as ISD, e.g. in no-scale F-theory compactifications.) As motivation for this starting point, we remark that KKLT compactifications [6] based on conformally Calabi-Yau spaces involve controllably small deviations from ISD backgrounds, as we will explain in §2.2.2.

Furthermore, we will assume that the background solution contains a warped throat region. Specifically, we consider a throat for which the internal metric takes the form of a Calabi-Yau cone C_6,

$$\mathrm{d}s^2_{C_6} = g_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n = \mathrm{d}r^2 + r^2\mathrm{d}s^2_{\mathcal{B}_5}\,, \quad m,n = 4,\ldots 9\,, \tag{2.19}$$

over some Sasaki-Einstein base $\mathcal{B}_5$ with metric $\tilde{g}_{ij}$,

$$\mathrm{d}s^2_{\mathcal{B}_5} = \tilde{g}_{ij}(\Psi)\,\mathrm{d}\Psi^i\mathrm{d}\Psi^j\,, \quad i,j = 5,\ldots 9\,. \tag{2.20}$$

(Throughout this paper, we use the letters i, j, k, l to represent angular values for the indices and m, n, p, q for general internal indices.) We will further assume that the geometry is approximately AdS, so that the background warp factor takes the form

$$e^{-4A} = \frac{C_1 + C_2 \ \ln r}{r^4}, \tag{2.21}$$

where the constants C_1 and C_2 are determined by the background fluxes F_5, F_3, and H_3.

In many solutions of interest, the throat terminates at a finite radial distance, either smoothly, as in the Klebanov-Strassler solution [9], or through the appearance of a horizon or singularity. In either case, the IR region of the throat, below some position $r = r_{\rm IR}$, will necessarily deviate from the approximately AdS form (2.19, 2.20, 2.21), and one will need to include corrections arising from the tip in a systematic expansion as well. Our approach yields a reliable description of the intermediate regime $r_{\rm IR} \ll r \ll r_{UV}$ that is far from the tip and far from the UV brane.

2.2.2 Perturbative expansion of the field equations

Our strategy is to approximate a highly warped region of a flux compactification in terms of a double expansion around an infinite throat geometry with ISD fluxes. The system of actual interest deviates in two ways from this simple background:

- The throat of interest has finite length: the UV region is glued into a compact space, with corresponding deviations from the infinite throat solution.
- Effects in the bulk of a stabilized compactification typically violate the ISD

conditions (3.16, 3.17).

Deviations of the first kind will be present even in compact models that everywhere satisfy the ISD conditions, e.g. in the warped compactifications of [5]. Moreover, where the throat is glued to the bulk, these deviations will generally be of order unity, reflecting the transition from the throat to the bulk. However, as one moves deeper and deeper into the throat, the bulk geometry has diminishing influence, and use of the infinite throat geometry should hold to better and better approximation. Thus, we can perform an expansion that is valid at some location $r = r_\star \ll r_{UV}$ far below the UV brane, with the infinite throat as the starting point and $r_\star/r_{\rm UV}$ controlling corrections.

Deviations of the second kind arise from sources in the bulk. Consider one well-motivated example: to obtain stabilized de Sitter vacua in the scenario of [6], one incorporates nonperturbative effects on four-cycles, and introduces one or more anti-D3-branes in warped throat regions. These sources lead to controllably small departures from the ISD conditions, and to controllably small breaking of supersymmetry. The nonperturbative contributions are exponential in the four-cycle volumes, while mass splittings due to a given anti-D3-brane are suppressed by the hierarchy of scales in the corresponding throat, $e^{A_{\rm min}} \equiv a_0$. Thus, both sorts of corrections are naturally small. Moreover, the requirement of a de Sitter vacuum links the scale of the nonperturbative effects and the infrared scale of the warped throat, so that all ISD-violating and supersymmetry-violating effects are controlled by the same small parameter, a_0. In summary, one has a double expansion in terms of the parameters $r_\star/r_{\rm UV}$ and a_0.

In practice, we will find it most convenient to use the magnitudes of the *harmonic* modes evaluated at $r = r_\star$ as our expansion parameters. Specifically, let ϕ be any one of the bosonic supergravity fields $\Phi_\pm$, $G_\pm$, τ, g_{mn}. The solution for field ϕ about the throat background will be given by a homogeneous piece plus an inhomogeneous piece,

$$\phi = \phi^{(0)} + \phi_{\mathcal{H}} + \phi_{\mathcal{IH}}\,, \tag{2.22}$$

where $\phi^{(0)}$ is the background value of the field. The homogeneous pieces obey simple harmonic equations and have solutions of the form

$$\phi_{\mathcal{H}} = \sum_I \left(c_0^I \left(\frac{r}{r_\star}\right)^{\Delta(I)-4} + c_1^I \left(\frac{r}{r_\star}\right)^{-\Delta(I)} \right) Y^I(\Psi)\,, \tag{2.23}$$

where I is a multi-index encoding the angular quantum numbers. The $Y^I(\Psi)$ are angular harmonics that are of order unity at a general point, while the c_i^I, with $i = 0, 1$, are numerical coefficients determined by the boundary conditions. The inhomogeneous piece of a given field then incorporates the effects of source terms in that field's equation of motion.

From (2.23), we see that the c_i^I give the sizes of the harmonic modes at $r = r_\star$. Provided that we work in a region where corrections to the background throat geometry are small, the c_i^I will likewise be small. In practice, we will use the c_i^I as our expansion parameters, i.e. we will develop solutions for the inhomogeneous pieces of the fields in terms of a multiple expansion in the c_i^I. Ultimately, the parametric sizes of the c_i^I can be expressed in terms of a_0 and $r_\star/r_{\rm UV}$, so that there are only two fundamental expansion parameters.

We now expand the fields around their values in the ISD background. For each field ϕ, we expand as

$$\phi = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \ldots = \phi^{(0)} + \phi_{\mathcal{H}} + \phi^{(1)}_{\mathcal{IH}} + \phi^{(2)}_{\mathcal{IH}} + \ldots\,. \tag{2.24}$$

where $\phi^{(0)}$ is the background value for the field, $\phi^{(1)}$ represents the sum of corrections to the field linear in the c^I, etc. It will also be convenient to use a notation where the homogeneous piece $\phi_{\mathcal{H}}$ and the inhomogeneous piece $\phi_{\mathcal{IH}}$ are split. Clearly $\phi_{\mathcal{H}}$ is linear in the c^I. The $\phi^{(n)}_{\mathcal{IH}}$ comprise the inhomogeneous piece of the correction: $\phi^{(1)}_{\mathcal{IH}}$ represents the sum of inhomogeneous corrections to the field linear in the c^I, $\phi^{(2)}_{\mathcal{IH}}$ represents the sum of corrections quadratic in the c^I, etc.

With these preliminaries, we can proceed to expand the supergravity equations (3.9-3.13) around the ISD background. We will examine the n-th order equations of motion, focusing for the moment on terms that involve the n-th order corrections, as opposed to products of lower order corrections. These terms are universal, in the sense that at any order n they take exactly the same form: since we are expanding to order n, whenever we take one of the fields in a term of an equation to be at order n, all other factors in the term must be taken to be at order zero.

The resulting equations for the n-th order perturbations around the ISD background following from equations (3.9-3.13) are

$$\nabla^2_{(0)}\,\Phi_-^{(n)} = \mathrm{Source}_{\Phi_-}(\phi^{(m<n)})\,, \tag{2.25}$$

$$\mathrm{d}\left(\Phi_+^{(0)}\,G_-^{(n)}\right) = -\mathrm{d}\left(\Phi_-^{(n)}\,G_+^{(0)} + \mathrm{Source}_{G_-,1}(\phi^{(m<n)})\right) + \mathrm{Source}_{G_-,2}(\phi^{(m<n)})\,, \tag{2.26}$$

$$(\star_6^{(0)} + i)\,G_-^{(n)} = \mathrm{Source}_{G_-,3}(\phi^{(m<n)})\,, \tag{2.27}$$

$$\nabla^2_{(0)}\tau_{(n)} = \frac{\Phi_+^{(0)}}{48i}\,G_+^{(0)}\cdot G_-^{(n)} + \mathrm{Source}_\tau(\phi^{(m<n)})\,, \tag{2.28}$$

$$\begin{aligned} -\frac{1}{2}\Delta_K^{(0)}\,g_{mn}^{(n)} = {}& -\frac{\Phi_+^{(0)}}{32\mathrm{Im}\,\tau}\left(G^{(0)}_{+\,(m}{}^{pq}\,\bar{G}^{(n)}_{-\,n)\,pq} + G^{(n)}_{-\,(m}{}^{pq}\,\bar{G}^{(0)}_{+\,n)\,pq}\right) \\ & + 2(\Phi_+^{-2})^{(0)}\,\nabla_{(m}\Phi_+^{(0)}\nabla_{n)}\Phi_-^{(n)} + \mathrm{Source}_g(\phi^{(m<n)})\,, \end{aligned} \tag{2.29}$$

$$\mathrm{d}\left(G_+^{(n)}\right) = \mathrm{d}\left(G_-^{(n)} - 2i\,\tau_{(n)}\,H_3^{(0)} - \mathrm{Source}_{G_+,1}(\phi^{(m<n)})\right)\,, \tag{2.30}$$

$$(\star_6^{(0)} - i)\,G_+^{(n)} = \mathrm{Source}_{G_+,2}(\phi^{(m<n)})\,, \tag{2.31}$$

$$\begin{aligned}-\nabla^2_{(0)}(\Phi_+^{-1})^{(n)} = {}& \nabla^2_{(n)}(\Phi_+^{-1})^{(0)} - \frac{g_s^2}{96}\mathrm{Im}\,\tau^{(n)}|G_+^{(0)}|^2 \\ &+ \frac{g_s}{96}\left(G_+^{(0)}\cdot\bar{G}_+^{(n)} + G_+^{(n)}\cdot\bar{G}_+^{(0)} + 3G^{(0)}_{+\,m_1n_1l_1}\bar{G}^{(0)}_{+\,m_2n_2l_2}g_{(0)}^{m_1m_2}g_{(0)}^{n_1n_2}g_{(n)}^{l_1l_2}\right) \\ &+ \left(\frac{g_s}{48}\,(\Phi_+^{-1})^{(0)}\,|G_+|^2_{(0)} - 2(\Phi_+^{-4})^{(0)}\,(\nabla\Phi_+)^2_{(0)}\right)\Phi_-^{(n)} + \mathrm{Source}_{\Phi_+}(\phi^{(m<n)})\,,\end{aligned} \tag{2.32}$$

where Δ_K denotes the metric kinetic operator

$$\Delta_K g_{mn}^{(n)} \equiv \nabla^2\, g_{mn}^{(n)} + \nabla_m\nabla_n g^{(n)} - 2\nabla^p\nabla_{(m}\, g^{(n)}_{n)p}\,, \quad g^{(n)} \equiv g_{(0)}^{pq}g_{pq}^{(n)}\,. \tag{2.33}$$

We have used the abbreviation "$\mathrm{Source}_\varphi(\phi^{(m<n)})$" to stand for all of the source terms in the equation for field φ involving the fields at previous orders $m < n$. As an illustrative example, we perform the τ expansion fully in Appendix A, giving the explicit form of $\mathrm{Source}_\tau(\phi^{(m<n)})$.

2.2.3 Method for generating solutions

We will now outline our algorithmic procedure for generating the solutions to equations (2.25-2.32) to an arbitrary order.

The order in which we arranged equations (2.25-2.32) is of critical significance: it reveals the triangular structure of the n-th order equations that will allow us to disentangle and solve the system. Let us emphasize that the equations of motion are triangular (in our chosen basis) whenever the background obeys the ISD conditions (3.16-2.18), i.e. whenever the background is conformally Calabi-Yau. In expanding around a background that is not ISD, the perturbed equations of motion will in general be intractably entangled, making an analytic solution impractical even at linear order.

Assuming that we have solved for the corrections at all orders before n, we see that in solving equation (2.25) for $\Phi_-^{(n)}$, $\mathrm{Source}_{\Phi_-}(\phi^{(m<n)})$ may be taken as given. Thus we can solve via the scalar Green's function, which we shall denote by $\mathcal{G}_s$. Having the solution for $\Phi_-^{(n)}$, we substitute it into equation (2.26) for $G_-^{(n)}$. Then all sources appearing in equations (2.26, 2.27) are given and we can solve for $G_-^{(n)}$ using the flux Green's functions $\mathcal{G}_G$. Continuing in this way, we can generate the n-th order solutions for all of the fields.[2] The result is an iterative procedure for generating the solutions, where the results from a lower order are fed into the next higher order. The seeds for this process are the harmonic modes, which obey simple equations without mixing between fields:

$$\nabla^2_{(0)}\,\Phi_-^{\mathcal{H}} = 0\,, \tag{2.34}$$

$$\mathrm{d}(\Phi_+^{(0)}\,G_-^{\mathcal{H}}) = 0\,, \tag{2.35}$$

$$(\star_6^{(0)} + i)\,(\Phi_+^{(0)}\,G_-^{\mathcal{H}}) = 0\,, \tag{2.36}$$

$$\mathrm{d}G_3^{\mathcal{H}} = 0\,, \tag{2.37}$$

$$\nabla^2_{(0)}\tau^{\mathcal{H}} = 0\,, \tag{2.38}$$

$$\Delta_K^{(0)}\,g_{mn}^{\mathcal{H}} = 0\,, \tag{2.39}$$

$$\nabla^2_{(0)}(\Phi_+^{-1})^{\mathcal{H}} = 0\,. \tag{2.40}$$

Note that when one divides a system of coupled partial differential equations into homogeneous and inhomogeneous pieces, the homogeneous equations are

[2]A similar method was used in [18] to find an all-orders local solution with dynamic SU(2) structure. We thank B. Heidenreich for helpful discussions of this point.

typically coupled. The fact that we can use the uncoupled system (2.34-2.40) is another fortuitous consequence of the triangular structure.

Let us explain how this works in detail. At first order, all of the $\text{Source}_\phi(\phi^{(m<n)}) = 0$. Then $\Phi_-^{(1)}$ simply obeys the harmonic equation (2.34), and thus

$$\Phi_-^{(1)} = \Phi_-^{\mathcal{H}} \,. \tag{2.41}$$

Substituting these harmonic modes as sources in the G_- equation (2.26), we find that, schematically,

$$G_-^{(1)} = \int \mathcal{G}_G \cdot \Phi_-^{\mathcal{H}} + G_-^{\mathcal{H}} \,. \tag{2.42}$$

Because we are solving equations (2.26, 2.27) with all source terms pre-specified, $G_-^{\mathcal{H}}$ is given by the uncoupled harmonic equations (2.35, 2.36). Working down the triangle in the same fashion, one obtains the solutions for all of the fields as functions of the harmonic solutions.

At order $n > 1$, the $\text{Source}_\phi(\phi^{(m<n)}) \neq 0$. One needs to carry out the expansion of the equations of motion to order n to determine the form of these terms. One next plugs in the solutions from previous orders for the $\text{Source}_\phi(\phi^{(m<n)})$, and then proceeds down the triangle just as in the linear case. In this way the solutions for the n-th order corrections are determined as functions of the harmonic modes. Moreover, one can use the same set of Green's functions at all orders, since the structure of the terms involving n-th order fields in equations (2.25-2.32) is the same for any n. Note that generally one would expect homogeneous contributions to the solutions at all orders:

$$\phi^{(n)} = \phi^{(n)}_{\mathcal{IH}} + \phi^{(n)}_{\mathcal{H}} \,. \tag{2.43}$$

However, since we are using the coefficients of the harmonic modes themselves as expansion parameters in our scheme, we have

$$\phi_{\mathcal{H}}^{(n)} \equiv \begin{cases} \phi_{\mathcal{H}} & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases} , \tag{2.44}$$

where $\phi_{\mathcal{H}}$ is the all-orders resummation of the harmonic modes.

The two key ingredients for our solutions are the seeding harmonic modes and the Green's functions for equations (2.25-2.32). We present the harmonic solutions in §2.3 and obtain the Green's functions in §2.4, relegating detailed derivations to Appendix B. Our results are presented in terms of the angular harmonics and associated spectroscopy on the base space $\mathcal{B}_5$: we expand all fields (and Green's functions) in these harmonics, separate the equations of motion, and solve the resulting radial equations. Thus, our solutions require the spectroscopy on $\mathcal{B}_5$ as input. For the case in which the base space is $\mathcal{B}_5 = T^{1,1}$ (i.e. the Klebanov-Strassler throat), all relevant eigenvalues and eigenfunctions are known [19, 20, 12, 21]. Moreover, the techniques applied in these works to $T^{1,1}$ can be extended to any homogeneous base space.

A primary goal of this paper is to characterize the effects of perturbations sourced in the bulk, and we have therefore emphasized non-normalizable perturbations in the discussions below. A general finite warped throat would involve normalizable perturbations sourced by effects in the IR (including, e.g., the deformation of the conifold, or a supersymmetry-breaking anti-D3-brane), in addition to the non-normalizable perturbations described in the preceding section. Moreover, boundary conditions at the tip will in general tie together normalizable and non-normalizable modes. Incorporating normalizable perturbations presents no technical challenge, and one can simply substitute normalizable modes along with non-normalizable modes when generating the Green's function solutions outlined in §2.2.3. Nevertheless, for simplicity of presentation we will restrict our attention to non-normalizable perturbations in this work.

Further details of our perturbative expansion are deferred to §2.5.

2.2.4 Matching solutions to boundary values

The method described so far takes solutions to the uncoupled homogeneous equations (2.34-2.40) as input, with the sizes of the corresponding harmonic modes serving as expansion parameters, and generates an inhomogeneous solution to any desired order. While this approach efficiently utilizes the triangular structure of the perturbed equations of motion (2.25-2.32), it is not yet adapted to solve a boundary value problem on the cone. We now remedy this.

Suppose that one would like to solve a boundary value problem in which the fields and their derivatives are specified on some slice $r = r_\star$, on which all corrections are small. To apply the method described above, one needs to extract the values of the c_i^I from the boundary data. We first expand the field value and the first radial derivative at $r = r_\star$:

$$\delta\phi(r_\star, \Psi) = \sum_I a^I\, Y^I(\Psi)\,, \tag{2.45}$$

$$\partial_r \delta\phi(r_\star, \Psi) = \sum_I \frac{b^I}{r_\star}\, Y^I(\Psi)\,, \tag{2.46}$$

with $\delta\phi = \phi - \phi^{(0)}$, so that the a^I, b^I parameterize the boundary conditions. Expanding $\phi_{\mathcal{IH}}$ in harmonics as

$$\phi_{\mathcal{IH}}(r, \Psi) = \sum_I \phi^I_{\mathcal{IH}}(r)\, Y^I(\psi)\,, \tag{2.47}$$

and using equation (2.23), (2.45, 2.46) give

$$c_0^I + c_1^I + \phi^I_{\mathcal{IH}}(r_\star) = a^I, \tag{2.48}$$

$$(\Delta(I) - 4)\ c_0^I - \Delta(I)\, c_1^I + r_\star\, \partial_r \phi^I_{\mathcal{IH}}(r_\star) = b^I\,. \tag{2.49}$$

We will see in §2.5 that $\phi^I_{\mathcal{IH}}(r_\star)$ and $r_\star\, \partial_r \phi^I_{\mathcal{IH}}(r_\star)$ are both given by power series in the c_i^I, with coefficients that are of order unity. Thus, we can obtain the c_i^I, which parameterize the homogeneous solutions, as power series in the a^I, b^I that parameterize the boundary conditions, by inverting the series (2.48, 2.49) to the desired order.

As each of the fields ϕ can be expanded in an infinite set of modes, equations (2.48, 2.49) represent an infinite system of coupled equations at each order. However, the triangular structure once again comes to our rescue, so that solving the system is a matter of straightforward[3] algebra, as we now explain. Suppose for simplicity that all normalizable modes are absent, in which case boundary condition (2.48) is by itself sufficient. Let us also suppose that there is some small parameter ϵ controlling the size of the perturbations on the boundary surface, so that we may expand

$$c^I_\phi = (c^I_\phi)^{(1)} + (c^I_\phi)^{(2)} + \dots \tag{2.50}$$

$$a^I_\phi = (a^I_\phi)^{(1)} + (a^I_\phi)^{(2)} + \dots \tag{2.51}$$

where $(c^I_\phi)^{(n)}$ and $(a^I_\phi)^{(n)}$ are the $O(\epsilon^n)$ parts of the nonnormalizable coefficient and boundary value, respectively, for field ϕ.

Now begin at first order and at the top level of the triangle. At this order, Φ_-

[3]No boundary value problem of interest will be specified in terms of an infinite number of *independent* coefficients of harmonics, as such a problem could not even be posed in finite time. Our approach is applicable when the harmonic expansion truncates, or when the coefficients of higher multipoles are simply related to the coefficients of lower multipoles, e.g. by a closed-form expression for the a^I, b^I for arbitrary I.

is harmonic, so (2.48) becomes

$$(c^I_{\Phi_-})^{(1)} = (a^I_{\Phi_-})^{(1)} . \tag{2.52}$$

Next, Φ_- acts as a source for G_-. When we expand the Green's function solution for this source in modes,

$$\left(G_-^{\mathcal{IH}}\right)^{(1)}(r,\Psi) = \int \mathcal{G}_G \cdot \Phi_-^{\mathcal{H}} = \sum_I \left(G_-^{\mathcal{IH}}\right)^{(1)}_I(r)\, Y^I(\Psi) , \tag{2.53}$$

we will generically find

$$\left(G_-^{\mathcal{IH}}\right)^{(1)}_I(r = r_\star) = \sum_J {n^J}_I\, (c^J_{\Phi_-})^{(1)} = \sum_J {n^J}_I\, (a^J_{\Phi_-})^{(1)} , \tag{2.54}$$

where the ${n^J}_I$ are numerical coefficients of order unity obtained by evaluating the Green's function solutions of §2.4 on the boundary surface. In the final equality we substituted the results from the previous level of the triangle. Equation (2.48) then gives

$$(c^I_{G_-})^{(1)} = (a^I_{G_-})^{(1)} - \sum_J {n^J}_I\, (a^J_{\Phi_-})^{(1)} . \tag{2.55}$$

One can continue in this way down the triangle. Then, moving to higher order poses no significant challenge. The contributions of the $\mathrm{Source}_\phi(\phi^{(n<m)})$ terms to (2.48) are determined by substituting from the previous orders. For instance, for Φ_- at second order, we could expand

$$\left(\Phi_-^{\mathcal{IH}}\right)^{(2)}(r,\Psi) = \int \mathcal{G}_s \cdot \mathrm{Source}_{\Phi_-}(\phi^{(n<2)}) = \sum_I \left(\Phi_-^{\mathcal{IH}}\right)^{(2)}_I(r)\, Y^I(\Psi) , \tag{2.56}$$

and would generically find

$$\left(\Phi_-^{\mathcal{IH}}\right)^{(2)}_I(r = r_\star) = \sum_{J,J',\phi,\phi'} \tilde{n}^{JJ'}_I\, (c^J_\phi)^{(1)}(c^{J'}_{\phi'})^{(1)} . \tag{2.57}$$

Equation (2.48) then gives for the second-order Φ_-

$$(c^I_{\Phi_-})^{(2)} = (a^I_{\Phi_-})^{(2)} - \sum_{J,J',\phi,\phi'} \tilde{n}^{JJ'}_I\, (c^J_\phi)^{(1)}(c^{J'}_{\phi'})^{(1)} . \tag{2.58}$$

The reader may inquire why we did not use the a^I as the parameters of our solution from the beginning. In this case the homogeneous piece of equation (2.43) would no longer vanish at order $n > 1$. At each order one would have to enforce boundary conditions tying the new harmonic modes to the inhomogeneous solutions, and the work done in imposing these boundary conditions would effectively amount to the algebraic steps described above. We find the above approach to be a more systematic way to organize the calculation.

2.3 Homogeneous Modes of the Supergravity Fields

The starting point of our expansion scheme is the set of homogeneous solutions to equations (2.34-2.40). The homogeneous modes are then fed into equations (2.25-2.32), sourcing the inhomogeneous solutions. As seen from equations (2.34-2.40), there are three distinct types of homogeneous equations:

Scalar The homogeneous modes of the scalar fields Φ_-, Φ_+^{-1} and τ obey the Laplace equation on the cone,

$$\nabla^2 \Phi^{\mathcal{H}} = 0\,, \tag{2.59}$$

where ∇^2 is constructed using the cone metric, equation (2.19).

Flux The homogeneous modes of the flux $G_\pm$ obey the system

$$\mathrm{d}(\Phi_+ G_-^{\mathcal{H}}) = 0\,, \tag{2.60}$$

$$\mathrm{d}G_3^{\mathcal{H}} = 0\,, \tag{2.61}$$

where Φ_+ is given by its background form, equation (2.21).

Metric The homogeneous modes of the metric perturbations obey

$$\Delta_K \, g^{\mathcal{H}}_{mn} = 0 \, . \tag{2.62}$$

The solutions below are presented in terms of various harmonics on the angular space $\mathcal{B}_5$. Details about these harmonics can be found in §B.1. Throughout this section, contractions, covariant derivatives, etc. are carried out with respect to the zeroth-order background metric, equations (2.19), (2.20). In §§2.1,2.2 we denoted the background by $g^{(0)}_{mn}$, but in this section we will drop the superscript for simplicity of notation. In addition, a tilde above the indices and the derivatives signifies contraction with and construction out of the angular metric $\tilde{g}_{ij}$ on $\mathcal{B}_5$.

2.3.1 Homogeneous solutions for the scalars

Consider first the Laplace equation (2.59). Using the cone structure of the background, we can expand Φ in scalar harmonics $Y^{I_s}(\Psi)$ on $\mathcal{B}_5$,

$$\Phi(r, \Psi) = \sum_{I_s} \Phi_{I_s}(r) \, Y^{I_s}(\Psi) \, , \tag{2.63}$$

where the $Y^{I_s}(\Psi)$ diagonalize the angular Laplacian

$$\tilde{\nabla}^2 \, Y^{I_s} \equiv \frac{1}{\sqrt{\tilde{g}}} \partial_i \Big(\sqrt{\tilde{g}} \, \tilde{g}^{ij} \partial_j Y^{I_s} \Big) = -\lambda^{I_s} \, Y^{I_s} \, . \tag{2.64}$$

Now using that the Laplacian decomposes,

$$\nabla^2 = \partial_r^2 + \frac{5}{r} \, \partial_r + \frac{1}{r^2} \tilde{\nabla}^2 \, , \tag{2.65}$$

the Laplace equation reduces to the following radial equation:

$$\partial_r^2 \, \Phi_{I_s} + \frac{5}{r} \, \partial_r \Phi_{I_s} - \frac{\lambda^{I_s}}{r^2} \Phi_{I_s} = 0 \, . \tag{2.66}$$

Thus, the homogeneous solutions for any of the fields $\Phi_-^{\mathcal{H}}, (\Phi_+^{-1})^{\mathcal{H}}, \tau^{\mathcal{H}}$ take the form

$$\Phi^{\mathcal{H}}(r,\Psi) = \sum_{I_s} \left(\Phi_0^{I_s}\, r^{\Delta(I_s)-4} + \Phi_1^{I_s}\, r^{-\Delta(I_s)}\right) Y^{I_s}(\Psi)\,, \tag{2.67}$$

where $\Phi_0^{I_s}$ and $\Phi_1^{I_s}$ are constants determined by the boundary conditions, and where we have defined

$$\Delta(I_s) \equiv 2 + \sqrt{4+\lambda^{I_s}}\,. \tag{2.68}$$

By comparison with the standard AdS form, equation (2.23), we see that for a canonically normalized scalar field, $\Delta(I_s)$ corresponds to the dimension of the operator dual to that mode. For the zero mode, $\lambda^{I_s} = 0$, we have $\Delta(I_s) = 4$, but for modes other than the zero mode we have $\lambda^{I_s} \geq 5$ (cf. §B.1), so that generically $\Delta(I_s) \geq 5$.

2.3.2 Homogeneous solutions for the fluxes

For the homogeneous perturbations of the three-form fluxes $G_\pm$, we have the system of equations (2.60), (2.61). The solution of this system is a slight generalization of that obtained in [12], now including logarithmic running of Φ_+, equation (2.21). Here we briefly outline the solution, leaving the details to §B.2.1.

Because $G_3^{\mathcal{H}}$ is closed by equation (2.61), it can be written locally in terms of a two-form potential A_2. We then expand the potential in terms of two-form harmonics and solve equation (2.60) for the coefficients of the harmonic expansion. The result is, cf. equation (B.60),

$$G_3^{\mathcal{H}} = \mathrm{d}A_2\,, \tag{2.69}$$

$$A_2 = \sum_{I_2} \left(A_-^{I_2}\, r^{-\delta^{I_2}} + A_+^{I_2}\left[(4-2\,\delta^{I_2})(C_1 + C_2\,\ln r) + C_2\right] r^{\delta^{I_2}-4}\right) Y^{I_2}\,, \tag{2.70}$$

where $A_{\pm}^{I_2}$ are constants of integration and $C_{1,2}$ are the coefficients of the running warp factor Φ_+, cf. equation (2.21). The $Y_{[ij]}^{I_2}(\Psi)$ are the transverse two-form harmonics on $\mathcal{B}_5$ that diagonalize the Laplace-Beltrami operator

$$\star_5 \mathrm{d}Y^{I_2} = i\,\delta^{I_2} Y^{I_2}\,. \tag{2.71}$$

The eigenvalues δ^{I_2} are real and are symmetric under $\delta^{I_2} \to -\delta^{I_2}$. In order for the radial scalings of the modes in equation (2.70) to take on the standard AdS form, equation (2.23), we identify $\Delta(I_2) = \max(\delta^{I_2}, 4 - \delta^{I_2})$. In §B.2.2 we give formulas expressing the resulting scaling dimensions of flux modes in terms of the dimensions $\Delta(I_s)$ of scalar modes.

2.3.3 Homogeneous solutions for the metric

The homogeneous part of the metric perturbation obeys (2.62). To fully utilize the cone structure of C_6 we decompose $g_{mn}^{\mathcal{H}}$ into irreducible pieces under general coordinate transformations of the base space $\mathcal{B}_5$. Then $g_{rr}^{\mathcal{H}}$ transforms as a scalar, $g_{ir}^{\mathcal{H}}$ transforms as a vector, and the trace, $\tilde{g}^{\mathcal{H}} \equiv \tilde{g}^{ij} g_{ij}^{\mathcal{H}}$, and the traceless part, $g_{\{ij\}}^{\mathcal{H}} \equiv g_{ij}^{\mathcal{H}} - \frac{1}{5}\tilde{g}_{ij}\tilde{g}^{\mathcal{H}}$, of $g_{ij}^{\mathcal{H}}$ transform as a scalar and a symmetric traceless two-tensor, respectively.

In what follows, we will find it convenient to impose a *transverse gauge*, i.e. we set

$$\tilde{\nabla}^{\tilde{k}} g_{k\,r}^{\mathcal{H}} = 0\,, \tag{2.72}$$

$$\tilde{\nabla}^{\tilde{k}} g_{\{k\,i\}}^{\mathcal{H}} = 0\,. \tag{2.73}$$

After imposing the transverse gauge, some residual gauge freedom remains, which we use to impose two additional conditions. First, we impose that the constant mode of the trace, $\tilde{g}^{\mathcal{H}}$, vanishes. Second, we impose that the Killing vector modes of $g^{\mathcal{H}}_{ir}$ vanish (cf. §B.3 for more details).

Solving the homogeneous equation (2.62) is the subject of §B.3.1. There it is found that in the transverse gauge specified above, equation (2.62) implies that the only nonvanishing metric component is $g^{\mathcal{H}}_{\{ij\}}$, i.e.

$$g^{\mathcal{H}}_{rr} = g^{\mathcal{H}}_{ir} = \tilde{g}^{\mathcal{H}} = 0\,. \tag{2.74}$$

Furthermore, when we expand $g^{\mathcal{H}}_{\{ij\}}$ in transverse-traceless, two-tensor harmonics, equation (2.62) is reduced to a radial equation for the coefficients with the solution (cf. equation (B.147)),

$$g^{\mathcal{H}}_{\{ij\}} = \sum_{I_t} \Big(g_0^{I_t}\, r^{\Delta(I_t)-2} + g_1^{I_t}\, r^{-\Delta(I_t)+2} \Big)\, Y^{I_t}_{\{ij\}}(\Psi)\,, \tag{2.75}$$

where $g_0^{I_t}$ and $g_1^{I_t}$ are integration constants determined by the boundary conditions, and where we have defined

$$\Delta(I_t) \equiv 2 + \sqrt{\lambda^{I_t} - 4}\,. \tag{2.76}$$

The $Y^{I_t}_{\{ij\}}$ are the transverse-traceless, symmetric, two-tensor harmonics on $\mathcal{B}_5$,

$$\tilde{\nabla}^{\tilde{k}} Y^{I_t}_{\{kj\}} = 0\,, \quad \tilde{g}^{ij} Y^{I_t}_{\{ij\}} = 0\,, \tag{2.77}$$

that diagonalize the angular Lichnerowicz operator

$$\tilde{\nabla}^2 Y^{I_t}_{\{ij\}} - 2\tilde{\nabla}^{\tilde{k}} \tilde{\nabla}_{(i} Y^{I_t}_{\{j)k\}} = -\lambda^{I_t} Y^{I_t}_{\{ij\}}\,. \tag{2.78}$$

2.3.4 Summary: radial scalings of the homogeneous modes

In this subsection we summarize the radial scalings of all supergravity fields ϕ and the dimensions $\Delta(\phi)$ of the dual operators, as obtained in §§2.3.1, 2.3.2, 2.3.3. The results are presented in Table 2.1, which we now explain.

Homogeneous Scalings of the Non-Normalizable Modes

Field	**Scaling**	**Dimension**
$r^{-4}\,\Phi_-^{\mathcal{H}}$	$r^{\Delta(\Phi_-)-4}$	$\Delta(\Phi_-) = \Delta(I_s) - 4\,,\ \lambda^{I_s} \neq 0$
$G_-^{\mathcal{H}}$	$r^{\Delta(G_-)-4}$	$\Delta(G_-) = \Delta(\delta^{I_2} \geq 2)$
$\tau^{\mathcal{H}}$	$r^{\Delta(\tau)-4}$	$\Delta(\tau) = \Delta(I_s)\,,\ \lambda^{I_s} \neq 0$
$r^{-2} g_{\{ij\}}^{\mathcal{H}}$	$r^{\Delta(g)-4}$	$\Delta(g) = \Delta(I_t)$
$G_+^{\mathcal{H}}$	$r^{\Delta(G_+)-4}$	$\Delta(G_+) = \Delta(\delta^{I_2} \geq 2),\ \Delta(\delta^{I_2} \leq -2)$
$r^4 (\Phi_+^{-1})^{\mathcal{H}}$	$r^{\Delta(\Phi_+^{-1})-4}$	$\Delta(\Phi_+^{-1}) = \Delta(I_s) + 4$

Homogeneous Scalings of the Normalizable Modes

Field	**Scaling**	**Dimension**
$r^{-4}\,\Phi_-^{\mathcal{H}}$	$r^{-\Delta(\Phi_-)}$	$\Delta(\Phi_-) = \Delta(I_s) + 4$
$G_-^{\mathcal{H}}$	$r^{-\Delta(G_-)}$	$\Delta(G_-) = \Delta(\delta^{I_2} \leq -2),\ \Delta(b_2)$
$\tau^{\mathcal{H}}$	$r^{-\Delta(\tau)}$	$\Delta(\tau) = \Delta(I_s)$
$r^{-2} g_{\{ij\}}^{\mathcal{H}}$	$r^{-\Delta(g)}$	$\Delta(g) = \Delta(I_t)$
$G_+^{\mathcal{H}}$	$r^{-\Delta(G_+)}$	$\Delta(G_+) = \Delta(\delta^{I_2} \geq 2),\ \Delta(\delta^{I_2} \leq -2),\ \Delta(b_2)$
$r^4 (\Phi_+^{-1})^{\mathcal{H}}$	$r^{-\Delta(\Phi_+^{-1})}$	$\Delta(\Phi_+^{-1}) = \Delta(I_s) - 4\,,\ \lambda^{I_s} \neq 0$

Table 2.1: The radial scalings of the homogeneous modes of the supergravity fields. Here $\Delta(I_s) = 2 + \sqrt{4 + \lambda^{I_s}}$, where the λ^{I_s} are the eigenvalues of the angular scalar Laplacian, cf. equation (2.64). Furthermore, $\Delta(I_t) = 2 + \sqrt{\lambda^{I_t} - 4}$, where the λ^{I_t} are the eigenvalues of the angular Lichnerowicz operator, cf. equation (2.78). The expressions $\Delta(\delta^{I_2} \geq 2)$, $\Delta(\delta^{I_2} \leq -2)$, and $\Delta(b_2)$ appearing in the flux dimensions can be found in equations (B.63, B.64, B.65). Although we have not explicitly displayed this in the tables, the modes of $G_\pm$ can have additional logarithmic running of the form $r^{\Delta_G - 4} \ln r$ and $r^{-\Delta_G} \ln r$ for the non-normalizable and normalizable modes, respectively; cf. equations (B.51, B.52).

For canonically normalized fields ϕ, the radial scalings of the modes and the dimensions of the operators of the dual field theory are related via the standard AdS formula (2.23). To start with, the scalar field τ is canonically normalized, so the dimension of the operator dual to τ is given by

$$\Delta(\tau) = \Delta(I_s) = 2 + \sqrt{4 + \lambda^{I_s}}\,. \tag{2.79}$$

The same is true for the potential A_2, and the dimensions $\Delta(G_\pm)$ can be read off from (B.63, B.64, B.65), taking into account the discussion at the end of §B.2.2. For the ISD flux G_+ both A_+ and A_- modes can be turned on, so that all modes are present except for non-normalizable Betti modes:

$$\textit{Non-normalizable:} \quad \Delta(G_+) = \Delta(\delta^{I_2} \geq 2),\ \Delta(\delta^{I_2} \leq -2)\,, \tag{2.80}$$

$$\textit{Normalizable:} \quad \Delta(G_+) = \Delta(\delta^{I_2} \geq 2),\ \Delta(\delta^{I_2} \leq -2),\ \Delta(b_2)\,, \tag{2.81}$$

while for the IASD flux G_- only A_+ can be turned on, and we have

$$\textit{Non-normalizable:} \quad \Delta(G_-) = \Delta(\delta^{I_2} \geq 2)\,, \tag{2.82}$$

$$\textit{Normalizable:} \quad \Delta(G_-) = \Delta(\delta^{I_2} \leq -2),\ \Delta(b_2)\,, \tag{2.83}$$

where the expressions for $\Delta(\delta^{I_2} \geq 2)$, $\Delta(\delta^{I_2} \leq -2)$, and $\Delta(b_2)$ are given in equations (B.63), (B.65), and (B.64), respectively.

Next, it is the warped internal metric $e^{-2A}\, g_{\{ij\}} \sim r^{-2}\, g_{\{ij\}}$ that is the canonical field [19, 20], corresponding to a dual operator with dimension

$$\Delta(g) = \Delta(I_t) = 2 + \sqrt{\lambda^{I_t} - 4}\,, \tag{2.84}$$

as anticipated by the notation. Finally, Φ_- and Φ_+^{-1} are not canonical fields, but as explained in [11], the combinations $r^{-4}\Phi_-$ and $r^4\Phi_+^{-1}$ exhibit the same radial scaling as do the corresponding canonical variables. Now comparing the non-normalizable and normalizable modes of $r^{-4}\Phi_-$ with equation (2.23) one can identify the operator dimensions

$$\textit{Non-normalizable:} \quad \Delta(\Phi_-) = \Delta(I_s) - 4\,, \tag{2.85}$$

$$\textit{Normalizable:} \quad \Delta(\Phi_-) = \Delta(I_s) + 4\,. \tag{2.86}$$

Similarly, by comparing the non-normalizable and normalizable modes of $r^4\Phi_+^{-1}$ with (2.23) one can identify the operator dimensions

$$\textit{Non-normalizable:} \quad \Delta(\Phi_+^{-1}) = \Delta(I_s) + 4\,, \tag{2.87}$$

$$\textit{Normalizable:} \quad \Delta(\Phi_+^{-1}) = \Delta(I_s) - 4\,. \tag{2.88}$$

Notice that $\Delta(\Phi_-)$ and $\Delta(\Phi_+^{-1})$ exchange roles in going from the normalizable modes to the non-normalizable modes.

In Table 2.1 we have excluded the zero modes of both τ and Φ_- for the non-normalizable modes (scaling like r^0) while for the normalizable modes we have excluded that of Φ_+^{-1} (scaling like r^{-4}). For τ, the non-normalizable zero mode corresponds to a constant shift of the axion $\mathrm{Re}\,\tau \equiv C_0$ and the dilaton $\mathrm{Im}\,\tau \equiv e^{-\phi}$. A constant shift of the dilaton can be absorbed in the background value of $g_s^{-1} \equiv \mathrm{Im}\,\tau^{(0)}$, while the axion C_0 is shift-symmetric. The non-normalizable zero mode of Φ_- can be gauged away using a constant shift of α, thus preserving the background $\Phi_-^{(0)} = 0$. The normalizable zero mode of Φ_+^{-1} corresponds to a shift of the constant C_1 in the warp factor (2.21), which we will also absorb into the background value.

2.4 Inhomogeneous Modes: Green's Function Solutions

The final ingredient of our expansion scheme is the set of inhomogeneous solutions to equations (2.25-2.32). In this section we will write down the Green's function solutions for the inhomogeneous scalar, flux and metric modes, again relegating detailed derivations to the appendix. As discussed in §2.2, the structure of the equations is the same at every order. Thus, we only need to write down one set of scalar, flux and metric Green's functions, $\mathcal{G}_s$, $\mathcal{G}_{G_\pm}$, and $\mathcal{G}_g$, which are used at all orders.

The initial seeds for the inhomogeneous pieces are the homogeneous solutions obtained in §2.3. The homogeneous modes are given by angular harmonics multiplying radial powers r^α (possibly including logarithmic running $(\ln r)^m$, in the case of flux). Thus, the source terms are of a non-localized nature, and the standard Green's functions for localized sources give divergences at the origin and at infinity when convoluted with the non-localized sources. One could introduce regulated Green's function with cutoffs at $r_{\rm IR}$ and $r_{\rm UV}$, but these introduce large counterterms, and in what follows we will take a more direct route by solving the equations explicitly.

2.4.1 Inhomogeneous solutions for the scalars

From equations (2.25, 2.28, 2.32) we see that n-th order perturbations of the scalar fields Φ_-, Φ_+^{-1} and τ obey Poisson's equation on the cone

$$\nabla^2_{(0)}\Phi^{(n)} = \mathcal{S}^{(n)}_\Phi\,, \tag{2.89}$$

where $\nabla^2_{(0)}$ is constructed from the background metric of the cone, equation (2.19). The source dependence on the fields at order n can be read off explicitly from equations (2.25, 2.28, 2.32), while the dependence on the fields at order

$m < n$ is left implicit:

$$\mathcal{S}^{(n)}_{\Phi_-} = \mathrm{Source}_{\Phi_-}(\phi^{m<n})\,, \tag{2.90}$$

$$\begin{aligned}\mathcal{S}^{(n)}_{\Phi_+^{-1}} &= \frac{g_s}{96}\left(G_+^{(0)}\cdot\bar{G}_+^{(n)} + G_+^{(n)}\cdot\bar{G}_+^{(0)} + 3G^{(0)}_{+\,m_1n_1l_1}\bar{G}^{(0)}_{+\,m_2n_2l_2}g^{m_1m_2}_{(0)}g^{n_1n_2}_{(0)}g^{l_1l_2}_{(n)}\right) \\ &\quad - \frac{g_s^2}{96}\mathrm{Im}\,\tau^{(t)}|G_+^{(0)}|^2 \;+\left[\frac{g_s}{48}\,(\Phi_+^{-1})^{(0)}\,|G_+|^2_{(0)} - 2(\Phi_+^{-4})^{(0)}\,(\nabla\Phi_+)^2_{(0)}\right]\Phi_-^{(n)} + \mathrm{Source}_{\Phi_+}(\phi^{m<n})\,,\end{aligned} \tag{2.91}$$

$$\mathcal{S}^{(n)}_{\tau} = \mathrm{Source}_{\tau}(\phi^{m<n}) - i\,\Phi_+^{(0)}\,G_+^{(0)}\cdot G_-^{(n)}\,. \tag{2.92}$$

We start by expanding the fields and the sources in terms of angular harmonics

$$\Phi^{(n)}(r,\Psi) = \sum_{I_s}\Phi^{(n)}_{I_s}(r)\,Y^{I_s}(\Psi)\,, \tag{2.93}$$

$$\mathcal{S}^{(n)}_{\Phi}(r,\Psi) = \sum_{I_s}\mathcal{S}^{(n)}_{I_s}(r)\,Y^{I_s}(\Psi)\,, \tag{2.94}$$

so that Poisson's equation (2.89) reduces to an equation for the radial coefficients

$$\left(\partial_r^2 + \frac{5}{r}\partial_r - \frac{\lambda^{I_s}}{r^2}\right)\Phi^{(n)}_{I_s}(r) = \mathcal{S}^{(n)}_{I_s}(r)\,. \tag{2.95}$$

As discussed above, the source $\mathcal{S}_{I_s}$ will involve a sum of various radial scalings due to the homogeneous modes

$$\mathcal{S}_{I_s}(r) = \sum_{\alpha,m}\mathcal{S}^{(n)}_{I_s}(\alpha,m)\,r^{\alpha}\,(\ln r)^m\,, \tag{2.96}$$

and the inhomogeneous solution to Poisson's equation (2.89) is

$$\Phi^{(n)}_{\mathcal{IH}}(r) = \sum_{I_s}\sum_{\alpha,m}\Phi^{(n)}_{I_s}(r;\alpha,m)\,Y^{I_s}(\Psi)\,, \tag{2.97}$$

where $\Phi^{(n)}_{I_s}(r;\alpha,m)$ is given in equations (2.98, 2.100). The solution for $\Phi^{(n)}_{I_s}(r;\alpha,m)$ depends on the value of α:

Case: $\alpha + 2 \neq -2 \pm (\Delta(I_s) - 2)$**.** The solution to equation (2.95) is given by

$$\Phi^{(n)}_{I_s}(r;\alpha,m) = \mathcal{S}^{(n)}_{I_s}(\alpha,m)\,r^{\alpha+2}\,(a_0 + a_1\ln(r) + \ldots + a_m\,(\ln r)^m)\,, \tag{2.98}$$

where the coefficients a_k are given by

$$a_k = (-1)^{k+m+1}\, \frac{m!/k!}{2\,\Delta(I_s) - 4} \left[(\alpha + 2 + \Delta(I_s))^{k-1-m} - (\alpha + 2 - \Delta(I_s) + 4)^{k-1-m} \right] . \tag{2.99}$$

Case: $\alpha + 2 = -2 \pm (\Delta(I_s) - 2)$**.** The solution to equation (2.95) is given by

$$\Phi^{(n)}_{I_s}(r;\alpha,m) = \mathcal{S}^{(n)}_{I_s}(\alpha,m)\, r^{\alpha+2}\left(b_0 + b_1 \ln(r) + \ldots + b_{m+1}\,(\ln r)^{m+1}\right), \tag{2.100}$$

where the coefficients b_k are given by

$$b_k = (-1)^{k+m+1}\,\frac{m!}{k!}\,(\pm 2\Delta(I_s) \mp 4)^{k-2-m}\,, \quad \alpha + 2 = -2 \pm (\Delta(I_s) - 2)\,. \tag{2.101}$$

2.4.2 Inhomogeneous solutions for the fluxes

We now find the inhomogeneous modes for $G_\pm$ solving equations (2.26, 2.27, 2.30, 2.31). The equations of motion for the n-th order perturbation of G_- take the form

$$\mathrm{d}\left(\Phi^{(0)}_+ G^{(n)}_- + \mathcal{S}^{(n)}_{G_-,1}\right) = \mathcal{S}^{(n)}_{G_-,3}\,, \tag{2.102}$$

$$(\star^{(0)}_6 + i)\,\Phi^{(0)}_+ G^{(n)}_- = \mathcal{S}^{(n)}_{G_-,2}\,. \tag{2.103}$$

Here the sources $\mathcal{S}^{(n)}_{G_-,1}, \mathcal{S}^{(n)}_{G_-,2}$ are three-forms and $\mathcal{S}^{(n)}_{G_-,3}$ is a four-form. The expressions for the sources in terms of the n-th order fields can be read off from equations (2.26, 2.27), where again the dependence on the fields at lower order is left implicit,

$$\mathcal{S}^{(n)}_{G_-,1} = \Phi^{(n)}_-\, G^{(0)}_+ + \mathrm{Source}_{G_-,1}(\phi^{m<n})\,, \tag{2.104}$$

$$\mathcal{S}^{(n)}_{G_-,2} = \mathrm{Source}_{G_-,2}(\phi^{m<n})\,, \tag{2.105}$$

$$\mathcal{S}^{(n)}_{G_-,3} = \mathrm{Source}_{G_-,3}(\phi^{m<n})\,. \tag{2.106}$$

The equations of motion for the n-th order perturbation of G_+ are similar to those of G_-:

$$\mathrm{d}\left(G_+^{(n)} + \mathcal{S}_{G_+,1}^{(n)}\right) = 0\,, \tag{2.107}$$

$$(\star_6^{(0)} - i)\,G_+^{(n)} = \mathcal{S}_{G_+,2}^{(n)}\,, \tag{2.108}$$

where the three-form sources $\mathcal{S}_{G_+,1}^{(n)}, \mathcal{S}_{G_+,2}^{(n)}$ can be read off from equations (2.30, 2.31)

$$\mathcal{S}_{G_+,1}^{(n)} = -G_-^{(n)} + 2i\tau^{(n)}\,H_3^{(0)} + \mathrm{Source}_{G_+,1}(\phi^{m<n})\,, \tag{2.109}$$

$$\mathcal{S}_{G_+,2}^{(n)} = \mathrm{Source}_{G_-,2}(\phi^{m<n})\,. \tag{2.110}$$

Both systems are of the form

$$\mathrm{d}\,(\Sigma_\pm + \mathcal{S}_1) = \mathcal{S}_3\,, \tag{2.111}$$

$$(\star_6^{(0)} \mp i)\,\Sigma_\pm = \mathcal{S}_2\,, \tag{2.112}$$

with $\Sigma_- = \Phi_+^{(0)} G_-^{(n)}$ and $\Sigma_+ = G_+^{(n)}$. We first solve the two simpler systems

$$\mathrm{I}: \quad \begin{array}{l} \mathrm{d}\left(\Sigma_\pm^{(\mathrm{I})} + \mathcal{S}_1\right) = 0 \\ (\star_6^{(0)} \mp i)\,\Sigma_\pm^{(\mathrm{I})} = \mathcal{S}_2 \end{array}\,, \qquad \mathrm{II}: \quad \begin{array}{l} \mathrm{d}\Sigma_\pm^{(\mathrm{II})} = \mathcal{S}_3 \\ (\star_6^{(0)} \mp i)\,\Sigma_\pm^{(\mathrm{II})} = 0 \end{array}\,. \tag{2.113}$$

By linearity the full solution is $\Sigma_\pm = \Sigma_\pm^{(\mathrm{I})} + \Sigma_\pm^{(\mathrm{II})}$. The solving of I and II is the subject of §B.2.3 and here we only present the results.

Flux Green's function I: From the first equation we see that $\Sigma_\pm^{(\mathrm{I})} + \mathcal{S}_1$ is closed and can locally be expressed as $\mathrm{d}\chi_\pm$ for some two-form $\chi_\pm$. The solution in terms of this potential is

$$\Sigma_\pm^{(\mathrm{I})} = -\mathcal{S}_1 + \mathrm{d}\chi_\pm\,, \tag{2.114}$$

$$\chi_\pm(y) = \int_{C_6} \mathcal{G}_G^{(\mathrm{I})}(y,y') \wedge \left(\mathcal{S}_2 + (\,\star_6^{(0)} \mp i)\mathcal{S}_1\right)(y')\,, \tag{2.115}$$

where the explicit form of $\mathcal{G}^{(\mathrm{I})}$ is given in equation (B.91). The indices of the above equation should be interpreted in the following way: the Green's function $(\mathcal{G}^{(\mathrm{I})})_{mn,p'q's'}$ has two legs in the y coordinate system and three legs in the y' coordinate system. When we wedge $\mathcal{G}_G^{(\mathrm{I})}$ with the three-form source $\mathcal{S}_2 + (\star_6^{(0)} \mp i)\mathcal{S}_1$ we produce a six-form in the y' coordinates which is integrated over the whole manifold C_6, resulting in a two-form $\chi_\pm(y)$ in the y coordinate system.

Flux Green's function II: In a similar way the solution to system II is given by

$$\Sigma_\pm^{(\mathrm{II})} = \int_{C_6} \mathcal{G}_G^{(\mathrm{II})}(y, y') \wedge \mathcal{S}_3(y') \,, \tag{2.116}$$

where the explicit form of $\mathcal{G}^{(\mathrm{II})}$ is given in equation (B.106). Here $\mathcal{S}_3$ is a four-form and $(\mathcal{G}^{(\mathrm{II})})_{mnp,q's'}$ is a $(3 + 2')$-form producing a three-form $\Sigma_\pm^{(\mathrm{II})}$.

2.4.3 Inhomogeneous solutions for the metric

The n-th order perturbations of the metric g_{mn} obey

$$\Delta_K^{(0)} g_{mn}^{(n)} = (\mathcal{S}_g^{(n)})_{mn} \,, \tag{2.117}$$

where the source can be read off from equation (2.29),

$$\begin{aligned}(\mathcal{S}_g^{(n)})_{mn} = \frac{\Phi_+^{(0)}}{16 \mathrm{Im}\,\tau} \Big(G^{(0)}_{+\,(m}{}^{pq}\, \bar{G}^{(n)}_{-\,n)\,pq} + G^{(n)}_{-\,(m}{}^{pq}\, \bar{G}^{(0)}_{+\,n)\,pq} \Big) \\ - 4(\Phi_+^{(0)})^{-2}\, \nabla^{(0)}_{(m} \Phi_+^{(0)} \nabla^{(0)}_{n)} \Phi_-^{(n)} + \mathrm{Source}_g(\phi^{m<n}) \,.\end{aligned} \tag{2.118}$$

As in the homogeneous case, we utilize the cone structure and decompose the metric perturbations into irreducible pieces under general coordinate transformations of the base space. We continue to impose a *transverse* gauge on the irreducible vector and tensor at each order in perturbation theory, i.e. we set

$$\tilde{\nabla}^{\tilde{k}} g^{(n)}_{k\,r} = 0\,, \tag{2.119}$$

$$\tilde{\nabla}^{\tilde{k}} g^{(n)}_{\{k\,i\}} = 0\,, \tag{2.120}$$

together with the additional constraint on the constant mode and Killing vector modes as discussed in §2.3.3. We end up with a Green's function solution of the form

$$(g^{(n)}_{mn})^{\mathcal{IH}}(y) = \int_{\mathcal{M}'} \mathrm{d}^6 y' \, \sqrt{g'} \, (\mathcal{G}_g)_{mn}{}^{m'n'}(y, y') \, (\mathcal{S}^{(n)}_g)_{m'n'}(y')\,. \tag{2.121}$$

The Green's function $(\mathcal{G}_g)_{mn}{}^{m'n'}(y, y')$ is valid only in the gauge specified above, cf. equations (B.174-B.176) in §B.3.2. Note that all components not listed in (B.180-B.184) vanish in this gauge.

2.5 Radial Scalings of Corrections

The results described above depend implicitly and explicitly on the angular harmonics, and corresponding eigenvalues, associated with the scalar, flux, and metric perturbations. Thus, although one can use our results to obtain an explicit solution to any desired order on a cone whose angular harmonics are known (e.g., the conifold), this is little consolation when one is faced with computing the eigenfunctions in a more general example. Fortunately, for many questions of physical interest[4] it suffices to determine how corrections scale with r, obviating the full Green's function solution. In this section we present results adapted to extracting radial scalings without obtaining the full angular dependence of the corresponding solutions.

[4]For example, one might want to estimate the scale of the mass term induced for some object, such as an anti-D3-brane [10] or a D3-brane [12], or determine the soft masses in a toy visible sector [22, 23, 24].

The main result of this section is equation (3.28), which qualitatively states that the n-th order correction $\phi^{(n)}$ of a field ϕ scales like a sum of products of n harmonic modes

$$\hat{\phi}^{(n)} \sim \sum_{i_1,\ldots,i_n} \hat{\phi}^{\mathcal{H}}_{i_1} \cdots \hat{\phi}^{\mathcal{H}}_{i_n}\,, \tag{2.122}$$

where the sum runs over subsets of the fields $\{\hat{\Phi}_+, \hat{G}_+, \hat{\tau}, \hat{g}_{\{ij\}}, \hat{G}_-, \hat{\Phi}_+^{-1}\}$, and the hatted variables are defined in equation (2.126). Throughout this section we will use $\sim$ to signify that two objects have the same radial scaling, but may differ by order-unity angular functions, e.g. we will write $r^\alpha \chi_1(\Psi) \sim r^\alpha \chi_2(\Psi)$, for angular functions $\chi_{1,2}(\Psi)$ that are of order unity at generic points.

One complication in equation (2.122) is that not every possible product of harmonic modes contributes in the sum, and one must trace through the expanded equations (2.25-2.32) to see which combinations appear for a given field. For example, from equation (2.25) for Φ_-, one sees that none of the harmonic modes apart from Φ_- itself contributes to the correction at first order. The results from the first and second order calculations are presented in Tables 2.2 and 2.3, respectively. We expect that at higher order in the expansion, all possible products will contribute, as the number of ways a particular combination can propagate through the equations of motion becomes large.

When checking which products of harmonics appear, we will not rule out the possibility that contractions of indices or convolutions of angular harmonics with Green's functions result in a vanishing contribution. If a particular mode is critical to an analysis, the associated product would need to be examined in detail by tracing through the equations of motion.

2.5.1 First-order and second-order scalings

We begin by determining the radial scalings of the inhomogeneous modes at first order, in terms of the first-order homogeneous modes obtained in §2.3.4. To make full use of the triangular structure of the equations of motion, we begin at the top of the triangle, with the scalar field $\Phi_-^{(1)}$, and work our way downward.

First level $\Phi_-^{(1)}$: At first order, equation (2.25) for $\Phi_-^{(1)}$ reads $\nabla^2_{(0)}\Phi_-^{(1)} = 0$, so that $\Phi_-^{(1)}$ is solely determined by its harmonic mode,

$$\Phi_-^{(1)} = \Phi_-^{\mathcal{H}} . \tag{2.123}$$

Second level $G_-^{(1)}$: From equation (2.26) we get at first order $\mathrm{d}(\Phi_+^{(0)} G_-^{(1)}) = -\mathrm{d}(\Phi_-^{(1)} G_+^{(0)})$, so that $G_-^{(1)}$ is sourced by $\Phi_-^{(1)}$. Using equation (2.123) for $\Phi_-^{(1)}$ together with the radial scalings of the background fields, $\Phi_+^{(0)} \sim r^{-4}$ and $G_+^{(0)} \sim r^0$, we infer that

$$G_-^{(1)} \sim r^{-4}\Phi_-^{\mathcal{H}} + G_-^{\mathcal{H}} , \tag{2.124}$$

where we also include the homogeneous contribution $G_-^{\mathcal{H}}$ in the first-order solution.

Third level $\tau^{(1)}$: Equation (2.28) for $\tau^{(1)}$ reads at first order $\nabla^2_{(0)}\tau^{(1)} = \Phi_+^{(0)}/(48i)\, G_+^{(0)} \cdot G_-^{(1)}$. To find the radial scaling for $\tau^{(1)}$ we substitute the radial scaling for $G_-^{(1)}$, equation (2.124), and the radial scalings for the background fields, yielding

$$\tau^{(1)} \sim r^{-4}\Phi_-^{\mathcal{H}} + G_-^{\mathcal{H}} + \tau^{\mathcal{H}} . \tag{2.125}$$

Thus, $\tau^{(1)}$ inherits a dependence on $r^{-4}\Phi_-^{\mathcal{H}}$ through the solution for $G_-^{(1)}$.

Higher levels $g^{(1)}_{\{ij\}}, G^{(1)}_+, (\Phi^{-1}_+)^{(1)}$: We continue in a similar manner, solving for the radial scalings of all the fields. The result is most efficiently presented in terms of new fields $\hat{\phi}$, which are defined such that they scale with r in the same way as the corresponding canonical degrees of freedom:

$$\hat{\Phi}_- \equiv r^{-4}\Phi_- \,, \quad \hat{G}_- \equiv G_- \,, \quad \hat{\tau} \equiv \tau \,, \quad \hat{g}_{mn} \equiv r^{-2} g_{mn} \,, \quad \hat{G}_+ \equiv G_+ \,, \quad \hat{\Phi}^{-1}_+ \equiv r^4 \Phi^{-1}_+ \,. \tag{2.126}$$

Then, the radial scalings at linear order are very simple:

$$\hat{\Phi}^{(1)}_- \sim \hat{\Phi}^{\mathcal{H}}_- \,, \tag{2.127}$$

$$\hat{G}^{(1)}_- \sim \hat{\Phi}^{\mathcal{H}}_- + \hat{G}^{\mathcal{H}}_- \,, \tag{2.128}$$

$$\hat{\tau}^{(1)} \sim \hat{\Phi}^{\mathcal{H}}_- + \hat{G}^{\mathcal{H}}_- + \hat{\tau}^{\mathcal{H}} \,, \tag{2.129}$$

$$\hat{g}^{(1)}_{ij} \sim \hat{\Phi}^{\mathcal{H}}_- + \hat{G}^{\mathcal{H}}_- + \qquad + \hat{g}^{\mathcal{H}}_{\{ij\}} \,, \tag{2.130}$$

$$\hat{G}^{(1)}_+ \sim \hat{\Phi}^{\mathcal{H}}_- + \hat{G}^{\mathcal{H}}_- + \hat{\tau}^{\mathcal{H}} + \qquad + \hat{G}^{\mathcal{H}}_+ \,, \tag{2.131}$$

$$(\hat{\Phi}^{-1}_+)^{(1)} \sim \hat{\Phi}^{\mathcal{H}}_- + \hat{G}^{\mathcal{H}}_- + \hat{\tau}^{\mathcal{H}} + \hat{g}^{\mathcal{H}}_{\{ij\}} + \hat{G}^{\mathcal{H}}_+ + (\hat{\Phi}^{-1}_+)^{\mathcal{H}} \,. \tag{2.132}$$

Notice that in terms of the fields $\hat{\phi}$, the first-order perturbation takes the simple form $\hat{\phi}^{(1)} \sim \sum_i \hat{\phi}^{\mathcal{H}}_i$. The content of equations (2.127-2.132) is also summarized in Table 2.2.

It is now easy to obtain the radial scaling for the first-order fields, using the results for the harmonic scalings obtained in §2.3.4. Restricting attention henceforth to the non-normalizable modes, we find that the radial scalings and the sizes of the modes at first order are

$$\hat{\phi}^{(1)}(r, \Psi) = \sum_{\phi} \sum_{\Delta(\phi)} c_0^{\Delta(\phi)} \left(\frac{r}{r_\star}\right)^{\Delta(\phi)-4} h_0^{\Delta(\phi)}(\Psi) \,. \tag{2.133}$$

where the first sum runs over contributing fields, and the explicit form of the angular functions $h_0^{\Delta(\phi)}(\Psi)$ can be obtained from the full Green's function analysis.

As an example, Table 2.2 together with equation (2.133) dictates that the solution for the first-order perturbation $\hat{G}_-^{(1)}(r,\Psi)$ takes the form

$$\hat{G}_-^{(1)}(r,\Psi) = \sum_{\Delta(\Phi_-)} \left(c_0^{\Delta(\Phi_-)} \left(\frac{r}{r_\star}\right)^{\Delta(\Phi_-)-4} h_0^{\Delta(\Phi_-)}(\Psi) \right) + \sum_{\Delta(G_-)} \left(c_0^{\Delta(G_-)} \left(\frac{r}{r_\star}\right)^{\Delta(G_-)-4} h_0^{\Delta(G_-)}(\Psi) \right),$$

for some order-unity angular functions $h_0^{\Delta(\Phi_-)}(\Psi)$, $h_0^{\Delta(G_-)}(\Psi)$.

Radial Scalings at 1^st Order

	Φ_-	G_-	τ	g	G_+	Φ_+^{-1}
$\Phi_-^{(1)}$	✓					
$G_-^{(1)}$	✓	✓				
$\tau^{(1)}$	✓	✓	✓			
$g^{(1)}$	✓	✓		✓		
$G_+^{(1)}$	✓	✓	✓		✓	
$(\Phi_+^{-1})^{(1)}$	✓	✓	✓	✓	✓	✓

Table 2.2: In this table we summarize the contents of equations (2.127-2.132). The fields in the leftmost column label the first-order modes in equations (2.127-2.132), while the fields in the shaded top row label the homogeneous modes. A checkmark (✓) indicates that the first-order mode receives a contribution with the corresponding homogeneous scaling, while an empty space indicates that no such scaling is present.

One can perform a similar exercise for the second-order perturbations. We omit the derivation and present the results in Table 2.3.

2.5.2 Higher-order scalings

We now go on to prove that the n-th order perturbation scales as a sum of products of n harmonic modes, as in (2.122). To see this, we introduce new coordinates $\hat{y}^{\hat{m}} = (\hat{r}, \hat{\Psi}^i)$ related to the coordinates $y^m = (r, \Psi^i)$ through

$$\hat{r} = \ln r\,, \quad \hat{\Psi}^i = \Psi^i\,. \tag{2.134}$$

Radial Scalings at 2^{nd} Order

$\Phi_-^{(2)}$	Φ_-	G_-	τ	g	G_+	Φ_+^{-1}	$G_-^{(2)}$	Φ_-	G_-	τ	g	G_+	Φ_+^{-1}
Φ_-	✓	✓		✓			Φ_-	✓	✓	✓	✓	✓	✓
G_-	✓	✓					G_-	✓	✓	✓	✓	✓	✓
τ							τ	✓	✓				
g	✓						g	✓	✓				
G_+							G_+	✓	✓				
Φ_+^{-1}							Φ_+^{-1}	✓	✓				
$\tau^{(2)}$	Φ_-	G_-	τ	g	G_+	Φ_+^{-1}	$g^{(2)}$	Φ_-	G_-	τ	g	G_+	Φ_+^{-1}
Φ_-	✓	✓	✓	✓	✓	✓	Φ_-	✓	✓	✓	✓	✓	✓
G_-	✓	✓	✓	✓	✓	✓	G_-	✓	✓	✓	✓	✓	✓
τ	✓	✓	✓	✓			τ	✓	✓	✓			
g	✓	✓	✓				g	✓	✓		✓		
G_+	✓	✓					G_+	✓	✓				
Φ_+^{-1}	✓	✓					Φ_+^{-1}	✓	✓				
$G_+^{(2)}$	Φ_-	G_-	τ	g	G_+	Φ_+^{-1}	$(\Phi_+^{-1})^{(2)}$	Φ_-	G_-	τ	g	G_+	Φ_+^{-1}
Φ_-	✓	✓	✓	✓	✓	✓	Φ_-	✓	✓	✓	✓	✓	✓
G_-	✓	✓	✓	✓	✓	✓	G_-	✓	✓	✓	✓	✓	✓
τ	✓	✓	✓	✓	✓		τ	✓	✓	✓	✓	✓	✓
g	✓	✓	✓				g	✓	✓	✓	✓	✓	✓
G_+	✓	✓	✓				G_+	✓	✓	✓	✓	✓	✓
Φ_+^{-1}	✓	✓					Φ_+^{-1}	✓	✓	✓	✓	✓	✓

Table 2.3: At second order, the perturbation $\hat{\phi}^{(2)}$ of a canonically normalized field $\hat{\phi}$ has the radial scaling of a sum of products of two canonically normalized homogeneous modes, i.e. $\hat{\phi}^{(2)} \sim \sum_{ij} \hat{\phi}_i^{\mathcal{H}} \hat{\phi}_j^{\mathcal{H}}$, where the range of ij is read off from the above table. The shaded rows label $\hat{\phi}_i$, the shaded columns label $\hat{\phi}_j$, and the fields inside white spaces label $\hat{\phi}$. For intersections indicated by a check mark (✓), the corresponding term is present in the sum, while for an empty space, no such term is present.

These coordinates are convenient because when taking derivatives with respect to them we do not change the scaling with r, i.e. $\partial_{\hat{m}} \phi \sim \phi$. This is obvious for angular derivatives, while for radial derivatives it follows from $\frac{\partial}{\partial \hat{r}} = r \frac{\partial}{\partial r}$.

When a tensor is expressed in this basis, the radial components and the angular components scale in the same way since $\mathrm{d}\hat{r} = \frac{\mathrm{d}r}{r}$, e.g. $(\hat{G}_\pm)_{\hat{r}jk} \sim (\hat{G}_\pm)_{ijk} \sim \hat{G}_\pm$. Furthermore, we observe that for all non-zero background fields $\phi^{(0)}$, the corresponding hatted variables $\hat{\phi}^{(0)}$ are of order unity in the entire background throat solution:

$$\hat{\phi}^{(0)} \sim r^0 \,. \tag{2.135}$$

The equations of motion (3.9-3.13) now take the form

$$r^{-4}\,(\hat{\nabla}^2 + 5\,\hat{g}^{\hat{r}\hat{m}}\,\partial_{\hat{m}})(r^4\,\hat{\Phi}_\pm) = \frac{(\hat{\Phi}_+ + \hat{\Phi}_-)^2}{96\,\mathrm{Im}\,\hat{\tau}}|\hat{G}_\pm|^{\hat{2}} + \frac{2}{(\hat{\Phi}_+ + \hat{\Phi}_-)}|\hat{\nabla}\hat{\Phi}_\pm|^{\hat{2}}\,, \tag{2.136}$$

$$\mathrm{d}\hat{\Lambda} + \frac{i}{2\,\mathrm{Im}\,\hat{\tau}}\mathrm{d}\hat{\tau} \wedge (\hat{\Lambda} + \bar{\hat{\Lambda}}) = 0\,, \tag{2.137}$$

$$\mathrm{d}\hat{G}_3 = -\mathrm{d}\hat{\tau} \wedge H_3\,, \tag{2.138}$$

$$(\hat{\nabla}^2 + 5\,\hat{g}^{\hat{r}\hat{m}}\,\partial_{\hat{m}})\,\hat{\tau} = \frac{\hat{\nabla}\hat{\tau}\,\hat{\cdot}\,\hat{\nabla}\hat{\tau}}{i\mathrm{Im}(\hat{\tau})} + \frac{\hat{\Phi}_+ + \hat{\Phi}_-}{48i}\,\hat{G}_+\,\hat{\cdot}\,\hat{G}_-\,, \tag{2.139}$$

$$\begin{aligned}\hat{R}^6_{\hat{m}\hat{n}} + \hat{\Xi}_{\hat{m}\hat{n}} = {} & \frac{\hat{\nabla}_{(\hat{m}}\hat{\tau}\,\hat{\nabla}_{\hat{n})}\bar{\hat{\tau}}}{2\,(\mathrm{Im}\,\hat{\tau})^2} + \frac{2}{(\hat{\Phi}_+ + \hat{\Phi}_-)^2}\hat{\nabla}_{(\hat{m}}\hat{\Phi}_+\hat{\nabla}_{\hat{n})}\hat{\Phi}_- \\ & - \frac{\hat{\Phi}_+ + \hat{\Phi}_-}{32\,\mathrm{Im}\,\hat{\tau}}\left(\hat{G}_{+\,(\hat{m}}{}^{\hat{p}\hat{q}}\,\bar{\hat{G}}_{-\,\hat{n})\,\hat{p}\hat{q}} + \hat{G}_{-\,(\hat{m}}{}^{\hat{p}\hat{q}}\,\bar{\hat{G}}_{+\,\hat{n})\,\hat{p}\hat{q}}\right)\,.\end{aligned} \tag{2.140}$$

In the above equations, a hat over a contraction, a modulus-squared, or a raised index indicates use of the metric $\hat{g}_{\hat{m}\hat{n}}$. Moreover, the Ricci tensor $\hat{R}^6_{\hat{m}\hat{n}}$ and all derivative operators $\hat{\nabla}_{\hat{m}}$ are constructed using the metric $\hat{g}_{\hat{m}\hat{n}}$. Furthermore, $\hat{\Xi}_{\hat{m}\hat{n}}$ represents the term generated by performing the conformal transformation from R^6_{mn} to $\hat{R}^6_{\hat{m}\hat{n}}$, which involves derivatives of the coordinate $\hat{r}$:

$$\begin{aligned}\hat{\Xi}_{\hat{m}\hat{n}} &\equiv -4\,\hat{\nabla}_{\hat{m}}\hat{\nabla}_{\hat{n}}\,\hat{r} - \hat{g}_{\hat{m}\hat{n}}\,\hat{\nabla}^2\,\hat{r} + 4\hat{\nabla}_{\hat{m}}\hat{\nabla}_{\hat{n}}\,\hat{r} - 4\,\hat{g}_{\hat{m}\hat{n}}\,\hat{g}^{\hat{p}\hat{q}}\,\hat{\nabla}_{\hat{p}}\,\hat{r}\hat{\nabla}_{\hat{q}}\,\hat{r} & (2.141)\\ &= 4\,\hat{\Gamma}^{\hat{r}}_{\hat{m}\hat{n}} + \hat{g}_{\hat{m}\hat{n}}\,\hat{g}^{\hat{p}\hat{q}}\,\hat{\Gamma}^{\hat{r}}_{\hat{p}\hat{q}} + 4\,\delta^{\hat{r}}_{\hat{m}}\,\delta^{\hat{r}}_{\hat{n}} + 4\,\hat{g}_{\hat{m}\hat{n}}\,\hat{g}^{\hat{r}\hat{r}}\,, & (2.142)\end{aligned}$$

where $\hat{\Gamma}^{\gamma}_{\hat{m}\hat{n}}$ is the Christoffel connection constructed from $\hat{g}_{\hat{m}\hat{n}}$. Finally, we have also defined

$$\hat{\Lambda} = \hat{\Phi}_-\hat{G}_+ + \hat{\Phi}_+\hat{G}_-\,. \tag{2.143}$$

From the form (2.136-2.140) of the supergravity equations in terms of the hatted fields and coordinates, we can deduce the desired result (2.122). Because all background fields $\hat{\phi}^{(0)}$ scale as r^0, all derivatives $\hat{\nabla}$ are logarithmic, and no coefficient in the equations depends on r, the n-th order perturbation $\hat{\phi}^{(n)}$ will inherit its radial scaling exclusively from the other perturbations. That is, if one were to expand any of the equations (2.136-2.140) to n-th order, then matching the radial scalings on either side of the equation one would find a relation of the form

$$\hat{\phi}^{(n)} \sim \sum_i \hat{\phi}_i^{(n)} + \sum_{p=1}^{n} \sum_{i,j} \hat{\phi}_i^{(p)} \hat{\phi}_j^{(n-p)} + \sum_{p,q=1}^{n} \sum_{i,j,k} \hat{\phi}_i^{(p)} \hat{\phi}_j^{(q)} \hat{\phi}_j^{(n-p-q)} + \dots \tag{2.144}$$

where the sums run over whichever fields appear in the equation under consideration. We have seen that the scalings of all the fields at linear order are given by the scalings of the homogeneous modes. Therefore, by iteratively applying equation (2.144), we deduce that n-th order perturbations scale as

$$\begin{aligned}\hat{\phi}^{(n)}(r,\Psi) &= \sum_{i_1,\dots,i_n} c_0^{\Delta(i_1)} \cdots c_0^{\Delta(i_n)} \left(\frac{r}{r_\star}\right)^{\Delta(i_1)+\dots+\Delta(i_n)-4n} \times h^{\Delta(i_1)\dots\Delta(i_n)}(\Psi)\,, \\ &\equiv \sum_{i_1,\dots,i_n} \hat{\phi}_0^{\Delta(i_1)}(r) \cdots \hat{\phi}_0^{\Delta(i_n)}(r) \times h^{\Delta(i_1)\dots\Delta(i_n)}(\Psi)\,,\end{aligned} \tag{2.145}$$

where the $h^{\Delta(i_1)\dots\Delta(i_n)}(\Psi)$ are angular functions that are of order unity at generic points in the angular space, and we have defined the running couplings

$$\hat{\phi}_0^{\Delta(\phi)}(r) \equiv c_0^{\Delta(\phi)} \left(\frac{r}{r_\star}\right)^{\Delta(\phi)-4}. \tag{2.146}$$

The formula (3.28) is one of our main results. It states that in the basis specified in (2.126), the size of the n-th order perturbation of any field $\hat{\phi}$ can be read off in terms of the sizes c_0^{Δ} of all the homogeneous modes at $r = r_\star$, and the dimensions Δ characterizing the spectrum of Kaluza-Klein masses. That corrections at order n are proportional to degree n products of the perturbation parameters is of course not surprising. However, equation (3.28) says more than this: it shows that in solving the equations of motion, no addition radial scaling is introduced that would affect the sizes of the corrections: the sizes of the n-th order inhomogeneous corrections at some point in the throat are immediately determined by n-th order products of harmonic modes *at that point*. It follows that throat perturbation theory is naturally organized as an expansion in the running sizes of the harmonic modes, and the expansion is convergent as long as the seeding harmonic modes are small.

2.5.3 Conditions for consistency

We now turn to explaining why our perturbative expansion can consistently describe a warped throat, despite the presence of relevant perturbations. On general grounds, one might expect the boundary conditions on the UV brane to activate all possible modes, with coefficients that are not much smaller than unity. In particular, any relevant modes will grow toward the infrared, and, given enough range of renormalization group evolution, would ultimately become large and destroy the IR region of the throat. This is a critical issue not just for our perturbation scheme, but for the existence of metastable vacua in which antibranes break supersymmetry. If effects in the bulk induce corrections to the throat geometry that grow precipitously large in the IR, then the vacuum energy of an antibrane at the tip of the throat is poorly approximated by the antibrane action in the uncorrected background,

$$V^{(0)}_{\overline{\mathrm{D3}}} = T_3\,\Phi_+^{(0)}\,, \tag{2.147}$$

and the vacuum energy will in general not remain small in string units, so that the compactification will be destabilized. This fundamental requirement that effects in the bulk do not destabilize the throat, and with it the entire compactification, therefore implies the existence of a perturbative expansion around a background throat geometry. Our task is to assess whether this requirement can be met without undue fine tuning.

In a finite warped throat, the hierarchy of scales is finite, so that if every relevant mode has a sufficiently small coefficient in the UV, all perturbations will remain small throughout the throat. If effects in the bulk source some relevant mode

$$\phi^{\mathcal{H}} = c_{\rm UV}^{\Delta} \left(\frac{r_\star}{r_{\rm UV}}\right)^{\Delta-4} \tag{2.148}$$

with $\Delta < 4$, then this mode will become dangerously large at the tip of the throat, $r = r_{\rm IR}$, if

$$c_{\rm UV}^{\Delta} \left(\frac{r_{\rm IR}}{r_{\rm UV}}\right)^{\Delta-4} \gtrsim 1\,. \tag{2.149}$$

Thus, using $\frac{r_{\rm IR}}{r_{\rm UV}} \sim e^{A_{\rm min}} \equiv a_0$, we see that the size of the mode in the UV must be

$$c_{\rm UV}^{\Delta} \lesssim a_0^{4-\Delta} \tag{2.150}$$

in order for the entire throat to be stable against corrections from this mode.

Let us now discuss the circumstances in which (2.150) can hold for all relevant modes. One obviously sufficient condition arises when there are no relevant modes (i.e. modes with $\Delta < 4$) that are sourced in the bulk. This can occur if an unbroken symmetry, such as supersymmetry, forbids all relevant modes.[5] In fact, a Klebanov-Strassler throat attached to a supersymmetric, ISD flux compactification is stable against compactification effects, because every relevant mode either violates the ISD conditions or violates the supersymmetry[6] of the

[5] See [25] for a construction utilizing discrete symmetries to protect a non-supersymmetric throat.

[6] To be precise, there are relevant perturbations that are consistent with four-dimensional $\mathcal{N} = 1$ supersymmetry, but the supercharges preserved are different from those preserved by the background.

background throat geometry. Thus, in a supersymmetric, ISD compactification, the existence of a Klebanov-Strassler throat does not require any unnatural fine-tuning of relevant perturbations.

However, in the same example there exist relevant modes that are incompatible with the supersymmetry of the background throat, but could be sourced by supersymmetry-breaking effects, e.g. by distant antibranes, fluxes, or nonperturbative effects. Thus, one should ask whether supersymmetry breaking in the compact space tends to induce perturbations that destroy the IR region of the throat.

Before proceeding, we emphasize that, by construction, in any stabilized vacuum in which an anti-D3-brane in a warped throat makes a dominant contribution to supersymmetry breaking, the scale of the moduli potential and of any bulk sources of supersymmetry breaking must obey

$$V_{bulk} \lesssim 2\, a_0^4\, T_3 \tag{2.151}$$

lest the supersymmetry-breaking energy drive decompactification. Crucially, this relationship links the scale of supersymmetry-breaking bulk perturbations to the IR scale of the throat. Arranging this near-equality between disparate contributions – e.g., anti-D3-brane supersymmetry breaking and gaugino condensation on D7-branes – obviously requires a degree of fine-tuning. We are asking whether *further* fine-tuning is required to subdue instabilities associated with relevant perturbations of the throat that are sourced in the bulk.[7]

If the scale of bulk supersymmetry breaking obeys (2.151), then every

[7]To differentiate these issues, imagine two warped throat backgrounds A, B with identical IR scales, with A admitting a large number of relevant modes, and B having no relevant modes whatsoever. Arranging for (2.151) to hold requires fine-tuning in either case, but throat A is vulnerable to large corrections from relevant modes sourced in the bulk, while B is not.

supersymmetry-breaking perturbation has a small coefficient, which by (2.151) can be expressed in terms of the IR scale a_0 of the throat. The particular power of a_0 multiplying a given mode,

$$\phi \propto a_0^Q \tag{2.152}$$

can be obtained by a spurion analysis, as in [12].

The dangerous modes in a general throat can be extracted by examining the homogeneous solutions presented in §2.3 (cf. Table 2.1). We easily see that the fields Φ_-, G_3, and $g_{\{ij\}}$ can all possess relevant (i.e. $\Delta \leq 4$) modes, while all modes of the remaining supergravity fields are irrelevant. Evidently, a throat is robust if

$$Q > 4 - \Delta \tag{2.153}$$

for all modes of Φ_-, G_3, and $g_{\{ij\}}$.

Let us now verify that the Klebanov-Strassler throat obeys (2.153), using the spectroscopic data for $T^{1,1}$ obtained in [19, 20, 12, 21]. First, as explained in [12], the harmonic modes of Φ_- have $Q = 4$, while G_3 perturbations that are not purely ISD have $Q = 2$, corresponding to double and single insertions, respectively, of the supersymmetry-breaking spurion $F_X \propto a_0^2$. As the lowest-dimension mode of flux has $\Delta = 5/2 > 2$, perturbations of Φ_- and G_3 are harmless. Finally, the two relevant modes of $g_{\{ij\}}$ with $\Delta = 2,\ 3$ are the bottom components of supermultiplets, and hence have $Q = 4$, completing the proof. Extending this argument to more general throats would be straightforward given the necessary spectroscopic data, but is beyond the scope of this work.

The arguments above refer only to harmonic modes. One might have worried that even if all harmonic modes remain small down to the tip, the solutions for the inhomogeneous modes could have scalings that are even more relevant than those of the harmonic modes. In fact, this is not a problem: our result (3.28) makes it evident that whenever the harmonic modes are small, the expansion is convergent. As we have just presented a spurion argument that shows that the harmonic modes remain small in a Klebanov-Strassler throat attached to a compactification with weakly broken supersymmetry, it follows that a consistent perturbation expansion exists in such a throat.

2.5.4 Truncation of the expansion: a worked example

The preceding sections have provided a perturbative solution near some location of interest, $r_\star$, in a *double* expansion in terms of a_0 and $r_\star/r_{UV}$. (In particular, the parametric sizes of the c_I^Δ can be expressed in terms of a_0 and $r_\star/r_{UV}$.) To make use of such a solution, we must consistently truncate the double expansion to some desired accuracy. The simplest way to accomplish this is to specify the relative sizes of the two expansion parameters,

$$\frac{r_\star}{r_{UV}} \sim a_0^P\,, \tag{2.154}$$

for some $P \in (0, 1]$, so that in practice there is a single expansion parameter, taken to be a_0 in the above. Then, if the size of some mode in the UV is

$$\phi^{\rm UV} \sim a_0^{Q_i}\,, \tag{2.155}$$

the size of the mode at $r = r_\star$ is

$$c_0^\Delta \equiv \phi(r_\star) \sim a_0^{Q_i}\left(\frac{r_\star}{r_{\rm UV}}\right)^{\Delta-4} \sim a_0^{Q_i+(\Delta-4)\cdot P}\,. \tag{2.156}$$

Truncation is then straightforward.

We will illustrate the necessary steps in the concrete example of the region near the tip of a Klebanov-Strassler throat, where $\frac{r_\star}{r_{\rm UV}} \sim a_0$, so that $P = 1$.[8] Suppose that we are interested in going up to an accuracy $\sim a_0^{1.5}$. The most relevant scalings of each field are [19, 20, 12, 21]

$$\Phi_- : \; \Delta_{\Phi_-} = 1.5, \; \ldots \tag{2.157}$$

$$G_- : \; \Delta_{G_-} = 2.5, \; 3, \; 3.5, \; \ldots \tag{2.158}$$

$$\tau : \; \Delta_\tau = 4 + \Delta_{\Phi_-} \tag{2.159}$$

$$g_{mn} : \; \Delta_g = 2, \; 3, \; 5.29, \; \ldots \tag{2.160}$$

$$G_+ : \; \Delta_{G_+} = \Delta_{G_-}, \; \ldots \tag{2.161}$$

$$\Phi_+^{-1} : \; \Delta_{\Phi_+} = 8, \; \ldots \tag{2.162}$$

Now we need the sizes of the modes in the UV. We have already seen that Φ_- scales as a_0^4 in the UV, while G_- scales as a_0^2. The relevant modes of G_+ come paired with modes of G_- and thus inherit the a_0^2 scaling. We have already shown that the two relevant modes of the metric scale like a_0^4 in the UV. The $\Delta_g = 5.29$ mode of the metric, the leading mode of τ, and the $\Delta_{\Phi_+} = 8$ mode of Φ_+ are all allowed by supersymmetry and the ISD conditions, and are therefore of order unity in the UV.

With the above data, we can estimate the sizes of the modes at the tip in terms of a_0. We find that the leading homogeneous modes of each field have scalings[9]

$$\hat{\Phi}_- \sim a_0^{1.5}, \tag{2.163}$$

$$\hat{G}_- \sim a_0^{0.5}, a_0^{1.0}, a_0^{1.5}, \tag{2.164}$$

[8] For simplicity we will neglect perturbations generated in the IR, even when studying the tip region. This is consistent, for example, if we are investigating the potential along a direction corresponding to an isometry preserved by the deformation of the tip, as in [10].

[9] One must be careful to compare the scaling of the hatted fields, as these modes are the proper perturbation variables.

$$\hat{\tau} \sim a_0^{1.5}\,, \tag{2.165}$$

$$\hat{g}_{mn} \sim a_0^{1.29}\,, \tag{2.166}$$

$$\hat{G}_+ \sim a_0^{0.5}, a_0^{1.0}, a_0^{1.5}\,, \tag{2.167}$$

$$\hat{\Phi}_+^{-1} \sim a_0^{8}\,. \tag{2.168}$$

Notice that there is a hierarchy between the various modes and therefore it would be inconsistent to truncate at the same order in each. To reach the desired accuracy of $a_0^{1.5}$, one considers combinations of the above modes whose net size is at least $a_0^{1.5}$, taking into account the restrictions presented in Tables 2.2 and 2.3. For example, the mode of $\hat{G}_-$ scaling as $a_0^{0.5}$ and the mode of $\hat{G}_+$ scaling as $a_0^{1.0}$ present a possible contribution. Consulting Table 2.3, we find that this combination of homogeneous modes can source second-order perturbations of all fields except Φ_-.

2.6 Chapter Summary

We have developed a method that yields local solutions of type IIB supergravity to any desired order in an expansion around a warped Calabi-Yau cone. Our approach relies on the observation that the equations of motion expanded to any order in perturbations around a background with ISD fluxes are easily disentangled. Specifically, we identified a basis of fields in which the equations for the n-th order perturbations take a triangular form. As a result, one can write down a Green's function solution to any desired order in a purely algebraic way. This is a striking simplification, as in expansion around a general background the equations of motion are typically intractably coupled.

Next, we obtained all necessary Green's functions, as functions of the angular harmonics on the Sasaki-Einstein base of the cone. For cones with known harmonics, such as the conifold, it is straightforward to obtain explicit solutions using the tools presented herein. We also presented a simple expression for the radial scaling of a general n-th order perturbation, so that the size of any desired perturbation is readily estimated. Our result demonstrates that the sizes of the harmonic modes at a given point in the throat serve as faithful expansion parameters. For the case of a Klebanov-Strassler throat attached to a KKLT compactification, we showed that our expansion is convergent above the tip, and we provided a spectroscopic criterion for assessing convergence in a more general throat.

We anticipate that our results will have applications to local model-building in flux compactifications of type IIB string theory. Our tools simplify the task of characterizing the effective action of a sector of fields localized on D-branes in a throat region, which is a common problem in the study of local models of particle physics and of inflation. In addition, the methods presented here could be useful in the study of the long-distance supergravity solutions induced by supersymmetry breaking on anti-D3-branes. Previous attempts in each direction have required considerable ingenuity in the choice of ansatz and the basis of fields, and in most cases it has not been evident whether one could in practice proceed to higher order. Our purely algebraic approach yields a solution to any desired order in terms of a single set of Green's functions.

A second application is to the construction of non-supersymmetric AdS/CFT dual pairs. Taking a supersymmetric warped Calabi-Yau cone as the background, one can construct families of non-supersymmetric solutions to any desired order in the supersymmetry breaking parameter, as functions of the harmonics on the base. This provides the prospect of exploring new aspects of non-supersymmetric, strongly coupled, approximately conformal field theories.[10]

The principal limitation of our approach is that the Green's functions and separable solutions that we have provided apply only in the approximately-AdS region of a warped Calabi-Yau cone. The triangular structure of the equations of motion, however, is far more general, applying in expansion around any conformally Calabi-Yau flux compactification. Extending our methods to more general supergravity backgrounds is a very interesting question for the future.

[10]Solutions making use of the expansion in $r_\star/r_{UV}$ would be dual to effective conformal field theories, in the spirit of [26].

CHAPTER 3

STABILIZING ANTIBRANE PSEUDOMODULI

Supersymmetry breaking by antibranes is a key element in many attempts to construct de Sitter vacua in string theory. In the type IIB compactifications proposed by KKLT [6], an anti-D3-brane at the tip of a Klebanov-Strassler throat region [9] provides controllably small supersymmetry breaking [27]. The angular coordinates of such an anti-D3-brane correspond to light moduli in the effective theory, and could have important phenomenological consequences. In this work we determine the leading contributions to the masses of these open string moduli.

The angular coordinates of an anti-D3-brane on the S^3 at the tip of a noncompact Klebanov-Strassler geometry are massless moduli: the noncompact throat enjoys exact isometries that translate the anti-D3-brane. These isometries are necessarily broken when the throat is glued into a compact bulk, so that the angular position moduli of the anti-D3-brane receive mass from what we may term 'compactification effects'. Aharony, Antebi and Berkooz (AAB) [10] argued that the resulting mass could be computed by determining the lightest Kaluza-Klein modes of $T^{1,1}$, corresponding to the most relevant perturbations of the dual CFT Lagrangian, and identifying the subset of perturbations that lift the anti-D3-brane moduli space. The lowest Kaluza-Klein mass (or equivalently, lowest operator dimension) could then provide a parametric estimate of the mass of the anti-D3-brane moduli.

In the analysis of AAB, perturbations lifting the moduli space of a probe D3-brane were forbidden. We will argue that this restriction, while appropriate for a no-scale flux compactification along the lines of [5], should not be imposed in a KKLT compactification. Moreover, we will find that relaxing this restriction, i.e. allowing perturbations that produce a force on a probe D3-brane, introduces a new contribution to the anti-D3-brane potential that is parametrically larger than the leading contribution obtained in prior work.

In the no-scale compactifications of [5], a D3-brane feels no force at tree level, and is free to explore the entire compactification without energy cost. However, the inclusion of an anti-D3-brane in a no-scale compactification leads to runaway decompactification. Therefore, any discussion of a metastable state involving anti-D3-branes must be in the context of a compactification with stabilized Kähler moduli, not a no-scale compactification.

The effective action for D3-branes in compactifications whose Kähler moduli are stabilized by nonperturbative effects is by now well understood [28, 29, 11, 12]. Notably, when nonperturbative effects stabilize the Kähler moduli, the D3-brane moduli space is lifted by these same effects. Thus, when studying the effective action of an anti-D3-brane in a stabilized compactification, one should not forbid perturbations to the supergravity solution on the sole ground that these perturbations lift the D3-brane moduli space: such lifting is generic in nonperturbatively-stabilized compactifications.

In this chapter, we obtain the potential for a probe[1] anti-D3-brane in a systematic expansion around a Klebanov-Strassler background. We allow arbitrary perturbations that lift the moduli space of a probe D3-brane, but we perform a spurion analysis that reflects the controllably small differences between a no-scale compactification and a KKLT compactification, at the level of the ten-dimensional solution. Our analysis is nonlinear in the perturbations sourced

[1]For advances in understanding the backreaction of an anti-D3-brane, see [30, 31, 32].

in the bulk of the compactification, and we find important contributions at quadratic order in perturbations.

We find that the dominant contributions to the anti-D3-brane potential are mediated by three-form flux, and arise from terms that are absent in the no-scale limit. The mass induced by fluxes is parametrically larger than the mass allowed in the no-scale limit. Nevertheless, the central qualitative conclusion of AAB is unchanged: the anti-D3-brane mass is small compared to the natural scale at the tip of the warped throat. We conclude that fluxes sourced in the bulk make the dominant contribution to the potential for angular motion, but there are light open string moduli in the four-dimensional theory.

In the course of our analysis, we obtain a technical result that could be of independent interest: we compute the spectrum of Kaluza-Klein excitations of the metric on $T^{1,1}$. In contrast to the 'method of exhaustion' used in the seminal works [19, 20], in which the spectrum of metric perturbations was inferred from the spectrum of more readily computed modes using superconformal symmetry, we directly compute the spectrum of the Lichnerowicz operator acting on symmetric two-tensors. Our results are consistent with small corrections to the results of [19, 20], for certain modes of the metric and three-form flux, pointed out in [12].

3.1 Preliminaries

We begin by describing the class of flux compactifications of interest, and then present the anti-D3-brane potential that will be our primary focus.

3.1.1 Equations of motion

Following [5],[2] we study field configurations of the form

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}g_{mn}dy^m dy^n\,, \tag{3.1}$$

$$\tilde{F}_5 = (1+\star_{10})d\alpha(y)\wedge dx^0\wedge dx^1\wedge dx^2\wedge dx^3\,, \tag{3.2}$$

$$G_{mnl} = G_{mnl}(y)\,,\quad m,n,l = 4,\ldots 9\,, \tag{3.3}$$

$$G_{\mu\nu\sigma} = 0\,,\quad \mu,\nu,\sigma = 0,\ldots 3\,, \tag{3.4}$$

$$\tau = \tau(y)\,. \tag{3.5}$$

It is convenient to define the quantities

$$G_\pm \equiv (\star_6 \pm i)G_3\,, \tag{3.6}$$

$$\Phi_\pm \equiv e^{4A}\pm\alpha\,, \tag{3.7}$$

$$\Lambda \equiv \Phi_+ G_- + \Phi_- G_+\,, \tag{3.8}$$

where we will refer to G_+ and G_- as imaginary self-dual (ISD) flux and imaginary anti-self-dual (IASD) flux, respectively. The equations of motion and Bianchi identities are

$$\nabla^2\Phi_\pm = \frac{(\Phi_+ + \Phi_-)^2}{96\,\mathrm{Im}\,\tau}|G_\pm|^2 + \frac{2}{\Phi_+ + \Phi_-}|\nabla\Phi_\pm|^2\,, \tag{3.9}$$

$$d\Lambda = -\frac{i}{2\,\mathrm{Im}\,\tau}d\tau\wedge(\Lambda+\bar{\Lambda})\,, \tag{3.10}$$

$$d(G_3) = -d(\tau H_3)\,, \tag{3.11}$$

$$\nabla^2\tau = \frac{\nabla\tau\cdot\nabla\tau}{i\mathrm{Im}(\tau)} + \frac{\Phi_+ + \Phi_-}{48i}G_+\cdot G_-\,, \tag{3.12}$$

$$R^6_{mn} = \frac{\nabla_{(m}\tau\nabla_{n)}\bar{\tau}}{2(\mathrm{Im}\,\tau)^2} + \frac{2}{(\Phi_+ + \Phi_-)^2}\nabla_{(m}\Phi_+\nabla_{n)}\Phi_- - \frac{\Phi_+ + \Phi_-}{32\,\mathrm{Im}\,\tau}\Big(G_{+(m}{}^{pq}\bar{G}_{-n)pq} + G_{-(m}{}^{pq}\bar{G}_{+n)pq}\Big), \tag{3.13}$$

where ∇^2 is constructed from g_{mn}, and we have omitted all contributions from localized sources.

[2]Our notation matches that of [5], except that $g^{\mathrm{here}}_{mn} = \tilde{g}^{\mathrm{there}}_{mn}$.

3.1.2 Background solution and perturbations

Our goal is to study an anti-D3-brane at the tip of a Klebanov-Strassler throat[3] region in a compactification of type IIB string theory with stabilized closed string moduli. Such a configuration is accurately modeled by an anti-D3-brane at the tip of a finite region of a noncompact Klebanov-Strassler geometry, subject to appropriate boundary conditions in the ultraviolet that encode the effects of sources in the compact bulk [10, 12, 33]. These boundary conditions induce perturbations of the supergravity fields within the throat.

Approximations for a finite throat

Concretely, we will study a finite segment of the infinite throat solution, subject to arbitrary non-normalizable[4] deformations. A striking simplification in this approach is that solutions that are completely general in the ultraviolet are well-approximated in the infrared by the handful of modes that diminish least rapidly for small values of the radial coordinate. (In the dual field theory, only the most relevant operators are important in the extreme infrared.) After discarding subleading modes, one is left with solutions parameterized by a finite set of coefficients. The precise values of these coefficients of course depend on the details of the entire compactification. However, one can make parametric estimates of these coefficients, leading to an estimate for the mass scale of the antibrane moduli.

The fact that we are only interested in the order of magnitude of the cor-

[3]The analysis presented in this paper could be extended to other types of throats given sufficient knowledge of the spectroscopy of the Sasaki-Einstein base of the corresponding cone.

[4]One could easily incorporate normalizable perturbations, corresponding to effects generated in the infrared (see [33]), but this is not of interest for the present work.

rections at the tip of the throat — concretely, we keep track only of powers of the warp factor — greatly simplifies matters. Most notably, we can approximate the Klebanov-Strassler geometry by a simpler warped solution. For the vast majority of the range of scales over which the corrections are to be run, the Klebanov-Strassler solution is well-approximated by the Klebanov-Tseytlin solution [34]. We may therefore study a Klebanov-Tseytlin throat that is cut off at a finite minimal value of the radial coordinate $r_{\rm IR} \sim a_0$. Moreover, logarithms make negligible corrections to our scaling estimates, and may be neglected. This leads to an even simpler model: the warped conifold solution,

$$\mathrm{d}s_6^2 = e^{-2A(y)} g_{mn} \mathrm{d}y^m \mathrm{d}y^n = r^4 \Big(\mathrm{d}r^2 + r^2\, \tilde{g}_{ij} \mathrm{d}\Psi^i \mathrm{d}\Psi^j\Big)\,, \tag{3.14}$$

truncated in the infrared at $r_{\rm IR} \sim a_0$. We have denoted by a_0 the minimum value of the warp factor, $a_0 \equiv e^{A_{\rm min}}$. Denoting by $r_{\rm UV}$ the radial location at which the throat is glued into the bulk, the tip is located at $r_{\rm IR} \sim a_0\, r_{\rm UV}$. For simplicity, we choose our units in this work such that $r_{\rm UV} = 1$, so that $r_{\rm IR} \sim a_0$. `fix me`

Here $\tilde{g}_{ij}$ is the metric on $T^{1,1}$, we use Ψ^i to denote the angular coordinates, and we use letters i, j, k to refer to indices running over the angular directions. It is important to note that although we use the $AdS_5 \times T^{1,1}$ form for the metric (3.14), we must still *retain the nonzero ISD flux,*

$$G_{ijk} \propto r^0, \quad G_{rij} \propto r^{-1}, \tag{3.15}$$

that occurs in the Klebanov-Tseytlin solution.

We now turn to perturbations of this background solution. It is useful to classify these perturbations according to whether they are allowed or forbidden in the no-scale compactifications of [5].

ISD perturbations

In the no-scale solutions of [5], the background fields satisfy the *ISD conditions*

$$G_- = 0\,, \tag{3.16}$$

$$\Phi_- = 0\,. \tag{3.17}$$

The Klebanov-Strassler geometry is itself an ISD solution.

Consider a finite Klebanov-Strassler throat attached to a no-scale compactification, as constructed in [5]. At the ultraviolet end of the throat where the throat is glued into the compact bulk, the metric deviates significantly from that of the noncompact warped deformed conifold, as do other supergravity modes including Φ_+ and G_+. However, as the entire compactification is ISD, the conditions (3.16, 3.17) are exactly satisfied both within the throat and in the gluing region. The deviations of the fields in the throat region from their profiles in the noncompact warped deformed conifold may be termed *ISD perturbations*.

The ISD perturbations in such a throat encode breaking of the continuous isometries of the noncompact throat by attachment to the compact bulk. Correspondingly, the ISD perturbations generically lift the flat directions for angular motion of a probe anti-D3-brane at the tip. However, ISD perturbations are *not* the most general isometry-breaking perturbations found in a metastable compactification containing anti-D3-branes, nor — as we shall see in §3.4 — are they the most important source of mass for the anti-D3-brane moduli.

Non-ISD perturbations

We first remark that compact solutions obeying the ISD conditions are incompatible with the presence of anti-D3-branes. An anti-D3-brane is a localized source for Φ_-, and moreover the inclusion of an anti-D3-brane introduces positive energy density and a decompactification instability: from the DBI and Chern-Simons actions one finds the anti-D3-brane potential

$$V_{\overline{D3}} = T_3 \int \sqrt{g}\, d^4x\, \Phi_+ \,. \tag{3.18}$$

where T_3 is the D3-brane tension.

We will instead study an anti-D3-brane in a KKLT compactification [6].[5] In a KKLT compactification, the ISD conditions (3.16, 3.17) are not satisfied. The nonperturbative effects that stabilize the Kähler moduli — either gaugino condensation on D7-branes, or Euclidean D3-branes — have been shown to source the IASD fields[6] G_- and Φ_- [12]. However, the departures from the conditions (3.16, 3.17) originate in nonperturbative effects, and will therefore be small in any controlled regime. In summary, a throat in a KKLT compactification is generically subject to non-ISD perturbations, but the sizes of these perturbations in the ultraviolet region are controlled by the smallness of the effects breaking no-scale symmetry. We will make this statement precise via a spurion analysis in §3.2.2.

[5]The general methods presented here would be applicable in any similar type IIB flux compactification in which nonperturbative and/or perturbative effects stabilize the Kähler moduli, such as the Large Volume Scenario [7], but for definiteness we will restrict to KKLT compactifications henceforth.

[6]See [35] for original results concerning the backreaction of nonperturbative effects on the metric, and [36] for a discussion of backreaction on fluxes.

3.1.3 The anti-D3-brane potential

In view of (3.18), the anti-D3-brane moduli masses are governed by the scalar Φ_+. In the noncompact background solution,

$$\Phi_+ = \Phi_+^{(0)}(r) = 2\, e^{4A^{(0)}}(r)\,, \tag{3.19}$$

with no dependence on the angular directions. (The superscripts denote the unperturbed background profile.)

At this level of approximation, the moduli corresponding to translations of the antibrane around the three-sphere at the tip of the throat receive no potential. This is not surprising, as the angular coordinates of an anti-D3-brane at the tip of a noncompact Klebanov-Strassler throat are the Goldstone bosons of the spontaneously broken rotational isometry. However, a compact Calabi-Yau possesses no exact continuous isometries, and so deviations from the infinite throat approximation—coming from gluing the throat into a compact bulk, as well as from fluxes, localized sources, and nonperturbative effects in that bulk—should generically lift these moduli.

We therefore consider small perturbations to the zeroth-order background,

$$\Phi_+ = \Phi_+^{(0)} + \delta\Phi_+\,, \tag{3.20}$$

where $\delta\Phi_+$ is the deviation from the infinite throat approximation generated by compactification effects. Suppose that the anti-D3-brane sits at a stable point of the fully corrected potential, and let X^i be canonically normalized coordinates parameterizing small displacements from this point in the angular directions. We wish to determine the mass term for the fields X^i. For this purpose it is convenient to expand the deviation $\delta\Phi_+$ evaluated at the tip of the throat,

$$\delta\Phi_+(r \sim a_0, X^i) = c_0 + c_2\, g_{ij} X^i X^j + \ldots \tag{3.21}$$

The ellipses represent higher terms in the Taylor series that do not contribute to the mass. Substituting into (3.18), we find a mass term

$$\delta V = T_3 \int \sqrt{g}\, d^4x \left\{ c_2\, g_{ij} X^i X^j + \ldots \right\} . \tag{3.22}$$

If we act on (3.21) with the Laplacian constructed from the background metric, $\nabla_0^2 \equiv g_0^{ij} \nabla_i \nabla_j$, we find

$$\nabla_0^2 \delta\Phi_+ \big|_{X^i=0} \sim c_2 \, . \tag{3.23}$$

Our strategy is to estimate the sizes of the corrections to the various supergravity fields at the tip of the throat and then use equation (3.9) to determine the size of c_2. At the tip, where $r \sim a_0$, the sizes of the various modes can be expressed as powers of a_0. It will be convenient to parameterize the contribution of a particular mode to the Laplacian as

$$\nabla_0^2 \delta\Phi_+ \big|_{X^i=0} \sim a_0^{\Delta-2} \, , \tag{3.24}$$

so that the induced mass (in units of the ultraviolet scale r_{UV}^{-1}, which we have set to unity) is

$$m^2 \sim a_0^{\Delta-2} \, , \tag{3.25}$$

by equation (3.22). With this parameterization the leading contribution found by Aharony, Antebi and Berkooz (AAB) [10] was $\Delta = 5.29$. We wish to find any possible larger contributions to the mass not identified in that analysis. Our task is then to enumerate all possible $\Delta < 5.29$.

3.2 Perturbations to the Supergravity Fields

We now determine the leading contributions to Φ_+, and correspondingly to the anti-D3-brane potential, in a stabilized compactification. A systematic approach to computing perturbations of a warped throat background was developed in [33]. In §3.2.1 we briefly summarize this method, referring the reader to [33] for further details. Then, in §3.2.2, we characterize the effective boundary conditions in the ultraviolet region of the throat that are induced by compactification.

3.2.1 Systematic perturbation of warped throats

In [33] we laid out a method for calculating perturbations of the supergravity fields on a Calabi-Yau cone. One begins with the harmonic modes for each field, which are the solutions to the equations obtained by ignoring the right-hand sides of equations (3.9-3.13) and setting the operators on the left-hand sides to zero. One assumes that these modes are turned on with perturbatively small coefficients in the ultraviolet. Then one generates solutions to the full equations of motion (3.9-3.13), to any order n in those small coefficients, by convolving products of n of the harmonic modes with Green's functions that are provided in [33].

We will not need the full machinery of this procedure here. Of present use is a simple formula obtained in [33] that provides the radial scaling of an arbitrary correction. The harmonic modes have simple power-law[7] dependences on the radial coordinate of the throat:

$$\delta\phi^{\rm Harmonic} = \sum_{\Delta(\phi)} c_{\Delta(\phi)} \left(\frac{r}{r_{\rm UV}}\right)^{\Delta(\phi)+\lambda(\phi)-4} h_{\Delta(\phi)}(\Psi)\,, \tag{3.26}$$

where $\delta\phi$ represents a perturbation to any of the supergravity fields, $r_{\rm UV}$ is the scale at which the throat is attached to the bulk, the $\Delta(\phi)$ represent the scaling

[7] In fact, the modes of flux introduce some logarithmic dependence, but these logarithms yield small corrections to the anti-D3-brane mass, and we ignore them throughout this work.

dimensions of the corresponding operators in the gauge theory, and the $h_\Delta(\Psi)$ are order-unity angular harmonics on the base space. The offsets in the exponents, $\lambda(\phi)$, come from rescalings necessary for canonical normalization. The unperturbed profile ϕ_0 of each field takes the form

$$\phi_0 = \alpha \left(\frac{r}{r_{\rm UV}}\right)^{\lambda(\phi)} , \tag{3.27}$$

where α is independent of r. In [33] we demonstrated that the general n-th order correction has the form

$$\delta\phi^{(n)}(r,\Psi) = \sum_{\Delta_1,\ldots,\Delta_n} c_{\Delta_1}\cdots c_{\Delta_n} \left(\frac{r}{r_{\rm UV}}\right)^{\Delta_1+\ldots+\Delta_n+\lambda(\phi)-4n} \times h_{\Delta_1\ldots\Delta_n}(\Psi) , \tag{3.28}$$

where the sum runs over products of n harmonic modes with dimensions Δ_i, and the $h_{\Delta_1\ldots\Delta_n}(\Psi)$ are order-unity angular functions that are determined via the Green's function solution. We will abbreviate the above relationship as

$$\delta\phi^{(n)}(r,\Psi) \sim c_{\Delta_1}\cdots c_{\Delta_n} \left(\frac{r}{r_{\rm UV}}\right)^{\Delta_1+\ldots+\Delta_n+\lambda(\phi)-4n} , \tag{3.29}$$

to signify that the left-hand side has a dependence on r as given by the right-hand side. This result allows us to determine the radial scaling of corrections given the values of Δ for the harmonic modes.

3.2.2 Sizes of perturbations in the ultraviolet

The next step is to estimate the sizes of the Wilson coefficients c_Δ. The coefficients of supersymmetric, ISD modes are of order unity: these modes are unsuppressed in the region where the throat is glued into the bulk [10].

Non-ISD modes are not strictly forbidden, but their coefficients in the ultraviolet will not be of order unity: a KKLT compactification is a small perturbation of a no-scale compactification. Similarly, non-supersymmetric perturbations are not forbidden, but their coefficients must reflect the controllably small breaking of supersymmetry in the compactification. We conclude that modes violating either supersymmetry or the ISD conditions should have suppressed coefficients. An efficient way to organize the suppressed Wilson coefficients is via a spurion analysis, with the c_Δ expressed as powers of small parameters ε_s and ε_{np} that measure the weak breaking of the supersymmetry and the no-scale symmetry, respectively, of the noncompact throat:

$$c_\Delta \sim \varepsilon_s^{Q_s(\Delta)} \varepsilon_{np}^{Q_{np}(\Delta)} \,. \tag{3.30}$$

The powers $Q_s(\Delta)$ and $Q_{np}(\Delta)$ are integers representing the number of spurion insertions required to obtain the corresponding perturbation in the field theory. We now turn to relating ε_s and ε_{np} to the warp factor a_0, and then to stating the rules for determining $Q_s(\Delta)$ and $Q_{np}(\Delta)$ for a given operator.

Recalling that the warp factor at the tip of the throat is $e^{A_{\rm tip}} \equiv a_0$, the antibrane contributes an amount

$$V_{\overline{D3}} = 2T_3 a_0^4 \,,$$

to the four-dimensional vacuum energy, where T_3 is the D3-brane tension. Thus, the scale of supersymmetry breaking is of order a_0^4, and it is convenient to use conventions for $Q_s(\Delta)$ in which $\varepsilon_s \sim a_0^2$.

Next, we consider the parameter ε_{np} measuring the breaking of no-scale symmetry. The ISD conditions are obeyed in the no-scale background of [5], so violations of the ISD conditions are controlled by the leading effect breaking no-scale symmetry: namely, by the nonperturbative superpotential W_{np} for the Kähler moduli.[8] In turn, W_{np} can be related to a_0. For simplicity we will assume

[8]This can be made very precise, as shown in [12]: gaugino condensation on D7-branes leads

that the positive energy from the anti-D3-brane is the dominant source of supersymmetry breaking in the compactification, so that in particular there are no antibranes in other throats with higher infrared scales $\tilde{a}_0 > a_0$. The total configuration will then be a metastable de Sitter vacuum, with vacuum energy that is small in string units, provided that the antibrane energy $V_{\overline{D3}}$ approximately cancels the negative energy density from the nonperturbatively-generated moduli potential. The key consequence of this relationship is that the nonperturbative superpotential W_{np} for the Kähler moduli obeys

$$|W_{np}| \sim a_0^2 \,,$$

because the moduli potential, and equivalently the potential for a probe D3-brane, scales as $|W_{np}|^2$. In summary, the fine-tuning needed to obtain a metastable vacuum in the KKLT scenario links the scale of violation of the ISD conditions to the infrared scale a_0 of the throat. Correspondingly, the departures from the ISD conditions (3.16, 3.17) are proportional to powers of the minimum warp factor a_0, and it is convenient to take $\varepsilon_{np} = a_0^2$.

As $\varepsilon_s = \varepsilon_{np} \sim a_0^2$, we arrive at the simple relationship

$$c_\Delta \sim a_0^{2Q_s(\Delta)+2Q_{np}(\Delta)} \equiv a_0^{2Q(\Delta)} \,, \tag{3.31}$$

with $Q = Q_s + Q_{np}$. We now turn to determining Q for each class of modes.

Non-ISD perturbations, i.e. perturbations of Φ_- and G_-, were extensively discussed in [12]. Based on the arguments above, the potential for a probe D3-brane, while nonvanishing, can at most be of order a_0^4. Correspondingly, any supergravity modes that contribute at linear order to the D3-brane potential, as

to a nonperturbative superpotential in the four-dimensional theory, but manifests in ten dimensions as a localized source for IASD flux G_-, so that perturbations of G_- are directly dictated by W_{np}.

Φ_- does, must have $Q \geq 2$, while modes that contribute at quadratic order, as G_- does, have $Q \geq 1$. See [12] for more details, and for a demonstration that modes of Φ_- in fact have $Q = 2$, while modes of G_- have $Q = 1$.

As noted above, perturbations that are ISD and supersymmetric have $Q = 0$, as they are allowed in the background solution. The only remaining cases are ISD perturbations that break the supersymmetry of the background solution. In our analysis of low-dimension operators, this question arises only for modes of the metric: the low-lying modes of Φ_+, G_+, and τ are supersymmetric. The supersymmetry-breaking spurion analysis for operators dual to metric modes is standard: for a chiral spurion superfield X, we may take $\varepsilon_s = F_X \sim a_0^2$, so that $Q(\Delta)$ counts the required number of insertions of X. A perturbation to the field theory Lagrangian by the bottom component of a chiral superfield then has $Q_s = 1$, while the bottom component of a non-chiral superfield has $Q_s = 2$.

3.2.3 Scaling of the anti-D3-brane mass

Using the fact that at the tip $\frac{r_{\rm tip}}{r_{\rm UV}} \sim a_0$, along with the result (3.31) in equation (3.26), we see that the general harmonic mode scales as

$$\delta\phi^{\rm Harmonic} \sim c_{\Delta(\phi)} \left(\frac{r_{\rm tip}}{r_{\rm UV}}\right)^{\Delta(\phi)+\lambda(\phi)-4} \sim a_0^{\Delta(\phi)+2\,Q(\Delta)+\lambda(\phi)-4} \equiv a_0^{\hat{\Delta}(\phi)+\lambda(\phi)-4}\,, \tag{3.32}$$

where we have defined the effective dimension $\hat{\Delta}(\phi) = \Delta(\phi) + 2\,Q(\Delta)$. Similarly, from equation (3.28) one sees that the inhomogeneous piece of the solutions can be decomposed into a sum of terms, each scaling as a power of a_0:

$$\delta\phi^{(n)} \sim a_0^{\Delta_1+\ldots+\Delta_n+2\,Q(\Delta_1)+\ldots+2\,Q(\Delta_n)+\lambda(\phi)-4n} \equiv a_0^{\hat{\Delta}(\phi)+\lambda(\phi)-4}\,, \tag{3.33}$$

where now $\hat{\Delta}(\phi) = \Delta_1 + \ldots + \Delta_n + 2\,Q(\Delta_1) + \ldots + 2\,Q(\Delta_n) - 4(n-1)$.

We now make the substitution

$$\phi = \phi_0 + \delta\phi \sim a_0^{\lambda(\phi)} \left(1 + a_0^{\hat{\Delta}(\phi)-4}\right) \tag{3.34}$$

in the equation of motion (3.9) and linearize in $a_0^{\hat{\Delta}(\phi)-4}$. For the left-hand side we find

$$\delta\left(\nabla^2\Phi_+\right) = \nabla_0^2\,\delta\Phi_+ + \delta\nabla^2\left(\Phi_+\right)_0 \ . \tag{3.35}$$

From the expression $\nabla^2\phi = -\frac{1}{\sqrt{g}}\partial_m\left(\sqrt{g}\,g^{mn}\,\partial_n\phi\right)$ we see that there will be an overall factor of $a_0^{\lambda(\Phi)-\lambda(g)}$ in $\delta\nabla^2\left(\Phi_+\right)_0$ along with a factor of `equation changed` $a_0^{\hat{\Delta}(g)-4}$ coming from the single variation of the metric δg_{mn}. For the first term on the right-hand side of equation (3.9),

$$\begin{aligned}\frac{(\Phi_+ + \Phi_-)^2}{96\mathrm{Im}\,\tau}|G_+|^2 = \frac{(\Phi_+ + \Phi_-)^2}{96\mathrm{Im}\,\tau}g^{mm'}g^{nn'}g^{kk'}\ (G_+)_{mnk}\ (G^*_+)_{m'n'k'} \\ \sim a_0^{(2\lambda(\Phi)+2\lambda(G)-3\lambda(g)-\lambda(\tau))}\times\left(1 + a_0^{\hat{\Delta}(\Phi_+)-4} + a_0^{\hat{\Delta}(\Phi_-)-4}\right)^2\left(1 + a_0^{\hat{\Delta}(G+)-4}\right)^2 \\ \times\left(1 + a_0^{\hat{\Delta}(g)-4}\right)^{-3}\left(1 + a_0^{\hat{\Delta}(\tau)-4}\right)^{-1} .\end{aligned} \tag{3.36}$$

Linearizing, this gives

$$\begin{aligned}\delta\left(\frac{(\Phi_+ + \Phi_-)^2}{96\mathrm{Im}\,\tau}|G_+|^2\right) \sim a_0^{(2\lambda(\Phi)+2\lambda(G)-3\lambda(g)-\lambda(\tau))}\times\left(a_0^{\hat{\Delta}(\Phi_+)-4} + a_0^{\hat{\Delta}(\Phi_-)-4}\right. \\ \left.+a_0^{\hat{\Delta}(G+)-4} + a_0^{\hat{\Delta}(g)-4} + a_0^{\hat{\Delta}(\tau)-4}\right) .\end{aligned} \tag{3.37}$$

A similar analysis applies to the second term, and we ultimately find

$$\nabla_0^2\,\delta\Phi_+ + a_0^{\lambda(\Phi)-\lambda(g)}\,a_0^{\hat{\Delta}(\Phi_+)-4} \tag{3.38}$$

$$\begin{aligned}&\sim a_0^{(2\lambda(\Phi)+2\lambda(G)-3\lambda(g)-\lambda(\tau))}\times\left(a_0^{\hat{\Delta}(\Phi_+)-4} + a_0^{\hat{\Delta}(\Phi_-)-4} + a_0^{\hat{\Delta}(G_+)-4} + a_0^{\hat{\Delta}(g)-4} + a_0^{\hat{\Delta}(\tau)-4}\right) \\ &+ a_0^{\lambda(\Phi)-\lambda(g)}\times\left(a_0^{\hat{\Delta}(\Phi_+)-4} + a_0^{\hat{\Delta}(\Phi_-)-4} + a_0^{\hat{\Delta}(g)-4}\right).\end{aligned} \tag{3.39}$$

Next, by examining the background profiles for each field, we determine that $\lambda(\Phi) - \lambda(g) = 2\lambda(\Phi) + 2\lambda(G) - 3\lambda(g) - \lambda(\tau) = 2$. We can therefore simplify

(3.38) to read

$$\nabla_0^2\,\delta\Phi_+ \sim a_0^{\mathbf{\Delta}-2} \sim \left(a_0^{\hat{\Delta}(\Phi_+)-2} + a_0^{\hat{\Delta}(\Phi_-)-2} + a_0^{\hat{\Delta}(G_+)-2} + a_0^{\hat{\Delta}(g)-2} + a_0^{\hat{\Delta}(\tau)-2}\right). \tag{3.40}$$

near the tip of the throat. This tells us that the set of values that $\mathbf{\Delta}$ can take is given by all possible effective dimensions $\hat{\Delta}$ from any mode (harmonic or inhomogeneous) coming from any of the fields $\Phi_\pm, G_+, \tau, g$.

In order to proceed further, we need to understand the spectrum of Kaluza-Klein modes of the throat, specifically the values of $\hat{\Delta}$ for the most relevant modes. We therefore turn to a study of the spectroscopy of $T^{1,1}$.

3.3 Spectroscopy of $T^{1,1}$

The spectrum of Kaluza-Klein excitations of $T^{1,1}$ was obtained in the pioneering works [19, 20]. In principle one could approach the problem directly by computing the eigenvalues of the Laplace, the Lichnerowicz, and the Laplace-Beltrami operators acting on scalar, symmetric two-tensor, and two-form harmonics, respectively: from these spectroscopic data the dimensions $\Delta(\phi)$ of the corresponding fields are readily obtained. The authors of [19, 20] instead employed a clever 'method of exhaustion' in which they partially filled supergravity multiplets with the modes of highest spin, and then used the superconformal algebra to predict the scaling dimensions for all vacant positions in the multiplets. From this they indirectly inferred all the eigenvalues without needing to compute the spectrum of the Lichnerowicz operator acting on symmetric two-tensors.

However, in the later work [12] it was found — through direct computation of the spectrum of the Laplace-Beltrami operator acting on two-forms — that certain eigenvalues occurring in the tables of supergravity multiplets in [19, 20] were assigned to modes of the metric, whereas they properly belong to modes of the two-form potential. A plausible reason for this disparity, explained to us by A. Dymarsky, is that a sign error in the spectrum of the Laplace-Beltrami operator in [19, 20] propagated through the exhaustion procedure and led to incorrect assignments of some of the modes of the metric.

As our present analysis crucially relies on the spectrum of low-lying operators, we need to perform an independent, explicit calculation of the spectrum of metric[9] modes. In the appendix we therefore obtain the spectrum of the Lichnerowicz operator on symmetric two-tensors. The changes to the metric spectrum required to accommodate the findings of [12] do indeed occur in our new results, providing a nontrivial check of our computation.

3.3.1 Lowest-dimension harmonic modes on $T^{1,1}$

In table 3.1 we have compiled the most relevant harmonic modes of the supergravity fields, which we now discuss in turn.

Scalar and two-form modes

Perturbations of Φ_+ are permitted in a supersymmetric ISD solution, and so have $Q = 0$. Even so, $\Delta(\Phi_+) \geq 8$, so that *harmonic* modes of Φ_+ provide negligi-

[9] The spectrum of the scalar Laplacian on $T^{1,1}$ was calculated in [37, 19, 20], while the spectrum of the Laplace-Beltrami operator acting on two-forms was first obtained in [19, 20], and was recomputed in [12], leading to minor corrections.

Modes of Φ_-

j_1	j_2	R	Multiplet	Type	Operator	λ	Q	Δ	$\hat{\Delta} = \Delta + 2Q$
$\frac{1}{2}$	$\frac{1}{2}$	1	V.I	chiral	$[\,\mathrm{Tr}\,(AB)\,]_b$	4	2	$\frac{3}{2}$	$\frac{11}{2}$
0	1	0	V.I	semi-long	$\left[\mathrm{Tr}\,(Ae^V\bar{A}e^{-V})\,\right]_b$	4	2	2	6
1	0	0	V.I	semi-long	$\left[\mathrm{Tr}\,(Be^V\bar{B}e^{-V})\,\right]_b$	4	2	2	6
1	1	2	V.I	chiral	$\left[\,\mathrm{Tr}\,(AB)^2\,\right]_b$	4	2	3	7

Modes of G_3 ***(ISD + IASD)***

j_1	j_2	R	Multiplet	Type	Operator	λ	Q	Δ	$\hat{\Delta} = \Delta + 2Q$
$\frac{1}{2}$	$\frac{1}{2}$	-1	V.I	chiral	$[\,\mathrm{Tr}\,(AB)\,]_{\theta^2}$	0	1	$\frac{5}{2}$	$\frac{9}{2}$
0	0	2	V.IV	chiral	$\left[\mathrm{Tr}\left(\mathcal{W}_1^2 + \mathcal{W}_2^2\right)\right]_b$	0	1	3	5
$\frac{1}{2}$	$\frac{1}{2}$	1	G.I	chiral	$[\,\mathrm{Tr}\,(\mathcal{W}_+^\alpha\,(AB))\,]_{\theta_\alpha}$	0	1	$\frac{7}{2}$	$\frac{11}{2}$

Modes of G_3 ***(Pure ISD)***

j_1	j_2	R	Multiplet	Type	Operator	λ	Q	Δ	$\hat{\Delta} = \Delta + 2Q$
$\frac{1}{2}$	$\frac{1}{2}$	-1	V.III	long	$\left[\,\mathrm{Tr}\left(e^V\overline{\mathcal{W}}_+^2 e^{-V}AB\right)\right]_{\theta^2\bar{\theta}^2}$	0	0	$\frac{13}{2}$	$\frac{13}{2}$

Modes of τ

j_1	j_2	R	Multiplet	Type	Operator	λ	Q	Δ	$\hat{\Delta} = \Delta + 2Q$
0	0	0	V.IV	chiral	$\left[\,\mathrm{Tr}\left(\mathcal{W}_+^2\right)\right]_{\theta^2}$	0	0	4	4
$\frac{1}{2}$	$\frac{1}{2}$	1	V.IV	chiral	$\left[\,\mathrm{Tr}\left(\mathcal{W}_+^2 AB\right)\right]_{\theta^2}$	0	0	$\frac{11}{2}$	$\frac{11}{2}$

Modes of Φ_+

j_1	j_2	R	Multiplet	Type	Operator	λ	Q	Δ	$\hat{\Delta} = \Delta + 2Q$
0	0	0	V.II	long	$\left[\,\mathrm{Tr}\left(\mathcal{W}_+^2 e^V\overline{\mathcal{W}}_+^2 e^{-V}\right)\right]_{\theta^2\bar{\theta}^2}$	4	0	8	8
$\frac{1}{2}$	$\frac{1}{2}$	1	V.II	long	$\left[\,\mathrm{Tr}\left(\mathcal{W}_+^2 e^V\overline{\mathcal{W}}_+^2 e^{-V}AB\right)\right]_{\theta^2\bar{\theta}^2}$	4	0	$\frac{19}{2}$	$\frac{19}{2}$

Modes of g_{mn}

j_1	j_2	R	Multiplet	Type	Operator	λ	Q	Δ	$\hat{\Delta} = \Delta + 2Q$
0	0	0	V.I	semi-conserved	$\left[\,\mathrm{Tr}\left(Ae^V\bar{A}e^{-V} - Be^V\bar{B}e^{-V}\right)\right]_b$	2	2	2	6
0	0	2	V.III	chiral	$\left[\mathrm{Tr}\left(\mathcal{W}_-^2\right)\right]_b$	2	2	3	7
1	1	0	V.I	long	$[\,\mathrm{Tr}(f)\,]_{\theta^2\,\bar{\theta}^2}$	2	0	5.29	5.29

Table 3.1: Summary of the leading harmonic modes of each field.

bly small corrections to the anti-D3-brane potential. Of course, the potential is dictated by the full Φ_+ solution, but in the infrared, the dominant terms in this solution are inhomogeneous terms sourced by the background profiles of other supergravity fields, via (3.9).

The perturbations of Φ_- were extensively explored in [12], where it was shown that two spurion insertions are necessary, i.e. $Q = 2$. The most relevant perturbation has a rather small dimension, $\Delta = 3/2$, but the 'effective' dimension, which encodes the smallness of the perturbation in the ultraviolet, is $\hat{\Delta} = 11/2$.

Harmonic modes of the dilaton τ are allowed in the supersymmetric ISD background, so that $Q = 0$. The nontrivial modes have $\Delta(\tau) \geq 5.5$, while the marginal mode with $\Delta(\tau) = 4$ simply corresponds to a shift in the string coupling, and can be absorbed into the background value.

Note that the flux G_3 possesses two types of modes, those with both ISD and IASD components ($G_\pm$), and those that are purely ISD (G_+). The data presented below for $G_\pm$ is taken from [12]. For the purely ISD modes G_+, it was shown in [33] that $\Delta(G_3) \geq 6.5$, and so these modes make small corrections to the potential. As explained in [12], modes of $G_\pm$ have $Q = 1$.

Evidently, the leading perturbations to the potential from harmonic modes of scalars come from the $\hat{\Delta} = 11/2$ modes of τ and Φ_-, but both are subleading in comparison to the contribution from the $\hat{\Delta} = 9/2$ mode of $G_\pm$.

Metric modes

Next, the dimensions $\Delta(g)$ of modes of the metric are related to the eigenvalues λ^{I_t} of the Lichnerowicz operator acting on traceless, symmetric two-tensors via

$$\Delta(g) = 2 + \sqrt{\lambda^{I_t} - 4}\,.$$

In the appendix, we obtain the eigenvalues λ^{I_t}. Using these results one finds that the three leading modes of the metric have $\Delta(g) = 2$, 3, and 5.29.

One can see from the functional form of the $\Delta(g) = 2$ mode presented in the appendix that it corresponds to the resolution of the tip of the conifold. It is known that the resolution of the conifold is associated with the bottom component of the baryon current multiplet $[\,\mathcal{J}_B\,]_b \equiv \left[\,\mathrm{Tr}\left(A\bar{A} - B\bar{B}\right)\right]_b$ of the Klebanov-Witten gauge theory [38]. Correspondingly, $Q = 2$ for this mode. Moreover, it is a singlet under all the angular isometries, and does not lift the anti-D3-brane moduli space in any case.

The $\Delta(g) = 3$ mode has quantum numbers matching the bottom component of the chiral multiplet $\left[\,\mathrm{Tr}(\mathcal{W}_1^2 - \mathcal{W}_2^2)\,\right]_b$ of the gauge theory, where $\mathcal{W}_i$ is the chiral field strength superfield of the $SU(N)_i$ gauge factor. Thus, $Q = 1$ for this mode. One can see from the functional form of the $\Delta(g) = 3$ mode presented in the appendix that it corresponds to the deformation of the tip of the conifold. As this mode is a singlet under the non-abelian symmetries that remain unbroken at the tip of the Klebanov-Strassler solution, it does not lift the anti-D3-brane moduli space.

Finally, the dual of the $\Delta(g) = 5.29$ mode can be identified as a top component of Vector Multiplet I of [19, 20] in its unshortened form, corresponding to an operator of the form $[\,\mathrm{Tr}(f)\,]_{\theta^2\bar{\theta}^2}$, where $f \equiv f\left(A, B, \bar{A}, \bar{B}\right)$ is a harmonic (but not holomorphic) function of the chiral superfields A, B. As this mode is supersymmetric and allowed in the ISD background, it has $Q = 0$.

Lowest-dimension modes

Let us now summarize the most relevant operators, ranked according to the effective dimension $\hat{\Delta}$ of the corresponding harmonic mode. All these operators have nontrivial quantum numbers[10] under the angular isometries, and hence generically lift the flat directions for angular displacements of the anti-D3-brane.

Most important is the $\hat{\Delta} = 9/2$ mode of $G_{\pm}$, corresponding to the superpotential perturbation $[\, \mathrm{Tr}\, AB \,]_{\theta^2}$ in the gauge theory. This perturbation is inconsistent with no-scale symmetry: it lifts the moduli space of a probe D3-brane. However, as explained in [12], this perturbation is generically present in a Klebanov-Strassler throat region of a stabilized compactification, with a small coefficient in the ultraviolet that we have encoded in the spurion exponent $Q = 1$.

Next in importance is the $\hat{\Delta} = 5.29$ mode of the metric, which is dual to $[\, \mathrm{Tr}(f) \,]_{\theta^2\bar{\theta}^2}$ as explained above. This metric perturbation is the leading mode that is allowed by no-scale symmetry ($Q = 0$), and correspondingly made the leading contribution in the analysis of AAB [10].

Of slightly lesser importance are the $\hat{\Delta} = 11/2$ modes of τ and Φ_{-}.

[10]There are two nontrivial modes with $\hat{\Delta} = 5$: the mode of $G_{\pm}$ dual to $\left[\mathrm{Tr}\left(\mathcal{W}_+^2\right)\right]_b$, and the mode of the metric dual to $\left[\mathrm{Tr}\left(\mathcal{W}_-^2\right)\right]_b$. However, both modes are singlets under the non-abelian symmetries, and so neither contributes to lifting the moduli space of angular displacements of the anti-D3-brane.

3.4 The Scale of the Anti-D3-brane Potential

Equipped with the above results, it is now straightforward to determine the leading contributions to the anti-D3-brane potential.

The dominant contribution arises from a linear-order perturbation by the $\hat{\Delta} = 9/2$ mode of $G_\pm$. This introduces a single insertion of a G_+ perturbation in (3.36); the remaining factor of G_+ is then a background flux. The result is a contribution to $m^2_{\overline{D3}}$ of size $\mathbf{\Delta} = 4.5$.

It might come as a surprise that the first subleading correction to the potential arises from the *second-order* contribution of the $\hat{\Delta} = 9/2$ mode of $G_\pm$, i.e. from two insertions of a G_+ perturbation in (3.36), rather than from a linear perturbation by some other mode. Consulting equation (3.33), we see that the effective dimension for a mode at second order is $\mathbf{\Delta} = 2\hat{\Delta}(\phi) - 4$, so that the $\hat{\Delta} = 9/2$ mode of $G_\pm$ makes a second-order contribution $\mathbf{\Delta} = 5$ to $m^2_{\overline{D3}}$.

The next correction to the potential comes from a linear perturbation by the $\mathbf{\Delta} = 5.29$ mode of the metric, as discussed in [10]. Further contributions arise at $\mathbf{\Delta} = 11/2$ and beyond.

In summary, and taking into account the factor of a_0 appearing in the canonical normalization for the angular fields, the anti-D3-brane potential takes the form

$$V_{\overline{D3}} = \sum_i c_i\, f_i(\Psi) a_0^{\Delta_i}\,, \tag{3.41}$$

where the c_i are coefficients of order unity, $f_i(\Psi)$ is a generic function of canonically-normalized angles on the S^3, and the leading dimensions Δ_i are

$$\Delta = 2.5, 3, 3.29, 3.5, \ldots \tag{3.42}$$

The leading mass term found in this work, $m^2 \sim a_0^{2.5}$, is parametrically larger, by a factor of $a_0^{-0.79} \gg 1$, than the largest contribution $m^2 \sim a_0^{3.29}$ consistent with no-scale symmetry [10]. Nevertheless, the scale of the angle-independent terms in the anti-D3-brane potential is $V_{\overline{D3}} \sim a_0^4$, and the natural scale for a mass-squared at the tip of the throat is $m^2 \sim a_0^2 \gg a_0^{2.5}$. We conclude that the mass terms of the angular displacement moduli of an anti-D3-brane are parametrically smaller than the infrared scale, by a factor of $a_0^{0.5} \ll 1$. Thus, the qualitative finding of [10] is unchanged: there are light open string moduli with masses that are small compared to the infrared scale.

We now ask the question of whether sufficient stabilization of the antibrane moduli can in principle be achieved without severe fine-tuning. In the previous section we assumed a strict KKLT scenario when obtaining the scalings for the c_Δ. That is, we assumed that moduli stabilization was accomplished by non-perturbative effects wrapping four-cycles in the bulk, that the primary uplifting source is the antibrane under question and that metastability is achieved by near equality of the bulk supersymmetry breaking and IR scale.

One could very well find other mechanisms to naturally stabilize the compactification, resulting in different values for the $Q(\Delta)$'s. For example in the Large Volume Scenario [7] the relation we quoted between the spurion scale and the IR scale of the throat no longer holds. We should explore whether other scenarios can result in a naturally large mass scale for the antibrane moduli.

This is easily addressed by referring to equation (3.25). The typical scale of physics at the tip is $m_{IR}^2 \sim a_0^2$. Thus the anti-D3-brane is free of light moduli if

$$m_{\overline{D3}}^2 \gtrsim a_0^2 \implies \boldsymbol{\Delta} \leq 4. \tag{3.43}$$

Then, just as the moduli mass is reaching the IR scale, modes are becoming effectively marginal and therefore order one at the tip. Thus corrections are becoming nonperturbative and the tip geometry is destroyed. More alarmingly, we see from equation (3.9) that all modes generate corrections to the antibrane potential Φ_+ and that, therefore, violent corrections at the tip will cause the antibrane tension to diverge and drive decompactification unless once again severe fine tuning of bulk effects is applied. We conclude that bulk effects cannot produce sufficient stabilization of the antibrane in practice.

3.5 Chapter Summary

We have seen that broadening the analysis of [10] to include, in particular, perturbations violating the ISD condition yields a mass for the anti-D3-brane moduli that can be parametrically larger than what was previously found. Under broad circumstances, these perturbations generate the dominant corrections to the antibrane effective action, despite the fact that they are nonperturbatively suppressed in the UV, since they are mediated by relevant operators that grow in the IR.

On the other hand, we have given conclusive evidence that sufficient stabilization of the anti-D3-brane moduli cannot be achieved by relying on bulk corrections. For the case of a KS throat in a strict KKLT scenario, the bulk induced anti-D3-brane mass is exponentially suppressed $m^2_{\overline{D3}} \sim a_0^{1/2}\, m^2_{\mathrm{IR}}$ compared to the typical scale of IR physics. More generally, achieving $m^2_{\overline{D3}} \sim m^2_{\mathrm{IR}}$ for antibranes at the tip of a warped throat will necessarily require severe fine tuning. We conclude that new elements must be introduced in the IR of the throat to stabilize the moduli.

APPENDIX A

STRUCTURE OF THE SOURCE TERMS

In §2.2.2 we left the source terms in the equations of motion implicit. In this appendix we will work out the source term for the dilaton as an example. Expanding the kinetic term, we find

$$(\nabla^2 \tau)_{(n)} = \sum_{l=0}^{n} \nabla^2_{(l)}\, \tau_{(n-l)} = \nabla^2_{(0)}\, \tau_{(n)} + \sum_{l=1}^{n-1} \nabla^2_{(l)}\, \tau_{(n-l)}\,. \tag{A.1}$$

For the first term on the right-hand side of equation (3.12) we have

$$\left(\frac{\nabla\tau\cdot\nabla\tau}{i\mathrm{Im}(\tau)}\right)_{(n)} = \sum_{l=0}^{n-2}\sum_{q=1}^{l-1} \frac{(-)^l\, l!\, g_s}{i}\, \mathrm{Im}\,\tau_{(l)}\, \partial_m\tau_{(q)}\, \partial^m\tau_{(n-l-q)}, \tag{A.2}$$

using the fact that $\partial_m\, \tau_{(0)} = 0$. For the second term on the right-hand side we get

$$\left(\frac{\Phi_+ + \Phi_-}{48i}\, G_+\cdot G_-\right)_{(n)} = -2i\, e^{4A_{(0)}}\, G_+^{(0)}\cdot G_-^{(n)} - 2i \sum_{l=0}^{n-1}\sum_{q=0}^{l-1} (\Phi_-^{(l)} + \Phi_+^{(l)})\, G_+^{(q)}\cdot G_-^{(n-l-q)}\,, \tag{A.3}$$

using the fact that $G_-^{(0)} = 0$. The n-th order equation of motion for τ is then

$$\begin{aligned} \nabla^2_{(0)}\tau_{(n)} ={}& \frac{\Phi_+^{(0)}}{48i}\, G_+^{(0)}\cdot G_-^{(n)} - \sum_{l=1}^{n-1} \nabla^2_{(l)}\, \tau_{(n-l)} + \sum_{l=0}^{n-2}\sum_{q=1}^{l-1} \frac{(-)^l\, l!\, g_s}{i}\, \mathrm{Im}\,\tau_{(l)}\, \partial_m\tau_{(q)}\, \partial^m\tau_{(n-l-q)} \\ & - 2i\sum_{l=0}^{n-1}\sum_{q=0}^{l-1} (\Phi_-^{(l)} + \Phi_+^{(l)})\, G_+^{(q)}\cdot G_-^{(n-l-q)}\,. \end{aligned} \tag{A.4}$$

This is then of the form (2.28), with

$$\begin{aligned} \mathrm{Source}_\tau(\phi^{(m<n)}) = {}& -\sum_{l=1}^{n-1} \nabla^2_{(l)}\, \tau_{(n-l)} + \sum_{l=0}^{n-2}\sum_{q=1}^{l-1} \frac{(-)^l\, l!\, g_s}{i}\, \mathrm{Im}\,\tau_{(l)}\, \partial_m\tau_{(q)}\, \partial^m\tau_{(n-l-q)} \\ & - 2i\sum_{l=0}^{n-1}\sum_{q=0}^{l-1} (\Phi_-^{(l)} + \Phi_+^{(l)})\, G_+^{(q)}\cdot G_-^{(n-l-q)}\,. \end{aligned} \tag{A.5}$$

The remaining $\mathrm{Source}_\varphi(\phi^{(m<n)})$ can be obtained in like fashion.

APPENDIX B

HARMONIC SOLUTIONS AND GREEN'S FUNCTIONS

In this appendix we derive the harmonic solutions and Green's functions that are needed in the main text. We separate the equations of motion for scalar, flux and metric perturbations on a Calabi-Yau cone into radial and angular equations, and then solve the resulting radial equations. This yields the homogeneous solutions and Green's functions on the cone, given the harmonics on the Sasaki-Einstein base as well as the associated spectrum of Hodge-de Rham eigenvalues (see [39, 40] for seminal related work). In the case that the base is $T^{1,1}$, the spectroscopy is well understood [20, 19] (see also [41, 12]), and is conveniently presented in [21].

In the main body of the text we have considered a six-dimensional cone, but many of the results of this appendix hold for any $(n+1)$-dimensional cone. However, in our treatment of fluxes in §B.2, we specialize to $n = 5$.

B.1 Angular harmonics on an Einstein manifold

We will begin by defining the angular harmonics and establishing their relevant properties. Some of the properties below are specific to $n = 5$, and we indicate this where applicable.

Consider a general $(n+1)$-dimensional Calabi-Yau cone C_{n+1}:

$$\mathrm{d}s^2_{C_{n+1}} = g_{mn}\mathrm{d}y^m\mathrm{d}y^n = \mathrm{d}r^2 + r^2\mathrm{d}s^2_{\mathcal{B}_n} \tag{B.1}$$

$$= \mathrm{d}r^2 + r^2\tilde{g}_{ij}\mathrm{d}\Psi^i\mathrm{d}\Psi^j\,, \tag{B.2}$$

where we use i, j, k, l for indices which lie in the angular space only, and m, n, p, q

for indices which run over both r and the angular directions. Here $\tilde{g}_{ij}$ is the metric on the base space $\mathcal{B}_n$, which must be a Sasaki-Einstein manifold, with

$$\tilde{R}_{ij} = (n-1)\tilde{g}_{ij}\,, \tag{B.3}$$

where $\tilde{R}_{ij}$ is the Ricci tensor built from $\tilde{g}_{ij}$. In the following, we will use a tilde above indices (derivative operators) to denote contraction with (construction from) the metric $\tilde{g}_{ij}$.

We now discuss the various tensor harmonics on the angular space $\mathcal{B}_n$. A complete basis for scalar functions on $\mathcal{B}_n$ are the scalar harmonics

$$Y^{I_s}(\Psi)\,. \tag{B.4}$$

A complete basis for one-forms on $\mathcal{B}_n$ are the transverse and longitudinal harmonics

$$Y_i^{I_v}(\Psi)\,, \quad \tilde{\nabla}_i Y^{I_s}(\Psi)\,. \tag{B.5}$$

A complete basis for two-forms on $\mathcal{B}_n$ are the transverse and longitudinal harmonics

$$Y_{[ij]}^{I_2}(\Psi)\,, \quad \tilde{\nabla}_{[i} Y_{j]}^{I_v}(\Psi)\,, \tag{B.6}$$

where square brackets denote antisymmetrization. A complete basis for symmetric, two-index tensors on $\mathcal{B}_n$ are the transverse and longitudinal harmonics

$$Y_{\{ij\}}^{I_t}(\Psi)\,, \quad \tilde{\nabla}_{\{i} Y_{j\}}^{I_v}(\Psi)\,, \quad \tilde{\nabla}_{\{i}\tilde{\nabla}_{j\}} Y^{I_s}(\Psi)\,, \quad \tilde{g}_{ij} Y^{I_s}(\Psi)\,, \tag{B.7}$$

where curly brackets around indices denote the symmetric traceless part:

$$A_{\{ij\}} = \frac{1}{2}\left(A_{ij} + A_{ji}\right) - \frac{\tilde{g}_{ij}}{n} A^{\tilde{k}}{}_k\,. \tag{B.8}$$

The transverse harmonics obey

$$\tilde{\nabla}^{\tilde{k}} Y_k^{I_v} \quad = \quad 0\,, \tag{B.9}$$

$$\tilde{\nabla}^{\tilde{k}} Y^{I_2}_{[ki]} = 0\,, \tag{B.10}$$

$$\tilde{\nabla}^{\tilde{k}} Y^{I_t}_{\{ki\}} = 0\,. \tag{B.11}$$

B.1.1 Eigenvalue properties

The zero-, one- and two-form harmonics Y^{I_s}, $Y^{I_v}_i$ and $Y^{I_2}_{[ij]}$ are eigenfunctions of the Hodge-de Rham operator $\tilde{\Delta} = \tilde{\delta}\mathrm{d}+\mathrm{d}\tilde{\delta}$, where d denotes the exterior derivative and $\tilde{\delta} = (-1)^{n(k+1)+1}\tilde{\star}_n \mathrm{d}\tilde{\star}_n$ denotes its adjoint acting on k-forms on $\mathcal{B}_n$. The symmetric two-index tensor harmonic $Y^{I_t}_{\{ij\}}$ is an eigenfunction of the Lichnerowicz operator $\tilde{\Delta}_L$ (cf. e.g. [42]). These equations are efficiently expressed as

$$\tilde{\Delta}_0 Y^{I_s} = \lambda^{I_s} Y^{I_s}\,, \tag{B.12}$$

$$\tilde{\Delta}_1 Y^{I_v}_i = \lambda^{I_v} Y^{I_v}_i\,, \tag{B.13}$$

$$\tilde{\Delta}_2 Y^{I_2}_{ij} = \lambda^{I_2} Y^{I_2}_{ij}\,, \tag{B.14}$$

$$\tilde{\Delta}_L Y^{I_t}_{ij} = \lambda^{I_t} Y^{I_t}_{ij}\,. \tag{B.15}$$

Using the relationships

$$\tilde{\delta}\mathrm{d}Y^{I_s} = -\nabla^2 Y^{I_s}\,, \tag{B.16}$$

$$(\tilde{\delta}\mathrm{d}Y^{I_v})_i = -2\nabla^k\nabla_{[k}Y^{I_v}_{i]}\,, \tag{B.17}$$

$$(\tilde{\delta}\mathrm{d}Y^{I_2})_{ij} = -3\nabla^k\nabla_{[i}Y^{I_2}_{jk]}\,, \tag{B.18}$$

together with

$$\tilde{\delta}Y^{I_s} = 0\,, \tag{B.19}$$

$$\tilde{\delta}Y^{I_v} = -\tilde{\nabla}^{\tilde{k}} Y^{I_v}_k\,, \tag{B.20}$$

$$(\tilde{\delta}Y^{I_2})_j = -\tilde{\nabla}^{\tilde{k}} Y^{I_2}_{kj}\,, \tag{B.21}$$

one can derive the explicit form of the Hodge-de Rham and Lichnerowicz operators:

$$\tilde{\Delta}_0 Y^{I_s} = -\tilde{\nabla}^2 Y^{I_s}\,, \tag{B.22}$$

$$\tilde{\Delta}_1 Y^{I_v}_i = -\tilde{\nabla}^{\tilde{2}} Y^{I_v}_i + \tilde{R}_i{}^{\tilde{j}} Y^{I_v}_j\,, \tag{B.23}$$

$$\tilde{\Delta}_2 Y^{I_2}_{ij} = -\tilde{\nabla}^2 Y^{I_2}_{ij} + 2\tilde{R}^{\tilde{k}}{}_{ij}{}^{\tilde{l}} Y^{I_2}_{kl} - 2\tilde{R}_{[i}{}^{\tilde{k}} Y^{I_2}_{j]k}\,, \tag{B.24}$$

$$\tilde{\Delta}_L Y^{I_t}_{ij} = -\tilde{\nabla}^{\tilde{2}} Y^{I_t}_{ij} + 2\tilde{R}^{\tilde{k}}{}_{ij}{}^{\tilde{l}} Y^{I_t}_{kl} + 2\tilde{R}_{(i}{}^{\tilde{k}} Y^{I_t}_{j)k}\,. \tag{B.25}$$

Notice that the transversality of the one- and two-form harmonics Y^{I_v} and Y^{I_2} is simply the statement that they are co-closed, $\tilde{\delta} Y^{I_v} = \tilde{\delta} Y^{I_2} = 0$. Using the transversality of the harmonics, the above eigenvalue equations can also be written as

$$\tilde{\nabla}^2 Y^{I_s} = -\lambda^{I_s} Y^{I_s}\,, \tag{B.26}$$

$$2\tilde{\nabla}^{\tilde{k}} \tilde{\nabla}_{[k} Y^{I_v}_{i]} = -\lambda^{I_v} Y^{I_v}_i\,, \tag{B.27}$$

$$3\tilde{\nabla}^{\tilde{k}} \tilde{\nabla}_{[i} Y^{I_2}_{jk]} = -\lambda^{I_2} Y^{I_2}_{[ij]}\,, \tag{B.28}$$

$$\tilde{\nabla}^2 Y^{I_t}_{\{ij\}} - 2\tilde{\nabla}^{\tilde{k}} \tilde{\nabla}_{(i} Y^{I_t}_{\{j)k\}} = -\lambda^{I_t} Y^{I_t}_{\{ij\}}\,. \tag{B.29}$$

We also note that when n is odd, the Hodge-de Rham operator for a tranverse $\left(\frac{n-1}{2}\right)$-form can be expressed in terms of the square of the first-order operator $\tilde{\star}_n \mathrm{d}$. In the case of interest for us, $n = 5$, the two-form $Y^{I_2}_{[ij]}$ is an eigenfunction of $\star_5 \mathrm{d}$,

$$\star_5\, \mathrm{d} Y^{I_2} = i\, \delta^{I_2}\, Y^{I_2}\,, \quad \delta^{I_2} \in \mathbb{R} \tag{B.30}$$

such that $\tilde{\delta}\mathrm{d} Y^{I_2} = -(\star_5 \mathrm{d})^2 Y^{I_2} = +(\delta^{I_2})^2 Y^{I_2}$, i.e.

$$\tilde{\Delta}_2\, Y^{I_2} = \lambda^{I_2}\, Y^{I_2}\,, \quad \lambda^{I_2} \equiv (\delta^{I_2})^2. \tag{B.31}$$

B.1.2 Orthogonality properties

We normalize the harmonics such that

$$\int d^n\Psi \sqrt{\tilde{g}}\, \bar{Y}_{I_s}\, Y^{I'_s} = \delta^{I'_s}_{I_s}, \tag{B.32}$$

$$\int d^n\Psi \sqrt{\tilde{g}}\, \bar{Y}^{\tilde{k}}_{I_v}\, Y^{I'_v}_{k} = \delta^{I'_v}_{I_v}, \tag{B.33}$$

$$\int d^n\Psi \sqrt{\tilde{g}}\, \bar{Y}^{[\tilde{k}\tilde{l}]}_{I_2}\, Y^{I'_2}_{[kl]} = \delta^{I'_2}_{I_2}, \tag{B.34}$$

$$\int d^n\Psi \sqrt{\tilde{g}}\, \bar{Y}^{\{\tilde{i}\tilde{j}\}}_{I_t}\, Y^{I'_t}_{\{ij\}} = \delta^{I'_t}_{I_t}. \tag{B.35}$$

Here we use a bar to denote complex conjugation, $\bar{Y} \equiv Y^*$. From the above orthonormality properties and equation (B.3) one can derive the remaining set of orthonormality conditions:

$$\int d^n\Psi \sqrt{\tilde{g}}\, \tilde{\nabla}^{\tilde{k}}\bar{Y}_{I_s}\, \tilde{\nabla}_k Y^{I'_s} = \lambda^{I_s}\, \delta^{I_s}_{I'_s}, \tag{B.36}$$

$$\int d^n\Psi \sqrt{\tilde{g}}\, (\tilde{g}^{\tilde{k}\tilde{l}}\bar{Y}_{I_s})\, (\tilde{g}_{kl} Y^{I'_s}) = n\, \delta^{I'_s}_{I_s}, \tag{B.37}$$

$$\int d^n\Psi \sqrt{\tilde{g}}\, \tilde{\nabla}^{[\tilde{k}}\bar{Y}^{\tilde{l}]}_{I_v}\, \tilde{\nabla}_{[k}Y^{I'_v}_{l]} = \frac{1}{2}\lambda^{I_v}\, \delta^{I'_v}_{I_v}, \tag{B.38}$$

$$\int d^n\Psi \sqrt{\tilde{g}}\, \tilde{\nabla}^{\{\tilde{k}}\bar{Y}^{\tilde{l}\}}_{I_v}\, \tilde{\nabla}_{\{k}Y^{I'_v}_{l\}} = \frac{1}{2}\Big(\lambda^{I_v} - 2(n-1)\Big)\, \delta^{I'_v}_{I_v}, \tag{B.39}$$

$$\int d^n\Psi \sqrt{\tilde{g}}\, \tilde{\nabla}^{\{\tilde{k}}\tilde{\nabla}^{\tilde{l}\}}\bar{Y}^{I_s}\, \tilde{\nabla}_{\{k}\tilde{\nabla}_{l\}}Y^{I'_s} = \frac{(n-1)}{n}\lambda^{I_s}(\lambda^{I_s} - n)\, \delta^{I'_s}_{I_s}. \tag{B.40}$$

All remaining inner products—those between transverse and longitudinal harmonics, or between longitudinal harmonics with different numbers of derivatives—vanish.

One can learn much from equations (B.36–B.40). Since the inner products must be positive definite, we see from equation (B.36) that $\tilde{\nabla}_i Y^{I_s}$ vanishes if and only if $\lambda^{I_s} = 0$. It is known (see [42]) that compact Einstein spaces always support exactly one zero mode—the constant mode $Y^{I_s}(\Psi) = \text{const}$. From equations (B.39) and (B.40) one deduces that $\lambda^{I_s} \geq n$ or $\lambda^{I_s} = 0$, while $\lambda^{I_v} \geq 2(n-1)$. Both of these conditions are known to hold for an Einstein space (with scaling as in equation (B.3)), see [42]. The value $\lambda^{I_s} = n$ occurs only for the trivial case of the sphere, $\mathcal{B}_n = \mathcal{S}^n$ [43]. This corresponds to the $(n+1)$-dimensional "cone" being merely flat Euclidean space. Next, the condition $\tilde{\nabla}_{\{i} Y^{I_v}_{j\}} = 0$ is just the condition that $Y^{I_v}_i$ is a Killing vector, and so there is one harmonic with $\lambda^{I_v} = 2(n-1)$ for each continuous isometry of $\mathcal{B}_n$.

For the two-form harmonics there is no lower bound on the eigenvalues δ^{I_2}. Indeed, by conjugation of equation (B.30) one sees that the spectrum is symmetric under $\delta^{I_2} \to -\delta^{I_2}$. Modes with $\delta^{I_2} = 0$ have a special significance: when $\delta^{I_2} = 0$, $\mathrm{d}Y^{I_2} = 0$. Combining this with the transversality condition (B.10), we see that such a Y^{I_2} must be harmonic, and is therefore a Betti form. We will denote these Betti two-forms as

$$\omega_2^i, \quad i = 1, 2, \ldots b_2\,, \tag{B.41}$$

where b_2 is the second Betti number of $\mathcal{B}_5$.

B.2 Flux solutions and Green's functions

The harmonic three-form flux solutions were obtained in [12]. In §B.2.1 we present a slight generalization of those solutions that allows for logarithmic running of the warp factor. Then, in §B.2.3 we derive the Green's functions for the three-form flux. In this section we specialize to the case of $n = 5$.

B.2.1 Homogeneous flux solutions

We wish to obtain the solution to the system of differential equations (2.35, 2.37), where the IASD part of the flux is given by $G_- = (\star_6 - i)G_3$, and the expression for $\Phi_+^{(0)}$ in an ISD background is given in terms of the warp factor (cf. equation (2.21)),

$$\frac{2}{\Phi_+^{(0)}} = e^{-4A^{(0)}} = \frac{C_1 + C_2 \ln r}{r^4} \,. \tag{B.42}$$

Because G_3 is closed, it can be written locally in terms of a two-form potential A_2 as $G_3 = \mathrm{d}A_2$. Generically, A_2 will have a harmonic expansion

$$\begin{aligned} A_2 = &\sum_{I_2} a^{I_2}(r)\, Y^{I_2}(\Psi) + \sum_{I_v} a^{I_v}(r)\, \mathrm{d}\, Y^{I_v}(\Psi) \\ &+ \sum_{I_v} b^{I_v}(r)\, \frac{\mathrm{d}r}{r} \wedge Y^{I_v}(\Psi) + \sum_{I_s} b^{I_s}(r)\, \frac{\mathrm{d}r}{r} \wedge \mathrm{d}\, Y^{I_s}(\Psi) \,. \end{aligned} \tag{B.43}$$

We have the obvious gauge symmetry $A_2 \to A_2 + \mathrm{d}\chi_1$, for a one-form gauge parameter χ_1, and by expanding χ_1 in harmonics, we can set $b^{I_v} = b^{I_s} = 0$:

$$A_2 = \sum_{I_2} a^{I_2}(r)\, Y^{I_2}(\Psi) + \sum_{I_v} a^{I_v}(r)\, \mathrm{d}\, Y^{I_v}(\Psi), \quad \text{gauge fixed} \,. \tag{B.44}$$

Now we insert this form of A_2 into equation (2.35). Since the equations are linear we can consider a single mode at a time, and we have two cases: non-exact and exact modes.

Non-exact modes: Consider the non-exact mode

$$A_2 = A^{I_2}(r)\, Y^{I_2}(\Psi) \,. \tag{B.45}$$

Using the identities

$$\star_6 \left(\frac{\mathrm{d}r}{r} \wedge \Omega_2 \right) = \star_5 \Omega_2 \,, \tag{B.46}$$

$$\star_6\,\Omega_3 = -\left(\frac{\mathrm{d}r}{r} \wedge \star_5\Omega_3\right), \tag{B.47}$$

for arbitrary two- and three-forms Ω_2 and Ω_3 on $\mathcal{B}_5$, together with $\star_5 \mathrm{d}Y^{I_2} = i\delta^{I_2} Y^{I_2}$, we get for the flux

$$G_\pm = \pm i\left(r\partial_r A^{I_2} \mp \delta^{I_2} A^{I_2}\right)\left(\frac{\mathrm{d}r}{r} \wedge Y^{I_2} \mp i \star_5 Y^{I_2}\right). \tag{B.48}$$

Inserting the above expression for G_- into equation (2.35) yields

$$r\partial_r f^{I_2}(r) - \delta^{I_2} f^{I_2}(r) = 0\,, \tag{B.49}$$

where $f^{I_2}(r) \equiv \Phi_+^{(0)}(r)\left(r\partial_r A^{I_2}(r) + \delta^{I_2} A^{I_2}(r)\right)$. Solving the above equation we find

$$A^{I_2}(r) = A_-^{I_2} r^{-\delta^{I_2}} + A_+^{I_2} r^{\delta^{I_2}-4}\left[(4-2\delta^{I_2})(C_1 + C_2 \log r) + C_2\right], \tag{B.50}$$

where $A_\pm^{I_2}$ are integration constants. The IASD/ISD components of this solution are

$$G_- = +i\,(2\delta^{I_2}-4)^2\, A_+^{I_2}\, r^{\delta^{I_2}-4}\,\left(C_1 + C_2\,\ln r\right)\left(\frac{\mathrm{d}r}{r} \wedge Y^{I_2} + i \star_5 Y^{I_2}\right), \tag{B.51}$$

$$G_+ = -i\left(2\delta^{I_2} A_-^{I_2} r^{-\delta^{I_2}} + 2A_+^{I_2} r^{\delta^{I_2}-4}\left[(8-4\delta^{I_2})\,(C_1 + C_2 \ln r) + \delta^{I_2} C_2\right]\right)\left(\frac{\mathrm{d}r}{r} \wedge Y^{I_2} - i \star_5 Y^{I_2}\right). \tag{B.52}$$

Notice that the mode $A_-^{I_2}$ does not contribute to G_-.

For the Betti modes with $\delta^{I_2} = 0$ the above solutions reduce to

$$A^{I_2}(r) = A_-^{I_2} + A_+^{I_2} r^{-4}\left(4(C_1 + C_2 \log r) + C_2\right), \tag{B.53}$$

with IASD/ISD flux components

$$G_\pm = \mp 32\, i\, \frac{A_+^{I_2}}{\Phi_+^{(0)}}\left(\frac{\mathrm{d}r}{r} \wedge Y^{I_2} \mp i \star_5 Y^{I_2}\right). \tag{B.54}$$

Exact modes: Consider the exact mode

$$A_2 = A^{I_v}(r)\, \mathrm{d}Y^{I_v} \,. \tag{B.55}$$

The flux is

$$G_\pm = \pm i r \partial_r A^{I_v} \left(\frac{\mathrm{d}r}{r} \wedge \mathrm{d}Y^{I_v} \mp i \star_5 \mathrm{d}Y^{I_v} \right) . \tag{B.56}$$

Plugging this expression into equation (2.35) and using $\star_5 \mathrm{d} \star_5 \mathrm{d}Y^{I_v} = \tilde{\delta}\mathrm{d}Y^{I_v} = \lambda^{I_v} Y^{I_v}$, we get

$$\mathrm{d}\left(\Phi_+^{(0)} G_-\right) = r\partial_r \left(\Phi_+^{(0)} r \partial_r A^{I_v}\right) \frac{\mathrm{d}r}{r} \wedge \star_5 \mathrm{d}Y^{I_v} + \lambda^{I_v} \left(\Phi_+^{(0)} r \partial_r A^{I_v}\right) \star_5 Y^{I_v} = 0. \tag{B.57}$$

From the discussion in §B.1.2, we know that $\lambda^{I_v} \geq 8$, so the second term on the right in equation (B.57) can only vanish if $A^{I_v}(r) = \text{const}$. Thus, for this mode the flux vanishes:

$$G_3 \propto \mathrm{d}(\mathrm{d}Y^{I_v}) = 0 \,. \tag{B.58}$$

Moreover, the mode is topologically trivial. Thus, the exact modes are unphysical.

Total solution: To summarize, our solution is

$$G_3 = \mathrm{d}A_2 \tag{B.59}$$

$$A_2 = \sum_{I_2} \left\{ A_-^{I_2} r^{-\delta^{I_2}} + A_+^{I_2} r^{\delta^{I_2}-4} \left[(4 - 2\delta^{I_2})(C_1 + C_2 \ln r) + C_2 \right] \right\} Y^{I_2} \,, \tag{B.60}$$

where the sum over I_2 runs over all non-exact modes, including the Betti modes with $\delta^{I_2} = 0$.

B.2.2 Scaling dimensions for modes of flux

In [12] explicit expressions for all possible closed IASD three-forms on a cone were given in terms of the scalar harmonic functions of the cone, the Kähler potential k, the Kähler form J, and the holomorphic three-form Ω. This in particular allows one to determine the set of radial scalings of flux modes in terms of the radial scalings of the scalar modes. One finds that the allowed Laplace-Beltrami eigenvalues are

$$\delta^{I_2} = \pm \begin{cases} -1 + \Delta(I_s) & \\ -2 + \Delta(I_s)\,, & \lambda^{I_s} \neq 0 \\ -3 + \Delta(I_s)\,, & \lambda^{I_s} \neq 0 \\ 0\,, & b_2 \neq 0 \end{cases}, \tag{B.61}$$

Now, since $-3 + \Delta(I_s) \geq 2$ for $\lambda^{I_s} \neq 0$ (see paragraph below equation (2.68)), we find that $|\delta^{I_2}| \geq 2$, apart from the Betti modes, that is

$$\delta^{I_2} \geq 2, \quad \text{or} \quad \delta^{I_2} = 0, \quad \text{or} \quad \delta^{I_2} \leq -2\,. \tag{B.62}$$

In order for the radial scalings of the modes in equation (2.70) to take on the standard AdS form, equation (2.23), we identify $\Delta(I_2) = \max(\delta^{I_2}, 4 - \delta^{I_2})$. Thus, the operator dimensions corresponding to modes with $\delta^{I_2} \geq 2$ are given by $\Delta(\delta^{I_2} \geq 2) = |\delta^{I_s}|$, i.e.

$$\Delta(\delta^{I_2} \geq 2) = \begin{cases} -1 + \Delta(I_s) & \\ -2 + \Delta(I_s)\,, & \lambda^{I_s} \neq 0 \\ -3 + \Delta(I_s)\,, & \lambda^{I_s} \neq 0 \end{cases}, \tag{B.63}$$

The dimensions of the Betti modes with $\delta^{I_2} = 0$ are given by

$$\Delta(b_2) = 4\,, \tag{B.64}$$

while the modes with $\delta^{I_2} \leq -2$ have $\Delta(\delta^{I_2} \leq -2) = 4 + |\delta^{I_s}|$, i.e.

$$\Delta(\delta^{I_2} \leq -2) = 4 + \begin{cases} -1 + \Delta(I_s) & \\ -2 + \Delta(I_s)\,, & \lambda^{I_s} \neq 0 \\ -3 + \Delta(I_s)\,, & \lambda^{I_s} \neq 0 \end{cases}. \tag{B.65}$$

The ISD/IASD parts $G_{\pm}^{\mathcal{H}}$ of the flux solutions are presented in equations (B.52, B.51) and in equation (B.54) for the Betti modes. From these expressions one can see that $G_{-}^{\mathcal{H}}$ always vanishes for the $A_{-}^{I_2}$ mode, which scales like $r^{-\delta^{I_2}}$. Whether this mode corresponds to the normalizable mode $r^{-\Delta(I_2)}$ or the non-normalizable mode $r^{\Delta(I_2)-4}$ depends on the value of δ^{I_2}. For $\delta^{I_2} \geq 2$ we have $r^{-\delta^{I_2}} = r^{-\Delta(I_2)}$ and this is the normalizable mode. For $\delta^{I_2} < 2$ we have $r^{-\delta^{I_2}} = r^{\Delta(I_2)-4}$ and this is the non-normalizable mode. For the Betti modes we see from equation (B.54) that both G_+ and G_- vanish for the non-normalizable mode, scaling like r^0. These modes are still physical, and they correspond to nontrivial topological configurations. So, to summarize,

- For $\delta^{I_2} \geq 2$, the IASD flux G_- vanishes in the normalizable mode.
- For $\delta^{I_2} \leq -2$, the IASD flux G_- vanishes in the non-normalizable mode.
- For $\delta^{I_2} = 0$, the total flux vanishes in the non-normalizable mode.

B.2.3 Flux Green's functions

We want to solve the system of equations

$$\mathrm{d}\left(\Sigma_{\pm} + \mathcal{S}_1\right) = \mathcal{S}_3\,, \tag{B.66}$$

$$(\star_6 \mp i)\,\Sigma_{\pm} = \mathcal{S}_2\,, \tag{B.67}$$

for two three-form sources $\mathcal{S}_1$ and $\mathcal{S}_2$ and a four-form source $\mathcal{S}_3$. We will do so in two steps.

System I: First, we will solve the system of equations with $\mathcal{S}_3 = 0$,

$$\mathrm{d}\left(\Sigma_{\pm}^{(\mathrm{I})} + \mathcal{S}_1\right) = 0\,, \tag{B.68}$$

$$(\star_6 \mp i)\,\Sigma_\pm^{(\mathrm{I})} \;=\; \mathcal{S}_2\,, \tag{B.69}$$

System II: Second, we will solve the system of equations with $\mathcal{S}_1, \mathcal{S}_2 = 0$,

$$\mathrm{d}\Sigma_\pm^{(\mathrm{II})} \;=\; \mathcal{S}_3\,, \tag{B.70}$$

$$(\star_6 \mp i)\;\Sigma_\pm^{(\mathrm{II})} \;=\; 0\,. \tag{B.71}$$

The solution to the original system (B.66), (B.67) is then obtained by adding the two solutions above,

$$\Sigma_\pm = \Sigma_\pm^{(\mathrm{I})} + \Sigma_\pm^{(\mathrm{II})}\,. \tag{B.72}$$

Solution to system I: We first note that equation (B.68) implies that the combination $\Theta_\pm \equiv \Sigma_\pm^{(\mathrm{I})} + \mathcal{S}_1$ is closed, so that we can locally solve equation (B.68) in terms of a two-form potential

$$\Theta_\pm = \mathrm{d}\chi_\pm\,, \tag{B.73}$$

where $\chi_\pm$ is defined only in one coordinate patch. In terms of $\Theta_\pm$, equation (B.69) becomes

$$(\star_6 \mp i)\,\Theta_\pm = \mathcal{S}_2 + (\star_6 \mp i)\,\mathcal{S}_1 \equiv \mathcal{S}_\pm\,, \tag{B.74}$$

where we defined the three-form $\mathcal{S}_\pm$ in the last line. To solve this equation we expand $\chi_\pm$ and $\mathcal{S}_\pm$ in harmonics and then equate the coefficients of the independent modes. Note that a three-form on $\mathcal{B}_5$ can always be dualized to give a two-form on $\mathcal{B}_5$. Thus we have

$$\mathcal{S}_\pm = \mathrm{d}r \wedge \mathcal{T}_\pm + \star_5 \tilde{\mathcal{T}}_\pm, \tag{B.75}$$

for $\mathcal{T}_\pm$ and $\tilde{\mathcal{T}}_\pm$ two-forms on $\mathcal{B}_5$. Now from the definition of $\mathcal{S}_\pm$, equation (B.74), we find that

$$(\star_6 \pm i)\,\mathcal{S}_\pm = 0\,, \tag{B.76}$$

which gives $\tilde{\mathcal{T}}_\pm = \pm\, ir\mathcal{T}_\pm$, so that we get

$$\mathcal{S}_\pm = \mathrm{d}r \wedge \mathcal{T} \pm i\, r \star_5 \mathcal{T}_\pm . \tag{B.77}$$

Thus, $\mathcal{S}_\pm$ has the harmonic expansion

$$\mathcal{S}_\pm = \sum_{I_2} r\, \mathcal{S}^{I_2}_\pm \left(\frac{\mathrm{d}r}{r} \wedge Y^{I_2} \pm i \star_5 Y^{I_2} \right) + \sum_{I_v} r\, \mathcal{S}^{I_v}_\pm \left(\frac{\mathrm{d}r}{r} \wedge \mathrm{d}Y^{I_v} \pm i \star_5 \mathrm{d}Y^{I_v} \right). \tag{B.78}$$

Just as for the potential A_2 of the previous subsection, we can choose a gauge in which $\chi_\pm$ has an expansion

$$\chi_\pm = \sum_{I_2} \chi^{I_2}_\pm(r)\, Y^{I_2} + \sum_{I_v} \chi^{I_v}_\pm(r)\, \mathrm{d}Y^{I_v} . \tag{B.79}$$

Therefore

$$\begin{aligned} (\star_6 \mp i)\mathrm{d}\chi_\pm \quad &= \quad \mp i \Bigg\{ \sum_{I_2} (r\partial_r \chi^{I_2}_\pm \pm \lambda^{I_2} \chi^{I_2}_\pm) \left(\frac{\mathrm{d}r}{r} \wedge Y^{I_2} \pm i \star_5 Y^{I_2} \right) \\ &\qquad + \sum_{I_v} r\partial_r \chi^{I_v}_\pm \left(\frac{\mathrm{d}r}{r} \wedge \mathrm{d}Y^{I_v} \pm i \star_5 \mathrm{d}Y^{I_v} \right) \Bigg\} . \end{aligned} \tag{B.80}$$

Inserting this into equation (B.74) we find the differential equations

$$\partial_r \chi^{I_2}_\pm \pm \frac{\lambda^{I_2}}{r} \chi^{I_2}_\pm \quad = \quad \pm\, i\mathcal{S}^{I_2}_\pm , \tag{B.81}$$

$$\partial_r \chi^{I_v}_\pm \quad = \quad \pm\, i\mathcal{S}^{I_v}_\pm , \tag{B.82}$$

with solutions

$$\chi^{I_2}_\pm(r) \quad = \quad \pm i \int_0^\infty dr'\, \vartheta(r - r') \left(\frac{r}{r'} \right)^{\pm\lambda^{I_2}} \mathcal{S}^{I_2}_\pm(r') , \tag{B.83}$$

$$\chi^{I_v}_\pm(r) \quad = \quad \pm i \int_0^\infty dr'\, \vartheta(r - r')\, \mathcal{S}^{I_v}_\pm(r') . \tag{B.84}$$

In writing down the above solutions we have introduced a modified step function ϑ suitable for non-localized sources $\mathcal{S}$ that takes care of the boundary behavior of the integrand in the IR and the UV:

$$\vartheta(r - r') = \begin{cases} \theta(r - r') & \text{for integrands that go to zero at zero,} \\ -\theta(r' - r) & \text{for integrands that go to zero at infinity.} \end{cases} \tag{B.85}$$

The orthonormality relations of §B.1.2 imply

$$\pm\, i\mathcal{S}_{\pm}^{I_2}\,\mathrm{d}r \;=\; \int_{\mathcal{B}_5} \frac{\mathrm{d}r}{r} \wedge 2\,\bar{Y}_{I_2} \wedge \mathcal{S}_{\pm}\,, \tag{B.86}$$

$$\pm\, i\mathcal{S}_{\pm}^{I_v}\,\mathrm{d}r \;=\; \int_{\mathcal{B}_5} \frac{\mathrm{d}r}{r} \wedge \lambda_{I_v}^{-1}\,\mathrm{d}\bar{Y}_{I_v} \wedge \mathcal{S}_{\pm}\,. \tag{B.87}$$

Using this together with the solutions (B.83) and (B.84), we can write down the Green's function solution for $\Sigma_{\pm}^{(\mathrm{I})}$ in terms of the sources $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\Sigma_{\pm}^{(\mathrm{I})} = \mathrm{d}\chi_{\pm} - \mathcal{S}_1\,, \tag{B.88}$$

$$\chi_{\pm}(y) = \int_{C_6} \mathcal{G}_{\pm}^{(\mathrm{I})}(y,y') \wedge \mathcal{S}_{\pm}(y')\,, \tag{B.89}$$

$$\mathcal{S}_{\pm} = \mathcal{S}_2 + (\star_6 \mp i)\,\mathcal{S}_1\,, \tag{B.90}$$

$$\begin{aligned}\mathcal{G}_{\pm}^{(\mathrm{I})}(y,y') = {} & \sum_{I_2} Y^{I_2}(\Psi)\left[\vartheta(r-r')\left(\frac{r'}{r}\right)^{\pm\lambda^{I_2}} \frac{\mathrm{d}r'}{r'} \wedge 2\,\bar{Y}^{I_2}(\Psi')\right] \\ & + \sum_{I_v} \mathrm{d}Y^{I_v}(\Psi)\left[\vartheta(r-r')\frac{\mathrm{d}r'}{r'} \wedge \lambda_{I_v}^{-1}\,\mathrm{d}\bar{Y}^{I_v}(\Psi')\right].\end{aligned} \tag{B.91}$$

The index structures of the above equations are as follows:

$$(\chi_{\pm}(y))_{ij} = \frac{1}{3!}\int \mathrm{d}^6y'\,\sqrt{g'}\,(\mathcal{G}_{\pm}^{(\mathrm{I})}(y,y'))_{ij}{}^{mnp}\left(\star_6^{-1}\mathcal{S}_{\pm}(y')\right)_{mnp}\,, \tag{B.92}$$

$$\begin{aligned}(\mathcal{G}_{\pm}^{(\mathrm{I})}(y,y'))_{ij,rkl} = {} & \sum_{I_2} Y_{ij}^{I_2}(\Psi)\vartheta(r-r')\left(\frac{r'}{r}\right)^{\pm\lambda^{I_2}} \frac{1}{r'}\,2\,\bar{Y}_{kl}^{I_2}(\Psi') \\ & + \sum_{I_v} 2\tilde{\nabla}_{[i}Y_{j]}^{I_v}(\Psi)\vartheta(r-r')\frac{1}{r'}\,\lambda_{I_v}^{-1}\,2\tilde{\nabla}_{[k}\bar{Y}_{l]}^{I_v}(\Psi')\,,\end{aligned} \tag{B.93}$$

where the full metric g_{mn} is used to raise and lower the indices, and the modified theta function ϑ was introduced in equation (B.85).

Solution to system II: We now solve the system (B.70), (B.71). Equation (B.71) tells us that

$$(\star_6 \mp i)\Sigma_{\pm}^{(\mathrm{I})} = 0\,. \tag{B.94}$$

The general solution to this equation is of the form of $S_\mp$ in equation (B.78), i.e.

$$\Sigma^{(\mathrm{I})}_\pm = \sum_{I_2} \sigma^{I_2}_\pm(r)\left(\frac{\mathrm{d}r}{r}\wedge Y^{I_2} \mp i\star_5 Y^{I_2}\right) + \sum_{I_v}\sigma^{I_v}_\pm(r)\left(\frac{\mathrm{d}r}{r}\wedge \mathrm{d}Y^{I_v} \mp i\star_5 \mathrm{d}Y^{I_v}\right). \tag{B.95}$$

A general four-form $\mathcal{S}_3$ can be expanded

$$\begin{aligned}\mathcal{S}_3 \;\; = \;\; & \sum_{I_2}\mathcal{S}^{I_2}_3(r)\,\mathrm{d}r\wedge\star_5 Y^{I_2} + \sum_{I_2}\mathcal{S}^{I_v}_3(r)\mathrm{d}r\wedge\star_5\mathrm{d}Y^{I_v} \\ & + \sum_{I_v}\tilde{\mathcal{S}}^{I_v}_3(r)\,\star_5 Y^{I_v} + \sum_{I_s}\mathcal{S}^{I_s}_3(r)\,\star_5\,\mathrm{d}Y^{I_s}\,.\end{aligned} \tag{B.96}$$

Equation (B.70) implies that $\mathcal{S}_3$ is closed. Upon imposing this, we find the constraints

$$\mathcal{S}^{I_s}_3 = 0\,, \tag{B.97}$$

$$\mathcal{S}^{I_v}_3 = \frac{1}{\lambda^{I_v}}\,\partial_r\tilde{\mathcal{S}}^{I_v}_3\,. \tag{B.98}$$

Substituting these expansions into equation (B.70) and collecting the coefficients of the independent harmonics, we find the radial equations

$$\partial_r\sigma^{I_2}_\pm \pm \frac{\lambda^{I_2}}{r}\,\sigma^{I_2}_\pm \;\; = \;\; \pm\, i\mathcal{S}^{I_2}_3\,, \tag{B.99}$$

$$\lambda^{I_v}\sigma^{I_v}_\pm \;\; = \;\; \pm\, i\tilde{\mathcal{S}}^{I_v}_3\,, \tag{B.100}$$

with solutions

$$\sigma^{I_2}_\pm(r) \;\; = \;\; \pm i\int_0^\infty \mathrm{d}r'\vartheta(r-r')\left(\frac{r}{r'}\right)^{\pm\lambda^{I_2}}\;\mathcal{S}^{I_2}_3(r')\,, \tag{B.101}$$

$$\sigma^{I_v}_\pm(r) \;\; = \;\; \pm i\int_0^\infty \mathrm{d}r'\delta(r-r')\,\lambda^{-1}_{I_v}\,\tilde{\mathcal{S}}^{I_v}_3(r')\,. \tag{B.102}$$

Using the orthonormality properties in §B.1.2,

$$\int_{\mathcal{B}_5} 2\,\bar{Y}_{I_2}\wedge\mathcal{S}_3 \;\; = \;\; \mathcal{S}^{I_2}_3\,\mathrm{d}r\,, \tag{B.103}$$

$$\int_{\mathcal{B}_5}\mathrm{d}r\wedge\bar{Y}_{I_v}\wedge\mathcal{S}_3 \;\; = \;\; \tilde{\mathcal{S}}^{I_v}_3\,\mathrm{d}r\,. \tag{B.104}$$

We can now use the solutions (B.101) and (B.102) to write down the Green's function solution for $\Sigma_\pm^{(\mathrm{II})}$ in terms of the source $\mathcal{S}_3$:

$$\Sigma_\pm^{(\mathrm{II})}(y) = \int_{C_6} \mathcal{G}_\pm^{(\mathrm{II})}(y,y') \wedge \mathcal{S}_3(y')\,, \tag{B.105}$$

$$\begin{aligned}\mathcal{G}_\pm^{(\mathrm{II})}(y,y') = &\sum_{I_2} \left(\frac{\mathrm{d}r}{r} \mp i\star_5\right) \wedge Y^{I_2}(\Psi) \left[\pm i\vartheta(r-r')\left(\frac{r'}{r}\right)^{\pm\lambda^{I_2}} 2\,\bar{Y}_{I_2}(\Psi')\right] \\ &+ \sum_{I_v} \left(\frac{\mathrm{d}r}{r} \mp i\star_5\right) \wedge \mathrm{d}Y^{I_v}(\Psi)\Big[\pm i\lambda_{I_v}^{-1}\delta(r-r')\,\mathrm{d}r' \wedge \bar{Y}_{I_v}(\Psi')\Big]\,.\end{aligned} \tag{B.106}$$

Total Solution: The total solution to the system (B.66, B.67) is just the sum of the pieces from each of the two steps:

$$\Sigma_\pm = \Sigma_\pm^{(\mathrm{I})} + \Sigma_\pm^{(\mathrm{II})} = \mathrm{d}\chi_\pm - \mathcal{S}_1 + \int_{C_6} \mathcal{G}_\pm^{(\mathrm{II})} \wedge \mathcal{S}_3\,, \tag{B.107}$$

$$\chi_\pm(y) = \int_{C_6} \mathcal{G}_\pm^{(\mathrm{I})}(y,y') \wedge \mathcal{S}_\pm(y')\,, \tag{B.108}$$

$$\mathcal{S}_\pm = (\star_6 \mp i)\,\mathcal{S}_1 + \mathcal{S}_2\,, \tag{B.109}$$

where the Green's functions $\mathcal{G}_\pm^{(\mathrm{I})}(y,y')$, $\mathcal{G}_\pm^{(\mathrm{I})}(y,y')$ are given in equations (B.91) and (B.106), respectively.

B.3 Metric solutions and Green's functions

Now we wish to solve the equations of motion for the metric perturbations $\delta g_{mn} \equiv h_{mn}$ on a general $(n+1)$-dimensional Calabi-Yau cone C_{n+1}. The linearized Einstein equations take the form

$$\Delta_K h_{mn} = \mathcal{S}_{mn}, \tag{B.110}$$

where $\mathcal{S}_{mn}$ denotes source terms, and the kinetic operator Δ_K defined in (2.33) is constructed using the background metric g_{mn}. The general solution takes the form

$$h_{mn}(y) = h^{\mathcal{H}}_{mn}(y) + \int \mathrm{d}^6y' \sqrt{g}\,(\mathcal{G}_g)_{mn}{}^{m'n'}(y,y')\,\mathcal{S}_{m'n'}(y'), \tag{B.111}$$

where $h^{\mathcal{H}}_{mn}$ is a homogenous solution (i.e., $\Delta_K h^{\mathcal{H}}_{mn} = 0$), and where $(\mathcal{G}_g)_{mn}{}^{m'n'}(y,y')$ denotes the metric Green's function. In §B.3.1 we will solve for the homogeneous perturbations in terms of angular harmonics on $\mathcal{B}_n$. In §B.3.2 we obtain the metric Green's function. To this end we separate the radial and angular variables in the operator Δ_K:

$$\begin{aligned}\Delta_K h_{ij} = &\left(\partial_r^2 + \frac{n-4}{r}\partial_r + \frac{4}{r^2}\right)h_{ij} + \frac{1}{r^2}\left(\tilde{\nabla}^2 h_{ij} - 2\tilde{\nabla}^{\tilde{k}}\tilde{\nabla}_{(i}h_{j)k}\right)\\ &- 2\left(\partial_r - \frac{2-n}{r}\right)\tilde{\nabla}_{(i}h_{j)r} - \frac{2}{r}\tilde{g}_{ij}\tilde{\nabla}^{\tilde{k}}h_{kr}\\ &+ \left[\frac{1}{r^2}\tilde{\nabla}_i\tilde{\nabla}_j + \tilde{g}_{ij}\frac{1}{r}\left(\partial_r - \frac{2}{r}\right)\right]h^{\tilde{k}}_k + \left[\tilde{\nabla}_i\tilde{\nabla}_j - \tilde{g}_{ij}r\left(\partial_r - \frac{2-2n}{r}\right)\right]h_{rr}\,, \end{aligned}\tag{B.112}$$

$$\begin{aligned}\Delta_K h_{ir} = &\frac{2n-2}{r^2}h_{ir} + \frac{1}{r^2}\left(\tilde{\nabla}^2 h_{ir} - \tilde{\nabla}^{\tilde{k}}\tilde{\nabla}_i h_{kr}\right)\\ &- \frac{1}{r^2}\left(\partial_r - \frac{2}{r}\right)\tilde{\nabla}^{\tilde{k}}h_{ik} + \frac{1}{r^2}\left(\partial_r - \frac{2}{r}\right)\tilde{\nabla}_i h^{\tilde{k}}_k + \frac{1-n}{r}\tilde{\nabla}_i h_{rr}\,,\end{aligned}\tag{B.113}$$

$$\Delta_K h_{rr} = \frac{1}{r^2}\left(\partial_r^2 - \frac{2}{r}\partial_r + \frac{2}{r^2}\right)h^{\tilde{k}}_k - \left(\frac{n}{r}\partial_r - \frac{1}{r^2}\tilde{\nabla}^2\right)h_{rr} - \frac{2}{r^2}\partial_r\tilde{\nabla}^{\tilde{k}}h_{kr}\,. \tag{B.114}$$

Throughout this work we impose a *transverse* gauge on the metric perturbations:

$$\tilde{\nabla}^{\tilde{k}}h_{\{ik\}} = 0\,, \tag{B.115}$$

$$\tilde{\nabla}^{\tilde{k}}h_{kr} = 0\,. \tag{B.116}$$

This gauge condition projects out the longitudinal harmonics $\tilde{\nabla}_{\{i}\tilde{\nabla}_{j\}}Y^{I_s}$, $\tilde{\nabla}_{\{i}Y^{I_v}_{j\}}$, and $\tilde{\nabla}_i Y^{I_s}$, and we get the following harmonic expansions

$$h_{\{ij\}} = \sum_{I_t}\phi^{I_t}(r)Y^{I_t}_{\{ij\}}(\Psi)\,, \tag{B.117}$$

$$\begin{aligned} h_{ir} &= \sum_{I_v} b^{I_v}(r) Y_i^{I_v}(\Psi)\,, && \text{(B.118)} \\ h_{\tilde{k}}^{\tilde{k}} &= \sum_{I_s} \pi^{I_s}(r) Y^{I_s}(\Psi)\,, && \text{(B.119)} \\ h_{rr} &= \sum_{I_s} \mathfrak{r}^{I_s}(r) Y^{I_s}(\Psi)\,. && \text{(B.120)} \end{aligned}$$

By expanding the gauge parameter ξ_m in angular harmonics, one can easily show that there always exists ξ_m such that the gauge (B.115, B.116) is attainable via

$$h_{mn} \longrightarrow h_{mn} + 2\nabla_{(m}\xi_{n)}\,. \qquad \text{(B.121)}$$

There is, however, a residual gauge freedom. The gauge conditions (B.115, B.116) are preserved under (B.121) if

$$\tilde{\nabla}^{\tilde{k}} \nabla_{\{k}\xi_{i\}} = 0\,, \qquad \text{(B.122)}$$

$$\tilde{\nabla}^{\tilde{k}} \nabla_{(k}\xi_{r)} = 0\,. \qquad \text{(B.123)}$$

The most general form for ξ is then

$$\begin{aligned} \xi_i &= \sum_{K_v} \Lambda^{K_v}(r)\, Y_i^{K_v}(\Psi)\,, && \text{(B.124)} \\ \xi_r &= \epsilon(r)\,, && \text{(B.125)} \end{aligned}$$

where the $Y_i^{K_v}(\Psi)$ are the Killing vectors on $\mathcal{B}_n$ with $\lambda^{K_v} = 2(n-1)$. The radial fields then transform as

$$\begin{aligned} \phi^{I_t} &\longrightarrow \phi^{I_t}\,, && \text{(B.126)} \\ b^{K_v} &\longrightarrow b^{K_v} + \left(\partial_r - \frac{2}{r}\right)\Lambda^{K_v}\,, && \text{(B.127)} \\ \pi_0 &\longrightarrow \pi_0 + nr\epsilon\,, && \text{(B.128)} \\ \mathfrak{r}_0 &\longrightarrow \mathfrak{r}_0 + \partial_r\epsilon\,, && \text{(B.129)} \end{aligned}$$

where π_0, $\mathfrak{r}_0$ are zero modes, i.e. correspond to harmonics with $\lambda^{I_s} = 0$. We will find it convenient to use the residual gauge symmetry to impose $\pi_0 = 0$ and

$b^{K_v} = 0$, i.e. we set

$$\int \mathrm{d}^n\Psi \ \sqrt{\tilde{g}}\, \bar{Y}^{\tilde{k}}_{K_v}(\Psi)\, h_{kr}(r,\Psi) \quad = \quad 0\,, \tag{B.130}$$

$$\int \mathrm{d}^n\Psi \ \sqrt{\tilde{g}}\, h^{\tilde{k}}_{k}(r,\Psi) \quad = \quad 0\,. \tag{B.131}$$

B.3.1 Homogeneous metric perturbations

Using the expansions (B.117–B.120) and the separation (B.112), and then collecting the coefficients of independent harmonics, the homogeneous equation

$$\Delta_K h_{ij} = 0 \tag{B.132}$$

gives the radial equations

$$\left(\partial_r^2 + \frac{n-4}{r}\partial_r + \frac{4-\lambda^{I_t}}{r^2}\right)\phi^{I_t}\, Y^{I_t}_{\{ij\}} \quad = \quad 0\,, \tag{B.133}$$

$$-2\left(\partial_r + \frac{2-n}{r}\right) b^{I_v}\, \tilde{\nabla}_{\{i}Y^{I_v}_{j\}} \quad = \quad 0\,, \quad \lambda^{I_v} \neq \lambda^{K_v} \tag{B.134}$$

$$\left(\frac{1}{r^2}\frac{n-2}{n}\pi^{I_s} + \mathfrak{r}^{I_s}\right)\tilde{\nabla}_{\{i}\tilde{\nabla}_{j\}}Y^{I_s} \quad = \quad 0\,, \quad \lambda^{I_s} \neq 0, n \tag{B.135}$$

$$\left[\frac{1}{n}\left(\partial_r^2 + \frac{2n-4}{r}\partial_r - \frac{2n-4}{r^2} - \frac{\lambda^{I_s}}{r^2}\frac{2n-2}{n}\right)\pi^{I_s} \quad - \quad \left(r\partial_r - (2-2n) + \frac{\lambda^{I_s}}{n}\right)\mathfrak{r}^{I_s}\right]\tilde{g}_{ij}Y^{I_s} \quad = \quad 0\,. \tag{B.136}$$

Note that equations (B.134) and (B.135) should not be applied for values of the quantum numbers I_v and I_s, respectively, for which the corresponding harmonics vanish identically, hence the restrictions listed. In a similar way,

$$\Delta_K h_{ir} = 0 \tag{B.137}$$

gives

$$\frac{1}{r^2}\left(2(n-1)-\lambda^{I_v}\right)b^{I_v}\,Y_i^{I_v} \;=\; 0\,, \quad \lambda^{I_v}\neq\lambda^{K_v}=2(n-1)\,, \tag{B.138}$$

$$\left(\frac{1}{r^2}\left(\partial_r-\frac{2}{r}\right)\pi^{I_s}+\frac{1-n}{r}\mathfrak{r}^{I_s}\right)\tilde{\nabla}_i Y^{I_s} \;=\; 0\,, \quad \lambda^{I_s}\neq 0\,, \tag{B.139}$$

and

$$\Delta_K h_{rr}=0 \tag{B.140}$$

gives

$$\left(\frac{1}{r^2}\left(\partial_r^2-\frac{2}{r}\partial_r+\frac{2}{r^2}\right)\pi^{I_s}-\frac{1}{r}\left(n\partial_r+\frac{\lambda^{I_s}}{r}\right)\mathfrak{r}^{I_s}\right)Y^{I_s}=0\,. \tag{B.141}$$

Solutions for π, $\mathfrak{r}$:

$\lambda^{I_s}\neq 0$**:** In this case we have four (three if $\lambda=n$) independent equations (B.135, B.136, B.139, B.141) for the two unknowns, π^{I_s}, $\mathfrak{r}^{I_s}$. Thus the only solutions are

$$\left.\begin{array}{rcl}\pi^{I_s}(r) &=& 0\\ \mathfrak{r}^{I_s}(r) &=& 0\end{array}\right\}\quad \text{if}\;\; \lambda^{I_s}\neq 0\,. \tag{B.142}$$

$\lambda^{I_s}=0$**:** Now we have only two equations — (B.136) and (B.141). We can nevertheless use the residual gauge freedom to set $\pi_0=0$. Equations (B.136) and (B.141) then give $\mathfrak{r}_0=0$:

$$\left.\begin{array}{rcl}\pi_0(r) &=& 0\\ \mathfrak{r}_0(r) &=& 0\end{array}\right\}\quad \text{gauge choice}\,. \tag{B.143}$$

Solutions for b^{I_v}:

$\lambda^{I_v} \neq \lambda^{K_v}$**:** Equation (B.138) immediately gives

$$b^{I_v}(r) = 0, \quad \lambda^{I_v} \neq \lambda^{K_v}. \tag{B.144}$$

$\lambda^{I_v} = \lambda^{K_v}$**:** We can use the residual gauge symmetry to eliminate the Killing modes

$$b^{K_v}(r) = 0, \quad \text{gauge choice}. \tag{B.145}$$

Solution for ϕ^{I_t}**:** The only nontrivial degrees of freedom in the homogeneous case are then the ϕ^{I_t}, obeying equation (B.133). The two independent solutions are

$$\phi^{I_t}_{\pm}(r) = r^{a_\pm(I_t)}, \quad a_\pm(I_t) = \frac{1}{2}\Big((5-n) \pm \sqrt{4\lambda^{I_t} + (n-1)(n-9)}\Big). \tag{B.146}$$

To summarize, the homogeneous solution is given by

$$h^{\mathcal{H}}_{\{ij\}}(y) = \sum_{I_t}\Big(h^{I_t}_{+}\, r^{a_+(I_t)} + h^{I_t}_{-}\, r^{a_-(I_t)}\Big)\, Y^{I_t}_{\{ij\}}(\Psi), \tag{B.147}$$

with all other components vanishing, where $h^{I_t}_{\pm}$ are constants of integration and the $a_\pm(I_t)$ are given by

$$a_\pm(I_t) = \frac{1}{2}\Big((5-n) \pm \sqrt{4\lambda^{I_t} + (n-1)(n-9)}\Big). \tag{B.148}$$

B.3.2 Metric Green's function

Now we wish to solve

$$\Delta_K h_{mn} = \mathcal{S}_{mn}. \tag{B.149}$$

We continue to impose the same gauge conditions as in the previous subsection, i.e. the transverse conditions (B.115, B.116) as well as the conditions $\pi_0 = 0$ and

$b^{K_v} = 0$. The symmetric tensor S_{mn} can in general be expanded as

$$\begin{aligned}
\mathcal{S}_{\{ij\}} &= \sum_{I_t} \mathcal{S}_t^{I_t}(r)\, Y_{\{ij\}}^{I_t}(\Psi) + \sum_{I_v} \mathcal{S}_t^{I_v}(r)\, \tilde{\nabla}_{\{i} Y_{j\}}^{I_v}(\Psi) + \sum_{I_s} \mathcal{S}_t^{I_s}(r)\, \tilde{\nabla}_{\{i}\tilde{\nabla}_{j\}} Y^{I_s}(\Psi)\,, && \text{(B.150)}\\
\mathcal{S}_{ir} &= \sum_{I_v} \mathcal{S}_v^{I_v}(r)\, Y_i^{I_v}(\Psi) + \sum_{I_s} \mathcal{S}_v^{I_s}(r)\, \tilde{\nabla}_i Y^{I_s}(\Psi)\,, && \text{(B.151)}\\
\mathcal{S}_{\tilde{k}}^{\tilde{k}} &= \sum_{I_s} \mathcal{S}_{\mathrm{tr}}^{I_s}(r) Y^{I_s}(\Psi)\,, && \text{(B.152)}\\
\mathcal{S}_{rr} &= \sum_{I_s} \mathcal{S}_s^{I_s}(r) Y^{I_s}(\Psi)\,. && \text{(B.153)}
\end{aligned}$$

In the above, the subscripts t, v, tr, s are used merely to distinguish the various radial functions and should not be interpreted as indices.

We will proceed similarly to the previous section. We will substitute the expansions for h_{mn} (B.117–B.120) and the expansions for $\mathcal{S}_{mn}$ (B.150–B.153) into the metric equation of motion (B.149) and make use of the decomposition of the operator Δ_K given in (B.112–B.114). We pick out the coefficient of each independent harmonic to obtain a set of radial equations.[1]

From the equation $\Delta_K h_{ij} = \mathcal{S}_{ij}$ one obtains the radial equations

$$\left(\partial_r^2 + \frac{n-4}{r}\partial_r + \frac{4-\lambda^{I_t}}{r^2}\right)\phi^{I_t}\, Y_{\{ij\}}^{I_t} \quad = \quad \mathcal{S}_t^{I_t}\, Y_{\{ij\}}^{I_t}\,, \qquad \text{(B.154)}$$

$$-2\left(\partial_r - \frac{2-n}{r}\right) b^{I_v}\, \tilde{\nabla}_{\{i} Y_{j\}}^{I_v} \quad = \quad \mathcal{S}_t^{I_v}\, \tilde{\nabla}_{\{i} Y_{j\}}^{I_v}\,, \quad \lambda^{I_v} \neq \lambda^{K_v} \qquad \text{(B.155)}$$

$$\left(\frac{1}{r^2}\frac{n-2}{n}\pi^{I_s} + \mathfrak{r}^{I_s}\right) \tilde{\nabla}_{\{i}\tilde{\nabla}_{j\}} Y^{I_s} \quad = \quad \mathcal{S}_t^{I_s}\, \tilde{\nabla}_{\{i}\tilde{\nabla}_{j\}} Y^{I_s}\,, \quad \lambda^{I_s} \neq 0, n \quad \text{(B.156)}$$

$$\left[\frac{1}{n}\left(\partial_r^2 + \frac{2n-4}{r}\partial_r - \frac{2n-4}{r^2} - \frac{\lambda^{I_s}}{r^2}\frac{2n-2}{n}\right)\pi^{I_s} \quad - \quad \left(r\partial_r - (2-2n) + \frac{\lambda^{I_s}}{n}\right)\mathfrak{r}^{I_s}\right] \tilde{g}_{ij} Y^{I_s}$$

[1] A subset of these radial equations represent constraints on the source $\mathcal{S}_{mn}$. These constraints must be satisfied in order for the solution derived below to be valid, but we will not present the explicit form of the constraints here: we assume that the constraints are automatically obeyed when the stress tensor is well-behaved.

$$= \frac{1}{n}\mathcal{S}^{I_s}_{\mathrm{tr}}\,\tilde{g}_{ij}Y^{I_s}\,. \tag{B.157}$$

From the equation $\Delta_K h_{ir} = \mathcal{S}_{ir}$ we get

$$\frac{1}{r^2}\left(2(n-1)-\lambda^{I_v}\right)b^{I_v}\,Y^{I_v}_i = \mathcal{S}^{I_v}_v\,Y^{I_v}_i\,, \quad \lambda^{I_v} \neq \lambda^{K_v} = 2(n-1)\,, \tag{B.158}$$

$$\left(\frac{1}{r^2}\left(\partial_r - \frac{2}{r}\right)\pi^{I_s} + \frac{1-n}{r}\mathfrak{r}^{I_s}\right)\tilde{\nabla}_i Y^{I_s} = \mathcal{S}^{I_s}_v\,\tilde{\nabla}_i Y^{I_s}\,, \quad \lambda^{I_s} \neq 0 \tag{B.159}$$

and from $\Delta_K h_{rr} = \mathcal{S}_{rr}$ we get

$$\left(\frac{1}{r^2}\left(\partial_r^2 - \frac{2}{r}\partial_r + \frac{2}{r^2}\right)\pi^{I_s} - \frac{1}{r}\left(n\partial_r + \frac{\lambda^{I_s}}{r}\right)\mathfrak{r}^{I_s}\right)Y^{I_s} = \mathcal{S}^{I_s}_s\,Y^{I_s}\,. \tag{B.160}$$

Solutions for π, $\mathfrak{r}$:

$\lambda^{I_s} = 0$**:** Since we have fixed to a gauge where $\pi_0 = 0$, equations (B.157, B.160) give

$$\mathfrak{r}_0 = \frac{r^2\,\mathcal{S}^0_s - \mathcal{S}^0_{\mathrm{tr}}}{2n(n-1)}\,. \tag{B.161}$$

$\lambda^{I_s} \neq 0$**:** Equations (B.156, B.159) give

$$\left(\partial_r + \frac{1}{r}\frac{n^2-5n+2}{n}\right)\pi^{I_s} = r^2\,\mathcal{S}^{I_s}_v + (n-1)r\mathcal{S}^{I_s}_t\,. \tag{B.162}$$

The regular solution to this equation is given by

$$\pi^I_s = \int_0^\infty \mathrm{d}r'\,\vartheta(r-r')\Big(\frac{r'}{r}\Big)^{\frac{n^2-5n+2}{n}}\left(r'^2\,\mathcal{S}^{I_s}_v(r') + (n-1)r'\mathcal{S}^{I_s}_t(r')\right), \tag{B.163}$$

where ϑ was introduced in equation (B.85). Equation (B.156) then gives the solution for $\mathfrak{r}^{I_s}$,

$$\mathfrak{r}^{I_s} = \mathcal{S}^{I_s}_t - \frac{1}{r^2}\frac{n-2}{n}\pi^{I_s}\,. \tag{B.164}$$

Solution for b^{I_v}:

$\lambda^{I_v} \neq \lambda^{K_v} = 2(n-1)$**:** Equation (B.158) gives

$$b^{I_v} = \frac{r^2}{2(n-1) - \lambda^{I_v}} \mathcal{S}_v^{I_v} \,. \tag{B.165}$$

$\lambda^{I_v} = \lambda^{K_v}$**:** We take $b^{K_v} = 0$ by gauge choice.

Solution for ϕ^{I_t}: Solving (B.154) is practically identical to solving the scalar Poisson equation (2.89). Thus we start by considering sources of the form

$$\mathcal{S}_t^{I_t}(r) = \mathcal{S}_t^{I_t}(\alpha, m)\, r^\alpha\, (\ln r)^m \,, \tag{B.166}$$

with $\mathcal{S}_t^{I_t}(\alpha, m) = \text{const.}$, and then generalize to a collection of such sources. For sources with $\alpha \neq -2 + a_\pm$, the solution to equation (B.154) is

$$\phi^{I_t}(r; \alpha, m) = \mathcal{S}_t^{I_t}(\alpha, m)\, r^{\alpha+2} \left(c_0 + c_1 \ln r + \ldots + c_m (\ln r)^m\right), \tag{B.167}$$

where the coefficients c_k are given by

$$c_k = (-1)^{m-k} \frac{m!/k!}{a_+ - a_-} \left[(\alpha + 2 - a_+)^{k-1-m} - (\alpha + 2 - a_-)^{k-1-m}\right], \quad \alpha \neq -2 + a_\pm \,, \tag{B.168}$$

while for sources with $\alpha = -2 + a_\pm$ the solution reads

$$\phi^{I_t}(r; \alpha, m) = \mathcal{S}_t^{I_t}(\alpha, m)\, r^{\alpha+2} \left(d_0 + d_1 \ln r + \ldots + d_{m+1} (\ln r)^{m+1}\right), \tag{B.169}$$

where the coefficients d_k are given by

$$d_k = (-1)^{m-k-1} \frac{m!}{k!} (\pm a_+ \mp a_-)^{k-1-m} \,, \quad \alpha = -2 + a_\pm \,. \tag{B.170}$$

For the general case

$$\mathcal{S}_t^{I_t}(r) \sum_{\alpha, m} \mathcal{S}_t^{I_t}(r; \alpha, m)\, r^\alpha\, (\ln r)^m \tag{B.171}$$

we get a solution

$$\phi^{I_t}(r) = \sum_{\alpha, m} \phi^{I_t}(r; \alpha, m) \,. \tag{B.172}$$

In this way ϕ^{I_t} becomes a function of the source $\mathcal{S}_t^{I_t}$, and we write the solution formally in terms of a Green's function G^{I_t} which we define by

$$\phi^{I_t}[\mathcal{S}_t^{I_t}](r) = \sum_{\alpha,m} \phi^{I_t}[\mathcal{S}_t^{I_t}](r;\alpha,m) \equiv \int_0^\infty \mathrm{d}r' G^{I_t}(r,r')\mathcal{S}_t^{I_t}(r)\,. \tag{B.173}$$

Summary: In the gauge given by

$$\begin{aligned} \tilde{\nabla}^{\tilde{k}} h_{\{ik\}} = \tilde{\nabla}^{\tilde{k}} h_{kr} &= 0\,, && \text{(B.174)}\\ \int \mathrm{d}^n\Psi\ \sqrt{\tilde{g}}\,\bar{Y}^i_{K_v}(\Psi)\,h_{ir}(r,\Psi) &= 0\,, && \text{(B.175)}\\ \int \mathrm{d}^n\Psi\ \sqrt{\tilde{g}}\,h_k^{\tilde{k}}(r,\Psi) &= 0\,, && \text{(B.176)} \end{aligned}$$

the general solution to (B.149) is

$$h_{ij}(y) = h_{ij}^{\mathcal{H}}(y) + \int \mathrm{d}^{n+1}y'\ \sqrt{g}\,\left((\mathcal{G}_g)_{ij}{}^{i'j'}(y,y')\,S_{i'j'}(y') + 2(\mathcal{G}_g)\ {}^{i'r}(y,y')\,S_{i'r}(y')\right), \tag{B.177}$$

$$h_{ir}(y) = \int \mathrm{d}^{n+1}y'\ \sqrt{g}\,2(\mathcal{G}_g)_{ir}{}^{i'r}(y,y')\,S_{i'r}(y')\,, \tag{B.178}$$

$$\begin{aligned} h_{rr}(y) = \int \mathrm{d}^{n+1}y'\ \sqrt{g}\Big(&(\mathcal{G}_g)_{rr}{}^{rr}(y,y')\,S_{rr}(y') + 2(\mathcal{G}_g)_{rr}{}^{i'r}(y;y')\,S_{i'r}(y')\\ &+ (\mathcal{G}_g)_{rr}{}^{i'j'}(y,y')\,S_{i'j'}(y')\Big)\,. \end{aligned} \tag{B.179}$$

The nonzero components of the metric Green's function $(\mathcal{G}_g)_{mn}{}^{m'n'}(y;y')$ are given by

$$\begin{aligned} (\mathcal{G}_g)_{ij}{}^{i'j'}(y,y') = (r')^{-n} \times \Bigg[&\sum_{I_t} G^{I_t}(r;r')\,Y^{I_t}_{\{ij\}}(\Psi)\,\bar{Y}^{\{i'j'\}}_{I_t}(\Psi') + \sum_{\lambda^{I_s}>n} \vartheta(r-r')\left(\frac{r'}{r}\right)^{\frac{n^2-5n+2}{n}}\\ &\times r'\left(\frac{\lambda^{I_s}}{n}(\lambda^{I_s}-n)\right)^{-1}\left(\frac{1}{n}\tilde{g}_{ij}(\Psi)Y^{I_s}(\Psi)\right)\tilde{\nabla}^{\{i'}\tilde{\nabla}^{j'\}}\bar{Y}^{I_s}(\Psi')\Bigg], \end{aligned} \tag{B.180}$$

$$2(\mathcal{G}_g)_{ij}{}^{i'r}(y,y') = (r')^{2-n} \times \Bigg[\sum_{\lambda^{I_s}>n} \vartheta(r-r')\left(\frac{r'}{r}\right)^{\frac{n^2-5n+2}{n}}\left(\lambda^{I_s}\right)^{-1}\left(\frac{1}{n}\tilde{g}_{ij}(\Psi)Y^{I_s}(\Psi)\right)\tilde{\nabla}^{i'}\bar{Y}^{I_s}(\Psi')\Bigg], \tag{B.181}$$

$$2(\mathcal{G}_g)_{ir}{}^{i'r}(y,y') = (r')^{-n} \times \sum_{\lambda^{I_v}>2(n-1)} \delta(r-r') \times \frac{r'^2}{2(n-1)-\lambda^{I_v}} \times Y_i^{I_v}(\Psi)\,\bar{Y}_{I_v}^{i'}(\Psi')\,, \quad \text{(B.182)}$$

$$\begin{aligned}(\mathcal{G}_g)_{rr}{}^{i'j'}(y,y') = (r')^{-n} \times \Bigg[\sum_{\lambda^{I_s}>n} \left(\delta(r-r') + \vartheta(r-r') \left(\frac{r'}{r}\right)^{\frac{n^2-5n+2}{n}} \times (n-1)r' \left(-\frac{1}{r}\frac{n-2}{n}\right)\right) \\ \times \left(\frac{n-1}{n}\lambda^{I_s}(\lambda^{I_s}-n)\right)^{-1} Y^{I_s}(\Psi)\,\tilde{\nabla}^{\{i'}\tilde{\nabla}^{j'\}}\bar{Y}^{I_s}(\Psi') \\ + \delta(r-r') \times \frac{-1}{2n(n-1)} \times Y^{\lambda^{I_s}=0}(\Psi)\,\tilde{g}^{i'j'}(\Psi')\bar{Y}_{\lambda^{I_s}=0}(\Psi')\Bigg], \quad \text{(B.183)}\end{aligned}$$

$$(\mathcal{G}_g)_{rr}{}^{rr}(y,y') = (r')^{-n} \times \delta(r-r') \times \frac{r'^2}{2n(n-1)} \times Y^{\lambda^{I_s}=0}(\Psi)\,\bar{Y}_{\lambda^{I_s}=0}(\Psi')\,. \quad \text{(B.184)}$$

APPENDIX C

SPECTROSCOPY ON $T^{1,1}$

The scalar, spinor, vector and two-form harmonics on $T^{1,1}$ were derived in [19, 20]. In this appendix we add to the spectroscopic data the spectrum of the Lichnerowicz operator Δ_L acting on symmetric two-tensors. The result is presented in equation (C.69). Along the way, we have independently reproduced the scalar, spinor and vector harmonics of [19, 20] with complete agreement.

C.1 Geometry of $T^{1,1}$

The manifold $T^{1,1}$ is defined as a coset space [44]

$$T^{1,1} = (SU(2)_1 \times SU(2)_2)\,/U(1)_H\,. \tag{C.1}$$

The two $SU(2)_{1,2}$ factors are generated by two independent sets of operators $T_A^{(1,2)}$, $A = 1,2,3$, with commutation relations

$$[T_A^{(1)}, T_B^{(1)}] = \epsilon_{AB}{}^{C} T_C^{(1)}\,, \quad [T_A^{(2)}, T_B^{(2)}] = \epsilon_{AB}{}^{C} T_C^{(2)}\,, \quad [T_A^{(1)}, T_B^{(2)}] = 0\,, \tag{C.2}$$

where $\epsilon_{12}{}^{3} = 1$, and the $U(1)_H$ and the orthogonal $U(1)_5$ are generated by[1]

$$T_H \equiv T_3^{(1)} - T_3^{(2)}, \quad T_5 \equiv T_3^{(1)} + T_3^{(2)}\,. \tag{C.3}$$

The standard commutation relations $[J_A, J_B] = i\epsilon_{AB}{}^{C} J_C$ for $SU(2)$ are obtained by identifying $J_A \equiv iT_A$.

[1]Notice that we have interchanged $T_H \leftrightarrow T_5$ compared to [19]. In our conventions the two $SU(2)_{1,2}$ factors are treated more symmetrically. We further differ from [19] in that we use an all plus sign metric, thus raising and lowering tangent space indices using the Kronecker delta instead of minus the Kronecker delta as in [19].

A general group element of $SU(2)_1 \times SU(2)_2$ can be parametrized in terms of Euler angles,

$$D(\varphi_1, \theta_1, \psi_1, \varphi_2, \theta_2, \psi_2) = e^{\varphi_1 T_3^{(1)}} e^{\theta_1 T_2^{(1)}} e^{\psi_1 T_3^{(1)}} e^{\varphi_2 T_3^{(2)}} e^{\theta_2 T_2^{(2)}} e^{\psi_2 T_3^{(2)}} . \tag{C.4}$$

For coset elements in $T^{1,1}$ we simply replace ψ_1 and ψ_2 with $(\psi_1 + \psi_2)/2$. Indeed, if we define $\psi_H \equiv (\psi_1 - \psi_2)/2$ and $\psi_5 \equiv (\psi_1 + \psi_2)/2$ we have that, up to $U(1)_H$ right transformations,

$$e^{\psi_1 T_3^{(1)}} e^{\psi_2 T_3^{(2)}} = e^{\psi_5 T_5} e^{\psi_H T_H} \cong e^{\psi_5 T_5} = e^{\psi_5 T_3^{(1)}} e^{\psi_5 T_3^{(2)}} . \tag{C.5}$$

Thus, coset elements in $T^{1,1}$ can be parametrized in terms of five angles $\Psi^\alpha = (\varphi_1, \theta_1, \varphi_2, \theta_2, \psi_5)$ in the following way

$$D(\varphi_1, \theta_1, \varphi_2, \theta_2, \psi_5) = e^{\varphi_1 T_3^{(1)}} e^{\theta_1 T_2^{(1)}} e^{\psi_5 T_3^{(1)}} e^{\varphi_2 T_3^{(2)}} e^{\theta_2 T_2^{(2)}} e^{\psi_5 T_3^{(2)}} . \tag{C.6}$$

C.1.1 Metric

We can construct a metric that is invariant under both $SU(2)_1 \times SU(2)_2$ left translations and $U(1)_5$ right translations[2], using the Maurer-Cartan one-forms $e^a = \mathrm{d}\Psi^\alpha \, e_\alpha{}^a$, $a = 1, \ldots, 5$, defined through

$$D^{-1}\mathrm{d}D \equiv e^a T_a + e^H T_H = e^A_{(1)} T_A^{(1)} + e^A_{(2)} T_A^{(2)} . \tag{C.7}$$

Here we have grouped the generators into $T_a = (T_i^{(1)}, T_r^{(2)}, T_5)$ and the one-forms into $e^a = (e^i_{(1)}, e^r_{(2)}, e^5)$, where we have split the indices up into $A_{(1)} = (i, 3)$ and $A_{(2)} = (r, 3)$ with $i, r = 1, 2$. Explicitly the one-forms of the two $SU(2)$ factors are given by

$$e^1_{(1)} = -\sin\theta_1 \cos\psi_5 \, \mathrm{d}\varphi_1 + \sin\psi_5 \, \mathrm{d}\theta_1 , \qquad e^1_{(2)} = -\sin\theta_2 \cos\psi_5 \, \mathrm{d}\varphi_2 + \sin\psi_5 \, \mathrm{d}\theta_2 ,$$

[2]Other right translations are ill-defined in the coset space since they do not commute with $U(1)_H$.

$$e^2_{(1)} = +\sin\theta_1 \sin\psi_5\, \mathrm{d}\varphi_1 + \cos\psi_5\, \mathrm{d}\theta_1\,, \qquad e^2_{(2)} = +\sin\theta_2 \sin\psi_5\, \mathrm{d}\varphi_2 + \cos\psi_5\, \mathrm{d}\theta_2\,,$$
$$e^3_{(1)} = \cos\theta_1\, \mathrm{d}\varphi_1 + \mathrm{d}\psi_5\,, \qquad e^3_{(2)} = \cos\theta_2\, \mathrm{d}\varphi_2 + \mathrm{d}\psi_5\,, \tag{C.8}$$

and the one-forms $e^H \equiv (e^3_{(1)} - e^3_{(2)})/2$ and $e^5 \equiv (e^3_{(1)} + e^3_{(2)})/2$, dual to T_H and T_5, are explicitly given by

$$e^H = \frac{1}{2}\,(\cos\theta_1\, \mathrm{d}\varphi_1 - \cos\theta_2\, \mathrm{d}\varphi_2)\,, \quad e^5 = \frac{1}{2}\,(\cos\theta_1\, \mathrm{d}\varphi_1 + \cos\theta_2\, \mathrm{d}\varphi_2 + 2\,\mathrm{d}\psi_5)\,. \tag{C.9}$$

Consistent with the isometries of $T^{1,1}$ we are free to rescale the one-forms, defining new one-forms $V^a = \mathrm{d}\Psi^\alpha V_\alpha{}^a$, in the following way

$$V^i = a^{-1}\, e^i_{(1)}\,, \quad V^r = b^{-1}\, e^r_{(2)}\,, \quad V^5 = c^{-1}\, e^5\,. \tag{C.10}$$

Using these as vielbeins the metric $\mathrm{d}s^2 = \delta_{ab} V^a V^b = \delta_{ab} V_\alpha{}^a V_\beta{}^b \mathrm{d}\Psi^\alpha \mathrm{d}\Psi^\beta$ takes the form

$$\mathrm{d}s^2 = \frac{1}{a^2}\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\varphi_1^2\right) + \frac{1}{b^2}\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\varphi_2^2\right) + \frac{1}{4c^2}\,(\cos\theta_1 \mathrm{d}\varphi_1 + \cos\theta_2 \mathrm{d}\varphi_2 + 2\mathrm{d}\psi_5)^2\,. \tag{C.11}$$

With the choice $a^2 = b^2 = 6$ and $c = 9/4$ the metric becomes Einstein $R_a^b = 4\delta_a^b$, and together with the identification $\psi \equiv 2\psi_5$ the metric takes the standard form [44]

$$\mathrm{d}s^2 = \frac{1}{6}\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\varphi_1^2\right) + \frac{1}{6}\left(\mathrm{d}\theta_2^2 + \sin^2\theta_2 \mathrm{d}\varphi_2^2\right) + \frac{1}{9}\,(\cos\theta_1 \mathrm{d}\varphi_1 + \cos\theta_2 \mathrm{d}\varphi_2 + \mathrm{d}\psi)^2\,. \tag{C.12}$$

C.2 Harmonics

The construction of tensor harmonics on group spaces, and more generally on coset spaces, is straightforward, see for example [45, 46, 47]. The Peter-Weyl theorem states that the collection of irreducible representations of the group elements forms a complete basis of functions on the space. For example, a function f on the group space $SU(2)$ can be expanded in terms of the matrix representations $\mathscr{D}^{(j)}_{nm}$, i.e.

$$f(\varphi,\theta,\psi) = \sum_{j,n,m} f^{(j)}_{nm}\, \mathscr{D}^{(j)}_{nm}(D(\varphi,\theta,\psi))\,, \tag{C.13}$$

where the $f^{(j)}_{nm}$ are constants and the $\mathscr{D}^{(j)}_{nm}$ are the familiar Wigner $\mathscr{D}$-matrices defined as

$$\mathscr{D}^{(j)}_{nm}(D(\varphi,\theta,\psi)) \equiv \langle j,n|D(\varphi,\theta,\psi)|j,m\rangle = \langle j,n|e^{\varphi T_3}e^{\theta T_2}e^{\psi T_3}|j,m\rangle\,. \tag{C.14}$$

Here $|j,m\rangle$ denotes a state in an $SU(2)$ representation with $J^2_A|j,m\rangle = j(j+1)|j,m\rangle$ and $J_3|j,m\rangle = m|j,m\rangle$, where $J_A = iT_A$.

In the case of $T^{1,1}$ we label the irreducible representations using their charges under the isometry group $SU(2)_1 \times SU(2)_2 \times U(1)_5$ and the charge under the $U(1)_H$. The representations of $SU(2)_1 \times SU(2)_2$ are labelled by the spin $J = (j_1, j_2)$ and magnetic quantum numbers $M = (m_1, m_2)$, while the representations of $U(1)_5$ and $U(1)_H$ are labeled by $R \equiv n_1 + n_2$ and $Q \equiv n_1 - n_2$. Thus we define irreducible harmonics $Y^{(J,M,R)}_{(Q)}$ on $T^{1,1}$

$$Y^{(J,M,R)}_{(Q)}(\varphi_1,\theta_1,\varphi_2,\theta_2,\psi_5) \equiv \mathscr{D}^{(j_1)}_{\frac{R+Q}{2}m_1}(D_1^{-1}(\varphi_1,\theta_1,\psi_5))\, \mathscr{D}^{(j_2)}_{\frac{R-Q}{2}m_2}(D_2^{-1}(\varphi_2,\theta_2,\psi_5))\,. \tag{C.15}$$

We choose to work with inverse representations $D^{-1}(\varphi,\theta,\psi) = D(-\psi,-\theta,-\varphi)$ for convenience. Explicitly we have that

$$Y^{(J,M,R)}_{(Q)}(\varphi_1,\theta_1,\varphi_2,\theta_2,\psi_5) = e^{iR\psi_5}e^{im_1\varphi_1}e^{im_2\varphi_2}\, d^{(j_1)}_{m_1\frac{R+Q}{2}}(\theta_1)\, d^{(j_2)}_{m_2\frac{R-Q}{2}}(\theta_2)\,, \tag{C.16}$$

where $d^{(j)}_{nm}(\theta) \equiv \langle j,n|e^{\theta T_3}|j,m\rangle$ denotes the little Wigner d-matrix which is orthogonal $d^{(j)}_{nm}(-\theta) = (d^{-1})^{(j)}_{nm}(\theta) = d^{(j)}_{mn}(\theta)$.

C.2.1 Scalar Harmonics

The scalar harmonics and the Laplacian eigenvalues can be determined without too much machinery. The scalar harmonics are given by $Y_{(Q)}^{(J,M,R)}$ in equation (C.16) with $Q = 0$, i.e.

$$\begin{aligned} Y_{(0)}^{(J,M,R)}(\varphi_1, \theta_1, \varphi_2, \theta_2, \psi_5) &= \mathscr{D}^{(j_1)}_{\frac{R}{2} m_1}(D_1^{-1}(\varphi_1, \theta_1, \psi_5))\, \mathscr{D}^{(j_2)}_{\frac{R}{2} m_2}(D_2^{-1}(\varphi_2, \theta_2, \psi_5)) \\ &= e^{i\frac{R}{2}2\psi_5} e^{im_1\varphi_1} e^{im_2\varphi_2}\, d^{(j_1)}_{m_1 \frac{R}{2}}(\theta_1)\, d^{(j_2)}_{m_2 \frac{R}{2}}(\theta_2)\,. \end{aligned} \tag{C.17}$$

Starting from equation (C.12) for the line element of $T^{1,1}$ the scalar Laplacian can be expressed in terms of Killing fields $\mathcal{K}_{A_1} = (\mathcal{K}_{A_1})^{\alpha_1}\partial_{\alpha_1}$ and $\mathcal{K}_{A_2} = (\mathcal{K}_{A_2})^{\alpha_2}\partial_{\alpha_2}$ on the two $SU(2)_{1,2}$ spaces together with the Killing field $\mathcal{K}_5 \equiv \partial/\partial\psi_5$

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_\alpha(\sqrt{g}g^{\alpha\beta}\partial_\beta) = 6\left[(\mathcal{K}_1^{(1)})^2 + (\mathcal{K}_2^{(1)})^2\right] + 6\left[(\mathcal{K}_1^{(2)})^2 + (\mathcal{K}_2^{(2)})^2\right] + \frac{9}{4}(\mathcal{K}_5)^2\,. \tag{C.18}$$

The Killing fields $\mathcal{K}_A = \mathcal{K}_A{}^\alpha\partial_\alpha$ on each of the two $SU(2)$'s are defined by the relation

$$\mathcal{K}_A D^{-1}(\varphi, \theta, \psi) = -T_A D^{-1}(\varphi, \theta, \psi)\,, \tag{C.19}$$

such that $[\mathcal{K}_A, \mathcal{K}_B] = \epsilon_{AB}{}^C\mathcal{K}_C$. From the defining relation (C.19) it follows that the Killing vectors are given by the vielbein inverses $\mathcal{K}_A{}^\alpha = \left[(D^{-1}\partial_\alpha D)^A\right]^{-1}$, so that

$$\begin{aligned} \mathcal{K}_1 &= -\frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\varphi} + \sin\psi\frac{\partial}{\partial\theta} + \cos\psi\cot\theta\frac{\partial}{\partial\psi}\,, \\ \mathcal{K}_2 &= +\frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\varphi} + \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\cot\theta\frac{\partial}{\partial\psi}\,, \\ \mathcal{K}_3 &= \frac{\partial}{\partial\psi}\,. \end{aligned} \tag{C.20}$$

In equation (C.18) we substitute into expression (C.20) $\varphi \to \varphi_\sigma$, $\theta \to \theta_\sigma$ and $\psi \to 2\psi_5$ for the two $SU(2)_\sigma$ copies, $\sigma = 1, 2$, and explicitly

$$(\mathcal{K}_1^{(\sigma)})^2 + (\mathcal{K}_2^{(\sigma)})^2 = \frac{1}{\sin\theta_\sigma}\frac{\partial}{\partial\theta_\sigma}\left(\sin\theta_\sigma\frac{\partial}{\partial\theta_\sigma}\right) + \frac{1}{\sin^2\theta_\sigma}\left(\frac{\partial}{\partial\varphi_\sigma} - \cos\theta_\sigma\frac{\partial}{\partial(2\psi_5)}\right)^2\,. \tag{C.21}$$

Now using $(\mathcal{K}_1^{(\sigma)})^2 + (\mathcal{K}_2^{(\sigma)})^2 = (\mathcal{K}_A^{(\sigma)})^2 - \partial^2/\partial(2\psi_5)^2$, and that when acting on the representations matrices $(\mathcal{K}_A^{(\sigma)})^2 = (T_A^{(\sigma)})^2 = -(J_A^{(\sigma)})^2 = -j_\sigma(j_\sigma + 1)$, we get

$$-\nabla^2 Y_{(0)}^{(J,M,R)} = \left(6\left[j_1(j_1+1) - (R/2)^2\right] + 6\left[j_2(j_2+1) - (R/2)^2\right] + 9R^2/4\right) Y_{(0)}^{(J,M,R)}. \tag{C.22}$$

Thus we conclude, in agreement with [19], that

$$\nabla^2 Y_{(0)}^{(J,M,R)} = -H_0(j_1, j_2, R) Y_{(0)}^{(J,M,R)}, \tag{C.23}$$

where

$$H_0(j_1, j_2, R) = 6\left(j_2(j_2+1) + j_2(j_2+1) - R^2/8\right). \tag{C.24}$$

C.2.2 Tensor Harmonics

The goal of this appendix is to calculate the spectrum of the Lichnerowicz operator acting on symmetric two-tensors,

$$\Delta_L Y_{ab} = -\mathcal{D}^2 Y_{ab} + 2R^c{}_{ab}{}^d Y_{cd} + 2R_{(a}{}^c Y_{b)c}, \tag{C.25}$$

where $R_{abc}{}^d$ and R_{ab} denote the tangent-space Riemann and Ricci tensor respectively, and $\mathcal{D}_\alpha$ denotes the tangent-space covariant derivative. The tangent-space covariant derivative is related to the standard world-tensor covariant derivative ∇_α in such a way that it commutes with conversion of tangent-to-world-space indices, e.g. for a vector $\nabla_\alpha Y_\beta = \nabla_\alpha(V_\beta{}^b Y_b) = V_\beta{}^b \mathcal{D}_\alpha Y_b$. The tangent-space covariant derivative $\mathcal{D} = \mathrm{d}\Psi^\alpha\, \mathcal{D}_\alpha$ can be written in the following useful form

$$\mathcal{D} = \mathrm{d} + \frac{1}{2}\omega^{ab}\Sigma_{ab}. \tag{C.26}$$

Here Σ_{ab} are the generators of tangent space-rotations with $[\Sigma_{ab}, \Sigma_{cd}] = g_{cb}\Sigma_{ad} - g_{ca}\Sigma_{bd} + g_{db}\Sigma_{ca} - g_{da}\Sigma_{cb}$. The generators carry appropriate tensor indices, e.g.

acting on a vector the generators $(\Sigma_{ab})^{cd} = 2\delta^c_{[a}\delta^d_{b]}$ carry vector indices. The tangent-space connection one-forms $\omega^a{}_b = \mathrm{d}\Psi^\alpha\,(\omega^a{}_b)_\alpha$ are related to the ordinary Christoffel connections $\Gamma^\alpha{}_{\beta\gamma}$ through $(\omega^a{}_b)_\alpha = V_\gamma{}^a\nabla_\alpha V_b{}^\gamma$ (which guarantees that $\nabla_\alpha Y_\beta = V_\beta{}^b\mathcal{D}_\alpha Y_b$). Using the torsion-free condition for the Christoffel connection $\Gamma^\alpha{}_{\beta\gamma} = \Gamma^\alpha{}_{\gamma\beta}$ this relation gives the defining equation for the connection one-forms

$$\mathrm{d}V^a + \omega^a{}_b \wedge V^b = 0\,. \tag{C.27}$$

Furthermore, using the connection we can determine the Riemann tensor $R_{abc}{}^d$, from which we obtain the components of the Riemann two-form,

$$R^a{}_b \equiv \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \equiv \frac{1}{2}R^a{}_{bcd}V^c \wedge V^c\,. \tag{C.28}$$

Connections We now calculate the connection and Riemann tensor on $T^{1,1}$. Consider the Maurer-Cartan equations for the two $SU(2)$ copies

$$\mathrm{d}e^A_{(\sigma)} + \frac{1}{2}\epsilon_{BC}{}^A e^B_{(\sigma)} \wedge e^C_{(\sigma)} = 0\,, \quad \sigma = 1, 2\,. \tag{C.29}$$

In terms of the vielbeins V^a obtained from the e^a by the rescalings $e^i_{(1)} = aV^i$, $e^r_{(2)} = bV^r$, $e^5 = cV^5$ these equations reduce to

$$\mathrm{d}V^i - \epsilon^i{}_j(cV^5 + e^H) \wedge V^j = 0\,, \tag{C.30}$$

$$\mathrm{d}V^r - \epsilon^r{}_s(cV^5 - e^H) \wedge V^s = 0\,, \tag{C.31}$$

$$\mathrm{d}V^5 + \frac{a^2}{4c}\epsilon_{ij}V^i \wedge V^j + \frac{b^2}{4c}\epsilon_{rs}V^r \wedge V^s = 0\,, \tag{C.32}$$

$$\mathrm{d}e^H + \frac{a^2}{4}\epsilon_{ij}V^i \wedge V^j - \frac{b^2}{4}\epsilon_{rs}V^r \wedge V^s = 0\,, \tag{C.33}$$

where $\epsilon_{ij} = \epsilon_{ij}{}^3$ and indices are raised and lowered using the unit matrix. We can now compare these equations to equation (C.27) for the connection one-forms $\omega^a{}_b$ and we can identify

$$\omega^5{}_i = -\frac{a^2}{4c}\epsilon_{ij}V^j\,, \qquad \omega^i{}_j = -\epsilon^i{}_j\left[+e^H + \left(c - \frac{a^2}{4c}\right)V^5\right], \tag{C.34}$$

$$\omega^5{}_r = -\frac{b^2}{4c}\epsilon_{rs}V^s\,, \qquad \omega^r{}_s = -\epsilon^r{}_s\left[-e^H + \left(c - \frac{b^2}{4c}\right)V^5\right]. \tag{C.35}$$

Notice that the connections ω^{ab} contain both the vielbeins V^a and e^H. In what follows it will be useful to split the the two kinds of contributions

$$\omega^{ab} \equiv M^{ab} + \Omega^{ab} \equiv M^{ab}_c\,V^c + \Omega^{ab}_H\,e^H\,. \tag{C.36}$$

From equations (C.34-C.35) it follows that

$$M^{ij}_5 = -\epsilon^{ij}\left(c - \frac{a^2}{4c}\right), \quad M^{rs}_5 = -\epsilon^{rs}\left(c - \frac{b^2}{4c}\right), \tag{C.37}$$

$$M^{5i}_j = -\frac{a^2}{4c}\epsilon^i{}_j\,, \qquad M^{5r}_s = -\frac{b^2}{4c}\epsilon^r{}_s\,, \tag{C.38}$$

$$\Omega^{ij}_H = -\epsilon^{ij}\,, \qquad \Omega^{rs}_H = +\epsilon^{rs}\,. \tag{C.39}$$

These results can also be derived using the general considerations of [48]. Starting from the Maurer-Cartan equations for the Maurer-Cartan one-forms $e^\Sigma = (e^a, e^H)$, $\Sigma = 1, 2, 3, 4, 5, H$,

$$\mathrm{d}e^\Sigma + \frac{1}{2}C_{\Lambda\Pi}{}^\Sigma\,e^\Lambda \wedge e^\Pi = 0\,, \tag{C.40}$$

where $C_{\Lambda\Pi}{}^\Sigma$ are the structure constants of the algebra $[T_\Lambda, T_\Pi] = C_{\Lambda\Pi}{}^\Sigma T_\Sigma$ for the generators $T_\Sigma = (T_a, T_H)$, one can derive

$$\Omega^{ab}_H = -C_H{}^{ab}\,, \quad M^{ab}_c = \frac{1}{2}\left(\frac{r_c r_b}{r_a}C_c{}^{ba} - \frac{r_c r_a}{r_b}C_c{}^{ab} - \frac{r_b r_a}{r_c}C^{ba}{}_c\right). \tag{C.41}$$

Here r_a denotes the rescaling parameters used in defining the vielbeins $V^\Sigma = (r_a^{-1}e^a, e^H)$, which in our case are given by $r_i = a$, $r_r = b$ and $r_5 = c$. Starting from equation (C.41) we reproduce equations (C.37-C.39).

Riemann tensor Using equation (C.28) it is now straightforward to go on and determine the components of the Riemann tensor $R_{abc}{}^d$:

$$R_{ijkl} = \left(a^2 - \frac{3a^4}{16c^2}\right)\epsilon_{ij}\epsilon_{kl}\,, \qquad R_{rsuv} = \left(b^2 - \frac{3b^4}{16c^2}\right)\epsilon_{rs}\epsilon_{uv}\,, \tag{C.42}$$

$$R_{ijrs} = -\frac{a^2b^2}{8c^2}\epsilon_{ij}\epsilon_{rs}\,, \qquad R_{irjs} = -\frac{a^2b^2}{16c^2}\epsilon_{ij}\epsilon_{rs}, \tag{C.43}$$

$$R^i{}_{5j5} = \frac{a^4}{16c^2}\delta^i_j\,, \qquad R^r{}_{5s5} = \frac{b^4}{16c^2}\delta^r_s\,, \tag{C.44}$$

and the Ricci tensor $R_{ac} = R_{abc}{}^b$

$$R^i{}_j = \left(a^2 - \frac{a^4}{8c^2}\right)\delta^i_j\,, \quad R^r{}_s = \left(b^2 - \frac{b^4}{8c^2}\right)\delta^r_s\,, \quad R^5{}_5 = \frac{a^4+b^4}{8c^2}\,. \tag{C.45}$$

For the choice $a = b = \sqrt{6}$ and $c = 3/2$ the metric becomes Einstein: $R^a{}_b = 4\delta^a_b$.

Covariant derivative Using that the connection splits as $\omega^{ab} = e^H\Omega^{ab}_H + V^cM^{ab}_c$ we can write the covariant derivative $\mathcal{D} = \mathrm{d} + \frac{1}{2}\omega^{ab}\Sigma_{ab}$ as $\mathcal{D} = \mathcal{D}^H + \frac{1}{2}V^cM^{ab}_c\Sigma_{ab}$, where we have defined the H-covariant derivative

$$\mathcal{D}^H \equiv \mathrm{d} + \frac{1}{2}e^H\Omega^{ab}_H\,\Sigma_{ab}\,. \tag{C.46}$$

The split is useful since, as we now show, when acting on harmonics $\mathcal{D}^H = -e^aT_a$. First, consider a left translation $gD = D'h$, where $g \in SU(2)_1 \times SU(2)_2$ and $h \in U(1)_H$. Then the Maurer-Cartan one-forms transform as

$$e'^a = (D'^{-1}\mathrm{d}D')^a = [h(D^{-1}\mathrm{d}D)h^{-1}]^a \equiv (D^{-1}\mathrm{d}D)^b(\mathrm{Adj}\,h^{-1})_b{}^a = e^b(\mathrm{Adj}\,h^{-1})_b{}^a\,, \tag{C.47}$$

that is, the action of T_H on vectors is given by the adjoint action $(T_H)_a{}^b = (\mathrm{Adj}\,T_H)_a{}^b$ or more generally on tensors by $T_H = \frac{1}{2}(\mathrm{Adj}\,T_H)_a{}^b\Sigma_b{}^a$. Now $(\mathrm{Adj}\,T_H)_a{}^b \equiv -C_{Ha}{}^b$ and from equation (C.41) we read off that $C_{Ha}{}^b = -(\Omega_H)_a{}^b$, so when acting on harmonics the H-covariant derivative takes the form

$$\mathcal{D}^H = \mathrm{d} + e^HT_H\,. \tag{C.48}$$

Then from the defining equations of the Maurer-Cartan one-forms follows the desired property $\mathcal{D}^HD^{-1} = (\mathrm{d} + e^HT_H)D^{-1} = -e^aT_aD^{-1}$, and the action of the covariant derivative $\mathcal{D} = V^a\mathcal{D}_a$ is determined completely by the group algebra

$$\mathcal{D}_c = -r_cT_c + \frac{1}{2}M^{ab}_c\Sigma_{ab}\,, \tag{C.49}$$

where the rescalings r_a are $r_i = a$, $r_r = b$ and $r_5 = c$. Using equation (C.37-C.38) for M_c^{ab} we explicitly get for the various components

$$\mathcal{D}_i = -aT_i^{(1)} + \frac{a^2}{4c}\epsilon_i{}^j\Sigma_{5j}\,, \tag{C.50}$$

$$\mathcal{D}_r = -bT_r^{(2)} + \frac{b^2}{4c}\epsilon_r{}^s\Sigma_{5s}\,, \tag{C.51}$$

$$\mathcal{D}_5 = -cT_5 - \left(c - \frac{a^2}{4c}\right)\Sigma_{12} - \left(c - \frac{b^2}{4c}\right)\Sigma_{34}\,. \tag{C.52}$$

Notice that the action on scalar harmonics with $\Sigma_{ab} = 0$ is extremely simple, i.e. $\mathcal{D}_a = r_a T_a$ and the scalar Laplacian is

$$\begin{aligned}\Box \equiv \sum_a (r_a T_a)^2 &= 6\left[(T_1^{(1)})^2 + (T_2^{(1)})^2\right] + 6\left[(T_1^{(2)})^2 + (T_2^{(2)})^2\right] + \frac{9}{4}T_5^2 \\ &= 6\left((T_A^{(1)})^2 + (T_A^{(2)})^2 - T_5^2/8 - T_H^2/2\right)\,, \end{aligned}\tag{C.53}$$

which reproduces the group theoretical structure for ∇^2 obtained in equation (C.18).

C.3 Symmetric Two-Tensor Harmonics

To simplify the action of the Lichnerowicz operator $(\Delta_L)_{ab}{}^{cd}$ we first go to a complex basis where the action of $T_3^{(\sigma)}$ is diagonal, i.e. we work with raising and lowering operators

$$T_\pm^{(\sigma)} = T_1^{(\sigma)} \pm iT_2^{(\sigma)}\,, \quad V_{(\sigma)}^\pm = \frac{1}{2}(V_{(\sigma)}^1 \mp iV_{(\sigma)}^2)\,. \tag{C.54}$$

That is, we do a transformation in the tangent space

$$T_{\bar{a}} = U_{\bar{a}}{}^b\, T_b\,, \quad V^{\bar{b}} = V^a\,(U^{-1})_a{}^{\bar{b}}\,, \tag{C.55}$$

where $a = 1, 2, 3, 4, 5$ and $\bar{a} = +_1, -_1, +_2, -_2, 5$. Lower indices are converted using ${U_{\bar{a}}}^{b}$ and upper indices are converted using ${(U^{-1})_a}^{\bar{b}}$, where

$$ {U_{\bar{a}}}^{b} = \begin{pmatrix} 1 & i & 0 & 0 & 0 \\ 1 & -i & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 \\ 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad {(U^{-1})_a}^{\bar{b}} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -i & i & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -i & i & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}. \tag{C.56} $$

In this basis the Lichnerowicz operator takes the form ${(\Delta_L)_{\bar{a}\bar{b}}}^{\bar{c}\bar{d}} = {U_{\bar{a}}}^{a}\, {U_{\bar{b}}}^{b}\, {(\Delta_L)_{ab}}^{cd} {(U^{-1})_c}^{\bar{c}}\, {(U^{-1})_d}^{\bar{d}}$.

We then go on and introduce a complete and orthogonal basis of symmetric matrices $E^N_{\bar{a}\bar{b}}$, labeled by $N = 1, \ldots, 15$, with the properties

$$ E^M_{\bar{a}\bar{b}} E^{\bar{a}\bar{b}}_N = \delta^M_N \,, \quad E^N_{\bar{a}\bar{b}} E^{\bar{c}\bar{d}}_N = \delta^{\bar{c}}_{(\bar{a}} \delta^{\bar{d}}_{\bar{b})} \,, \tag{C.57} $$

where we have defined $E^{\bar{a}\bar{b}}_N$ numerically as $E^{\bar{a}\bar{b}}_N \equiv E^N_{\bar{a}\bar{b}}$. We then use $E^N_{\bar{a}\bar{b}}$ to convert all pairs of symmetric indices $\bar{a}\bar{b}$ into an index N, such that the Lichnerowicz operator becomes a 15×15 matrix

$$ {(\Delta_L)_M}^N = -{(\mathcal{D}^2)_M}^N + 2{R_M}^N + 8{\delta_M}^N \,, \tag{C.58} $$

where ${(\mathcal{D}^2)_M}^N = E^{\bar{a}\bar{b}}_M\, {(\mathcal{D}^2)_{\bar{a}\bar{b}}}^{\bar{c}\bar{d}}\, E^N_{\bar{c}\bar{d}}$ and ${R_M}^N = E^{\bar{a}\bar{b}}_M {R^{\bar{c}}{}_{\bar{a}\bar{b}}}^{\bar{d}} E^N_{\bar{c}\bar{d}}$, also we used that ${R_a}^b = 4\delta^b_a$. We choose $E^N_{\bar{a}\bar{b}}$ such that the basis of symmetric two-tensors $W^N \equiv E^N_{\bar{a}\bar{b}}\, V^{\bar{a}} \otimes V^{\bar{b}}$ takes the form

$$
\begin{aligned}
W^1 &= V^+_{(1)} \otimes V^+_{(1)} \,, & W^6 &= \frac{1}{\sqrt{2}}(V^+_{(1)} \otimes V^-_{(1)} + V^-_{(1)} \otimes V^+_{(1)}) \,, & W^{11} &= \frac{1}{\sqrt{2}}(V^-_{(1)} \otimes V^-_{(2)} + V^-_{(2)} \otimes V^-_{(1)}) \,, \\
W^2 &= V^-_{(1)} \otimes V^-_{(1)} \,, & W^7 &= \frac{1}{\sqrt{2}}(V^+_{(1)} \otimes V^+_{(2)} + V^+_{(2)} \otimes V^+_{(1)}) \,, & W^{12} &= \frac{1}{\sqrt{2}}(V^-_{(1)} \otimes V^5 + V^5 \otimes V^-_{(1)}) \,, \\
W^3 &= V^+_{(2)} \otimes V^+_{(2)} \,, & W^8 &= \frac{1}{\sqrt{2}}(V^+_{(1)} \otimes V^-_{(2)} + V^-_{(2)} \otimes V^+_{(1)}) \,, & W^{13} &= \frac{1}{\sqrt{2}}(V^+_{(2)} \otimes V^-_{(2)} + V^-_{(2)} \otimes V^+_{(2)}) \,, \\
W^4 &= V^-_{(2)} \otimes V^-_{(2)} \,, & W^9 &= \frac{1}{\sqrt{2}}(V^+_{(1)} \otimes V^5 + V^5 \otimes V^+_{(1)}) \,, & W^{14} &= \frac{1}{\sqrt{2}}(V^+_{(2)} \otimes V^5 + V^5 \otimes V^+_{(2)}) \,, \\
W^5 &= V^5 \otimes V^5 \,, & W^{10} &= \frac{1}{\sqrt{2}}(V^-_{(1)} \otimes V^+_{(2)} + V^+_{(2)} \otimes V^-_{(1)}) \,, & W^{15} &= \frac{1}{\sqrt{2}}(V^-_{(2)} \otimes V^5 + V^5 \otimes V^-_{(2)}) \,.
\end{aligned}
$$

Then if we want to construct a symmetric two-tensor $Y^{(J,M,R)} = Y_N^{(J,M,R)} W^N$ with charges (J, M, R) we pair up basis tensors W^N with irreducible harmonics $Y_{(Q_N)}^{(J,M,R_N)}$ of appropriately chosen charges R_N and Q_N, i.e.

$$Y^{(J,M,R)} = \sum_N c_N Y_{(Q_N)}^{(J,M,R_N)} W^N, \tag{C.59}$$

where c_N are constants and the $Y_{(Q_N)}^{(J,M,R_N)}$ are chosen according to the charges ${(T_5)_M}^N$ and ${(T_H)_M}^N$ of the basis tensors W_N, where

$$\begin{aligned}
{(T_3^{(1)})_M}^N &= -{(\Sigma_{12})_M}^N = -i \operatorname{diag}(-2, +2, 0, 0, 0, 0, -1, -1, -1, +1, +1, +1, 0, 0, 0)\,, \\
{(T_3^{(2)})_M}^N &= -{(\Sigma_{34})_M}^N = -i \operatorname{diag}(0, 0, -2, +2, 0, 0, -1, +1, 0, -1, +1, 0, 0, -1, +1)\,, \\
{(T_5)_M}^N &= {(T_3^{(1)} + T_3^{(2)})_M}^N = -i \operatorname{diag}(-2, +2, -2, +2, 0, 0, -2, 0, -1, 0, +2, +1, 0, -1, +1)\,, \\
{(T_H)_M}^N &= {(T_3^{(1)} - T_3^{(2)})_M}^N = -i \operatorname{diag}(-2, +2, +2, -2, 0, 0, 0, -2, -1, +2, 0, +1, 0, +1, -1)\,.
\end{aligned}$$

Explicitly, in this basis, the components of the symmetric two-tensor take the form

$$Y_N^{(J,M,R)} = \begin{pmatrix} c_1 Y_{(-2)}^{(J,M,R-2)} \\ c_2 Y_{(+2)}^{(J,M,R+2)} \\ c_3 Y_{(+2)}^{(J,M,R-2)} \\ c_4 Y_{(-2)}^{(J,M,R+2)} \\ c_5 Y_{(0)}^{(J,M,R)} \\ c_6 Y_{(0)}^{(J,M,R)} \\ c_7 Y_{(0)}^{(J,M,R-2)} \\ c_8 Y_{(-2)}^{(J,M,R)} \\ c_9 Y_{(-1)}^{(J,M,R-1)} \\ c_{10} Y_{(+2)}^{(J,M,R)} \\ c_{11} Y_{(0)}^{(J,M,R+2)} \\ c_{12} Y_{(+1)}^{(J,M,R+1)} \\ c_{13} Y_{(0)}^{(J,M,R)} \\ c_{14} Y_{(+1)}^{(J,M,R-1)} \\ c_{15} Y_{(-1)}^{(J,M,R+1)} \end{pmatrix} . \tag{C.60}$$

To determine the Lichnerowicz operator we also need the Riemann tensor and the Laplacian. In the above specified basis the components of the Riemann tensor are given by

$$
R_M{}^N = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & -2\sqrt{2} & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \tag{C.61}
$$

and row 1 through 15 and column 1 through 8 of the Laplacian $(\mathcal{D}^2)_M{}^N$ are given by

$$
(\mathcal{D}^2)_{1-15}{}^{1-8} = \begin{pmatrix}
\Box-3-3iT_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Box-3+3iT_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Box-3-3iT_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Box-3+3iT_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Box-8 & \sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 4\sqrt{2} & \Box-2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Box-3-3iT_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Box-2 \\
2\sqrt{3}iT_-^{(1)} & 0 & 0 & 0 & 4\sqrt{3}iT_+^{(1)} & -\sqrt{6}iT_+^{(1)} & \sqrt{6}iT_-^{(2)} & -\sqrt{6}iT_+^{(2)} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2\sqrt{3}iT_+^{(1)} & 0 & 0 & -4\sqrt{3}iT_-^{(1)} & \sqrt{6}iT_-^{(1)} & 0 & 0 \\
0 & 0 & 0 & 0 & 4\sqrt{2} & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{3}iT_-^{(2)} & 0 & 4\sqrt{3}iT_+^{(2)} & 0 & \sqrt{6}iT_-^{(1)} & 0 \\
0 & 0 & 0 & -2\sqrt{3}iT_+^{(2)} & -4\sqrt{3}iT_-^{(2)} & 0 & 0 & \sqrt{6}iT_-^{(1)}
\end{pmatrix}
$$

while row 1 through 15 and column 9 through 15 of the Laplacian $(\mathcal{D}^2)_A{}^B$ are given by

$$
(\mathcal{D}^2)_{1-15}{}^{9-15} = \begin{pmatrix}
4\sqrt{3}iT_+^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -4\sqrt{3}iT_-^{(1)} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4\sqrt{3}iT_+^{(2)} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -4\sqrt{3}iT_-^{(2)} \\
2\sqrt{3}iT_-^{(1)} & 0 & 0 & -2\sqrt{3}iT_+^{(1)} & \sqrt{2} & 2\sqrt{3}iT_-^{(2)} & -2\sqrt{3}iT_+^{(2)} \\
-2\sqrt{6}iT_-^{(1)} & 0 & 0 & 2\sqrt{6}iT_+^{(1)} & 0 & 0 & 0 \\
2\sqrt{6}iT_+^{(2)} & 0 & 0 & 0 & 0 & 2\sqrt{6}iT_+^{(1)} & 0 \\
-2\sqrt{6}iT_-^{(2)} & 0 & 0 & 0 & 0 & 0 & 2\sqrt{6}iT_+^{(1)} \\
\Box-\frac{29}{4}-\frac{3}{2}iT_5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \Box-2 & 0 & 2\sqrt{6}iT_+^{(2)} & 0 & -2\sqrt{6}iT_-^{(1)} & 0 \\
0 & 0 & \Box-3+3iT_5 & -2\sqrt{6}iT_-^{(2)} & 0 & 0 & -2\sqrt{6}iT_-^{(1)} \\
0 & \sqrt{6}iT_-^{(2)} & -\sqrt{6}T_+^{(2)} & \Box-\frac{29}{4}+\frac{3}{2}iT_5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \Box-2 & -2\sqrt{6}iT_-^{(2)} & 2\sqrt{6}iT_+^{(2)} \\
0 & -\sqrt{6}iT_+^{(1)} & 0 & 0 & -\sqrt{6}iT_+^{(2)} & \Box-\frac{29}{4}-\frac{3}{2}iT_5 & 0 \\
0 & 0 & -\sqrt{6}iT_+^{(1)} & 0 & \sqrt{6}iT_-^{(2)} & 0 & \Box-\frac{29}{4}+\frac{3}{2}iT_5
\end{pmatrix},
$$

where $\Box$ is defined in equation (C.53).

Now for fixed indices M, N the operator $(\Delta_L)_M{}^N$ takes $Y^{(J,M,R_N)}_{(Q_N)}$ to $Y^{(J,M,R_M)}_{(Q_M)}$[3] with some numerical coefficient $\mathcal{M}_M{}^N$, i.e.

$$
(\Delta_L)_M{}^N Y^{(J,M,R_N)}_{(Q_N)} \equiv \mathcal{M}_M{}^N Y^{(J,M,R_M)}_{(Q_M)}, \quad \text{no sum over } M, N \tag{C.62}
$$

Then we diagonalize $(\Delta_L)_M{}^N$ by finding the eigenvectors c_N and eigenvalues λ of $\mathcal{M}_M{}^N$, with $\sum_N \mathcal{M}_M{}^N c_N = \lambda\, c_M$, such that the vector with coefficients $Y^{(J,M,R)}_N = c_N Y^{(J,M,R_N)}_{(Q_N)}$ has the eigenvalue λ, indeed

$$
\Delta_L Y^{(J,M,R)}_M = \sum_N (\Delta_L)_M{}^N c_N Y^{(J,M,R_N)}_{(Q_N)} = \sum_N \mathcal{M}_M{}^N\, c_N\, Y^{(J,M,R_M)}_{Q_M} = \lambda\, Y^{(J,M,R)}_M\,. \tag{C.63}
$$

[3] We hope that there is no confusion in using both $M = 1, \ldots, 15$ as an index and $M = (m_1, m_2)$ as a charge label.

When deriving the numerical matrix $\mathcal{M}_M{}^N$ we use that all terms in the Lichnerowicz operator give purely group theoretical factors familiar from $SU(2)$ group theory

$$T_\pm^{(1)} Y_{(Q)}^{(J,M,R)} = -iJ_\mp\left(j_1, \frac{R+Q}{2}\right) Y_{(Q\mp 1)}^{(J,M,R\mp 1)}\,, \tag{C.64}$$

$$T_\pm^{(2)} Y_{(Q)}^{(J,M,R)} = -iJ_\mp\left(j_2, \frac{R-Q}{2}\right) Y_{(Q\pm 1)}^{(J,M,R\mp 1)}\,, \tag{C.65}$$

$$T_5\, Y_{(Q)}^{(J,M,R)} = -iR\, Y_{(Q)}^{(J,M,R)}\,, \tag{C.66}$$

$$T_H\, Y_{(Q)}^{(J,M,R)} = -iQ\, Y_{(Q)}^{(J,M,R)}\,, \tag{C.67}$$

$$\Box\, Y_{(Q)}^{(J,M,R)} = -6\left(j_1(j_1+1) + j_2(j_2+1) - R^2/8 - Q^2/2\right) Y_{(Q)}^{(J,M,R)}\,. \tag{C.68}$$

where $J_\pm(j,m) = \sqrt{j(j+1) - m(m\pm 1)}$ and where we used expression (C.53) for $\Box$.

When we diagonalize $\mathcal{M}_M{}^N$ we find 15 eigenvalues: one scalar trace mode, five longitudinal modes of which one corresponds to a scalar and four correspond to transverse vectors, and nine transverse-traceless modes. In terms of the quantity H_0 defined in (C.24), the nine eigenvalues corresponding to transverse-traceless modes are

$$\begin{aligned}\lambda_{\text{Transverse-Traceless}} = \{&H_0(j_1, j_2, R) + 8\,,\ H_0(j_1, j_2, R\pm 4) + 8\,,\\ &9 + H_0(j_1, j_2, R\pm 2) - 2\sqrt{H_0(j_1, j_2, R\pm 2) + 4}\,,\\ &9 + H_0(j_1, j_2, R\pm 2) + 2\sqrt{H_0(j_1, j_2, R\pm 2) + 4}\,,\\ &12 + H_0(j_1, j_2, R) \pm 4\sqrt{H_0(j_1, j_2, R) + 4}\}\,.\end{aligned} \tag{C.69}$$

C.4 Smallest Eigenvalues

For the lowest quantum numbers (J, M, R) the Lichnerowicz operator reduces considerably in size and the formula (C.69) for the eigenvalues is not applicable. The result from studying the individual cases $0 \leq j_1, j_2 \leq 1$ is summarized in Table C.1.

Eigenvalues of the Lichnerowicz operator

j_1	j_2	$\lvert R\rvert$	$\lambda_{\text{Longitudinal/Trace}}$	$\lambda_{\text{Transverse-Traceless}}$
0	0	0	0	4, 20
0	0	2	—	5
$\frac{1}{2}$	$\frac{1}{2}$	1	8.25, 8.25, 11.25, 19.25,	24.25, 34.25
$\frac{1}{2}$	$\frac{1}{2}$	3	—	16.25
1	0	0	12, 12, 24	20, 40
1	0	2	15	29
1	1	0	$17.42^{\dagger}$, 24, 24, 24, 24, 38.58^{*}	$14.83^{\S}$, 32, 40, 40, $57.17^{\dagger\dagger}$
1	1	2	21, 21, 27, 35	22.42^{**}, 29, $43.58^{\S\S}$, 53
1	1	4	—	32

$^{\dagger}28-4\sqrt{7}\approx 17.42$, $^{*}28+4\sqrt{7}\approx 38.58$, $^{\S}36-8\sqrt{7}\approx 14.83$, $^{\dagger\dagger}36+8\sqrt{7}\approx 57.17$, $^{**}33-4\sqrt{7}\approx 22.42$, $^{\S\S}33+4\sqrt{7}\approx 43.58$

Table C.1: The eigenvalues of the Lichnerowicz operator for the modes with the lowest quantum numbers. The two smallest eigenvalues $\lambda = 4, 5$ correspond to modes with $j_1 = j_2 = 0$ and $R = 0, 2$. The third-smallest eigenvalue is $\lambda = 36 - 8\sqrt{7}$ and corresponds to a mode with $j_1 = j_2 = 1$ and $R = 0$.

C.4.1 Modes with $j_1 = j_2 = 0$

As a concrete example we discuss the most degenerate case when $j_1 = j_2 = 0$. In this case only the vielbeins can be used to construct tensors, and there are only five tensors with $Q = 0$:

$$W^5 = V^5 \otimes V^5 \,, \tag{C.70}$$

$$W^{6,13} = \frac{1}{\sqrt{2}}(V^+_{(1,2)} \otimes V^-_{(1,2)} + V^-_{(1,2)} \otimes V^+_{(1,2)}) \,, \tag{C.71}$$

$$W^{7,11} = \frac{1}{\sqrt{2}}(V^\pm_{(1)} \otimes V^\pm_{(2)} + V^\pm_{(2)} \otimes V^\pm_{(1)}) \,. \tag{C.72}$$

Of these, only W^7 has $R = 2$ and only W^{11} has $R = -2$, so for these two modes the Lichnerowicz operator reduces to 1×1 matrices and the eigenvalues can be read off directly from the expressions for $(\Delta_L)_7{}^7$ and $(\Delta_L)_{11}{}^{11}$:

$$\Delta_L W^{7,11}_{ab} = 5\, W^{7,11}_{ab} \,. \tag{C.73}$$

Both modes $W^{7,11}$ are transverse-traceless $g^{ab} W^{7,11}_{ab} = \mathcal{D}^a W^{7,11}_{ab} = 0$.

The modes $W^{5,6,13}$ all have $R = 0$, so the Lichnerowicz operator is reduced to a 3×3 matrix which has three eigenvalues $\lambda = 0, 4, 20$. Of these, the $\lambda = 0$ eigenvalue corresponds to a trace mode:

$$\begin{aligned} Y_0^{\text{Trace}} &= 2\sqrt{2}(V^6 + V^{13}) + V^5 \\ &= 2V^+_{(1)} \otimes V^-_{(1)} + 2V^-_{(1)} \otimes V^+_{(1)} + 2V^+_{(2)} \otimes V^-_{(2)} + 2V^-_{(2)} \otimes V^+_{(2)} + V^5 \otimes V^5 \\ &\equiv g_{ab} V^a \otimes V^b \,. \end{aligned} \tag{C.74}$$

Note that $\Delta_L g_{ab} = 0$ trivially. The two other modes are transverse-traceless and the mode with $\lambda = 4$ is given by

$$\begin{aligned} Y_4^{\text{Transverse-Traceless}} &= 2\sqrt{2}\,(V^6 - V^{13}) \\ &= \left[2V^+_{(1)} \otimes V^-_{(1)} + 2V^-_{(1)} \otimes V^+_{(1)}\right] - \left[2V^+_{(2)} \otimes V^-_{(2)} + 2V^-_{(2)} \otimes V^+_{(2)}\right] \,, \end{aligned} \tag{C.75}$$

while the the mode with $\lambda = 20$ is given by

$$\begin{aligned} Y_{20}^{\text{Transverse-Traceless}} &= 2\sqrt{2}\,(V^6 + V^{13}) + 4\,V^5 \\ &= 2V^+_{(1)} \otimes V^-_{(1)} + 2V^-_{(1)} \otimes V^+_{(1)} + 2V^+_{(2)} \otimes V^-_{(2)} + 2V^-_{(2)} \otimes V^+_{(2)} + 4V^5 \otimes V^5 \,. \end{aligned} \tag{C.76}$$

C.4.2 Higher quantum numbers

From Table C.1 we see that the smallest transverse-traceless eigenvalues are

$$\lambda_{\text{transverse-traceless}} = 4, 5, 14.83, 16.25, \ldots \tag{C.77}$$

The modes with $\lambda = 4, 5$ have $j_1 = j_2 = 0$ and are discussed in the above section. The third-smallest eigenvalue is $\lambda = 36 - 8\sqrt{7} \approx 14.83$ and corresponds to a transverse-traceless mode with $j_1 = j_2 = 1$ and $R = 0$. This mode has eigenvector $c_5 = -\sqrt{2}(8 - \sqrt{28})$, $c_6 = c_{13} = 8 - \sqrt{28}$, $c_9 = c_{14} = \sqrt{3}(2 - \sqrt{28})$, $c_{12} = c_{15} = -\sqrt{3}(2 - \sqrt{28})$, $c_7 = c_{11} = -10 - \sqrt{28}$, $c_8 = c_{10} = 6$, and all other c_N's zero.

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