COSMOLOGICAL MODELS OF MODIFIED GRAVITY

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The recent discovery of dark energy has prompted an investigation of ways in which the accelerated expansion of the universe can be realized. In this dissertation, we present two separate projects related to dark energy. The first project analyzes a class of braneworld models in which multiple branes float in a five-dimensional anti-de Sitter bulk, while the second investigates a class of dark energy models from an effective field theory perspective.

Investigations of models including extra dimensions have led to modifications of gravity involving a number of interesting features. In particular, the Randall-Sundrum model is well-known for achieving an amelioration of the hierarchy problem. However, the basic model relies on Minkowski branes and is subject to solar system constraints in the absence of a radion stabilization mechanism. We present a method by which a four-dimensional low-energy description can be obtained for braneworld scenarios, allowing for a number of generalizations to the original models. This method is applied to orbifolded and uncompactified N-brane models, deriving an effective four-dimensional action. The parameter space of this theory is constrained using observational evidence, and it is found that the generalizations do not weaken solar system constraints on the original model. Furthermore, we find that general N-brane systems are qualitatively similar to the two-brane case, and do not naturally lead to a viable dark energy model.

We next investigate dark energy models using effective field theory techniques. We describe dark energy through a quintessence field, employing a derivative expansion. To the accuracy of the model, we find transformations to write the description in a form involving no higher-order derivatives in the equations of motion. We use a pseudo-Nambu-Goldstone boson construction

to motivate the theory, and find the regime of validity and scaling of the operators using this. The regime of validity is restricted to a class of models for which both the derivative expansion and EFT construction is valid, which forces the quintessence potential to be the dominant source of energy-density in this class of model. The resulting effective theory is described by nine free functions.

Biographical Sketch

Jolyon Bloomfield was born in 1984 in Sydney, Australia. His family moved a number of times before settling down in a tiny village called Uki, nestled in the picturesque Tweed Valley on the East coast of Australia. He completed his schooling at Wollumbin High School in 2002, the year he represented Australia at the International Physics Olympiad, where he was awarded a silver medal.

In 2003, Jolyon went to the Australian National University in Canberra to pursue a Bachelor of Philosophy (Honors) degree, majoring in physics and math. In 2006, he completed his degree, and was awarded First Class Honors as well as the University Medal. In 2007, he travelled to the USA to begin his graduate studies in theoretical physics at Cornell University.

This dissertation is dedicated to my grandparents Audrey Woodland and Geoffrey Bloomfield.

Acknowledgements

It has been a pleasure to work with my research advisor Professor Éanna É. Flanagan over the past few years. I have benefited from his outstanding breadth of knowledge repeatedly, and it is certainly to him that I owe my development as a researcher. His skill at identifying interesting problems to work on has kept me enthusiastic about my work, and led to much stimulating discussion. I am also delighted to join his academic genealogy, with advisors including Kip Thorne, John Wheeler, Joseph Fourier, Simeon Poisson, Carl Friedrich Gauss, Joseph-Louis Lagrange, and Pierre-Simon Laplace.

I would like to thank my colleagues, particularly Kristofer Henriksson, Curran Muhlberger, Justin Vines and Leo Stein, for their insights, commentary, and assistance in many a calculation. Although we never formally collaborated, I appreciate all the help that they have given me, and I hope that I've helped them in return.

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In my final few years at Cornell, my social and exercise life was dominated by dancing. Anna Viau and Mary Gooding in particular have shared many dances, coffees, dinners, and talks, and helped me through a number of hard times with their kind words and support. Thank you both, from the bottom of my heart. For the short time that she was here, Alex Francis-Dixon, a fellow Australian, was a wonderful dance partner who ensured that I routinely got enough exercise and socializing in my life.

I would like to thank my undergraduate advisors, John Close and David Williams, who encouraged me to broaden my horizons and apply to America for my PhD. I certainly wouldn't be here today if it wasn't for them (especially John, who told me I should apply to Cornell!).

My parents have been ever-encouraging and supportive, no matter what I do. I'm particularly thankful for their visits early in my time in the USA, helping me to purchase my first car, and to get set up for the snowy Ithaca winter. I must also thank my siblings, Felix, Jarrah, Celeste and Nathaniel, who have variously kept me distracted during my PhD. I may not have been able to convince them to study physics at university, but I seem to have convinced most of them to go to my undergraduate institution. A special thank you to my sister, who came and visited around Christmas in my final year.

A particularly special thanks must go to all of my relatives who came to Ithaca for my graduation ceremony. My grandmother, Audrey Woodland, aunt and uncle Anne and Noel Hall, aunt Sue Woodland, and parents Peter and Marie Bloomfield all traveled from Australia to be here for my special day. Furthermore, my uncle, aunt and cousin Gilles, Jane and Carina Charbonneau, my aunt and cousins Lucie, Tschunna and Thia Charbonneau all traveled from Canada to be there for me.

I would like to thank the anonymous referees of my publications [1, 2, 3], who challenged me to strive towards better and clearer scholarship.

Finally, I must thank Tracy Slatyer, who has given me constant support and companionship throughout my time at Cornell University. Her achievements have been an inspiration to me for many years now, and I am very glad to have her presence in my life. Thank you, Tracy, for everything.

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List of Abbreviations

The following abbreviations are commonly used in this document.

- ADD – *Arkani-Hamed, Dimopoulos and Dvali*, referring to their model described in [4].
- AdS – *Anti de-Sitter*, referring to a maximally symmetric space with negative curvature, typically involving a negative cosmological constant.
- BAO – *Baryon Acoustic Oscillations*.
- CMB – *Cosmic Microwave Background*, referring to the cosmic microwave background radiation.
- DGP – *Dvali, Gabadadz, Porrati*, referring to their model [5].
- EFT – *Effective Field Theory*.
- FRW – *Friedmann-Robertson-Walker*, referring to the metric or cosmology.
- KK – *Kaluza-Klein*, typically referring to the Kaluza-Klein modes of a field.
- ΛCDM – Λ *Cold Dark Matter*, referring to the concordance model of cosmology.
- pNGB – *pseudo-Nambu-Goldstone Boson*.
- PPN – *Parameterized Post-Newtonian*, referring to a number of terms to describe deviations from general relativity.
- RS – *Randall-Sundrum*, referring to the Randall-Sundrum model (RS-I [6] or RS-II [7]).
- WEP – *Weak Equivalence Principle*.
- WMAP – *Wilkinson Microwave Anisotropy Probe*, referring to the satellite or data obtained therefrom.

Preface

Citations to Previously Published Work

This dissertation includes material from work that has been previously published or submitted as follows.

Jolyon K. Bloomfield and Éanna É. Flanagan. A Four-dimensional description of five-dimensional N-brane models [1]. Physical Review D, 82:124013, 2010.

Jolyon K. Bloomfield and Éanna É. Flanagan. Gravitational Interactions in a General Multibrane Model [2]. Physical Review D, 84:104016, 2011.

Jolyon K. Bloomfield and Éanna É. Flanagan. A Class of Effective Field Theory Models of Cosmic Acceleration [3]. 2011.
Submitted to Journal of Cosmology and Astroparticle Physics.

Permission from coauthors has been granted for these works to be included in this dissertation.

Notation

This dissertation describes two major projects. While efforts have been made to standardize notation throughout this document, some idiosyncrasies are inevitable. The following conventions are common for both projects.

- We use natural units with $c = \hbar = 1$.
- We define the reduced Planck mass $m_P^2 = 1/8\pi G$.
- We use the $\{-,+,+,+\}$ metric signature ($\{-,+,+,+,+\}$ in five dimensions) and the sign conventions $(+,+,+)$ in the notation of Ref. [8].

Braneworld Models

These conventions are specific to Chapters 2 and 3, which investigate braneworld models.

- The metric g refers to a five-dimensional metric, while the metric h refers to a four-dimensional metric.
- Many functions, coordinates and parameters will be indexed by some index n in this work. For coordinates and parameters, the index will always be in the lower right, e.g, x_n. For functions, the index will be in the upper left, e.g, ${}^n g_{\alpha\beta}$.
- We use capital Greek letters (Γ, Σ, Θ) to index five-dimensional tensors in arbitrary coordinate systems. When we specialize our coordinate system, we will use lowercase Greek letters (α, β, γ) to index five-dimensional tensors. We use Roman letters (a, b, c) for four-dimensional tensors.
- The stress-energy tensor for matter on a brane is defined as the following.

$$ {}^n S_m[{}^n h_{ab} + \delta {}^n h_{ab}, {}^n \phi] = {}^n S_m[{}^n h_{ab}, {}^n \phi] - \frac{1}{2} \int d^4 w_n \sqrt{-{}^n h}\, {}^n T_{ab} \delta {}^n h^{ab} \qquad (0.0.1) $$

Dark Energy Models

These conventions are specific to Chapter 4, which investigates dark energy models.

- We use lowercase Greek letters (α, β, γ) to index all four-dimensional tensors.
- The Einstein and Jordan frame metrics are $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ respectively, and the corresponding derivative operators are ∇_μ and $\bar{\nabla}_\mu$.
- We use the usual abbreviations $(\nabla\phi)^2 = g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ and $\Box\phi = \nabla_\mu\nabla^\mu\phi$.
- Primes denote derivatives with respect to the appropriate scalar field (almost always ϕ), as in $U'(\phi)$.
- We take $\epsilon^{\mu\nu\lambda\rho}$ to be the antisymmetric tensor with $\epsilon^{0123} = 1/\sqrt{-g}$.
- We define the (Jordan-frame) stress-energy tensor $T_\mu{}^\nu$ in the usual way in terms of the Jordan-frame metric $\bar{g}_{\mu\nu}$ that appears in the matter action $S_{\rm m}$:

$$ S_{\rm m}[\bar{g}_{\mu\nu} + \delta\bar{g}_{\mu\nu}, \psi_{\rm m}] - S_{\rm m}[\bar{g}_{\mu\nu}, \psi_{\rm m}] = \frac{1}{2} \int d^4x \sqrt{-\bar{g}}\, T_\mu{}^\nu \bar{g}^{\mu\lambda} \delta\bar{g}_{\lambda\nu} + O(\delta\bar{g}^2). \qquad (0.0.2) $$

  Note that this definition differs from the definition used in the braneworld project.

  We then define $T = T_\mu{}^\mu$, and define the quantities $T_{\mu\nu}$ and $T^{\mu\nu}$ by raising and lowering indices with the Einstein-frame metric $g_{\mu\nu}$, which is related to $\bar{g}_{\mu\nu}$ via Eq. (4.2.4). To zeroth-order in ϵ this stress energy tensor obeys the conservation law

$$ e^{-2\alpha}\nabla_\lambda(e^{2\alpha}T^{\lambda\sigma}) = \frac{1}{2}\alpha' T \nabla^\sigma \phi + O(\epsilon). \qquad (0.0.3) $$

Chapter 1

Introduction

Contents

The recent discovery of the accelerating expansion of the Universe [9, 10] has prompted many theoretical speculations about the underlying mechanism. The most likely mechanism is a cosmological constant, which is the simplest model and is in good agreement with observational data. More complicated models involve new dynamical sources of gravity that act as dark energy, and/or modifications to general relativity on large scales.

1.1 Theoretical Underpinnings of Cosmology

We begin with a very brief review of ΛCDM cosmology (see, e.g., [11]). On the largest scales, the universe appears to be very homogeneous and isotropic. Modelling the universe as a homogeneous and isotropic background with perturbations, it is straightforward to show that the background metric must be that of a Friedmann-Robertson-Walker (FRW) universe.

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \tag{1.1.1}$$

Here, $a(t)$ is known as the scale factor of the universe scaled such that $a(t) = 1$ today, and k describes the curvature of the universe. Current observations suggest that the universe is very close to flat, corresponding to $k \sim 0$.

The Einstein equations, using this metric and the assumptions of homogeneity and isotropy lead to the Friedmann equations.

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{1.1.2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) + \frac{\Lambda}{3} \tag{1.1.3}$$

Here, $H = \dot{a}/a$ is the Hubble factor, ρ and P are the average energy density and pressure of everything in the universe (excluding curvature), and Λ represents the cosmological constant.

The statement that the universe's expansion is accelerating corresponds to $\ddot{a}/a$ in Eq. (1.1.3) being positive. Evidently, this requires either the existence of a (positive) cosmological constant, or a dominant form of matter with equation of state $w = P/\rho < -1/3$.

From Eq. (1.1.2), we can define a critical density today for which there is no curvature

$$\rho_c = \frac{8\pi G\rho}{3H_0^2} \tag{1.1.4}$$

where H_0 is the Hubble factor today, also known as the "Hubble constant". Dividing the Friedmann equation by H_0^2, splitting the energy density into a cosmological constant, matter (scaling as $a(t)^{-3}$) and photons (scaling as $a(t)^{-4}$), and writing these densities in terms of the density fractions $\Omega_X = \rho_X/\rho_c$, we have

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\text{matter}}a^{-3} + \Omega_\gamma a^{-4} + \Omega_\Lambda + \Omega_k a^{-2} \tag{1.1.5}$$

where $\Omega_k = -3k/8\pi G\rho_c$. Evaluating this equation today yields

$$1 = \Omega_{\text{matter}} + \Omega_\gamma + \Omega_\Lambda + \Omega_k, \tag{1.1.6}$$

and so we may think of Ω_X as the fraction of the universe made up of X today. WMAP 7 year results [12] indicate that $\Omega_{\text{matter}} \sim 0.27$, $\Omega_\gamma \sim 0$, $\Omega_\Lambda \sim 0.73$, and $\Omega_k \sim 0$. The universe is thus presently dominated by the presence of dark energy, which will continue to become more important in the future. It can be seen that the large scale future of the universe is intimately related to the behavior of dark energy.

1.2 Experimental Evidence for Dark Energy

Following the initial announcements of the accelerating expansion of the universe in 1998, a number of separate experimental signatures of dark energy have been discovered. We briefly review the different experimental evidence for dark energy to date. Figure 1 shows how different methods combine to produce strong evidence for the phenomenon.

I Type 1a Supernovae

Supernovae are very bright explosions of stars, and have been classified into different classes depending on their properties, which in turn correspond to the original composition of the star. Type Ia supernovae occur when white dwarfs accrete matter beyond the Chandrasekhar mass and explode. Because the mass of all such objects is the same when it explodes, Type Ia supernovae are expected to explode with almost identical signatures, leading them to be called "standard candles". In particular, the luminosity L is constant between such events[1], and so measuring the flux from the supernova allows the calculation of the luminosity distance from $f = L/4\pi d_L^2$. Because the luminosity distance relation depends on the integrated cosmological history, measurements of Type 1a supernovae at different redshifts allow for the recent cosmological history to be ascertained. Figure 2 presents early evidence of the accelerated expansion of the universe.

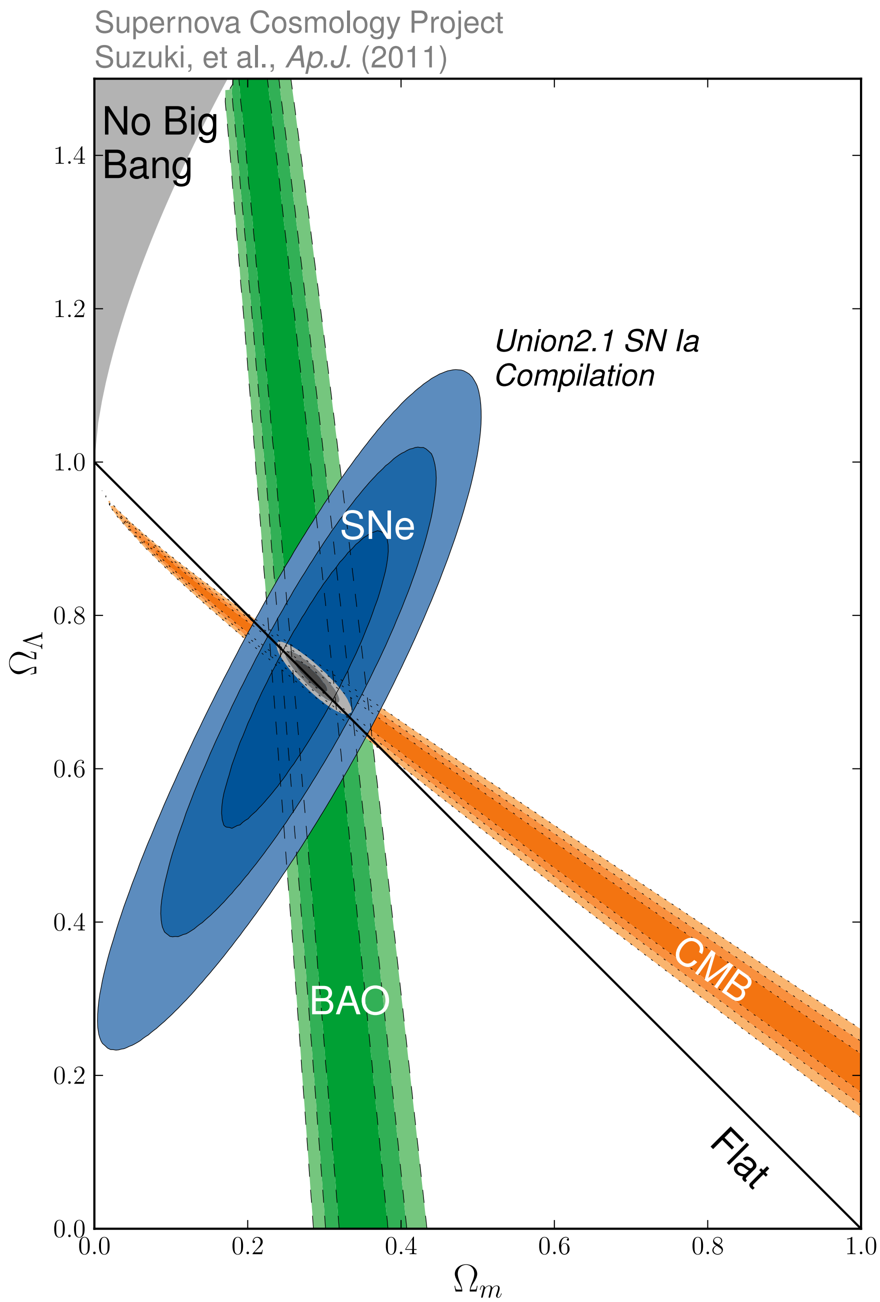

Figure 1: Constraints on the Ω_Λ vs Ω_{matter} plot, showing contributions from the cosmic microwave background, Type Ia supernovae, and baryon acoustic oscillations (excluding systematic errors). Note that the three methods are highly complementary. Figure from Ref. [13]. Reproduced by permission of the AAS.

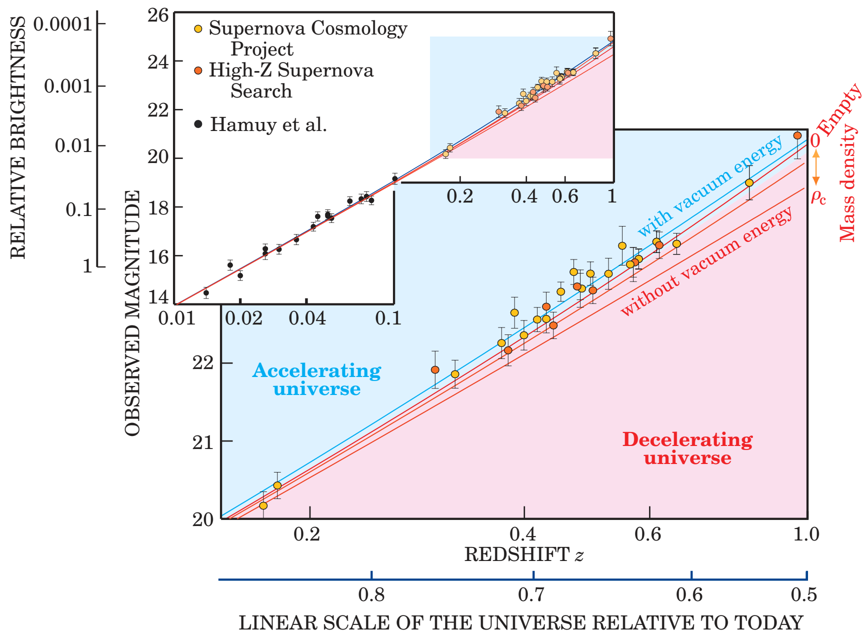

Figure 2: Luminosity of observed Type Ia supernovae plotted against redshift, including a comparison to dependence on expansion history of the universe. Figure from Ref. [14]. Reprinted by permission of the AIP.

II Baryon Acoustic Oscillations

In the early hot universe, pressure waves from density fluctuations were able to travel through the primordial medium only a certain distance before decoupling, at which point photons decoupled from the newly-formed neutral hydrogen. This sound horizon imprints a characteristic scale on the matter distribution, and can be measured from perturbations in the cosmic microwave background (CMB) radiation. Assuming that these initial perturbations seed galaxy formation, this characteristic scale can then be inferred today from a statistical analysis of galaxy surveys. Identifying the sound horizon scale as a function of redshift of the galaxies allows the identification of the expansion history of that scale, and therefore the expansion history of H as a function of redshift. This method is particularly useful as a complementary probe to supernova measurements, as can be seen in Fig. 3.

[1]Or at least, is standardizable between such events.

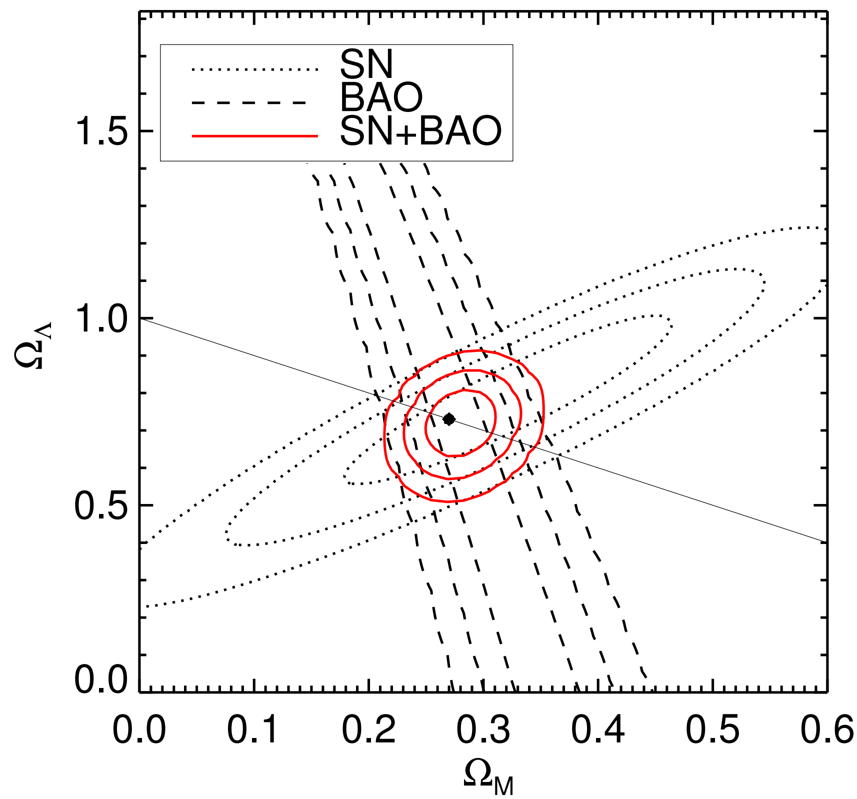

Figure 3: Recent constraints on the Ω_Λ vs Ω_{matter} plot, comparing results from Type Ia supernovae and baryon acoustic oscillations. Figure from Ref. [15]. Reproduced by permission of the AAS.

III Weak Lensing Surveys

When light from a distant galaxy passes a large mass, such as a galaxy cluster, the light is deflected. This phenomenon is known as gravitational lensing (see Fig. 4). The angle of deflection depends on the mass of the cluster, and the ratios of distances between the observer, lens and source. While the deflection angle cannot be inferred directly, such lensing tends to distort the picture of a galaxy, shearing its image by $\sim 2\%$. When large numbers of galaxies are observed, a bias in nearby galaxies to have aligned shapes leads to a statistical picture of the deflection angle. Knowledge of how the deflection angle behaves leads to a probe of the expansion history from its dependence on proper distances.

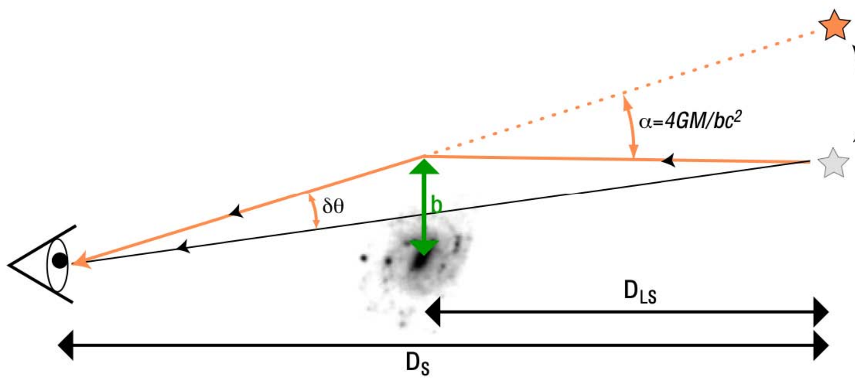

Figure 4: Diagrammatic overview of weak lensing phenomenon. Figure from Ref. [16]. Reproduced by permission of authors.

IV Cluster Surveys

The largest structures in the universe are galaxy clusters. It is possible to predict a mass function $dN/(dMdV)$ for the abundance of such clusters, particularly with the aid of N-body simulations. These predictions can be compared to observations from galaxy surveys. The dependence on dark energy can be extracted from the comoving volume element, which depends on the scale factor, which thus traces the cosmological history. A further dependence on the expansion rate comes from the way the mass function depends on the growth of perturbations, which is in turn sensitive to the Hubble factor. Comparisons to the size of perturbations in the CMB allow this dependency to be accounted for in predicting the expected mass function.

V Current Constraints on Dark Energy

The most current constraints on dark energy come from the WiggleZ survey [17]. For the equation of state parameter $w = P/\rho$ for dark energy, they find $w = -1.03 \pm 0.08$, consistent with a cosmological constant ($w_\Lambda = -1$). Allowing for an equation of state that varies with

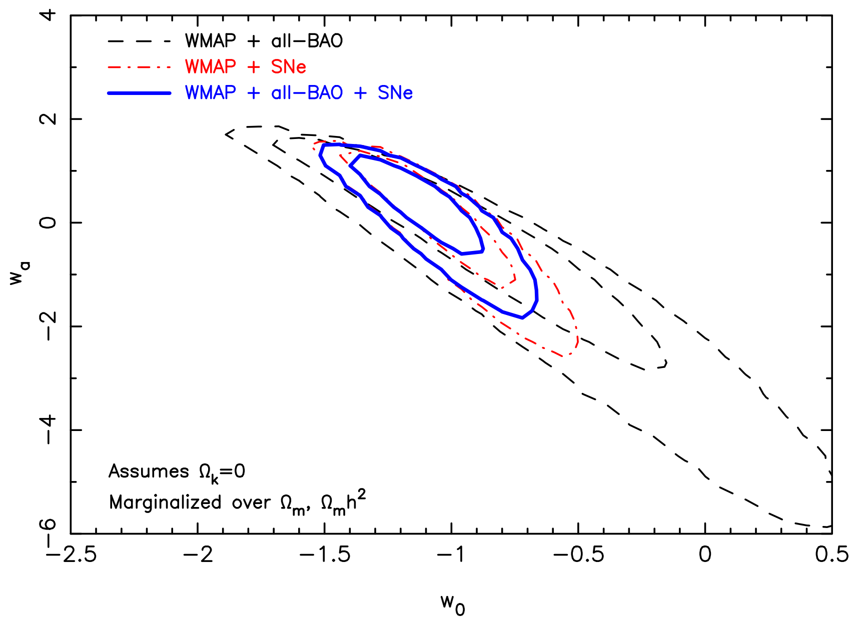

Figure 5: Current constraints on the equation of state of dynamical dark energy, using the parametrization given in Eq. 1.2.1. Figure rom Ref. [17]. Reproduced by permission of John Wiley and Sons.

redshift as[2]

$$w(a) = w_0 + (1 - a)w_a, \tag{1.2.1}$$

they find $w_0 = -1.09 \pm 0.17$ and $w_a = 0.19 \pm 0.69$, also consistent with a cosmological constant. Their fitting curves are shown in Fig. 5.

1.3 Theory Space

Since the discovery of the accelerated expansion of the universe, a large number of models have been proposed to give rise to this phenomenon.

The simplest model is a cosmological constant, with an energy density $\rho \sim (10^{-3}\text{ eV})^4$. While all current data is satisfied by a cosmological constant, the value it appears to take is in gross conflict with theoretical estimates. Assuming that the cosmological constant is the

[2]Note that redshift z is related to the scale factor by $1 + z = 1/a$.

vacuum energy density of spacetime, this corresponds to $\rho \sim m_P^4$, 120 orders of magnitude away from its measured value. This is known as the "cosmological constant problem".

A number of other models have been proposed, which typically attempt to avoid the cosmological constant problem by assuming that the vacuum energy of spacetime doesn't gravitate (i.e., $\Lambda = 0$), and searching for a dynamical field to emulate the desired behavior.

The next simplest model, dubbed "quintessence", involves a minimally coupled scalar field rolling in a potential. The present energy density of the universe is then dominated by energy stored in the scalar field potential. It can be shown that such a model can yield any desired cosmological evolution through fine-tuning the quintessence potential. Various quintessence potentials have been shown to be attractor solutions, so that models can be relatively agnostic with regards to the initial conditions in the universe. Quintessence models suffer from two major problems. Firstly, it is often difficult to protect the quintessence potential from quantum loop corrections. To be effective, the quintessence mass must be on the order of the Hubble scale ($\sim 10^{-33}$ eV), which is difficult to protect without invoking a broken symmetry, such as for pseudo-Nambu-Goldstone bosons (pNGBs). Secondly, the light mass of quintessence fields mean that any coupling to standard model fields will give rise to a long range force, which has not been observed in nature.

A variation on quintessence called k-essence [18, 19] is based on using functions of $(\nabla\phi)^2$ in the action to generate the desired energy density based on kinetic energy rather than potential energy.

Further afield, modifications to gravity such as extra dimensions, Ghost Condensates [20], DGP gravity [5], and $f(R)$ gravity, to name but a few, have been proposed over the past decade. See Refs. [21, 22, 23, 24, 25, 26, 27] for detailed reviews of these and other models.

While many models of dark energy have been shown to produce an acceptable cosmological history, the greatest discriminating factor for such models will come from understanding the perturbative behavior of the model. For this reason, it is of great interest to construct a generic manner in which dark energy models may be tested against observations. We begin to address this question in Chapter 4, and discuss future work in Chapter 5.

1.4 Other Issues in Theoretical Physics

There are a number of other issues in theoretical physics which motivate the exploration of modified gravity models.

It turns out that constructing a consistent modification to gravity is surprisingly difficult. In the low-energy limit, a theorem due to Weinberg [28] requires that the behavior of massless spin-two fields is that of general relativity, which entails that any modification is equivalent to the introduction of new fields. For such fields to be observationally consistent often requires that they are either too weakly-coupled or too massive to mimic dark energy. A few exceptions exist (e.g. Galileon [29, 30, 31], but are plagued with issues such as superluminal propagation.

Circumventing Weinberg's theorem by looking at massive gravity has historically suffered from the infamous vDVZ discontinuity [32, 33] and the Boulware-Deser ghost [34], although recent attempts at constructing a consistent bimetric massive gravity theory have been able to overcome this issue [35]. However, they in turn suffer from arbitrariness of the background metric.

On the high-energy side, it is universally accepted that gravity must become modified at energies approaching the Planck scale, as naïve scattering amplitudes diverge. However, there are numerous difficulties involved in constructing a consistent theory of quantum gravity. The current leading candidate is string theory, although there is a long way to go to connect the ideas of string theory to our present universe.

Related to the high energy scale is the hierarchy problem of particle physics. In a quantum field theory, the mass of a scalar field is not protected by a symmetry, and typically receives loop corrections, driving it up to the cutoff scale of the theory. Given a cutoff scale of the Planck mass, it is an unsolved question as to why the recently-discovered Higgs boson [36, 37] has a mass ~ 125 GeV. The presently favoured mechanism for doing so is supersymmetry at a TeV scale. However, it is possible that the four-dimensional gravitational constant is only an effective scale derived from some more fundamental scale, such as in the RS-I model [6]. The issue of scalar field masses particularly plagues quintessence models, which require a mass to be protected at around the present-day Hubble scale (10^{-33} eV).

The final issue we discuss is that of dark matter. One possibility for the weakness of the interaction strength between normal matter and dark matter comes from the idea of sequestration, or physically removing the standard model and dark matter fields. This idea has been of particular interest in braneworld models.

Along with dark energy, these issues provide a number of reasons to investigate various modifications to gravity.

1.5 Structure of this Dissertation

This dissertation is a combination of two separate investigations. The first looks at a class of braneworld models, with interest in fields that may give rise to dark energy-like behavior. The second investigates a broad class of dark energy models, using the tools of effective field theory to construct a generic model of dark energy that can describe a large amount of theory-space.

In Chapter 2, we introduce the idea of extra-dimensional models of the universe. We begin by reviewing the historical evolution of ideas in this field, and describe the significant results. One of the most important aspects of a model involving extra dimensions is the manner in which those extra dimensions are hidden from us. The mechanism used to do so will inevitably leave an imprint on the resulting four-dimensional universe that we observe, and it is thus of great interest to understand the four-dimensional universe that one would expect to observe, given a model involving extra dimensions. This chapter focuses on extensions to the Randall-Sundrum (RS) braneworld models, and the task of calculating an effective four-dimensional description for them. A computational method is proposed and described in detail through the implementation of the method for an uncompactified N-brane model in five dimensions.

Chapter 3 builds upon the results of the previous chapter. Having derived a four-dimensional effective description for a class of braneworld models, it is of interest to understand the physics of those models. We begin by investigating the conditions under which no ghosts appear in the theory, and focus our attention on the subclass of theories that satisfy this condition. We then investigate gravitational interactions between different branes, and identify the behavior of the Parameterized Post-Newtonian (PPN) γ parameter. Next, we look at the possibility of using the discussed models to sequester dark matter on a separate physical brane from standard model fields, in order to give a physical reason for the weak interaction strength between standard model and dark matter fields. Unfortunately, the models we investigated did not give rise to dark energy behavior, and we discuss this in conclusion.

We then turn to a very different approach. Rather than investigating specific models or classes of models, we develop an inclusive approach to investigating dark energy models in Chapter 4, where we employ an effective field theory approach to quintessence. Such an approach is of particular interest in putting observational constraints on possible terms in the lagrangian for dynamical dark energy. An appropriate expansion method is identified, and the operators in the action are written down. Next, a number of field redefinitions are employed to simplify the action. We then investigate a possible UV motivation for the resulting theory using a pseudo-Nambu-Goldstone boson construction. This construction allows us to determine the scaling of each of the operators in our effective action, and also to identify the regime of validity of the description.

In Chapter 5, we conclude by describing the overlap between these two approaches. We demonstrate the "middle ground" in which our four-dimensional effective description of a braneworld model is described in the more general approach of an effective field theory construction. We discuss possibilities for future theoretical work, and briefly describe upcoming experiments and the scientific impact that these experiments are expected to have on the field of dark energy.

A number of appendices are included. Appendix A describes the exact five-dimensional equations of motion for the braneworld models of Chapter 2, which are used to motivate the approximation scheme. Appendix B briefly outlines the application of our method for finding the four-dimensional effective description to orbifolded models. Appendix C describes the Kaluza-Klein (KK) modes of our braneworld models, complementing the analysis included in Chapter 3.

One of the requirements we impose on our effective field theory of dark energy is that it must maintain the weak equivalence principle (WEP). In Appendix D, we describe various aspects of the WEP, and show how it is obeyed within the regime of validity of our analysis. The reduction of order technique used in our EFT is described in detail in Appendix E. The work described in Chapter 4 builds on previous work; Appendix F provides a comparison between this and the work presented here. The full equations of motion for our effective theory are presented in Appendix G. Finally, we provide details on how the scaling of operators in the EFT is derived from the pNGB perspective in Appendix H.

Chapter 2

The Low-Energy Effective Scalar Sector of Multibrane-Worlds

Contents

The idea of extra-dimensional models of the universe dates back to at least Kaluza and Klein in the 1920's [38, 39]. Due to issues with the basic model presented there, the idea was largely ignored until its revival with string theory, which depending upon the field content, requires anywhere from 10 to 26 dimensions. In the late 1990's, motivated by ideas from string theory, the notion of constraining matter fields to a membrane ("brane") in a higher-dimensional spacetime was used to resurrect the ideas of Kaluza and Klein. This led to a series of models designed to address a variety of theoretical issues, in particular, the hierarchy problem, the cosmological constant problem, and dark matter. Braneworld models have become a very active field of research with many papers investigating extensions to the basic ideas (see, e.g., [40, 41, 42] and citations therein).

In this chapter, we develop a method to acquire a low-energy effective description of a five-dimensional braneworld model which contains an arbitrary (but finite) number of branes. Our goal is to devise a simple method that yields a four-dimensional action which captures the leading-order effects of braneworld models. This chapter is based on work presented in [1].

2.1 Braneworld Models

We begin with a brief overview of the features of significant extra-dimensional models which are relevant to this work.

I Kaluza-Klein Model

The first important extra-dimensional model is that of Kaluza and Klein [38, 39], dating back to the 1920s. They had noticed that if a five-dimensional metric ansatz is decomposed into a four-dimensional metric, a four-vector, and a scalar, one recovers a four-dimensional Ricci scalar, and a Maxwell-like term for the four-vector. Based on this observation, they proposed a mechanism to unify electromagnetism with gravity based on a fifth dimension.

The first step in any extra-dimensional model is to hide the fifth dimension from current observations. They proposed doing so by having the fifth dimension curled up on itself so tightly that it is effectively invisible at low energies. To do so, they introduced a circular compactification with periodic boundary conditions, and proposed the radius L of this circle to be sufficiently small.

Assuming the extra dimension is flat, one can then decompose a field into Fourier modes over the extra dimension. We demonstrate with a massive scalar field. Call the dimensions

x^a and y. The scalar field then obeys a wave equation

$$\Box^{(5)}\phi = m^2\phi. \tag{2.1.1}$$

Identifying $y \Leftrightarrow y + 2\pi nL$, $n \in \mathbb{Z}$, we can then decompose the field into a fourier expansion over the fifth dimension,

$$\phi(x^a, y) = \sum_n e^{iyn/L}\phi_n(x^a). \tag{2.1.2}$$

Looking at an individual mode ϕ_n, we find from the scalar equation of motion

$$\Box^{(4)}\phi_n(x^a) = \left(m^2 + \frac{n^2}{L^2}\right)\phi_n(x^a). \tag{2.1.3}$$

These mode functions $\phi_n(x^a)$ are called "Kaluza-Klein" (KK) modes, formed from a decomposition of the field over the extra dimension, and collectively form the "Kaluza-Klein tower". The four dimensional modes have a modified mass, given by

$$m_n^2 = m^2 + \frac{n^2}{L^2}. \tag{2.1.4}$$

The spacing of these modes is characteristic of compactification over a flat extra dimension.

The idea of Kaluza and Klein suffered from a number of drawbacks. Firstly, it proposed the existence of a Kaluza-Klein tower of modes for each known particle, such as the electron, and nobody had ever observed a "heavy" electron. Secondly, it is difficult to include fermions in such a model, because of the different types of fermions which exist in different dimensions. Thirdly, the scalar mode, now called the "radion" mode, which governs the size of the extra dimension, needs be fixed in some manner. Naïve estimates for the size of the extra dimension suggest $L \sim m_P^{-1}$. Finally, the strength with which the electromagnetic field couples to matter was the same as the gravitational coupling, in stark contrast to experiment. The inability to change this final problem led to the idea being dropped for seventy years.

Nevertheless, the Kaluza-Klein model remains an important model, as it introduces the basic ideas of compactification, Kaluza-Klein modes, radion modes and the need for radion stabilization, which are all important in recent investigations of extra-dimensional models.

II ADD Model

In 1998, Arkani-Hamed, Dvali and Dimopoulos resurrected the Kaluza-Klein model, by borrowing the idea of a brane from string theory. They suggested that if standard model fields were constrained to live on a brane in some number of extra dimensions, then only gravity would develop Kaluza-Klein modes. Estimates for the effect of the gravitational coupling of such modes suggested that extra dimensions as large as one millimetre might be feasible. This model became known as the ADD model [4, 43].

The important aspect of the ADD model is that four-dimensional observers experience an effective Planck scale that is derived from a more fundamental, higher-dimensional gravitational scale, based on the size of the extra dimensions. For simple estimates, the effective Planck scale is given by

$$m_P^2 \sim M_\star^{2+n} V_{(n)} \tag{2.1.5}$$

where m_P is the four-dimensional Planck mass, $M_\star$ is the fundamental gravitational scale, n is the number of extra dimensions, and $V_{(n)}$ is the volume of the extra dimensions. This implies that it may be possible to significantly alleviate the hierarchy problem of particle physics through the use of extra-dimensional models.

One of the more exciting predictions from this model is that a reduced fundamental gravitational scale would make the production of black holes in collider experiments possible upon reaching energies $\sim M_\star$. This in turn led to fears that the LHC would produce a black hole that would destroy the world.

III Randall-Sundrum Model

Following the ADD model, Randall and Sundrum proposed that it is possible to give rise to the desired hierarchy without small compactification of the extra dimension, by using warping in the extra dimension from a bulk cosmological constant. Using two branes at the fixed points of an orbifold with a negative bulk cosmological constant was shown to provide exponential enhancement of the effective Planck scale on the brane with the smaller warp factor (called the TeV brane). Furthermore, hints that four-dimensional gravity was shown to be recovered on the brane were shown, and the KK modes were shown to be sufficiently weakly coupled that they did not change the $1/r^2$ force law within the regime in which gravity has been experimentally tested. This model is known as the Randall-Sundrum model, or RS-I [6].

A second model, involving only one brane in an infinite five-dimensional anti de-Sitter (AdS) bulk, was shown to be able to do away with compactification entirely, relying on the curvature of AdS space to confine gravity to the brane. While this model did not give rise to a useful hierarchy, it did demonstrate a mechanism for infinite extra dimensions. This model is known as the RS-II model [7].

Building on the success of the Randall-Sundrum model, many papers have considered various extensions to it, including bulk fields [44], radion stabilization mechanisms [45, 46], and models including more than one or two branes [47, 48, 49, 50, 51, 52]. A wealth of knowledge of the phenomenology of these models has been accumulated (see [40, 41, 42, 53, 54] and citations therein, for example).

2.2 Previous Work

Having discussed historical developments in this field, we now move on to detail previous work pertinent to the results that will be presented here.

I Four-Dimensional Effective Descriptions

Several different approximation and computational methods have been used to extract physical predictions from extra-dimensional models. In particular, many models have an effective four-dimensional regime at low energies, where the radius of curvature of spacetime measured by four-dimensional observers is much larger than a certain microphysical lengthscale. We review some of the computational methods that have been used to obtain a four-dimensional description of five-dimensional braneworld models, in order to place our results in context.

One method is to linearize the higher dimensional equations of motion about simple background solutions, then specialize to the long-lengthscale limit in order to obtain the linearized four-dimensional effective theory (which roughly corresponds to discarding the Kaluza-Klein modes). This method was used by Garriga and Tanaka [55] in their analysis of the RS-I model [6], who showed that linearized Einstein gravity is recovered on one of the branes in a particular regime. Further analyses to quadratic order and analyses on other backgrounds have also been performed; see, for example, Refs. [56, 57, 58, 59]. Linearized analyses have many advantages: they are quick and simple, and serve to identify all of the dynamical degrees of freedom in the theory, particularly the Kaluza-Klein modes. However, the linearized method is inherently limited and cannot describe strong field phenomena such as cosmology and black holes.

A second method is to project the five-dimensional equations of motion onto a brane; see, for example, Ref. [60]. This "covariant curvature" formalism fully incorporates the nonlinearities of the theory. However, the projected description includes nonlocal terms, and the truncation to a low-energy effective theory is nontrivial, except in cases with high degrees of symmetry.

In order to overcome some of these shortcomings, Kanno and Soda [61, 62, 63] suggested a perturbation expansion of the covariant curvature formalism known as the "gradient expansion method", which involves expanding the theory in powers of the ratio between a microphysical scale and the four-dimensional curvature lengthscale. This approach allows a low-energy description of the model to be found, while retaining the nonlinearities of the theory. This method has been particularly successful in investigating the cosmology of braneworld models

[61, 63, 47, 64] and has the benefit of providing an explicit calculation of the five-dimensional metric, but is algebraically complex and requires assumptions on the form of the metric.

An alternative approach to obtaining a four-dimensional effective action, discussed by Wiseman [65], focusses on the radion mode of the RS-I model. Treating the radion mode as a deflection of the branes, the approach uses a derivative expansion to calculate its nonlinear behavior. Although this method nicely captures the nonlinearities of the theory, it is highly nontrivial, and guesses the four-dimensional effective action, based on the first-order equations of motion the method finds.

A final method involves making an ansatz for the form of the five-dimensional metric in terms of four-dimensional fields and integrating over the fifth dimension to obtain a four-dimensional action. Examples of this method in the literature include Refs. [64, 66, 46, 48, 67]. The benefits of this method are the automatic truncation of the massive Kaluza-Klein modes, and the computational efficiency in dealing strictly at the level of the action. The main drawback is that the five-dimensional metric ansatz must usually be found (or guessed) using another method.

II Extensions to Multiple Branes

One common extension of the RS models is to consider models with more than one or two branes. A variety of papers have considered three-brane models, usually on an orbifold (see, e.g., [47, 48, 49, 50, 51]). Some special cases have been considered for arbitrary N-brane models, mostly to investigate their cosmological properties [50, 52]. A few papers comment that their methods should extend to arbitrary N-brane situations (e.g., [68]), but little analysis has actually been performed in this regard.

Four-dimensional effective descriptions typically contain moduli fields (radion modes) which describe the distances between branes. Often, such modes appear as massless scalar fields which couple to gravity in a Brans-Dicke like manner (see, e.g., Ref. [66]). This occurs in the RS-I model of two branes in a compactified bulk with orbifold symmetry, for example. In this model, the radion mode must be stabilized by some mechanism (for example, by using a bulk scalar field as in the Goldberger-Wise mechanism [45]), or else the theory is ruled out for observers on the TeV brane (see, e.g., Refs. [55, 69]). In theories including multiple branes, one expects several radion modes which may have nontrivial couplings to one another and to the four-dimensional metric at the nonlinear level.

2.3 Construction of the Model

In this chapter, we present a new method to obtain a four-dimensional effective theory from an N-brane model in five dimensions. We assume that matter is confined to branes with the only bulk field being gravity, and we do not invoke mechanisms to stabilize the radion modes. The method utilizes a two-lengthscale expansion to find solutions to the five-dimensional equations of motion in a low-energy regime. We do not require assumptions about the form of the metric, or the existence of Gaussian normal coordinates. The method is computationally efficient and does not require the explicit use of the five-dimensional Einstein equations or the Israel junction conditions. Instead, one always works at the level of the action. Furthermore, our method is very general and can be applied to various models. The method has similarities to the gradient expansion method (see especially [67]), but is computationally much simpler, and can deal with multiple branes in a straightforward manner. A particular strength of the method is that it performs a rigorous treatment of all radion modes, and automatically truncates massive modes. We present a brief example of the method for the case of the RS-I model [6], before illustrating the method in detail for the case of N four-dimensional branes in an uncompactified extra dimension, deriving the four-dimensional effective action for a general configuration.

I Applicable Models

We begin by defining the model we use to illustrate our method, and introduce the parameters, metrics, and coordinate systems used to describe it. The most basic model assumes that the extra dimension is infinite and not compactified, but the generalization to circularly compactified and orbifolded systems is straightforward, and is described briefly in Section 2.4 and in more detail in Appendix B.

We consider a system of N four-dimensional branes in a five-dimensional universe with one temporal dimension, with coordinates $x^\Gamma = (x^0, \ldots, x^4)$. We denote the bulk metric by $g_{\Gamma\Sigma}(x^\Theta)$ and the associated five-dimensional Ricci scalar by $R^{(5)}$. For simplicity, we assume that there are no physical singularities in the spacetime.

The N branes are labeled by an index $n = 0, 1, \ldots, N-1$, so that adjacent branes are labeled by successive values of n. We assume that the branes are nonintersecting. Denote the n^{th} brane by $\mathcal{B}_n$. On $\mathcal{B}_n$, we introduce a coordinate system $w_n^a = (w_n^0, \ldots, w_n^3)$. The location of the branes in the five-dimensional spacetime is described by N embedding functions ${}^n x^\Gamma(w_n^a)$. From these embedding functions, we can calculate the induced metric ${}^n h_{ab}$ on $\mathcal{B}_n$,

$$ {}^n h_{ab}(w_n^c) = \frac{\partial\, {}^n x^\Gamma}{\partial w_n^a} \frac{\partial\, {}^n x^\Sigma}{\partial w_n^b} g_{\Gamma\Sigma} \left[{}^n x^\Theta \right] \bigg|_{w_n^c} . \tag{2.3.1} $$

We associate a nonzero brane tension σ_n with each brane $\mathcal{B}_n$, and we also take there to be matter fields ${}^n\phi(w_n^a)$ which live on $\mathcal{B}_n$, with their own matter action ${}^n S_m[{}^n h_{ab}, {}^n\phi]$.

In between each brane there exists a bulk region of spacetime, which we denote $\mathcal{R}_0, \ldots, \mathcal{R}_N$, with $\mathcal{R}_n$ lying between branes $n-1$ and n. The first (last) bulk region describes the region between the first (last) brane and spatial infinity in the bulk. In each bulk region $\mathcal{R}_n$ we allow for a bulk cosmological constant Λ_n (see Ref. [52] for a possible microphysical origin for such piecewise constant cosmological constants).

Finally, the action for the model is

$$S\left[g_{\Gamma\Sigma}, {}^{n}x^{\Gamma}, {}^{n}\phi\right] = \int d^5x\sqrt{-g}\left(\frac{R^{(5)}}{2\kappa_5^2} - \Lambda(x^\Gamma)\right) - \sum_{n=0}^{N-1}\sigma_n \int_{\mathcal{B}_n} d^4w_n\sqrt{-{}^{n}h} + \sum_{n=0}^{N-1} {}^{n}S_m[{}^{n}h_{ab}, {}^{n}\phi], \tag{2.3.2}$$

where κ_5^2 is the five-dimensional Newton's constant, and $\Lambda(x^\Gamma)$ takes the value Λ_n in $\mathcal{R}_n$.

II Overview of the Method and Results

Our method works in five steps.

Step 1: Gauge specialize. From the general action (Eq. (2.3.2) in the model we discuss here), we perform a gauge transformation to specialize the metric to the straight gauge [58], illustrated in Fig. 6.

Step 2: Separate lengthscales in the action. There are two characteristic lengthscales in the model. The first, which we call the microphysical lengthscale, is the lengthscale associated with the bulk cosmological constants, which is typically assumed to be on the order of the micron scale or smaller. The second lengthscale is the four-dimensional radius of curvature felt on the branes. When the ratio of the microphysical lengthscale to the four-dimensional radius of curvature is small (the low-energy limit), the dynamics of the extra dimension effectively decouples from the four-dimensional dynamics, leading to a four-dimensional effective theory. We introduce a small parameter to tune this ratio, and use this parameter to perform a two-lengthscale expansion of the action.

Step 3: Solve equations of motion. The equations of motion at zeroth-order in this small parameter are calculated and explicitly solved. As expected in this type of model, all of the bulk cosmological constants must be negative, and at this order, the brane tensions are required to be tuned to a specific value[1] [6] in order to avoid an effective cosmological

[1] We consider small deviations from this value in Section 2.10.II.

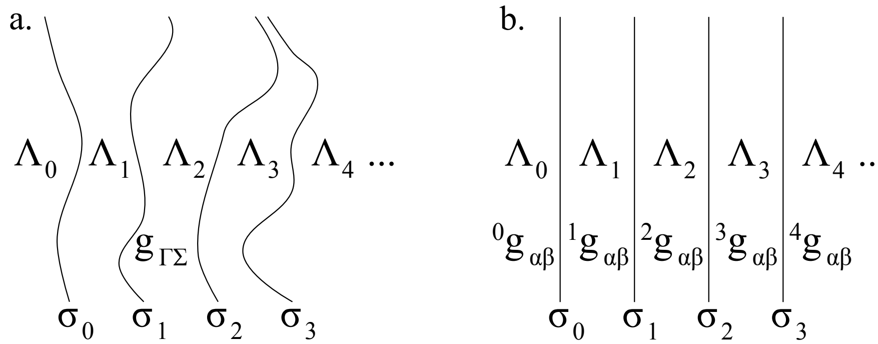

Figure 6: An illustration of the model a) before and b) after gauge fixing. The bulk cosmological constants, brane tensions, and metrics are labeled.

constant on the branes. The solution to the zeroth-order equations of motion provides a background metric solution, which is perturbed at the next order in our small parameter (the metric is an exact solution if the four-dimensional space is flat).

Step 4: Integrate five-dimensional dynamics. The five-dimensional dynamics of the theory are integrated out by substituting the metric into the action, and integrating over the extra dimension. The action to zeroth order in the small parameter is minimized by the ansatz, leaving only the four-dimensional terms in the action.

Step 5: Redefine fields. The final step is to redefine fields in order to cast the four-dimensional effective action in the form of a four-dimensional multiscalar-tensor theory in a nonlinear sigma model. In the Einstein conformal frame, the general form of the four-dimensional effective action is given by

$$\begin{aligned} S[g_{ab}, \Phi^A, {}^n\phi] = & \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa_4^2}R[g_{ab}] - \frac{1}{2}g^{ab}\gamma_{AB}\left(\Phi^C\right)\nabla_a\Phi^A\nabla_b\Phi^B\right\} \\ & + \sum_{n=0}^{N-1} {}^nS_m\left[e^{2\alpha_n\left(\Phi^C\right)}g_{ab}, {}^n\phi\right] \end{aligned} \tag{2.3.3}$$

where Φ^A, $1 \leq A \leq N-1$, are massless scalar fields (radion modes), which encode the interbrane distances. Also, κ_4^2 $(= 8\pi G_N)$ is the effective four-dimensional Newton's constant, which is a function of κ_5^2 and the bulk cosmological constants. Finally, $\gamma_{AB}(\Phi^C)$ is the field space metric of the nonlinear sigma model, and $\alpha_n(\Phi^C)$ are the brane coupling functions. The functional form of both of these depends on the specifics of the model.

One of the features of the method used here is that five-dimensional gravitational perturbations, which give rise to massive four-dimensional fields, are automatically truncated. The mass scales of these fields are typically of order $\hbar/\mathcal{L}$, where $\mathcal{L}$ is the microphysical lengthscale of the theory. However, Damour and Kogan [49, 50, 68] have shown that it is possible to have graviton Kaluza-Klein modes where masses are of order $\hbar/\mathcal{L}\exp(-l/\mathcal{L})$, where l is an interbrane separation. Because of the exponential factor, these second graviton modes can be ultralight and observationally relevant. Although the models we consider are likely to contain such ultralight graviton modes, our method excludes their possible contributions to a four-dimensional effective theory.

Our method has similarities to the gradient expansion method of Kanno and Soda [61, 62, 63]. Our small expansion parameter coincides with theirs, and the zeroth-order solutions from both methods agree in cases where both methods are applicable. However, beyond this point, the methods diverge. Our method Taylor expands the action, but not the metric as in the covariant curvature formalism. Although higher-order corrections to the metric do exist, they are intrinsically five-dimensional interactions that are unnecessary for the construction of a four-dimensional effective theory; their contributions to the effective theory are exponentially suppressed within the low-energy regime. Furthermore, our method arrives at a four-dimensional effective action, rather than working only at the level of the equations of motion. This provides for computational efficiency and a more intuitive understanding of the final result.

2.4 Application of the Method to the Randall Sundrum Model

To briefly illustrate an application of our method, we apply it to the well-known case of the Randall Sundrum (RS-I) model with a general background. The derivation of results in this section follows the details on the uncompactified model treated in the remainder of this chapter closely.

Many papers have used a metric ansatz for the RS-I model (e.g. [6, 66]), guessing at the form of the five-dimensional metric, and using this to compute the four-dimensional effective action. Such metrics are typically of the form

$$ds^2 = e^{\chi(x^c,y)}\gamma_{ab}(x^c)dx^a dx^b + \left(\frac{\chi_{,y}(x^c,y)}{2k}\right)^2 dy^2 \tag{2.4.1}$$

where $k = \sqrt{-\kappa_5^2\Lambda/6}$. Rather than guessing at the form of the five-dimensional metric, our method derives a five-dimensional metric solution, from which the four-dimensional action is calculated.

The RS-I model contains two branes on a circular orbifold. We consider the circle of circumference $2L$, with the branes at $y = 0$ and $y = L$, with $-L < y < L$. We let the $y = 0$ brane be the Planck brane and the $y = L$ brane be the TeV brane. The points y and $-y$ are identified. To write this in the language of regions described previously, we treat the regions $-L < y < 0$ and $0 < y < L$ as two distinct regions, but identify fields by using $\phi(-y) = \phi(y)$, where ϕ is representative of an arbitrary field.

We now follow the computational steps outlined in Section 2.3.II.

Step 1. Write the action in the straight gauge [58]. In this gauge, the general metric is given by

$$ds^2 = e^{\chi(x^c,y)}\gamma_{ab}(x^c,y)dx^a dx^b + \Phi^2(x^c,y)dy^2 \tag{2.4.2}$$

where $\det\gamma = -1$, and we take Φ to be positive. For this model, the general action (2.3.2) specializes to

$$\begin{aligned} S = &\int d^4x \left(\int_{0^+}^{L^-} dy + \int_{-L^+}^{0^-} dy\right)\sqrt{-g}\left(\frac{R^{(5)}}{2\kappa_5^2} - \Lambda\right) \\ &- \sigma_0 \int_{\mathcal{B}_0} d^4x\sqrt{-{}^0h} - \sigma_L \int_{\mathcal{B}_L} d^4x\sqrt{-{}^Lh} \\ &+ \frac{1}{\kappa_5^2}\int_{\mathcal{B}_0} d^4x\sqrt{-{}^0h}\left({}^0K^+ + {}^0K^-\right) + \frac{1}{\kappa_5^2}\int_{\mathcal{B}_L} d^4x\sqrt{-{}^Lh}\left({}^LK^+ + {}^LK^-\right) \\ &+ {}^0S_m\left[{}^0h_{ab}, {}^0\phi\right] + {}^LS_m\left[{}^Lh_{ab}, {}^L\phi\right]. \end{aligned} \tag{2.4.3}$$

The indices 0 and L refer to the Planck and TeV branes, respectively. h_{ab} is the four-dimensional induced metric on a brane, and σ is the brane tension. K^+ and K^- are the extrinsic curvature tensors on either side of the branes, and S_m is the matter action on each brane.

Step 2. Now, expand the action (2.4.3) to lowest order in the two-lengthscale expansion detailed in Section 2.6. The action to lowest order in this model is given by

$$\begin{aligned} S = & \int d^4x \left(\int_{0^+}^{L^-} dy + \int_{-L^+}^{0^-} dy \right) \sqrt{-\gamma} \frac{e^{2\chi}}{2\kappa_5^2 \Phi} \left(-\frac{1}{4} \gamma^{ab} \gamma_{bc,y} \gamma^{cd} \gamma_{da,y} - 5 \left(\chi_{,y} \right)^2 \right. \\ & \left. \qquad - 4\chi_{,yy} + 4 \frac{\Phi_{,y}}{\Phi} \chi_{,y} - 2\kappa_5^2 \Phi^2 \Lambda \right) \\ & + \int d^4x \left(\int_{0^+}^{L^-} dy + \int_{-L^+}^{0^-} dy \right) \lambda(x^a, y) \left(\sqrt{-\gamma} - 1 \right) \\ & + \int_{\mathcal{B}_0} d^4x \sqrt{-\gamma} e^{2\chi(0)} \left[\frac{2}{\kappa_5^2} \left(\left. \frac{\chi_{,y}}{\Phi} \right|_{y=0^-} - \left. \frac{\chi_{,y}}{\Phi} \right|_{y=0^+} \right) - \sigma_0 \right] \\ & + \int_{\mathcal{B}_L} d^4x \sqrt{-\gamma} e^{2\chi(L)} \left[\frac{2}{\kappa_5^2} \left(\left. \frac{\chi_{,y}}{\Phi} \right|_{y=L^-} - \left. \frac{\chi_{,y}}{\Phi} \right|_{y=-L^+} \right) - \sigma_L \right] \end{aligned} \tag{2.4.4}$$

Here, $\chi(0)$ denotes $\chi(x^a, 0)$, and similarly for $\chi(L)$. The third line in this action includes a Lagrange multiplier (λ) to enforce the condition $\det \gamma = -1$.

Step 3. Varying the action (2.4.4) with respect to the three fields χ, γ, and Φ, the following equations of motion are obtained.

$$0 = \frac{1}{4} \gamma^{ab} \gamma_{bc,y} \gamma^{cd} \gamma_{da,y} - 3\chi_{,y}^2 - 2\kappa_5^2 \Phi^2 \Lambda \tag{2.4.5}$$

$$\gamma_{ad,yy} = \gamma_{ab,y} \gamma^{bc} \gamma_{cd,y} - \gamma_{ad,y} \left(2\chi_{,y} - \frac{\Phi_{,y}}{\Phi} \right) \tag{2.4.6}$$

$$0 = \frac{1}{12} \gamma^{ab} \gamma_{bc,y} \gamma^{cd} \gamma_{da,y} + \chi_{,y}^2 + \chi_{,yy} - \frac{\Phi_{,y}}{\Phi} \chi_{,y} + \frac{2}{3} \kappa_5^2 \Phi^2 \Lambda \tag{2.4.7}$$

The following boundary conditions at the branes are also obtained.

$$\gamma_{ab,y}(y = 0, L) = 0 \tag{2.4.8}$$

$$\chi_{,y}(y = 0^+) = -\frac{1}{3} \kappa_5^2 \sigma_0 \Phi \tag{2.4.9}$$

$$\chi_{,y}(y = L^-) = \frac{1}{3} \kappa_5^2 \sigma_L \Phi \tag{2.4.10}$$

We now solve the equations of motion. The solution to (2.4.6) is given by (in matrix notation)

$$\boldsymbol{\gamma}(x^a, y) = \mathbf{A}(x^a) \exp\left(\mathbf{B}(x^a) \int_0^y \Phi(x^a, y') e^{-2\chi(x^a, y')} dy'\right) \tag{2.4.11}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary 4×4 real matrix functions of x^a, subject to the constraint that $\boldsymbol{\gamma}$ is a metric. This can be combined with (2.4.8) to yield $\mathbf{B} = \mathbf{0}$, and so γ is a function of x^a only. The only remaining equation of motion is then $\chi_{,y}^2 = -2\kappa_5^2 \Phi^2 \Lambda/3$. Defining $k = \sqrt{-\kappa_5^2 \Lambda/6}$, this gives $\chi_{,y} = \pm 2k\Phi$. Choose the negative solution, so that the brane at $y = 0$ corresponds to the Planck brane. The other boundary conditions (2.4.9) and (2.4.10) yield

$$\sigma_0 = \frac{6k}{\kappa_5^2} \quad \text{and} \quad \sigma_L = -\frac{6k}{\kappa_5^2} \tag{2.4.12}$$

which are the well-known brane-tuning conditions. Combining these solutions, the metric solution is then

$$ds^2 = e^{\chi(x^c, y)} \gamma_{ab}(x^c) dx^a dx^b + \left(-\frac{\chi_{,y}(x^c, y)}{2k}\right)^2 dy^2. \tag{2.4.13}$$

Step 4. We now have the zeroth-order metric solution, which has solved the five-dimensional dynamics. The next step is to use this metric in the original action and integrate over the fifth dimension (c.f. [66]). The zeroth-order part of the action integrates to exactly zero, while the remainder of the action (the original second-order terms) yields the following four-dimensional effective action.

$$\begin{aligned} S = &\int d^4x \frac{\sqrt{-\gamma}}{2k\kappa_5^2} \left[\left(1 - e^{\chi(L)}\right) R^{(4)} - \frac{3}{2} e^{\chi(L)} (\nabla^a \chi(L))(\nabla_a \chi(L))\right] \\ &+ {}^0 S_m \left[\gamma_{ab}, {}^0\phi\right] + {}^L S_m \left[e^{\chi(L)} \gamma_{ab}, {}^L\phi\right] \end{aligned} \tag{2.4.14}$$

The constraint $\det \gamma = -1$ has been relaxed, instead choosing $\chi(0) = 0$.

Step 5. Transforming to the Einstein frame, let $g_{ab} = \left(1 - e^{\chi(L)}\right) \gamma_{ab}$, and define $\exp(\chi(x^a, L)/2) = \tanh\left(\kappa_4 \varphi(x^a)/\sqrt{6}\right)$. Let $\kappa_4^2 = k\kappa_5^2$ be the four-dimensional gravitational scale. The action in the Einstein frame is then given by

$$
\begin{aligned}
S = & \int d^4x\sqrt{-g}\left[\frac{R^{(4)}}{2\kappa_4^2} - \frac{1}{2}(\nabla^a\varphi)(\nabla_a\varphi)\right] \\
& + {}^0S_m\left[\cosh^2\left(\frac{\kappa_4\varphi}{\sqrt{6}}\right)g_{ab}, {}^0\phi\right] + {}^LS_m\left[\sinh^2\left(\frac{\kappa_4\varphi}{\sqrt{6}}\right)g_{ab}, {}^L\phi\right].
\end{aligned} \tag{2.4.15}
$$

This action corresponds to the four-dimensional effective action arrived at by other means, such as the covariant curvature formalism [62, 66].

For the rest of this chapter, we confine our discussions to uncompactified N-brane models. In Appendix B, we revisit orbifold models in more detail.

2.5 The Five-Dimensional Action in a Convenient Gauge

We now begin to derive the result (2.3.3), starting from the action (2.3.2). We start by making coordinate choices to simplify the expression, and separate out contributions due to discontinuities in the connection across branes. We specialize the coordinate system to that of the straight gauge [58] and give the action corresponding to (2.3.2) in this gauge. Again, while the details presented here are specific to an uncompactified extra dimension, they generalize straightforwardly to the other situations described previously.

In general, the five-dimensional Ricci scalar can have distributional components at the branes, as the metric will have a discontinuous first derivative due to the brane tensions. It is convenient to separate these distributional components from the continuous parts. It is further convenient to use separate bulk coordinates x_n^Γ in each bulk region $\mathcal{R}_n$, rather than using a single global coordinate system. We will therefore have a bulk metric in each region $\mathcal{R}_n$, rather than one global metric. We note that the n^{th} brane will then have two embedding functions: ${}^nx_n^\Gamma(w_n^a)$ in the coordinates x_n^Γ of $\mathcal{R}_n$, and ${}^nx_{n+1}^\Gamma(w_n^a)$ in the coordinates x_{n+1}^Γ of $\mathcal{R}_{n+1}$.

Combining these modifications, we can write Eq. (2.3.2) as

$$
\begin{aligned}
S\left[g_{\Gamma\Sigma}, {}^{n}x^{\Gamma}, {}^{n}\phi\right] = & \sum_{n=0}^{N} \int_{\mathcal{R}_n} d^5x_n \sqrt{-{}^{n}g} \left(\frac{{}^{n}R^{(5)}}{2\kappa_5^2} - \Lambda_n \right) \\
& + \sum_{n=0}^{N-1} \frac{1}{\kappa_5^2} \int_{\mathcal{B}_n} d^4w_n \sqrt{-{}^{n}h} \left({}^{n}K^{+} + {}^{n}K^{-} \right) \\
& - \sum_{n=0}^{N-1} \sigma_n \int_{\mathcal{B}_n} d^4w_n \sqrt{-{}^{n}h} + \sum_{n=0}^{N-1} {}^{n}S_m[{}^{n}h_{ab}, {}^{n}\phi]
\end{aligned} \tag{2.5.1}
$$

where ${}^{n}K^{+}$ is the trace of the extrinsic curvature tensor of the n^{th} brane in the bulk region $\mathcal{R}_{n+1}$, and ${}^{n}K^{-}$ is the trace of the extrinsic curvature tensor of the n^{th} brane in the bulk region $\mathcal{R}_n$, where the normals are always defined to be pointing away from the bulk region and towards the brane [see Eqs. (2.5.13) and (2.5.14) below]. These terms are just the usual Gibbons-Hawking terms [70].

I Specializing the Coordinate System

We begin by specializing the coordinate systems in each bulk region. Denote the coordinates by $x_n^{\Gamma} = (x_n^a, y_n)$, where a indicates one temporal and three spatial dimensions. Without loss of generality, we can choose the coordinates such that the branes bounding the region are located at fixed y_n. Next, choose the y_n coordinates such that the branes are located at $y_n = n-1$ and $y_n = n$. In other words, in the brane embedding functions ${}^{n}x_n^{\Gamma}(w_n^a)$,

$$ {}^{n-1}y_n(w_{n-1}^a) = n-1, \tag{2.5.2} $$

$$ {}^{n}y_n(w_n^a) = n. \tag{2.5.3} $$

In this way, the first brane will be located at $y_0 = y_1 = 0$, and the last brane located at $y_{N-1} = y_N = N-1$. The n^{th} bulk region $\mathcal{R}_n$ then extends from $y_n = n-1$ to $y_n = n$, with the exceptions of the first and last bulk regions, which extend away from the branes to $\mp\infty$ respectively.

Next, we use some of the available gauge freedom to remove off-diagonal elements of the metrics. Carena *et al.* [58] have shown that it is always possible to find a coordinate transformation in $\mathcal{R}_n$ of the form $x_n^a \to f_n^a(x_n^b, y_n)$ to make ${}^n g_{ya} = 0$ while simultaneously maintaining that the branes be located at $y_n = n-1$ and $y_n = n$. After such a transformation, the metric in $\mathcal{R}_n$ can be written as

$$ {}^n ds^2 = {}^n\gamma_{ab}(x_n^c, y_n) dx_n^a dx_n^b + {}^n\Phi^2(x_n^c, y_n) dy_n^2 \tag{2.5.4} $$

where the sign of ${}^n g_{yy}$ is known from the signature of the metric. We choose the sign of ${}^n\Phi$ to be positive.

The brane positions are now hyperplanes located at $y_n =$ integer. It is obvious that only coordinate transformations for which $y \to g(y)$ (with no x^a dependence) can preserve this form for the hyperplanes. With this condition, only coordinate transformations for which $x^a \to f^a(x^b)$ will preserve the form of the metric. Thus, the remaining gauge freedom lies in coordinate transformations of the form $x^a \to f^a(x^b)$ and $y \to g(y)$ such that the positions of the branes are preserved.

For later simplicity, we choose the following parameterization of the four-dimensional metric ${}^n\gamma_{ab}$. In each bulk region, let

$$ {}^n\gamma_{ab}(x_n^c, y_n) = e^{{}^n\chi(x_n^c, y_n)}\, {}^n\hat{\gamma}_{ab}(x_n^c, y_n) \tag{2.5.5} $$

such that the determinant of ${}^n\hat{\gamma}_{ab}$ is constrained to be -1. The function $\exp({}^n\chi)$ is sometimes called the warp factor. The metric in $\mathcal{R}_n$ is then

$$ {}^n ds^2 = e^{{}^n\chi(x_n^c, y_n)}\, {}^n\hat{\gamma}_{ab}(x_n^c, y_n) dx_n^a dx_n^b + {}^n\Phi^2(x_n^c, y_n) dy_n^2. \tag{2.5.6} $$

II Embedding Functions, Coordinate Systems on the Branes, and Induced Metrics

We now specialize the coordinate system w_n^a on the n^{th} brane $\mathcal{B}_n$. We choose the coordinate system on $\mathcal{B}_0$ to coincide with the first four coordinates of the bulk coordinate system of $\mathcal{R}_0$, evaluated on the brane. Thus,

$$
{}^{0}x_{0}^{\Gamma}(w_0^a) = ({}^{0}x_0^a(w_0^a), {}^{0}y_0(w_0^a)) \tag{2.5.7a}
$$

$$
= (w_0^a, 0). \tag{2.5.7b}
$$

Now, transform the coordinates in the second bulk region by transforming x_1^a such that

$$
{}^{0}x_{1}^{\Gamma}(w_0^a) = (w_0^a, 0). \tag{2.5.7c}
$$

Such a transformation only requires a mapping of the form $x_1^a \to f^a(x_1^b)$, and so the locations of the branes are preserved. Next, choose a coordinate system w_1^a on $\mathcal{B}_1$ such that

$$
{}^{1}x_{1}^{\Gamma}(w_1^a) = (w_1^a, 1) \tag{2.5.7d}
$$

and continue this process until all branes and bulk regions have related coordinate systems. The coordinate systems we acquire have the property that for a point $\mathcal{P}$ on $\mathcal{B}_n$, we have

$$
{}^{n}x_{n}^{\Gamma}(\mathcal{P}) = {}^{n}x_{n+1}^{\Gamma}(\mathcal{P}). \tag{2.5.7e}
$$

Note that while the condition (2.5.7e) implies that the coordinate patches can be joined continuously from one region to another in a straightforward manner, they need not form a global coordinate system because they may not join smoothly across the branes.

From the embedding functions in these coordinate systems we can calculate the induced metric on the branes, using Eq. (2.3.1). As each brane is adjacent to two bulk regions, there will be two induced metrics, one from each bulk region. For $\mathcal{B}_n$, the induced metric from $\mathcal{R}_n$ is

$$
{}^{n}h_{ab}^{-}(w_n^c) = e^{{}^{n}\chi(w_n^c,n)}\, {}^{n}\hat{\gamma}_{ab}(w_n^c, n) \tag{2.5.8}
$$

while the induced metric from $\mathcal{R}_{n+1}$ is

$$
{}^{n}h_{ab}^{+}(w_n^c) = e^{{}^{n+1}\chi(w_n^c,n)}\, {}^{n+1}\hat{\gamma}_{ab}(w_n^c, n) \tag{2.5.9}
$$

We will restrict attention to configurations where the two induced metrics coincide (as would be enforced by the first Israel junction condition [71]). We then have

$$
{}^{n}h_{ab}(w_n^c) = {}^{n}h^{-}_{ab}(w_n^c) = {}^{n}h^{+}_{ab}(w_n^c) \tag{2.5.10a}
$$

$$
{}^{n}h_{ab}(w_n^c) = e^{{}^{n}\chi(w_n^c,n)}\,{}^{n}\hat{\gamma}_{ab}(w_n^c,n) = e^{{}^{n+1}\chi(w_n^c,n)}\,{}^{n+1}\hat{\gamma}_{ab}(w_n^c,n). \tag{2.5.10b}
$$

Taking the determinant of this expression and using the fact that the determinants of $\hat{\gamma}_{ab}$ are constrained to be -1, we find that

$$
{}^{n}\chi(w_n^c,n) = {}^{n+1}\chi(w_n^c,n). \tag{2.5.11}
$$

Then by Eqs. (2.5.10), it follows that

$$
{}^{n}\hat{\gamma}_{ab}(w_n^c,n) = {}^{n+1}\hat{\gamma}_{ab}(w_n^c,n). \tag{2.5.12}
$$

III The Action

Now that we have specialized the coordinate systems for every region and brane in our model, we can rewrite our action (2.5.1) in terms of these coordinates.

We can evaluate the extrinsic curvature tensor terms as follows. Each brane has two normal vectors, one each from the two adjacent bulk regions. We define the normal vectors ${}^{n}\vec{n}^{\pm}$ at $\mathcal{B}_n$ to be the inward pointing normals from $\mathcal{R}_{n+1}$ and $\mathcal{R}_n$. Since the branes are at fixed values of the coordinates y_n, this gives

$$
{}^{n}\vec{n}^{-}(w_n^a) = \frac{1}{{}^{n}\Phi(w_n^a,n)}\partial_{y_n} \tag{2.5.13}
$$

as the normal vector from $\mathcal{R}_n$ and

$$
{}^{n}\vec{n}^{+}(w_n^a) = -\frac{1}{{}^{n+1}\Phi(w_n^a,n)}\partial_{y_{n+1}} \tag{2.5.14}
$$

as the normal vector from $\mathcal{R}_{n+1}$. The vector ${}^{n}\vec{n}^{-}$ points to the right of bulk region n towards brane n, while ${}^{n}\vec{n}^{+}$ points to the left of region $n+1$ towards brane n, using the layout illustrated in Fig. 6.

For the extrinsic curvature tensors, we have by definition

$$ {}^{n}K^{-}_{ab}(w^c_n) = \left. \frac{\partial({}^{n}x^{\alpha})}{\partial w^a_n}\frac{\partial({}^{n}x^{\beta})}{\partial w^b_n}\nabla_{\beta}{}^{n}n^{-}_{\alpha}\right|_{x^c_n=w^c_n, y_n=n}, \tag{2.5.15} $$

$$ {}^{n}K^{+}_{ab}(w^c_n) = \left. \frac{\partial({}^{n+1}x^{\alpha})}{\partial w^a_n}\frac{\partial({}^{n+1}x^{\beta})}{\partial w^b_n}\nabla_{\beta}{}^{n}n^{+}_{\alpha}\right|_{x^c_{n+1}=w^c_n, y_{n+1}=n}. \tag{2.5.16} $$

Evaluating these using the explicit form of the normals, we have

$$ {}^{n}K^{-}_{ab}(w^c_n) = \frac{1}{2}\frac{1}{{}^{n}\Phi}\left({}^{n}\chi_{,y}e^{{}^{n}\chi}\,{}^{n}\hat{\gamma}_{ab} + e^{{}^{n}\chi}\,{}^{n}\hat{\gamma}_{ab,y}\right)(w^c_n, n), \tag{2.5.17} $$

$$ {}^{n}K^{+}_{ab}(w^c_n) = -\frac{1}{2}\frac{1}{{}^{n+1}\Phi}\left({}^{n+1}\chi_{,y}e^{{}^{n+1}\chi}\,{}^{n+1}\hat{\gamma}_{ab} + e^{{}^{n+1}\chi}\,{}^{n+1}\hat{\gamma}_{ab,y}\right)(w^c_n, n). \tag{2.5.18} $$

To take the trace of the extrinsic curvature tensor, we contract with the inverse induced metric

$$ {}^{n}h^{ab} = e^{-{}^{n}\chi}\,{}^{n}\hat{\gamma}^{ab} = e^{-{}^{n+1}\chi}\,{}^{n+1}\hat{\gamma}^{ab}. \tag{2.5.19} $$

We find

$$ {}^{n}K^{+}(w^c_n) = \left. -\frac{2\,{}^{n+1}\chi_{,y}}{{}^{n+1}\Phi}\right|_{w^c_n, n}, \tag{2.5.20} $$

$$ {}^{n}K^{-}(w^c_n) = \left. \frac{2\,{}^{n}\chi_{,y}}{{}^{n}\Phi}\right|_{w^c_n, n}. \tag{2.5.21} $$

In deriving these equations, we used the fact that ${}^{n}\hat{\gamma}^{ab}\ {}^{n}\hat{\gamma}_{ab,y} = 0$, which follows from $\det({}^{n}\hat{\gamma}_{ab}) = -1$.

From Eq. (2.5.6), the determinant of the five-dimensional metric can be written as

$$ \sqrt{-{}^{n}g} = {}^{n}\Phi e^{2\,{}^{n}\chi}\sqrt{-{}^{n}\hat{\gamma}}. \tag{2.5.22} $$

We do not substitute $\sqrt{-{}^{n}\hat{\gamma}} = 1$ at this stage; instead we choose to enforce this at the level of the action by a Lagrange multiplier (see Appendix A). Using Eqs. (2.5.20), (2.5.21), and

(2.5.22), the action (2.5.1) can be written as

$$\begin{aligned}
S\left[{}^n\hat{\gamma}_{ab}, {}^n\Phi, {}^n\chi, {}^n\phi\right] = & \sum_{n=0}^{N} \int_{\mathcal{R}_n} d^5x_n \, {}^n\Phi e^{2\,{}^n\chi} \sqrt{-{}^n\hat{\gamma}} \left(\frac{{}^nR^{(5)}}{2\kappa_5^2} - \Lambda_n \right) \\
& + \sum_{n=0}^{N-1} \frac{2}{\kappa_5^2} \int_{\mathcal{B}_n} d^4w_n e^{2\,{}^n\chi(n)} \sqrt{-{}^n\hat{\gamma}} \left(\left. \frac{{}^n\chi_{,y}}{{}^n\Phi} \right|_{y_n=n} - \left. \frac{{}^{n+1}\chi_{,y}}{{}^{n+1}\Phi} \right|_{y_{n+1}=n} \right) \\
& - \sum_{n=0}^{N-1} \sigma_n \int_{\mathcal{B}_n} d^4w_n e^{2\,{}^n\chi(n)} \sqrt{-{}^n\hat{\gamma}} + \sum_{n=0}^{N-1} {}^nS_m[{}^nh_{ab}, {}^n\phi]. \qquad (2.5.23)
\end{aligned}$$

2.6 Separation of Lengthscales

We now describe the approximation method, based on a two-lengthscale expansion, which we use to obtain a four-dimensional description of the system. We begin by defining the appropriate lengthscales, and then detail how the theory simplifies in the regime where the ratio of lengthscales is small.

I Two Lengthscales

There are three groups of parameters in our model: the five-dimensional gravitational scale κ_5^2, the brane tensions $\{\sigma_n\}$, and the bulk cosmological constants $\{\Lambda_n\}$. We assume that all parameters in a group are of the same order of magnitude, and so will just consider typical parameters σ and Λ. Working with units in which $c = 1$, the dimensionality of these parameters in terms of mass units M and length units L are $[\kappa_5^2] = L^2/M$, $[\sigma] = M/L^3$, and $[\Lambda] = M/L^4$.

We assume that the dimensionless combination $\sigma^2\kappa_5^2/\Lambda$ is approximately of order unity; this will be enforced by the brane-tuning conditions we derive below [see Eq. (2.7.17)]. Eliminating κ_5^2, we can then define a lengthscale by

$$\mathcal{L} = \sigma/\Lambda \qquad (2.6.1)$$

and a mass scale by

$$\mathcal{M} = \sigma^4/\Lambda^3. \tag{2.6.2}$$

For a given configuration, we also define a *four-dimensional curvature lengthscale* $\mathcal{L}_c(y)$ on each slice of constant y, as follows. We take the minimum of the transverse lengthscale over which the induced metric varies, and the transverse lengthscale over which the metric coefficient Φ varies. In other words,

$$\mathcal{L}_c(y) \sim \min \left\{ \left| R^{(4)}_{\hat{a}\hat{b}\hat{c}\hat{d}} \right|^{-1/2}, \left| \nabla_{\hat{a}} R^{(4)}_{\hat{b}\hat{c}\hat{d}\hat{e}} \right|^{-1/3}, \ldots, \frac{|\Phi|}{|\nabla_{\hat{a}}\Phi|}, \frac{|\Phi|^{1/2}}{\left|\nabla_{\hat{a}}\nabla_{\hat{b}}\Phi\right|^{1/2}}, \ldots \right\} \tag{2.6.3}$$

where $\hat{a}, \hat{b}, \ldots$ denotes an orthonormal basis of the induced metric, $R_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is the Riemann tensor of the induced metric, and dots denote similar terms with more derivatives.

Thus, for a given configuration, we have two natural lengthscales: the microphysical lengthscale $\mathcal{L} = \sigma/\Lambda$ (the same for all configurations), and the macrophysical curvature lengthscale $\mathcal{L}_c$ (where the c is intended to denote "curvature").

II Separating the Lengthscales

We now evaluate the action (2.5.23) in the low-energy regime $\mathcal{L}_c \gg \mathcal{L}$, in which the theory admits a four-dimensional description. We will find that there is a leading order term of order $\sim \mathcal{M}\mathcal{L}$, and a subleading term of order $\sim \mathcal{M}\mathcal{L}(\mathcal{L}/\mathcal{L}_c)^2$. Our strategy will be to separate the contributions to the action at each order, minimize the leading order piece of the action, and then substitute the general solutions obtained from that minimization into the subleading piece of the action. The result will be a four-dimensional action that gives the effective description of the system in the low-energy regime.

We write the action (2.5.23) as a sum $S = S_g + S_m$ of a gravitational part S_g and a matter part S_m, where the matter part is the last term in Eq. (2.5.23) and the gravitational part comprises the remaining terms.

We first discuss the expansion of the gravitational action S_g, which is a functional of a bulk metric $g_{\alpha\beta}$ and brane embedding functions ${}^n x^\Gamma$. We define a mapping T_ϵ that acts on these variables

$$T_\epsilon : (g_{\alpha\beta}, {}^n x^\Gamma) \to (g^\epsilon_{\alpha\beta}, {}^n x^\Gamma_\epsilon), \tag{2.6.4}$$

where $\epsilon > 0$ is a dimensionless parameter, as follows: (i) We specialize to our chosen gauge, (ii) replace the metric (2.5.6) with the rescaled version

$$ds^2_\epsilon = \frac{1}{\epsilon^2} e^{\chi(x^c,y)} \hat{\gamma}_{ab}(x^c, y) dx^a dx^b + \Phi^2(x^c, y) dy^2, \tag{2.6.5}$$

where indices indicating regions have been suppressed, and (iii) leave the embedding functions in our chosen gauge unaltered. We may think of ϵ as a parameter that tunes the ratio of the microphysical lengthscale to the macrophysical lengthscale. As ϵ is decreased, lengthscales on the brane are inflated, and so $\mathcal{L}_c$ increases. Thus, as ϵ decreases, so does the ratio $\mathcal{L}/\mathcal{L}_c$. In particular, we have

$$\left(\frac{\mathcal{L}}{\mathcal{L}_c}\right)_\epsilon = \epsilon \frac{\mathcal{L}}{\mathcal{L}_c}. \tag{2.6.6}$$

It is important to note that this ϵ scaling does not map solutions to solutions, but just provides a means of keeping track of the dependence on the various lengthscales.

We can construct a one-parameter family of action functionals by using these rescaled metrics in our original action (2.5.23)[2]:

$$S_{g,\epsilon}\left[g_{\alpha\beta}, {}^n x^\Gamma\right] \equiv \epsilon^4 S_g \left[g^\epsilon_{\alpha\beta}, {}^n x^\Gamma_\epsilon\right]. \tag{2.6.7}$$

We can expand this action in powers of ϵ by

$$S_{g,\epsilon}\left[g_{\alpha\beta}, {}^n x^\Gamma\right] = S_{g,0}\left[g_{\alpha\beta}\right] + \epsilon^2 S_{g,2}\left[g_{\alpha\beta}\right], \tag{2.6.8}$$

where on the right hand side we omit the dependence on the embedding functions since we have used the gauge freedom to fix those. The expansion (2.6.8) truncates after two terms;

[2]The factor of ϵ^4 in Eq. (2.6.7) is for convenience, so that Eq. (2.6.8) contains terms of $O(1)$ and $O(\epsilon^2)$. This is explicitly shown in Section 2.7.

there are no higher-order terms in ϵ. Note that there is no $O(\epsilon)$ term, as when the action (2.6.7) is evaluated, terms of $O(\epsilon^2)$ arise from contractions in the Ricci scalar using g^{ab} ($O(1)$ terms arise from g^{yy} contractions). Terms of order $O(\epsilon)$ would arise from contractions using g^{ay}, but as these components of the metric have been gauge-fixed to zero, they are not present. This can be seen explicitly in the expansion of the Ricci scalar (A.3). As we tune $\epsilon \to 0$, we move further into the low-energy regime, and so we identify the zeroth-order term as the dominant contribution to the action, and the second-order term as the subleading term. This provides the separation of lengthscales we desire.

Let us now turn to the matter contribution to the action, S_m. We expect the matter action to contribute at $O(\epsilon^2)$, the same order as the subleading gravitational term. To see this, note that the brane tensions scales as $\sigma \sim \mathcal{M}/\mathcal{L}^3$, where the scales $\mathcal{M}$ and $\mathcal{L}$ were defined in Eqs. (2.6.1) and (2.6.2). The matter action will be roughly $S_m \sim \int \rho \, d^4x$, where ρ is a four-dimensional energy density. The four-dimensional Newton constant $\kappa_4^2 = 8\pi G$ is of order $\kappa_4^2 \sim \mathcal{L}/\mathcal{M}$ by dimensional analysis [c.f. Eq. (2.8.14) below], and so ρ will be of order

$$\rho \sim \frac{1}{\kappa_4^2 \mathcal{L}_c^2} \sim \frac{\mathcal{M}}{\mathcal{L}\mathcal{L}_c^2}. \tag{2.6.9}$$

Taking the ratio ρ/σ now gives

$$\frac{\rho}{\sigma} \sim \frac{\mathcal{M}/\mathcal{L}\mathcal{L}_c^2}{\mathcal{M}/\mathcal{L}^3} \sim \frac{\mathcal{L}^2}{\mathcal{L}_c^2} \propto \epsilon^2. \tag{2.6.10}$$

Formally, the scaling (2.6.10) can be achieved by replacing the matter action S_m with a rescaled action $S_{m,\epsilon}$ given by (i) multiplying by ϵ^4 as in Eq. (2.6.7), (ii) rescaling all fields and dimensional constants with dimensions (mass)^r(length)^s by factors of $\epsilon^{-(r+s)}$. The expansion of the full action is then

$$\begin{aligned} S_\epsilon = S_{g,\epsilon} + S_{m,\epsilon} &= S_{g,0} + \epsilon^2 \left[S_{g,2} + S_m \right] \\ &= S_0 + \epsilon^2 S_2. \end{aligned} \tag{2.6.11}$$

It can be seen that given brane tensions tuned to the bulk cosmological constants, $\sigma^2 \sim \Lambda/\kappa_5^2$, we require that the matter density on a brane should be small, so as not to spoil the tuning.

This also yields $\rho \ll \sigma$, which roughly corresponds to the separation of lengthscales condition $\mathcal{L} \ll \mathcal{L}_c$.

We perform this ϵ scaling separately in each bulk region of the model. The contribution to the action from each region will separate into zeroth- and second-order terms.

III The Low-Energy Regime

Now that the contributions to each order have been identified, we can minimize the leading order term in the action, S_0. Once general solutions to the equations of motion have been found, we can use these solutions in the second-order term in the action. Thus, we solve for the high-energy (short lengthscale) dynamics first, and use the solution to this as a background solution for the low-energy (long lengthscale) dynamics. At this point, we may let $\epsilon \to 1$, and rely on the ratio $(\mathcal{L}/\mathcal{L}_c)^2$ being sufficiently small to provide the separation of lengthscales.

The effect of this separation of lengthscales is to enforce a decoupling of the high-energy dynamics from the low-energy dynamics. We will see below that the equation of motion for the high-energy dynamics contains y derivatives, but no x^a derivatives. The theory at this order thus reduces to a set of uncoupled theories, one along each fiber $x^a = \text{const}$ in the bulk. These theories are coupled together at $O(\epsilon^2)$, and thus in the regime of interest, the coupling is minimal. After solving the high-energy dynamics along these fibers, a four-dimensional effective description of the system remains.

The low-energy regime, in which the theory admits a four-dimensional description, is the regime

$$\mathcal{L}_c \gg \mathcal{L}. \tag{2.6.12}$$

This regime is also frequently characterized in the literature by the condition

$$\rho \ll \sigma, \tag{2.6.13}$$

where ρ is the mass density on a brane and σ is a brane tension [c.f. Eq. (2.6.10) above]. One can interpret the condition (2.6.13) as saying that the mass density on the brane must be sufficiently small that the brane-tuning conditions [Eq. (2.7.17) below which enforces $\sigma^2 \sim \Lambda/\kappa_5^2$] are not appreciably modified. However, the condition (2.6.13) is less general than the condition (2.6.12), and although necessary, is actually insufficient. First, as discussed above, (2.6.13) only applies to branes, whereas (2.6.12) applies at each value of y, including away from the branes. Second, even when the density on a given brane vanishes, four-dimensional gravitational waves on that brane can give rise to radii of curvature $\mathcal{L}_c$ that are comparable to $\mathcal{L}$. In this case, the separation of lengthscales will not apply and the four-dimensional effective theory will not be valid, despite the fact that the condition (2.6.13) is satisfied. Curvature associated with the metric coefficient Φ can also yield similar results.

Finally, we discuss a subtlety in our definition of the "low-energy regime". As noted in the previous paragraph, $\mathcal{L}_c$ varies with position in the five-dimensional universe. Our separation of lengthscales will break down when the induced metric on any slice of constant y has a radius of curvature $\mathcal{L}_c$ comparable to that of the microphysical lengthscale $\mathcal{L}$; it is insufficient to require that $\mathcal{L}_c \gg \mathcal{L}$ on each brane. When this happens, the terms of order ϵ^2 will couple strongly to the $O(1)$ terms, and our approximate solutions for the five-dimensional metric will no longer be valid. This will generically occur at sufficiently large distances from the branes, as $\exp({}^n\chi)$ typically grows exponentially small away from the branes, and $\mathcal{L}_c^{-2} \propto \exp(-{}^n\chi)R^{(4)}$. Despite this breakdown, the contribution to the action from these regimes is exponentially suppressed by the warp factor, and thus provides only a small deviation from the effective theory. It is unlikely that the warp factor can grow without bound after encountering this regime while maintaining a globally hyperbolic spacetime.

2.7 The Action to Lowest Order

In this section, we calculate and explicitly solve the equations of motion to lowest order in the two-lengthscale expansion. First, however, we write out the complete, rescaled action showing explicitly the dependence on ϵ. Inserting the decomposition (A.3) of the Ricci scalar and the rescaled metric (2.6.5) into the action (2.5.23) [following the prescription of Eq. (2.6.7)], we obtain

$$\begin{aligned} S_\epsilon = \sum_{n=0}^{N} \int_{\mathcal{R}_n} d^5x_n \Bigg[& \sqrt{-{}^n\hat{\gamma}} \frac{e^{2\,{}^n\chi}}{2\kappa_5^2\,{}^n\Phi} \Bigg(-\frac{1}{4} {}^n\hat{\gamma}^{ab\,n}\hat{\gamma}_{bc,y}\,{}^n\hat{\gamma}^{cd\,n}\hat{\gamma}_{da,y} - 5({}^n\chi_{,y})^2 - 4\,{}^n\chi_{,yy} \\ & + 4\frac{{}^n\Phi_{,y}}{{}^n\Phi}\,{}^n\chi_{,y} - 2\kappa_5^2\,{}^n\Phi^2\Lambda_n \Bigg) + {}^n\lambda(x^a, y)\left(\sqrt{-{}^n\hat{\gamma}} - 1\right) \Bigg] \\ & + \sum_{n=0}^{N-1} \int_{\mathcal{B}_n} d^4w_n e^{2\,{}^n\chi(n)} \sqrt{-{}^n\hat{\gamma}} \left[\frac{2}{\kappa_5^2} \left(\frac{{}^n\chi_{,y}}{{}^n\Phi}\Bigg|_{y_n=n} - \frac{{}^{n+1}\chi_{,y}}{{}^{n+1}\Phi}\Bigg|_{y_{n+1}=n} \right) - \sigma_n \right] \\ & + \sum_{n=0}^{N} \epsilon^2 \int_{\mathcal{R}_n} d^5x_n \sqrt{-{}^n\hat{\gamma}} \frac{e^{{}^n\chi}}{2\kappa_5^2} \Bigg({}^n\Phi\,{}^nR^{(4)} - 3\,{}^n\Phi \nabla^a \nabla_a {}^n\chi - \frac{3}{2}{}^n\Phi (\nabla^{a\,n}\chi)(\nabla_a {}^n\chi) \\ & - 2\nabla^a\nabla_a {}^n\Phi - 2(\nabla^{a\,n}\chi)(\nabla_a {}^n\Phi) \Bigg) + \sum_{n=0}^{N-1} \epsilon^2\,{}^nS_m[{}^nh_{ab}, {}^n\phi] \end{aligned} \tag{2.7.1}$$

where we include the Lagrange multiplier terms (A.2) discussed in Appendix A, and the factor of ϵ^2 in front of the matter action comes from the process described in the previous section (functional dependence of the action on $[{}^n\hat{\gamma}_{ab}, {}^n\Phi, {}^n\chi, {}^n\phi]$ has been suppressed to save space). This form explains the choice of the ϵ^4 factor in Eq. (2.6.7), and shows the decomposition into $O(1)$ and $O(\epsilon^2)$ terms, as claimed in Eq. (2.6.11).

From the form of Eq. (2.7.1), we see that we can neglect the last two lines in the limit $\epsilon \to 0$. We can obtain a more precise characterization of the domain of validity of this low-energy approximation by estimating the ratio between the terms dropped and the terms retained. As an example, consider the first term on the 4$^{\text{th}}$ line and the first term on the first line. Their ratio is (dropping the 'n' labels)

$$\left[e^{\chi}\Phi R^{(4)}\right]\left[\frac{e^{2\chi}}{\Phi}\hat{\gamma}^{ab}\hat{\gamma}_{bc,y}\hat{\gamma}^{cd}\hat{\gamma}_{da,y}\right]^{-1} \sim \left[e^{\chi}\Phi R^{(4)}\right]\left[\frac{e^{2\chi}}{\Phi\tilde{y}^2}\right]^{-1}$$
$$\sim \left[e^{-\chi}R^{(4)}\right]\left[\Phi^2\tilde{y}^2\right] \qquad (2.7.2)$$

where $\tilde{y}$ is the coordinate lengthscale over which $\hat{\gamma}_{ab}$ varies. We recognize the first factor as essentially the Ricci scalar of the induced metric $e^{\chi}\hat{\gamma}_{ab}$, which is of order $\mathcal{L}_c^{-2}$. We recognize the second factor as the square of the physical lengthscale in the y direction over which $\hat{\gamma}$ varies, which is always $\sim \mathcal{L}^2$ (see the explicit solution (2.7.20) below). Thus, the ratio is $(\mathcal{L}/\mathcal{L}_c)^2$, confirming the identification of the low-energy regime as $\mathcal{L} \ll \mathcal{L}_c$.

I Varying the Action

In the action (2.7.1) at zeroth-order in ϵ, we have three fields to vary (in N regions): ${}^n\Phi(x^c, y)$, ${}^n\chi(x^c, y)$, and ${}^n\hat{\gamma}_{ab}(x^c, y)$. There is a subtlety in the variation however. The constraint that $\det\left({}^n\hat{\gamma}_{ab}\right) = -1$ must be imposed either at the level of the equations of motion, or by a Lagrange multiplier. The Lagrange multiplier is explicitly included in Eq. (2.7.1). Further details are provided in Appendix A.

We begin by varying with respect to ${}^n\Phi$. From this variation, we find a single equation of motion in each region,

$$\frac{1}{4}\,{}^n\hat{\gamma}^{ab}\;{}^n\hat{\gamma}_{bc,y}\,{}^n\hat{\gamma}^{cd}\;{}^n\hat{\gamma}_{da,y} - 3\;{}^n\chi_{,y}^2 - 2\kappa_5^2\,{}^n\Phi^2\Lambda_n = 0. \qquad (2.7.3)$$

Next, we vary with respect to ${}^n\hat{\gamma}_{ab}$. Note that in varying the action, we obtain boundary terms from neighboring regions from the relationship (2.5.12). The variation produces an equation of motion in each bulk region,

$${}^n\hat{\gamma}_{ad,yy} = {}^n\hat{\gamma}_{ab,y}\;{}^n\hat{\gamma}^{bc}\;{}^n\hat{\gamma}_{cd,y} - {}^n\hat{\gamma}_{ad,y}\left(2\;{}^n\chi_{,y} - \frac{{}^n\Phi_{,y}}{{}^n\Phi}\right). \qquad (2.7.4)$$

(If using Lagrange multipliers, this equation results after the Lagrange multiplier is eliminated by tracing the equation using ${}^n\hat{\gamma}^{ab}$ and back substituting). Note that tracing over the indices in Eq. (2.7.4) and using Eq. (A.4) leads to Eq. (A.5) as expected. We also find a boundary condition to be satisfied at each brane,

$$\frac{1}{{}^{n}\Phi}\,{}^{n}\hat{\gamma}_{ab,y}(y_n = n) = \frac{1}{{}^{n+1}\Phi}\,{}^{n+1}\hat{\gamma}_{ab,y}(y_{n+1} = n). \tag{2.7.5}$$

Finally, we vary with respect to ${}^{n}\chi$. Here, we once again have boundary terms arising from integrating bulk terms by parts in neighboring regions. There is an equation of motion in each bulk region,

$$\frac{1}{12}\,{}^{n}\hat{\gamma}^{ab}\;{}^{n}\hat{\gamma}_{bc,y}\,{}^{n}\hat{\gamma}^{cd}\;{}^{n}\hat{\gamma}_{da,y} + {}^{n}\chi_{,y}^{2} + {}^{n}\chi_{,yy} - \frac{{}^{n}\Phi_{,y}}{{}^{n}\Phi}\,{}^{n}\chi_{,y} + \frac{2}{3}\kappa_5^2\,{}^{n}\Phi^2\Lambda_n = 0. \tag{2.7.6}$$

We also find a boundary condition at each brane,

$$\left.\frac{{}^{n}\chi_{,y}}{{}^{n}\Phi}\right|_{y_n=n} - \left.\frac{{}^{n+1}\chi_{,y}}{{}^{n+1}\Phi}\right|_{y_{n+1}=n} = \frac{2}{3}\kappa_5^2\sigma_n. \tag{2.7.7}$$

II Solving the Equations of Motion

We have three equations of motion for each bulk region, as well as numerous boundary conditions for the fields at the branes [Eqs. (2.5.11), (2.5.12), (2.7.3), (2.7.4), (2.7.5), (2.7.6), and (2.7.7)]. Note that these equations all describe the dynamics along a fiber of constant x^a which doesn't couple to any other fibers, and so solving these equations of motion consists of solving the dynamics of the extra dimension of the model.

We begin by solving Eq. (2.7.4). It is convenient to solve this equation in matrix notation. Let

$$[\hat{\gamma}_{ab}] = \hat{\boldsymbol{\gamma}} \tag{2.7.8}$$

where we suppress indices n. Then in matrix notation, Eq. (2.7.4) is

$$\ddot{\hat{\boldsymbol{\gamma}}} = \dot{\hat{\boldsymbol{\gamma}}}\;\hat{\boldsymbol{\gamma}}^{-1}\dot{\hat{\boldsymbol{\gamma}}} - \dot{\hat{\boldsymbol{\gamma}}}\left(2\chi_{,y} - \frac{\Phi_{,y}}{\Phi}\right), \tag{2.7.9}$$

where dots denote derivatives with respect to y. It is straightforward to check that a solution to this differential equation in region n is

$$\hat{\boldsymbol{\gamma}}(x^a, y) = \mathbf{A}(x^a) \exp\left(\mathbf{B}(x^a) \int_{n-1}^{y} \Phi(x^a, y') e^{-2\chi(x^a, y')} dy'\right). \tag{2.7.10}$$

where $\mathbf{A}$ and $\mathbf{B}$ are arbitrary 4×4 real matrix functions of x^a. The lower limit on the integral is chosen so that the boundary conditions may be matched at the previous brane (obviously, care must be taken in the first region). The expression (2.7.10) has the correct number of integration constants to satisfy arbitrary boundary conditions. From our knowledge of $\hat{\gamma}_{ab}$, $\mathbf{A}$ must be a symmetric matrix with determinant -1. The exponential has unit determinant, and so $\mathbf{B}$ must be traceless. The condition that $\hat{\boldsymbol{\gamma}}$ is symmetric implies that $\mathbf{B}^T = \mathbf{A}\ \mathbf{B}\ \mathbf{A}^{-1}$. The quantity that appears in Eqs. (2.7.3) and (2.7.6) is

$$\begin{aligned} {}^n\hat{\gamma}^{ab}\ {}^n\hat{\gamma}_{bc,y}\ {}^n\hat{\gamma}^{cd}\ {}^n\hat{\gamma}_{da,y} &= {}^n\gamma^{ab}\ {}^n\gamma_{ab,yy} \\ &= \mathrm{Tr}\left(\mathbf{B}^2(x^a)\right) \Phi^2 e^{-4\chi}. \end{aligned} \tag{2.7.11}$$

We define

$$b(x^a) = \frac{1}{12} \mathrm{Tr}\left(\mathbf{B}^2(x^a)\right) \tag{2.7.12}$$

where the factor of 12 has been chosen for later convenience. From combining Eq. (2.7.5) with Eqs. (2.5.11) and (2.5.12), we see that $\mathbf{B}$ (and thus $b(x^a)$) is independent of region, while $\mathbf{A}$ will change with each region according to Eq. (2.5.12).

From Eq. (2.7.3), we find

$$\begin{aligned} {}^n\chi_{,y} &= \pm\sqrt{b\ {}^n\Phi^2 \exp(-4\,{}^n\chi) - \frac{2}{3}\kappa_5^2\,{}^n\Phi^2 \Lambda_n} \\ &= P_n\,{}^n\Phi \sqrt{b \exp(-4\,{}^n\chi) - \frac{2}{3}\kappa_5^2 \Lambda_n} \end{aligned} \tag{2.7.13}$$

where P_n is either ± 1 and is constant in each bulk region. Differentiating Eq. (2.7.13) gives

$${}^n\chi_{,yy} = \frac{{}^n\Phi_y}{\Phi}\chi_{,y} - 2b\,{}^n\Phi^2 e^{-4^n\chi}. \tag{2.7.14}$$

The same result is obtained by substituting Eq. (2.7.3) into Eq. (2.7.6), and so we see that these equations of motion are degenerate. This leaves only one equation of motion (Eq. (2.7.13)) and one boundary condition (Eq. (2.7.7)) to satisfy.

III Classes of Solutions

If $\mathbf{B}(x^a) \equiv \mathbf{0}$, then the induced metric on all the branes are related to one another by conformal transformations, and a four-dimensional effective action is easily calculated. On the other hand, when $\mathbf{B}(x^a) \neq \mathbf{0}$, the induced metrics on each brane are not simply related conformally, but through Eqs. (2.5.12) and (2.7.10). If solutions with $\mathbf{B}(x^a) \neq \mathbf{0}$ were to exist, the four-dimensional effective theory would contain more than one massless tensor degree of freedom; i.e., it would constitute a multigravity theory (see Damour and Kogan [68]). No such degrees of freedom have been seen in any linearized analyses[3]. It is important to note that this is not a Kaluza-Klein mode. We believe that solutions with $\mathbf{B}(x^a) \neq \mathbf{0}$ are ruled out due to divergences at $y \to \pm\infty$, leading to a lack of global hyperbolicity in the spacetime, although we have been unable to prove this rigorously. We will restrict attention to the case $\mathbf{B}(x^a) = \mathbf{0}$ for the remainder of this work.

IV General Solutions at Leading Order

With $\mathbf{B}(x^a) \equiv \mathbf{0}$, the field ${}^n\hat{\gamma}_{ab}$ becomes independent of y [see Eq. (2.7.10)], and also independent of n by Eq. (2.5.12). This means that we can drop the index n from x^a_n, w_n, and ${}^n\hat{\gamma}_{ab}$ without causing confusion. With $b(x^a) = 0$, the remaining equation of motion and boundary condition simplify somewhat. Equation (2.7.13) becomes

$$ {}^n\chi_{,y} = P_n {}^n\Phi \sqrt{-\frac{2}{3}\kappa_5^2 \Lambda_n}, \tag{2.7.15} $$

[3]In addition, it can be shown that in orbifolded models, there are no solutions with $\mathbf{B}(x^a) \neq \mathbf{0}$; see Appendix B.

which implies that $\Lambda_n < 0$, and so the bulk regions must be slices of anti de-Sitter space. Define

$$k_n = \sqrt{\frac{-\kappa_5^2 \Lambda_n}{6}}. \tag{2.7.16}$$

We can use Eq. (2.7.15) for ${}^n\chi$ in Eq. (2.7.7) to obtain

$$k_n P_n - k_{n+1} P_{n+1} = \frac{1}{3}\kappa_5^2 \sigma_n. \tag{2.7.17}$$

These relations are the well-known "brane-tunings", which determine the branes tensions required in order to avoid a cosmological constant on the branes [6].

We may integrate Eq. (2.7.15) to find

$${}^n\chi(x^a, y) = \begin{cases} 2k_0 P_0 \int_0^y {}^n\Phi(x^a, y')dy' + f(x^a) & n = 0, \\ {}^{n-1}\chi(x^a, n-1) + 2k_n P_n \int_{n-1}^y {}^n\Phi(x^a, y')dy' & n > 0 \end{cases} \tag{2.7.18}$$

where $f(x^a)$ is an arbitrary function. Note that the field ${}^n\chi$ is related to the distance from the previous brane to y along a geodesic normal to the branes, made dimensionless by the appropriate lengthscale in the bulk. In particular, χ describes the number of e-foldings the warp factor in the metric provides between two points in the five-dimensional spacetime. Assuming that Φ is not divergent, if $\exp({}^n\chi(y))$ approaches zero or ∞ anywhere, it can only occur as $y \to \pm\infty$. We will restrict attention to the cases

$$P_0 = +1, \quad \text{and} \quad P_N = -1. \tag{2.7.19}$$

When these signs fail to hold, then the warp factor increases monotonically as one goes to infinity, and it seems likely that the spacetime cannot be globally hyperbolic. We exclude cases where $\exp({}^n\chi(y)) \to 0$ at finite y by restricting ourselves to topologically connected spacetimes [52, 72].

V Summary

We summarize our results so far. We have N branes, each with a brane tension which has been carefully adjusted, according to (2.7.17). The branes divide our system into $N+1$ regions. Our coordinates are x^a, describing four-dimensional space, and y, describing the extra dimension.

We expanded the action in terms of our ϵ scaling parameter to separate the high- and low-energy contributions. Specializing to a low-energy regime, we solved for the high-energy dynamics, arriving at the metric for each region of our system:

$$
{}^n ds^2 = e^{{}^n\chi(x^c,y)}\hat{\gamma}_{ab}(x^c)dx^a dx^b + \frac{{}^n\chi_{,y}^2(x^c,y)}{4k_n^2}dy^2, \tag{2.7.20}
$$

with ${}^n\chi$ given by Eq. (2.7.18), where ${}^n\Phi(x^a,y)$ can be chosen freely. The parameters k_n are determined by the bulk cosmological constants and the five-dimensional Newton's constant, by Eq. (2.7.16). The derivative $\chi_{,y}$ has fixed sign $P_n = \pm 1$ in each region, although the derivative may approach zero as $y \to \pm\infty$.

As an aside, when the metric in each region is in the form (2.7.20), the zeroth-order action $S_0[g_{ab}]$ [Eq. (2.6.8)] evaluates to exactly zero. This can be seen by substituting the metric (2.7.20) into the action and explicitly evaluating the integral over the y dimension. All of the integrals become total derivatives whose boundary terms exactly cancel the boundary terms present in the action at this order.

The background metric ansatz (2.7.20) is essentially the same as the zeroth-order metric calculated by Kanno and Soda [62], taking $\Phi^2(x^a,y) = \exp(2\phi(y,x))$ in their notation. However, from here, we proceed without their assumption that $\phi(y,x) = \phi(x)$. The "naïve" ansatz and the CGR ansatz of Chiba [66] are also in the form of our metric (2.7.20).

2.8 The Action to Second Order

In this section, we investigate the action to second order in ϵ. By integrating out the previously determined high-energy dynamics, we find the four-dimensional effective action.

I Acquiring the Four-Dimensional Effective Action

Using the metric (2.7.20) in Eqs. (2.5.23) and (2.6.8), we can calculate the second-order contribution to the action, S_2. The result is

$$\begin{aligned} S_2\left[{}^n\hat{\gamma}_{ab}, {}^n\chi, {}^n\phi\right] = \sum_{n=0}^{N} \int_{\mathcal{R}_n} d^5x_n \sqrt{-\hat{\gamma}} \frac{e^{{}^n\chi}}{4\kappa_5^2 k_n P_n} \Big[& {}^n\chi_{,y} R^{(4)} - 3\ {}^n\chi_{,y} \nabla^{2\,n}\chi - 2\nabla^{2\,n}\chi_{,y} \\ & - \frac{3}{2} {}^n\chi_{,y} (\nabla^{a\,n}\chi)(\nabla_a{}^n\chi) - 2(\nabla^{a\,n}\chi)(\nabla_a{}^n\chi_{,y}) \Big] \\ & + \sum_{n=0}^{N-1} {}^nS_m\left[e^{{}^n\chi(x^a,n)}\hat{\gamma}_{ab}, {}^n\phi\right]. \end{aligned} \tag{2.8.1}$$

Note that covariant derivatives written here are associated with the metric $\hat{\gamma}_{ab}$, as is the four-dimensional Ricci scalar $R^{(4)}$.

To obtain the effective four-dimensional action, we integrate over y in the five-dimensional action (2.8.1), as the dynamics of this dimension have already been solved. The term involving the Ricci scalar can be integrated straightforwardly, as $R^{(4)}$ has no y dependence, but the other terms require more manipulation. We can combine the last four terms in the five-dimensional integral in the following way:

$$\begin{aligned} -3e^{{}^n\chi}\ {}^n\chi_{,y} \nabla^{2\,n}\chi - \frac{3}{2} e^{{}^n\chi}\ {}^n\chi_{,y} (\nabla^{a\,n}\chi)(\nabla_a{}^n\chi) - 2e^{{}^n\chi} \nabla^{2\,n}\chi_{,y} - 2e^{{}^n\chi}(\nabla^{a\,n}\chi)(\nabla_a{}^n\chi_{,y}) \\ = \frac{3}{2}\frac{\partial}{\partial y}\left(e^{{}^n\chi}(\nabla^{a\,n}\chi)(\nabla_a{}^n\chi)\right) - \nabla^a\left(3e^{{}^n\chi}\ {}^n\chi_{,y}\nabla_a{}^n\chi + 2e^{{}^n\chi}\nabla_a{}^n\chi_{,y}\right) \end{aligned} \tag{2.8.2}$$

The covariant derivative commutes with the integration over the fifth dimension in the action, and thus gives rise to a boundary term at $x^a \to \infty$, which we discard. We obtain

$$S_2\left[{}^n\hat{\gamma}_{ab}, {}^n\chi, {}^n\phi\right] = \sum_{n=0}^{N} \int_{\mathcal{R}_n} d^5x_n \frac{\sqrt{-\hat{\gamma}}}{4\kappa_5^2 k_n P_n} \frac{\partial}{\partial y} \left\{ e^{{}^n\chi} R^{(4)} + \frac{3}{2} e^{{}^n\chi} (\nabla^a {}^n\chi)(\nabla_a {}^n\chi) \right\}$$
$$+ \sum_{n=0}^{N-1} {}^nS_m \left[e^{{}^n\chi(x^a,n)} \hat{\gamma}_{ab}, {}^n\phi \right]. \tag{2.8.3}$$

Integrating over y, we find boundary terms at each brane and at $y = \pm\infty$. We note that the integral converges in the first and last regions because of the choices $P_0 = +1$ and $P_N = -1$, and so the terms evaluated at $\pm\infty$ vanish. We can rearrange the remaining terms into a sum over the branes.

$$S_2\left[{}^n\hat{\gamma}_{ab}, {}^n\chi, {}^n\phi\right] = \sum_{n=0}^{N-1} \int d^4x \sqrt{-\hat{\gamma}} \frac{1}{4\kappa_5^2} \left(\frac{1}{k_n P_n} - \frac{1}{k_{n+1} P_{n+1}} \right) \times$$
$$\left[e^{{}^n\chi} R^{(4)} + \frac{3}{2} e^{{}^n\chi} (\nabla^a {}^n\chi(x^a,n))(\nabla_a {}^n\chi(x^a,n)) \right]_{y=n}$$
$$+ \sum_{n=0}^{N-1} {}^nS_m \left[e^{{}^n\chi(x^a,n)} \hat{\gamma}_{ab}, {}^n\phi \right] \tag{2.8.4}$$

II Field Redefinitions

For convenience, we define the following quantities.

$$A_n = \left| \frac{1}{k_n P_n} - \frac{1}{k_{n+1} P_{n+1}} \right|, \tag{2.8.5}$$

$$\epsilon_n = \operatorname{sgn}\left(\frac{1}{k_n P_n} - \frac{1}{k_{n+1} P_{n+1}} \right), \tag{2.8.6}$$

for $0 \leq n \leq N-1$. It is useful to note that ϵ_n can be written as, from Eq. (2.7.17),

$$\epsilon_n = -\operatorname{sgn}\left(\sigma_n P_n P_{n+1}\right). \tag{2.8.7}$$

We now have a four-dimensional Ricci scalar, and a number of scalar fields. The values of the function $\chi(x^a, n)$ evaluated on the branes become N scalar fields in the four-dimensional action, and we denote these by

$$\Psi_n = \sqrt{A_n e^{\chi_n}}, \tag{2.8.8}$$

where we use $\chi_n = \chi(x^a, n)$. The values of Ψ_n encode the distance between the branes, along with some physical parameters. Note that the domain of Ψ_n is the positive reals. There is a residual parameterization freedom which implies that one of the fields Ψ_n is nondynamical, but before we fix this freedom, we first give the four-dimensional low-energy action using the definitions so far. It is given by[4]

$$\begin{aligned} S\left[\hat{\gamma}_{ab}, \Psi_n, {}^n\phi\right] = & \int d^4x \frac{\sqrt{-\hat{\gamma}}}{4\kappa_5^2}\left[R^{(4)}\left[\hat{\gamma}_{ab}\right]\left(\sum_{n=0}^{N-1}\epsilon_n\Psi_n^2\right) + 6\sum_{n=0}^{N-1}\epsilon_n(\hat{\nabla}^a\Psi_n)(\hat{\nabla}_a\Psi_n)\right] \\ & + \sum_{n=0}^{N-1}{}^nS_m\left[\frac{\Psi_n^2}{A_n}\hat{\gamma}_{ab}, {}^n\phi\right] \end{aligned} \tag{2.8.9}$$

where we have suppressed the subscript “2”, and will continue to do so from now on. Here, we have used the four-dimensional metric $\hat{\gamma}_{ab}$ to raise and lower indices, and $\hat{\nabla}_a$ is the covariant derivative associated with this same metric.

The residual parameterization freedom is

$$\chi(x^a, y) \to \chi(x^a, y) + \delta\chi(x^a) \tag{2.8.10}$$

$$\hat{\gamma}_{ab}(x^a) \to \hat{\gamma}_{ab}e^{-\delta\chi(x^a)}, \tag{2.8.11}$$

under which the metric (2.7.20) is invariant. We can fix this freedom by specifying the value of $\chi(x^a, n)$ for any n. In order to remain general, let us choose $\chi(x^a, T) = 0$, for some T with $0 \le T \le N-1$. This causes the field Ψ_T to become non-dynamical. We note that this means that the determinant of $\hat{\gamma}$ is no longer constrained to be -1.

Some further field redefinitions now simplify the action. Let

$$B_n = \frac{A_n}{A_T}, \tag{2.8.12}$$

$$\psi_n = \sqrt{B_n e^{\chi_n}} = \frac{\Psi_n}{\sqrt{A_T}}. \tag{2.8.13}$$

[4]In Appendix B, we show that an orbifolded N-brane model gives rise to this same four-dimensional low-energy action with a rescaling of some parameters. Most of what follows from here onwards is the same for orbifolded and uncompactified models.

Our dynamical scalar fields are now ψ_n, $0 \leq n \leq N-1, n \neq T$. Again, the domain of each ψ_n is the positive reals. Finally, we can define a four-dimensional effective Newton's constant as

$$\frac{1}{2\kappa_4^2} = \frac{1}{4\kappa_5^2} A_T. \tag{2.8.14}$$

The action with these definitions is

$$\begin{aligned} S = &\int d^4x \sqrt{-\hat{\gamma}} \frac{\epsilon_T}{2\kappa_4^2} \left[R^{(4)} \left[\hat{\gamma}_{ab}\right] \left(1 + \sum_{\substack{n=0 \\ n \neq T}}^{N-1} \epsilon_T \epsilon_n \psi_n^2 \right) + 6 \sum_{n=1}^{N-1} \epsilon_T \epsilon_n (\hat{\nabla}^a \psi_n)(\hat{\nabla}_a \psi_n) \right] \\ &+ {}^T S_m [\hat{\gamma}_{ab}, {}^T\phi] + \sum_{\substack{n=0 \\ n \neq T}}^{N-1} {}^n S_m \left[\frac{\psi_n^2}{B_n} \hat{\gamma}_{ab}, {}^n\phi \right] \end{aligned} \tag{2.8.15}$$

where the functional dependence of the action on $[\hat{\gamma}_{ab}, \psi_n, {}^n\phi]$ has been suppressed to save space. This is the four-dimensional effective action in the Jordan conformal frame of the T^{th} brane, $\mathcal{B}_T$. Note that the target space metric, determined by the kinetic energy term for the scalar fields, is flat, and the target space manifold is a subset of the quadrant of $\mathbb{R}^{N-1}$ in which all the coordinates ψ_n are positive, bearing in mind that each ψ_n will be bounded either above or below by their definition (2.8.13) and Eq. (2.7.18).

III Transforming to the Einstein Conformal Frame

The Einstein conformal frame is defined by an action in which the Ricci scalar (the Einstein-Hilbert term) is canonically normalised, i.e., has a coefficient of $m_P^2/2$. It is typically possible to transform to the Einstein conformal frame by use of a conformal transformation[5].

Defining the function

$$\Theta = 1 + \sum_{\substack{n=0 \\ n \neq T}}^{N-1} \epsilon_T \epsilon_n \psi_n^2, \tag{2.8.16}$$

we transform to the Einstein conformal frame using the conformal transformation $g_{ab} = \hat{\gamma}_{ab}|\Theta|$.

[5]Exceptions exist in two spacetime dimensions, and points at which the coefficient is vanishing in field space.

The four-dimensional effective action becomes

$$S\left[g_{ab}, \psi_n, {}^n\phi\right] = \int d^4x\sqrt{-g}\,\frac{\epsilon_T\,\mathrm{sgn}(\Theta)}{2\kappa_4^2}\left[\tilde{R}^{(4)}[g] - \frac{3}{2\Theta^2}(\tilde{\nabla}^a\Theta)(\tilde{\nabla}_a\Theta)\right.$$
$$\left. + 6\sum_{\substack{n=0\\ n\neq T}}^{N-1}\frac{\epsilon_T\epsilon_n}{\Theta}(\tilde{\nabla}^a\psi_n)(\tilde{\nabla}_a\psi_n)\right]$$
$$+ {}^TS_m\left[\frac{1}{|\Theta|}g_{ab}, {}^T\phi\right] + \sum_{\substack{n=0\\ n\neq T}}^{N-1}{}^nS_m\left[\frac{\psi_n^2}{B_n|\Theta|}g_{ab}, {}^n\phi\right] \tag{2.8.17}$$

where tildes refer to the metric g_{ab}. Note that the kinetic energy terms in this action (2.8.17) have apparent divergences at $\Theta = 0$. However, for any given set of signs ϵ_n (which correspond to a choice of model), it can be shown that $|\Theta| > 0$. This occurs because of the way each ψ_n is bounded either above or below.

2.9 Analysis of the Action

In this section, we analyze the four-dimensional effective action (2.8.17) in a variety of cases. We begin with the cases of one and two branes, which serve to highlight some features of the model in the general case. In these special cases, our results reduce to previously known results. We then analyze the general situation.

I One-Brane Case

In the one brane case, the effective action simplifies greatly.

$$S[g_{ab}, {}^0\phi] = \int d^4x\sqrt{-g}\frac{\epsilon_0}{2\kappa_4^2}\tilde{R}^{(4)}[g] + {}^0S_m\left[g_{ab}, {}^0\phi\right]. \tag{2.9.1}$$

The four-dimensional effective action is just general relativity ($\epsilon_0 = +1$ if the brane has positive tension). This corresponds to the RS-II model [7].

II Two-Brane Case

Here the parameter of importance is $\epsilon_0\epsilon_1$, which from Eqs. (2.7.17), (2.7.19) and (2.8.6) is given by

$$\epsilon_0\epsilon_1 = -\operatorname{sgn}(\sigma_0\sigma_1)\,. \tag{2.9.2}$$

With P_0 and P_2 predetermined, it is possible for one brane tension to be negative, but not both. Therefore $\epsilon_0\epsilon_1$ is positive if there is a negative tension brane, and is negative if both branes have positive tension. Without loss of generality, we choose $T = 0$. Using the definition (2.8.16) of Θ, the action (2.8.17) becomes

$$\begin{aligned} S = &\int d^4x\sqrt{-g}\,\frac{\epsilon_0\operatorname{sgn}(1+\epsilon_0\epsilon_1\psi_1^2)}{2\kappa_4^2}\left[\tilde{R}^{(4)}[g] + 6\frac{\epsilon_0\epsilon_1}{(1+\epsilon_0\epsilon_1\psi_1^2)^2}(\tilde{\nabla}^a\psi_1)(\tilde{\nabla}_a\psi_1)\right] \\ &+ {}^0S_m\left[\frac{1}{|1+\epsilon_0\epsilon_1\psi_1^2|}g_{ab},\,{}^0\phi\right] + {}^1S_m\left[\frac{\psi_1^2}{B_1|1+\epsilon_0\epsilon_1\psi_1^2|}g_{ab},\,{}^1\phi\right]. \end{aligned} \tag{2.9.3}$$

The action is a functional of g_{ab}, ψ_1, ${}^0\phi$, and ${}^1\phi$.

II.a Positive Brane Tensions

When both branes have positive tension, $\epsilon_0\epsilon_1 = -1$. Which of ϵ_0 and ϵ_1 is negative depends on the sign of Θ. Combining Eqs. (2.8.16) and (2.8.13),

$$\Theta = 1 - B_1 e^{\chi_1}. \tag{2.9.4}$$

From Eqs. (2.7.19) and (2.8.7), we see that

$$\epsilon_0 = -\epsilon_1 = -\operatorname{sgn}(P_1). \tag{2.9.5}$$

Combining this with Eq. (2.7.18) and recalling that $\chi_0 = 0$ if $T = 0$, we see that the exponential function in Eq. (2.9.4) is greater than unity for $P_1 = +1$, and less than unity for $P_1 = -1$. If $P_1 = +1$, then the brane tensions [Eq. (2.7.17)] require that $k_0 > k_1$, and we see that $B_1 > 1$, giving $\Theta < 0$ for $\epsilon_0 = -1$, $\epsilon_1 = +1$. If $P_1 = -1$, then the brane tensions dictate that $k_0 < k_1$. Thus, in this case, $B_1 < 1$, and so $\Theta > 0$ for $\epsilon_0 = +1$, $\epsilon_1 = -1$.

Assuming that $0 < \psi_1 < 1$ ($\Theta > 0$, $P_1 = -1$, $\epsilon_0 = +1$), we define

$$\varphi = \mu \tanh^{-1}(\psi_1) \tag{2.9.6}$$

where

$$\mu = \frac{\sqrt{6}}{\kappa_4}. \tag{2.9.7}$$

The domain of φ is 0 to ∞. The action (2.9.3) then becomes

$$\begin{aligned} S[g_{ab}, \varphi, {}^0\phi, {}^1\phi] = & \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa_4^2}\tilde{R}^{(4)}[g] - \frac{1}{2}(\tilde{\nabla}^a\varphi)(\tilde{\nabla}_a\varphi)\right] \\ & + {}^0S_m\left[\cosh^2\left(\frac{\varphi}{\mu}\right)g_{ab}, {}^0\phi\right] + {}^1S_m\left[\frac{1}{B_1}\sinh^2\left(\frac{\varphi}{\mu}\right)g_{ab}, {}^1\phi\right]. \end{aligned} \tag{2.9.8}$$

Requiring that the branes do not intersect or overlap gives

$$0 < \psi_1 < \sqrt{B_1} = \sqrt{\frac{1 - k_1/k_2}{1 + k_1/k_0}}. \tag{2.9.9}$$

Note that $k_1 < k_2$ to satisfy Eq. (2.7.17), and that $\sqrt{B_1} < 1$ (responsible for $\Theta > 0$). Thus, Eq. (2.9.9) is a more stringent constraint than $0 < \psi_1 < 1$.

In the situation where $\psi_1 > 1$ ($\Theta < 0$, $P_1 = +1$, $\epsilon_1 = +1$), we define

$$\varphi = \mu \tanh^{-1}\left(\frac{1}{\psi_1}\right). \tag{2.9.10}$$

The domain of φ is from 0 to ∞. The action (2.9.3) then becomes

$$\begin{aligned} S[g_{ab}, \varphi, {}^0\phi, {}^1\phi] = & \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa_4^2}\tilde{R}^{(4)}[g] - \frac{1}{2}(\tilde{\nabla}^a\varphi)(\tilde{\nabla}_a\varphi)\right] \\ & + {}^0S_m\left[\sinh^2\left(\frac{\varphi}{\mu}\right)g_{ab}, {}^0\phi\right] + {}^1S_m\left[\frac{1}{B_1}\cosh^2\left(\frac{\varphi}{\mu}\right)g_{ab}, {}^1\phi\right], \end{aligned} \tag{2.9.11}$$

which coincides with the previous action (2.9.8) if we swap the actions ${}^0S_m$ and ${}^1S_m$ and rescale units in each matter action by factors of $B_1^{\pm 1/2}$.

The constraint on the radion field we impose to ensure that the branes do not overlap in this case is

$$\psi_1 > \sqrt{B_1} = \sqrt{\frac{1 + k_1/k_2}{1 - k_1/k_0}} > 1, \tag{2.9.12}$$

where $k_1 < k_0$ from the brane-tunings (Eq. (2.7.17)).

The actions (2.9.8) and (2.9.11) coincide with formulae in the literature for the action for the RS-I model, up to a rescaling of units [6, 66, 46] [also, c.f. Eq. (2.4.15)]. They describe a Brans-Dicke like scalar-tensor theory of gravity, with matter on each brane having a different coupling strength to the scalar component.

II.b One Negative Brane Tension

If $\epsilon_0\epsilon_1 = 1$ then $\Theta > 0$ always, and by requiring the conditions (2.7.19), both ϵ_0 and ϵ_1 must be positive. We define

$$\varphi = \mu \tan^{-1}(\psi_1), \tag{2.9.13}$$

where the domain of φ is 0 to $(\pi/2)\mu$. The action (2.9.3) becomes

$$\begin{aligned} S[g_{ab}, \varphi, {}^0\phi, {}^1\phi] = & \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa_4^2}\tilde{R}^{(4)}[g] + \frac{1}{2}(\tilde{\nabla}^a\varphi)(\tilde{\nabla}_a\varphi)\right] \\ & + {}^0S_m\left[\cos^2\left(\frac{\varphi}{\mu}\right)g_{ab}, {}^0\phi\right] + {}^1S_m\left[\frac{1}{B_1}\sin^2\left(\frac{\varphi}{\mu}\right)g_{ab}, {}^1\phi\right]. \end{aligned} \tag{2.9.14}$$

Note that φ is a ghost field, which gives rise to the usual instability associated with a negative tension brane.

III General Case of N branes

In the general case of N branes, it is convenient to redefine our fields in fieldspace. Let P be the number of elements of the set $\{\epsilon_T\epsilon_n, 0 \leq n \leq N-1, n \neq T\}$ for which $\epsilon_T\epsilon_n = +1$, corresponding to the number of scalar fields with positive coefficients in the action (2.8.17) (ignoring the sign of Θ, and the kinetic-looking term for the same). Note that $0 \leq P \leq N-1$. Also, let $M = N-1-P$ be the number of elements with $\epsilon_T\epsilon_n$ negative, corresponding to the number of scalar fields with negative coefficients. It is convenient to relabel the fields $\{\psi_n\}$ based on which have positive kinetic coefficient $(\psi_1, \ldots, \psi_P)$ and which have negative kinetic coefficient ($\psi_{P+1}, \ldots, \psi_{P+M}$), based on the action 2.8.15 (the coefficient for each term was $\epsilon_T \epsilon_n$). We now define new coordinates $\zeta, \theta_1, \ldots, \theta_{P-1}$ and $\eta, \lambda_1, \ldots, \lambda_{M-1}$, such that

$$(\psi_1, \ldots, \psi_P) = \zeta \left(\cos(\theta_1), \sin(\theta_1)\cos(\theta_2), \ldots, \sin(\theta_1)\sin(\theta_2)\cdots\sin(\theta_{P-1})\right), \quad \text{(2.9.15a)}$$

$$(\psi_{P+1}, \ldots, \psi_{P+M}) = \eta \left(\cos(\lambda_1), \sin(\lambda_1)\cos(\lambda_2), \ldots, \sin(\lambda_1)\sin(\lambda_2)\cdots\sin(\lambda_{M-1})\right). \quad \text{(2.9.15b)}$$

We choose η, $\zeta > 0$. All of the angular fields (θ_i and λ_j) have a domain of 0 to $\pi/2$, as each ψ_n is positive. The fields ζ and η have domains of $0 < \eta, \zeta < \infty$. This is essentially a transformation to spherical polar coordinates in fieldspace, with one sphere for the positive-coefficient fields, and a separate sphere for the negative-coefficient fields. The function Θ now becomes

$$\Theta = 1 + \zeta^2 - \eta^2. \quad \text{(2.9.16)}$$

Using these field definitions, the four-dimensional low-energy action can be written in as

$$\begin{aligned} S[g_{ab}, \Phi^A, {}^n\phi] = &\int d^4x \sqrt{-g}\epsilon_T \mathrm{sgn}\left(\Theta\right) \left[\frac{R^{(4)}[g_{ab}]}{2\kappa_4^2} - \frac{1}{2}\gamma_{AB}(\Phi^C) g^{ab} \nabla_a \Phi^A \nabla_b \Phi^B\right] \\ &+ \sum_{n=0}^{N-1} {}^nS_m \left[e^{2\alpha_n(\Phi^C)} g_{ab}, {}^n\phi\right]. \end{aligned} \quad \text{(2.9.17)}$$

Here, $\left\{\Phi^A\right\} \equiv \{\zeta, \eta, \theta_1, \ldots, \theta_{P-1}, \lambda_1, \ldots, \lambda_{M-1}\}$, and $\gamma_{AB}(\Phi^C)$ is the metric on field space, given by

$$\begin{aligned} d\sigma^2 = \gamma_{AB} d\Phi^A d\Phi^B = \frac{\mu^2}{\Theta} \Bigg[&-d\zeta^2 \left(\frac{1-\eta^2}{\Theta}\right) - \zeta^2 d\Omega_p^2 + d\eta^2 \left(\frac{1+\zeta^2}{\Theta}\right) \\ &+ \eta^2 d\Omega_m^2 - \frac{2\eta\zeta}{\Theta} d\eta d\zeta \Bigg], \end{aligned} \quad \text{(2.9.18)}$$

where $d\Omega_p^2 = d\theta_1^2 + \sin^2(\theta_1) d\theta_2^2 + \ldots$ is the metric on the unit $(P-1)$-sphere, and similarly for $d\Omega_m^2$. The parameter μ is defined by $\mu = \sqrt{6}/\kappa_4$. The coupling functions $\alpha_n(\Phi^C)$ are given by

$$e^{2\alpha_T} = \frac{1}{|\Theta|}, \quad \text{(2.9.19a)}$$

$$e^{2\alpha_n} = \frac{1}{|\Theta|}\frac{\psi_n^2}{B_n}, \; 0 \le n \le N-1, n \ne T, \quad \text{(2.9.19b)}$$

where B_n is given by Eq. (2.8.12), and $\psi_n(\Phi^C)$ is defined by the relevant expression in Eq. (2.9.15).

We have now arrived at the explicit form of the theory originally given in Eq. (2.3.3). There are $N-1$ scalar fields, with a field space metric given by (2.9.18). The matter coupling functions are given by Eqs. (2.9.19). The relationship between the five-dimensional gravitational constant and the four-dimensional effective gravitational constant is given by Eq. (2.8.14). There are no mass terms for the scalar fields, so the theory forms a massless multiscalar-tensor theory in a nonlinear sigma model.

2.10 Discussion

This completes the explicit derivation of the low-energy effective action (2.3.3) in the case of a specific model, and the illustration of our method of acquiring the four-dimensional effective action. Although only the one application was demonstrated, the method is generally applicable to compactified and orbifolded models[6]. Before analyzing the physics of the four-dimensional effective action, we discuss various aspects of the method and its results.

I Domain of Validity of the Four-Dimensional Description

We begin our discussion of the domain of validity of the four-dimensional description given by Eq. (2.8.17) by recapping the method of computation discussed in Section 2.6.III. Starting from the five-dimensional action S, we define a rescaled action S_ϵ which has the expansion

$$S_\epsilon = S_0 + \epsilon^2 S_2. \tag{2.10.1}$$

In Section 2.7 we found the most general solution of $\delta S_0 = 0$, and substituting that solution into S_2, gave the four-dimensional action functional of Section 2.8.III[7].

[6]See Appendix B.

[7]The action S_0 for the solution is zero, assuming the brane-tunings (2.7.17).

The basis of our approximation method is the smallness of the bulk radius of curvature $1/k_n$ compared to the radius of curvature $\mathcal{L}_c$ of the four-dimensional metric $e^{\chi}\hat{\gamma}_{ab}$. However, although this approximation is valid on all the branes, it inevitably breaks down as $y \to \pm\infty$, far from the branes, as $\mathcal{L}_c \to 0$, as discussed in Section 2.6.III. It is worth noting that in the special case where all of the induced metrics on the branes are flat and there are no matter fields, the metric ansatz (with $\Phi = \text{const}$) is an exact solution to the five-dimensional Einstein equations, and this breakdown does not occur.

One might expect contributions from the regime far from the branes to invalidate our four-dimensional effective description. However, we expect that the contribution to the action far from the brane will negligibly change the calculation, as in the region in which we expect large departures from the derived metric, the warp factor exponentially suppresses any contributions.

It is possible for our two-lengthscale expansion to break down not only asymptotically, but also in between branes. A number of models (e.g., [58, 52, 68, 72] to cite but a few) discuss bounce behavior in the warp factor, where it decreases and increases again in between branes, as with a $\cosh^2$ dependence. Typically, this behavior appears when the metric $\hat{\gamma}$ is a curved FRW metric. It is a limitation of our method that this bounce is not evident in our solutions, as it explicitly requires coupling between the $O(1)$ and $O(\epsilon^2)$ components (in particular, the four-dimensional Ricci scalar). Thus, this behavior is excluded by the underlying assumptions of our method, as near the turning point of these bounces, the separation of lengthscales has broken down. We note, however, that $\cosh^2$ behavior is likely to be forbidden in the first or last ($y \to \pm\infty$) regions by global hyperbolicity. It is also possible to produce $\sinh^2$ behavior in the warp factor. In between branes, this can lead to topologically disconnected regions of spacetime as discussed in [52], which we have excluded by assumption. In the first or last regions, correctly accounting for this behavior requires that the integration over the fifth dimension be truncated. However, the contributions to our effective action from integrating beyond these regions is again exponentially suppressed and negligible. In the regime in which the separation of lengthscales is valid, our solutions are in agreement with models displaying these types of behavior.

For black holes, the solution given by our effective action is subject to the Gregory-Laflamme instability [73] and the final outcome is uncertain (see [74] and citations thereof). The five-dimensional stability of solutions for which the induced metric on the branes is not nearly flat (e.g., black holes and neutron stars) is an interesting open question, although recent numerical results [75] suggest that such solutions exist. We conjecture that all the solutions without horizons are stable and are reasonably described by our four-dimensional effective action.

We may also consider the regime in which $\mathcal{L}_c \ll \mathcal{L}$, such as will occur a long way away from the branes. In this limit, the physical description would change from being that of decoupled fibers to that of decoupled four-dimensional hypersurfaces [one should solve the $O(\epsilon^2)$ contribution to the action first, and substitute that into the $O(1)$ contribution to the action]. This approach may yield a matched asymptotic expansion approach to obtaining a solution far from the branes. Our method may therefore be useful for investigating the regime between Minkowski space on a brane and a black hole on a brane.

It is important to note that our method does not yield the leading order five-dimensional metric. This can be seen from the fact that our four-dimensional action depends only on the fields ${}^n\chi$ evaluated on the branes, and the values of these fields between the branes are not determined. However, knowledge of the leading order five-dimensional metric is, rather surprisingly, *not* a prerequisite for correctly capturing the leading order four-dimensional dynamics. Most other methods rely on knowledge of the five-dimensional behavior of the metric to calculate the effective four-dimensional equations of motion, and our method is somewhat unique in this regard.

Our method of computation correctly captures the leading order dynamics of the system. However, there will be higher-order corrections, suppressed by powers of ϵ^2. In particular, the fields ${}^n\chi$ and ${}^n\Phi$ can be expanded as

$$
{}^n\chi = {}^n\chi^{(0)} + \epsilon^2\, {}^n\chi^{(2)} + O(\epsilon^4), \tag{2.10.2a}
$$

$$
{}^n\Phi = {}^n\Phi^{(0)} + \epsilon^2\, {}^n\Phi^{(2)} + O(\epsilon^4). \tag{2.10.2b}
$$

Throughout this chapter, we have dealt only with the fields ${}^n\chi^{(0)}$ and ${}^n\Phi^{(0)}$. The necessity of higher-order terms can be seen from the exact, five-dimensional equations of motion, which are derived in Appendix A. For example, the exact Israel junction conditions are given by Eq. (A.10). If we substitute the expansions (2.10.2) into Eq. (A.10), and use (2.7.7) [with ${}^n\chi$ and ${}^n\Phi$ replaced by ${}^n\chi^{(0)}$ and ${}^n\Phi^{(0)}$] together with the brane-tuning conditions (2.7.17), we find that the higher-order corrections ${}^n\chi^{(2)}$ and ${}^n\Phi^{(2)}$ are related to the matter stress energy tensors on the brane. Our results confirm the suggestion of Kanno and Soda that these higher-order corrections do not affect the four-dimensional effective action to leading order [67].

II Models That Violate the Brane Tension Tunings

If a brane's tension is adjusted so as to violate the tuning condition (2.7.17), then it is possible to view the situation as having either detuned brane tensions or detuned bulk cosmological constants. For accounting purposes, it is simpler to think of the bulk cosmological constants as being detuned. When this occurs, the exact equations of motion in the bulk (A.6) to (A.10) imply that a nonzero Ricci curvature is induced to compensate for the detuning. Exact solutions have been calculated in highly symmetric cases, see for example Ref. [72]. In general, the exact nature of the perceived detuning is nontrivial, as the bulk cosmological constants on either side of the offending brane(s) can appear detuned by different amounts to compensate.

If the deviation from the brane-tuning conditions is small [$\Delta\sigma/\sigma^T = O(\epsilon^2)$], then we can approximate the contribution to the four-dimensional effective action as

$$\Delta S = -\sum_{n=0}^{N-1} \int d^4x \sqrt{-{}^n h}(\sigma_n - \sigma_n^T), \tag{2.10.3}$$

where σ_n^T is the tuned value for the n^{th} brane, given by (2.7.17). This approximation is of the same order as the other approximations we have made in our method. The net result is then an effective cosmological constant on each brane, given by

$$\Lambda_n^{(4)} = \sigma_n - \sigma_n^T, \tag{2.10.4}$$

which vanishes when the brane tensions are tuned. [Note that this expression differs from results given in the literature for the RS-II model, see for example Ref. [42], but the difference is $O(\epsilon^4)$].

If the detuning of a brane's tension from its tuned value should become too large [$O(1)$ rather than $O(\epsilon^2)$], then the curvature induced by the four-dimensional effective cosmological constant can cause the radius of curvature on a slice of constant y close to the branes to violate the approximations used in our method, which implies that our four-dimensional effective action will not be a good description of a system in this regime.

III Multigravity

Theories with more than one independent dynamical tensor field are called multigravity theories; see the general discussion in Damour and Kogan [68]. The models in this work may exhibit two forms of multigravity, although we have ignored one of them entirely.

The first form of multigravity is the possible existence of a second tensor field, given by the matrix $\mathbf{B}(x^a)$ in Eq. (2.7.10). We argued in Section 2.7.III that this form of multigravity is likely forbidden.

The second form of multigravity arises from the the fact that outside of the low-energy regime, the models will contain Kaluza-Klein graviton modes. These modes will have masses that are formally of order $\mathcal{L}^{-1}$, but may be much lighter due to exponential suppression factors, and so may be phenomenologically important (so-called "ultra-light modes") [49, 50]. Our method of analysis automatically excludes all massive fields (formally, we take ϵ sufficiently small to overcome any large exponential factors), so we have neglected all graviton Kaluza-Klein modes. It is likely that some of these modes are in fact ultralight in our model, as in the analyses of Damour and Kogan [49, 50, 68]. We discuss Kaluza-Klein modes in Appendix C.

IV Evaluation

Our goal in this chapter was to devise a simple method by which to obtain an effective four-dimensional action to capture the leading-order effects of braneworld models. Although the description of the method became reasonably long and rather mathematical, most of the effort contained here involved book-keeping and justifying approximations. The application of the method is actually reasonably quick and straightforward, as we demonstrate in Appendix B, where we apply the method to an orbifold model.

In the appropriate limits, the method yields results that are consistent with the literature, and we are satisfied that it captures the leading order dynamics of five-dimensional models, especially with regards to the radion structure. One feature of this method is that the Kaluza-Klein gravitational modes have been truncated. However, this is also a drawback in that there is no simple manner in which to reincorporate their effects. However, given that our four-dimensional metric has been left arbitrary and dynamical (within the regime of validity), no general expansion for the Kaluza-Klein tower is possible, and so acquiring a four-dimensional effective description for these modes as massive gravitational modes cannot be performed anyway.

Having developed a four-dimensional effective action, it is now time to investigate the physics of this model, and apply both theoretical and experimental constraints on the parameter space. We perform this undertaking in the following chapter.

Chapter 3

Gravitational Interactions in Multibrane-Worlds

Contents

In the previous chapter, we proposed an approximation scheme based upon a two-lengthscale expansion which can be used to evaluate a four-dimensional low-energy action for five-dimensional braneworld models, and demonstrated its application to an uncompactified N-brane model. We now turn to analyzing the physics of the four-dimensional effective action for this model.

We investigate the parameter space of the general model, and find regions in which the theory has no ghosts. The parameter space is further refined by imposing observational constraints from Solar System tests of gravity. We consider the possibility of placing dark matter and Standard Model fields on separate branes, and by comparing to observational data, find that the vast majority of the dark matter must reside on our brane in the models considered.

Our motivation in analyzing general N-brane models is to determine whether the presence of extra branes may overcome some of the constraints the RS-I and RS-II models have, particularly with regards to radion stabilization requirements for experimentally viable models. We are also interested in potential applications to models of dark matter and energy.

Repeated here for convenience are the low-energy effective action, target space metric, and coupling functions, for convenience. The low-energy effective action, as derived in the previous chapter [Eq. (2.9.17)], is

$$S[g_{ab}, \Phi^A, {}^n\phi] = \int d^4x\sqrt{-g}\epsilon_T \mathrm{sgn}\,(\Theta)\left[\frac{R^{(4)}[g_{ab}]}{2\kappa_4^2} - \frac{1}{2}\gamma_{AB}(\Phi^C)g^{ab}\nabla_a\Phi^A\nabla_b\Phi^B\right] + \sum_{n=0}^{N-1} {}^nS_m\left[e^{2\alpha_n(\Phi^C)}g_{ab}, {}^n\phi\right]. \tag{3.0.1}$$

The field space metric [Eq. (2.9.18)] is

$$d\sigma^2 = \gamma_{AB}d\Phi^A d\Phi^B = \frac{\mu^2}{\Theta}\left[-d\zeta^2\left(\frac{1-\eta^2}{\Theta}\right) - \zeta^2 d\Omega_p^2 + d\eta^2\left(\frac{1+\zeta^2}{\Theta}\right) + \eta^2 d\Omega_m^2 - \frac{2\eta\zeta}{\Theta}d\eta d\zeta\right]. \tag{3.0.2}$$

Finally, the coupling functions [Eqs. (2.9.19)] are

$$e^{2\alpha_T} = \frac{1}{|\Theta|}, \tag{3.0.3a}$$

$$e^{2\alpha_n} = \frac{1}{|\Theta|}\frac{\psi_n^2}{B_n},\ \ 0 \le n \le N-1, n \ne T. \tag{3.0.3b}$$

This chapter is based on work originally presented in [2].

3.1 Parameterization of Field Space

We begin our investigation by finding coordinates on field space which diagonalize the field space metric, Eq. (3.0.2). This is particularly useful for identifying the presence of any unphysical ghost modes. We look at two special cases before analyzing the general case. Recall that P is the number of terms with positive $\epsilon_T\epsilon_n$, and M the number of terms for which it is negative. The two must add to give $P + M = N - 1$.

I Negative Definite Field Space Metric

In the case $M = 0$, the general metric reduces to

$$\frac{(1+\zeta^2)}{\mu^2} d\sigma^2 = -\frac{1}{1+\zeta^2} d\zeta^2 - \zeta^2 d\Omega_p^2. \tag{3.1.1}$$

This can be rewritten as

$$d\sigma^2 = -da^2 - \mu^2 \sin^2\left(\frac{a}{\mu}\right) d\Omega_p^2, \tag{3.1.2}$$

where $a = \mu \tan^{-1}(\zeta)$, with $0 \leq a \leq \pi\mu/2$.

II Positive Definite Field Space Metric

In the case of $P = 0$, the general metric reduces to

$$\frac{(1-\eta^2)}{\mu^2} d\sigma^2 = d\eta^2 \frac{1}{1-\eta^2} + \eta^2 d\Omega_n^2. \tag{3.1.3}$$

For the case where $\eta < 1$, this can be rewritten as

$$d\sigma^2 = da^2 + \mu^2 \sinh^2\left(\frac{a}{\mu}\right) d\Omega_n^2, \tag{3.1.4}$$

where $a = \mu \tanh^{-1}(\eta)$, with $0 < a < \infty$. This is shown in Section 3.2 to be the only physically relevant case.

For the case of $\eta > 1$, the metric (3.0.2) can be rewritten as

$$d\sigma^2 = da^2 - \mu^2 \cosh^2\left(\frac{a}{\mu}\right) d\Omega_n^2, \tag{3.1.5}$$

where $a = \mu \coth^{-1}(\eta)$, and $0 < a < \infty$.

We see that the two cases $\eta > 1$ and $\eta < 1$ are topologically disconnected, one being a metric on elliptic space and the other being a metric on de Sitter space, and so the divergence at $\eta = 1$ in the metric (3.1.3) is simply a coordinate singularity.

III General Case

In the general case with $M > 0, P > 0$, the metric (3.0.2) is non-diagonal. It can be diagonalized using suitable coordinate transformations in the three different cases $\Theta < 0$, $0 < \Theta \leq 1$, and $\Theta \geq 1$. Recall that

$$\Theta = 1 + \zeta^2 - \eta^2. \tag{3.1.6}$$

III.a $\Theta < 0$

For Θ to be negative, we require from Eq. (3.1.6) that $\eta^2 - \zeta^2 > 1$. Recall that η and ζ are non-negative. We define new coordinates (a, b) by

$$\eta = a \cosh\left(\frac{b}{\mu}\right), \tag{3.1.7a}$$

$$\zeta = a \sinh\left(\frac{b}{\mu}\right), \tag{3.1.7b}$$

where $a > 1$, $b \geq 0$. The metric (3.0.2) becomes

$$d\sigma^2 = \frac{a^2}{a^2-1}\left[db^2 + \frac{\mu^2}{a^2(a^2-1)}da^2 + \mu^2\sinh^2\left(\frac{b}{\mu}\right)d\Omega_p^2 - \mu^2\cosh^2\left(\frac{b}{\mu}\right)d\Omega_m^2\right]. \tag{3.1.8}$$

Defining c by $a = \mathrm{cosec}(c/\mu)$ with $0 < c < \pi\mu/2$, the metric becomes

$$d\sigma^2 = \sec^2\left(\frac{c}{\mu}\right)\left(db^2 + dc^2 + \mu^2\sinh^2\left(\frac{b}{\mu}\right)d\Omega_p^2 - \mu^2\cosh^2\left(\frac{b}{\mu}\right)d\Omega_m^2\right). \tag{3.1.9}$$

III.b $0 < \Theta \leq 1$

In this regime, $\eta > \zeta$ as previously, but with $\eta^2 - \zeta^2 \leq 1$. We use the same coordinate definitions (3.1.7), but with $0 \leq a < 1$ and $b \geq 0$. The metric is the same as Eq. (3.1.8). This time, define $c = \mu\ \mathrm{sech}^{-1}(a)$ with $0 < c < \infty$, which gives

$$d\sigma^2 = \mathrm{cosech}^2\left(\frac{c}{\mu}\right)\left[-db^2 + dc^2 - \mu^2\sinh^2\left(\frac{b}{\mu}\right)d\Omega_p^2 + \mu^2\cosh^2\left(\frac{b}{\mu}\right)d\Omega_m^2\right] \tag{3.1.10}$$

as the metric.

III.c $1 \leq \Theta$

In this region of field space, $\zeta \geq \eta$. We define coordinates (a, b) by

$$\eta = a \sinh\left(\frac{b}{\mu}\right), \tag{3.1.11}$$

$$\zeta = a \cosh\left(\frac{b}{\mu}\right), \tag{3.1.12}$$

with domains of $a \geq 0$, $b \geq 0$. The metric (3.0.2) in these coordinates is

$$d\sigma^2 = \frac{a^2}{1+a^2}\left[-\frac{\mu^2}{a^2(1+a^2)}da^2 + db^2 - \mu^2 \cosh^2\left(\frac{b}{\mu}\right) d\Omega_p^2 + \mu^2 \sinh^2\left(\frac{b}{\mu}\right) d\Omega_m^2\right]. \tag{3.1.13}$$

If we define $c = \mu \operatorname{cosech}^{-1}(a)$ with $0 < c < \infty$, the metric becomes

$$d\sigma^2 = \operatorname{sech}^2\left(\frac{c}{\mu}\right)\left[-db^2 + dc^2 - \mu^2 \sinh^2\left(\frac{b}{\mu}\right) d\Omega_p^2 + \mu^2 \cosh^2\left(\frac{b}{\mu}\right) d\Omega_m^2\right]. \tag{3.1.14}$$

The two cases $0 < \Theta \leq 1$ and $\Theta \geq 1$ are two coordinate patches on the same manifold. We see that the apparent divergence in the metric (3.0.2) at $\eta^2 - \zeta^2 = 1$ is just a coordinate divergence; it delineates the boundary between topologically disconnected spaces ($\Theta > 0$ and $\Theta < 0$). We show in Section 3.2 that only one of these cases is physically viable, and corresponds to case 3.1.II with a different choice of T.

3.2 Physically Viable Models

In this section, we impose the constraint that all kinetic terms in the Einstein conformal frame have the correct signs, in order to exclude ghosts. This requires that the field space metric have positive definite signature. Of the field space configurations, only those giving rise to the metrics (3.1.4) and (3.1.9) (with $M = 1$) meet this condition. We investigate the constraints this imposes on the parameters of the model.

Recall that P is the number of parameters in the set $\{\epsilon_T \epsilon_n, n \neq T\}$ which are positive, and $M = N - 1 - P$ is the number which are negative. The metric (3.1.4) occurs when $P = 0$

and $M = N - 1$. This requires all ϵ_n to have the same sign, except for ϵ_T which has the opposite sign. It also requires $\Theta > 0$.

The metric (3.1.9) occurs with the correct signature when $M = 1$ and $P = N - 2$. This requires all ϵ_n (including ϵ_T) to have the same sign except for one (not ϵ_T), which has the opposite sign. This metric also requires $\Theta < 0$.

Combining these two cases, we see that all ϵ_n (including ϵ_T) must have the same sign except one, which must be opposite. If this special n is labelled S, then evidently the first case [with metric (3.1.4)] corresponds to the choice $S = T$, while the second case corresponds to $S \neq T$ [with metric (3.1.9)]. We now investigate what constraints the requirements for these metrics impose.

At brane $\mathcal{B}_n$, where the bulk regions n and $n+1$ meet, there are four possible combinations for the parameters P_n and P_{n+1}, namely $(P_n, P_{n+1}) = (-,-)$, $(-,+)$, $(+,-)$ and $(+,+)$. Furthermore, the bulk cosmological constant can either increase or decrease across the brane. The sign of the brane tension σ_n and the sign of ϵ_n for each of these eight cases is given in Fig. 7, where the warp factor is plotted for each situation. Below, we refer to these eight possibilities as cases 1 through 8. We begin by looking at the situation where a single ϵ_n is positive ($0 \leq n \leq N - 1$), and then look at the situation where a single ϵ_n is negative.

I A single brane with ϵ_n positive

Recall that P_n is the sign of the slope of the warp factor in $\mathcal{R}_n$. Using $P_0 = +1$ and $P_N = -1$ (which was assumed in deriving the four-dimensional low-energy action), we need a turning point in the warp factor somewhere in the progression of branes, which restricts us to either case 2 or case 6. Both of these cases have positive ϵ, and so we require that all other ϵ_n are negative. Given that if the warp factor turns back upwards after turning downwards, it would need to turn around again using another case 2 or 6 which would introduce a second positive ϵ, we see that the warp factor is only allowed to increase, turn around, and then

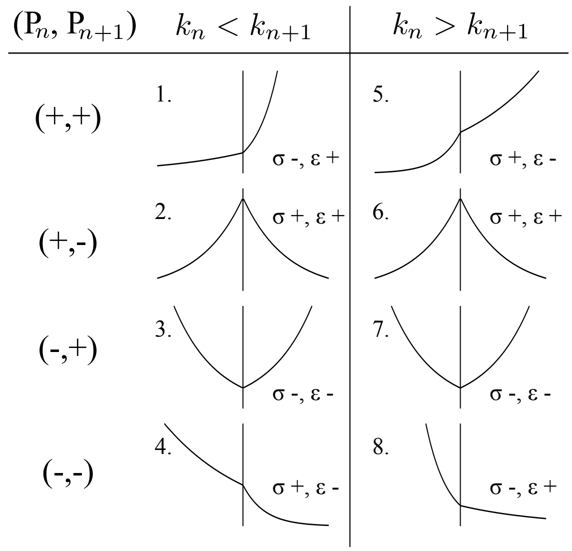

Figure 7: The behavior of the warp factor at a brane interface in the eight possible configurations. An increasing warp factor in a region has $P_n = +1$, while a decreasing warp factor has $P_n = -1$. In cases 2, 3, 6 and 7, the adjacent bulk cosmological constants can be equal. The horizontal axis in all plots is the y coordinate. Note that cases 2 and 6 are equivalent, for all intents and purposes, as are cases 3 and 7.

decrease. The only way to continue increasing with negative ϵ is using case 5, and the only way to decrease with negative ϵ is using case 4. Thus, the progression of cases across the branes must go

$$5, \ldots, 5, (2 \text{ or } 6), 4, \ldots, 4. \tag{3.2.1}$$

It is unnecessary to have any branes with case 5 or 4 (i.e., the first or last case may be 2/6). Note that cases 2, 4, 5 and 6 all correspond to positive tension branes.

Given the growth and fall of the warp factor, there can only be one brane on which the warp factor is a maximum. We call this the "central" brane. Choose T to be this brane, such that $\chi(x^a, T) = 0$, and so the warp factor is unity on the brane where the warp factor is a maximum. With the progression (3.2.1), $\epsilon_T = +1$, and all other $\epsilon_n = -1$. We have $P = 0$ and $M = N - 1$, and so we require that $\Theta > 0$ using these field definitions.

We are interested in the sign of Θ, to see if the requirement that $\Theta > 0$ is met for the metric (3.1.4). As $A_n > 0$, it is sufficient to know the sign of $A_T\Theta$. We have

$$A_T\Theta = A_T - \sum_{n \neq T} A_n e^{\chi_n}. \qquad (3.2.2)$$

Now, given that the warp factor is a maximum on $\mathcal{B}_T$ and we know that $P_n = -1$ for $n > T$, it follows that $\chi_n > \chi_{n+1}$ for $n > T$. Similarly, we have $\chi_n < \chi_{n+1}$ for $n < T$. We now consider the expression for A_n [Eq. (2.8.5)] based on what we know about P_n and k_n from the progression (3.2.1).

$$\begin{aligned} A_T &= 1/k_T + 1/k_{T+1} \\ A_n &= 1/k_n - 1/k_{n+1} \quad (n > T) \\ A_n &= 1/k_{n+1} - 1/k_n \quad (n < T) \end{aligned} \qquad (3.2.3)$$

Thus, Θ may be written as

$$A_T\Theta = \sum_{\substack{n \leq T \\ n \neq 0}} \frac{1}{k_n}\left(e^{\chi_n} - e^{\chi_{n-1}}\right) + \frac{1}{k_0}e^{\chi_0} + \sum_{\substack{n \geq T \\ n \neq N-1}} \frac{1}{k_{n+1}}\left(e^{\chi_n} - e^{\chi_{n+1}}\right) + \frac{1}{k_N}e^{\chi_{N-1}}. \qquad (3.2.4)$$

Each term in both sums is positive, and so $\Theta > 0$.

Thus, we see that a situation with all ϵ_n parameters negative bar one produces an action with no incorrectly signed kinetic terms. Furthermore, this choice of parameters requires all the brane tensions to be positive. Finally, the Ricci scalar in the action has positive coefficient, as $\epsilon_T \, \mathrm{sgn}(\Theta) = +1$. We investigate the properties of models in this parameter space in the remainder of this chapter.

II A single brane with ϵ_n negative

Here, the number of possibilities is larger than in the previous case. By using the same logic as above, we find that the following progressions of cases are the only ways to meet the required conditions:

$$\text{Option 1:}\;\; 1,\ldots,1,5,1,\ldots,1,(2\text{ or }6),8,\ldots,8 \tag{3.2.5a}$$

$$\text{Option 2:}\;\; 1,\ldots,1,(2\text{ or }6),8,\ldots,8,4,8,\ldots,8 \tag{3.2.5b}$$

$$\text{Option 3:}\;\; 1,\ldots,1,(2\text{ or }6),8,\ldots,8,(3\text{ or }7),1,\ldots,1,(2\text{ or }6),8,\ldots,8 \tag{3.2.5c}$$

Each of these cases requires one or more negative tension branes. We consider each of these cases in turn.

Option 1:

Let the one negative ϵ_n be ϵ_T, corresponding to case 5. One brane will have the maximum warp factor; call this brane X. Note that $X \neq T$, as brane T, being case 5, does not have the maximum warp factor. We now have $\epsilon_T = -1$, and all other $\epsilon_n = +1$, and so we have $P = 0$ once again, which requires $\Theta > 0$. Consider the sign of $A_T\Theta$. We have

$$A_T\Theta = A_T - \sum_{n\neq T} A_n e^{\chi_n}. \tag{3.2.6}$$

We can once again calculate A_n explicitly.

$$\begin{aligned} A_T &= 1/k_{T+1} - 1/k_T, \quad A_n = 1/k_n - 1/k_{n+1} \quad (0 \leq n \leq X-1, n \neq T), \\ A_X &= 1/k_X + 1/k_{X+1}, \quad A_n = 1/k_{n+1} - 1/k_n \quad (n > X) \end{aligned} \tag{3.2.7}$$

$A_T\Theta$ can then be expressed as

$$A_T\Theta = -\frac{1}{k_0}e^{\chi_0} - \sum_{n=1}^{X} \frac{1}{k_n}\left(e^{\chi_n} - e^{\chi_{n-1}}\right) - \frac{1}{k_N}e^{\chi_{N-1}} - \sum_{n=X}^{N-2} \frac{1}{k_{n+1}}\left(e^{\chi_n} - e^{\chi_{n+1}}\right). \tag{3.2.8}$$

Here, all bracketed terms are positive. Thus, $\Theta < 0$, in contradiction of the requirement that $\Theta > 0$ necessary for this situation.

Option 2:

This case proceeds in exactly the same manner as Option 1, and we again find $\Theta < 0$, in contradiction of the requirements for this situation.

Option 3.

This case is a little more complicated. Let T be the one brane with negative ϵ, corresponding to case 3 or 7. Two branes will have a local maximum warp factor; let them be L and R (to the left and right of brane T). Now, consider $A_T\Theta$, which we require to be positive in this situation (as we once again have $P = 0$).

$$A_T\Theta = A_T - \sum_{n\neq T} A_n e^{\chi_n}. \tag{3.2.9}$$

This time, we have

$$\begin{aligned}
A_n &= \frac{1}{k_n} - \frac{1}{k_{n+1}}, \quad 0 \leq n < L, \quad T < n < R, \\
A_n &= \frac{1}{k_{n+1}} - \frac{1}{k_n}, \quad L < n < T, \quad R < n, \\
A_L &= \frac{1}{k_L} + \frac{1}{k_{L+1}}, \qquad A_T = \frac{1}{k_T} + \frac{1}{k_{T+1}}, \qquad A_R = \frac{1}{k_R} + \frac{1}{k_{R+1}}.
\end{aligned} \tag{3.2.10}$$

Combining these, we find

$$\begin{aligned}
A_T\Theta = & -\frac{e^{\chi_0}}{k_0} - \sum_{n=1}^{L} \frac{1}{k_n}\left(e^{\chi_n} - e^{\chi_{n-1}}\right) - \sum_{n=L}^{T-1} \frac{1}{k_{n+1}}\left(e^{\chi_n} - e^{\chi_{n+1}}\right) \\
& - \sum_{n=T+1}^{R} \frac{1}{k_n}\left(e^{\chi_n} - e^{\chi_{n-1}}\right) - \sum_{n=R}^{N-1} \frac{1}{k_{n+1}}\left(e^{\chi_n} - e^{\chi_{n+1}}\right) - \frac{e^{\chi_{N-1}}}{k_N}.
\end{aligned} \tag{3.2.11}$$

Once again, Θ is negative, and so this configuration also creates a contradiction.

III The Effect of Negative Tension Branes

From the above arguments, we see that the only ghost-free configurations are those which do not have any negative tension branes. This is consistent with the well-known local arguments for the instability of a negative tension brane. We note that by just using positive tension branes with the assumption that $P_0 = +1$ and $P_N = -1$ (and ignoring the requirement of the different ϵ_n parameters having specific signs), the only possible combination is (3.2.1), and so it is the presence of negative tension branes which are giving rise to the instability. Any valid configuration which only has positive tension branes will not have this instability.

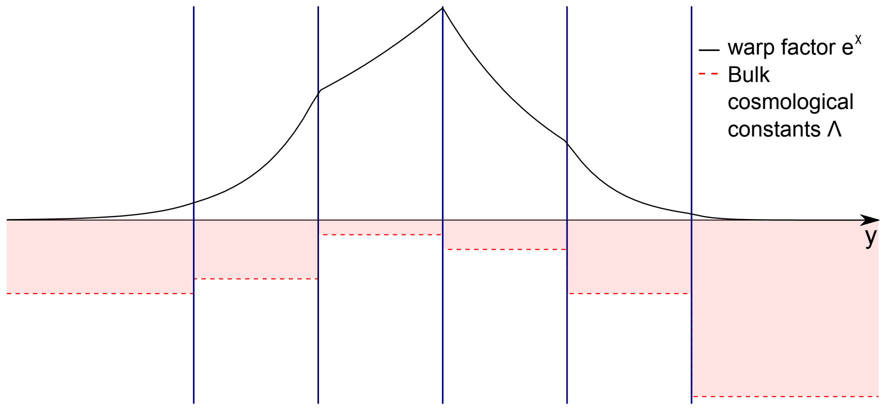

Figure 8: Diagram of a physically allowable warp factor between branes, and the associated bulk cosmological constants (dashed). Branes are represented as vertical lines. The bulk cosmological constants are negative, while the warp factor lies between 0 and 1.

The combination of cases (3.2.1) provides a rather tight restriction on the progressions of the bulk cosmological constant which can give rise to physically viable scenarios. Recalling that the bulk cosmological constants are negative, we require the bulk cosmological constants to increase across the branes monotonically to a maximum, and then decrease monotonically (see Fig. 8). Note that in the special case where the first (last) brane has the maximum warp factor, then $|\Lambda|$ can be monotonically increasing (decreasing).

3.3 Specializing to Physically Viable Cases

In this section, we specialize to the physically viable cases discussed above, and find a set of variables which simplifies the action.

I The Physical Action

We previously found that the only physically viable configuration for the model is the configuration (3.2.1), in which the warp factor increases to a maximum, and then decreases again, with all brane tensions positive. We denote by $n = T$ the index of the brane with the maximum warp factor, and call this brane the "central brane". Specializing Eq. (2.8.15) to these parameters, we find

$$S[\hat{\gamma}_{ab}, \psi_n, {}^n\phi] = \int d^4x \sqrt{-\hat{\gamma}} \frac{1}{2\kappa_4^2} \left[R^{(4)} \left[\hat{\gamma}_{ab}\right] \left(1 - \sum_{\substack{n=0 \\ n\neq T}}^{N-1} \psi_n^2 \right) - 6 \sum_{\substack{n=0 \\ n\neq T}}^{N-1} (\hat{\nabla}^a \psi_n)(\hat{\nabla}_a \psi_n) \right]$$
$$+ {}^T S_m \left[\hat{\gamma}_{ab}, {}^T\phi\right] + \sum_{\substack{n=0 \\ n\neq T}}^{N-1} {}^n S_m \left[\frac{\psi_n^2}{B_n} \hat{\gamma}_{ab}, {}^n\phi \right]. \tag{3.3.1}$$

This is the action in the Jordan conformal frame of the central brane.

As $P = 0$, $M = N - 1$, the function Θ is now given by

$$\Theta = 1 - \sum_{\substack{n=0 \\ n\neq T}}^{N-1} \psi_n^2 = 1 - \eta^2, \tag{3.3.2}$$

and we know that $\Theta > 0$ from the arguments of the previous section. We now follow the field redefinitions (2.9.15b) exactly, transforming into spherical polar coordinates. Let $(\lambda_1, \ldots, \lambda_{N-2})$ be angular coordinates such that

$$\frac{\psi_0}{\eta} = \cos(\lambda_1) = f_0 \tag{3.3.3a}$$
$$\frac{\psi_1}{\eta} = \sin(\lambda_1)\cos(\lambda_2) = f_1 \tag{3.3.3b}$$
$$\vdots$$
$$\frac{\psi_{T-1}}{\eta} = \sin(\lambda_1)\ldots\sin(\lambda_{T-1})\cos(\lambda_T) = f_{T-1} \tag{3.3.3c}$$
$$\frac{\psi_{T+1}}{\eta} = \sin(\lambda_1)\ldots\sin(\lambda_T)\cos(\lambda_{T+1}) = f_{T+1} \tag{3.3.3d}$$
$$\vdots$$
$$\frac{\psi_{N-2}}{\eta} = \sin(\lambda_1)\ldots\sin(\lambda_{N-3})\cos(\lambda_{N-2}) = f_{N-2} \tag{3.3.3e}$$
$$\frac{\psi_{N-1}}{\eta} = \sin(\lambda_1)\ldots\sin(\lambda_{N-3})\sin(\lambda_{N-2}) = f_{N-1}. \tag{3.3.3f}$$

Defining $a = \mu \tanh^{-1}(\eta)$ with $a > 0$ as in Section 3.1.II, we have our final four-dimensional low-energy action, written in the Einstein conformal frame, where $g_{ab} = \Theta\hat{\gamma}_{ab}$.

$$S = \int d^4x\sqrt{-g}\left[\frac{R^{(4)}[g]}{2\kappa_4^2} - \frac{(\nabla a)^2}{2} - \frac{\mu^2}{2}\sinh^2\left(\frac{a}{\mu}\right)\sum_{n=1}^{N-2}\left\{\prod_{m=1}^{n-1}\sin^2(\lambda_m)\right\}(\nabla\lambda_n)^2\right]$$
$$+ {}^T S_m\left[\cosh^2\left(\frac{a}{\mu}\right)g_{ab}, {}^T\phi\right] + \sum_{\substack{n=0\\n\neq T}}^{N-1} {}^n S_m\left[\sinh^2\left(\frac{a}{\mu}\right)\frac{f_n^2}{B_n'}g_{ab}, {}^n\phi\right] \tag{3.3.4}$$

The functional dependence of the action on $[g_{ab}, a, \lambda_n, {}^n\phi]$ has been suppressed for space, and $(\nabla X)^2 = (\nabla^a X)(\nabla_a X)$. In a more convenient notation, the field space metric is

$$d\sigma^2 = da^2 + \mu^2\sinh^2\left(\frac{a}{\mu}\right)d\Omega_n^2, \tag{3.3.5}$$

where $d\Omega_n^2 = d\lambda_1^2 + \sin^2(\lambda_1)d\lambda_2^2 + \ldots$ is the metric on the unit $N-2$ sphere. This is the metric on hyperbolic space.

The target space will not be all of the quadrant of $(N-1)$-dimensional hyperbolic space for which all the field coordinates are positive, as we have yet to impose the constraint of having no branes intersecting, which was implicit in the derivation of the action. In the general case, these constraints are

$$\chi_n < \chi_{n+1}, \quad n < T, \tag{3.3.6a}$$

$$\chi_n > \chi_{n+1}, \quad n > T, \tag{3.3.6b}$$

where χ_n is related to ψ_n by Eq. (2.8.13).

II The Effect of One Brane on Another

Given the low-energy action (3.3.4), it is interesting to ask about the effect one brane has on another, depending on how they are located. To investigate this, we consider two separate scenarios, one with N branes, and one with $N+1$ branes, where an extra brane has been added after the last brane in the original scenario. The effect of this extra brane on η^2 is to add an extra term to the sum (3.3.2). In the scenario with $N+1$ branes,

$$\eta^2 = \eta_0^2 + B_{N+1}e^{\chi_N}, \tag{3.3.7}$$

where η_0 is the value of η in the scenario with N branes.

The continuity of $\chi(x^a, y)$ across branes requires that

$$e^{\chi_N} = e^{\chi_{N-1}} e^{-2k_N d_N}, \tag{3.3.8}$$

where d_N is the geodesic distance between the now second last and last (newly added) branes. As $\exp(\chi_{N-1}) \leq 1$ ($\chi_T = 0$ is the maximum χ), this contribution to η^2 becomes exponentially small as the distance to the new brane increases. Looking at Eqs. (3.3.3), we see that the change to the angular fields is also exponentially suppressed, and so the contribution of this new brane to the gravitational coupling is exponentially suppressed on all other branes. We therefore infer that the effect of the position of one brane on another, insofar as that information is coded into the radion fields, grows exponentially small as the distance between the branes increases. Given that the interbrane distances must be large compared to the AdS radii of curvature in order to meet the constraint from γ (see Section 3.4.I), this implies that the physics of a model with a large number of branes will dominated by the central brane and those branes nearest to it.

3.4 Observational Constraints

The theories (3.3.4) that are not ruled out by instabilities contain several massless radion fields, which will mediate long range forces and give rise to corrections to general relativity. Therefore, these theories will be subject to constraints arising from Solar System and other tests of general relativity. The nature of these constraints depends on which brane normal visible matter is assumed to reside. In this section, we investigate the extent to which these radion fields modify general relativity, and determine the corresponding observational constraints on the parameters of the theory.

I Eddington PPN Parameter

The Eddington parameterized post-Newtonian (PPN) parameter γ, which measures deviations from general relativity, is one of the most tightly constrained numbers from Solar System measurements of gravity. In this section, we compute this parameter from the action (3.3.4).

As shown in Ref. [76], for a theory of the form

$$S[g_{ab}, \Phi^A, {}^n\phi] = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa_4^2}R^{(4)}[g_{ab}] - \frac{1}{2}\gamma_{AB}(\Phi^C)g^{ab}\nabla_a\Phi^A\nabla_b\Phi^B\right\} + \sum_{n=0}^{N-1} {}^nS_m\left[e^{2\alpha_n(\Phi^C)}g_{ab}, {}^n\phi\right] \tag{3.4.1}$$

where Φ^A are scalar fields and $\gamma_{AB}(\Phi^C)$ is the metric on field space, the Eddington PPN γ parameter for observers on brane n is given by

$$1-\gamma = \frac{2\ {}^n\alpha_0^2}{1+{}^n\alpha_0^2} \tag{3.4.2}$$

where

$${}^n\alpha_0^2 = \frac{2}{\kappa_4^2}\gamma^{AB}\frac{\partial\alpha_n}{\partial\Phi^A}\frac{\partial\alpha_n}{\partial\Phi^B} \tag{3.4.3}$$

and γ^{AB} is the inverse field space metric. For our theory (3.3.4), we have $\Phi^A \equiv (a, \lambda_1, \ldots, \lambda_{N-2})$, the field space metric is given by Eq. (3.3.5), and the functions α_n are given by Eqs. (3.3.3) and (3.3.4).

We calculate γ for each of our branes. On the central brane, we find that

$${}^T\alpha_0^2 = \frac{1}{3}\eta^2, \tag{3.4.4}$$

where $\eta = \tanh(a/\mu)$ has been used. As $0 < \eta < 1$, it is possible for ${}^T\alpha_0^2$ to be sufficiently small on this brane to meet experimental constraints, which require that [77]

$$|\gamma - 1| \leq 2.3 \times 10^{-5}. \tag{3.4.5}$$

This constraint implies that the brane which is closest to the central brane must be at least 5 times the bulk curvature scale away from it [from Eqs. (2.7.18), (2.8.13) and (3.3.2)].

For the other branes, let

$$p(n) = \begin{cases} n, & n < T \\ n-1, & n > T \end{cases} \tag{3.4.6}$$

in order to account for the hole in the sum over the matter actions in Eq. (3.3.4). For brane n, we calculate ${}^p\alpha_0^2$ to find

$$\begin{aligned} {}^p\alpha_0^2 &= \frac{1}{3\eta^2}\left[1 + (1-\eta^2)\left\{\sum_{j=1}^{p} \frac{\cot^2(\lambda_j)}{\prod_{m=1}^{j-1}\sin^2(\lambda_m)} + (1-\delta_{p,N-2})\frac{\tan^2(\lambda_{p+1})}{\prod_{m=1}^{p}\sin^2(\lambda_m)}\right\}\right] \qquad (3.4.7) \\ &> \frac{1}{3\eta^2}. \qquad (3.4.8) \end{aligned}$$

As $0 < \eta < 1$, none of these branes can give rise to a γ parameter consistent with our observed Universe, and thus for this type of model not to be observationally excluded requires that we live on the central brane, where the warp factor is maximized. This implies that models of the form we are considering are unsuitable for explanations of the hierarchy problem, as no hierarchy can be obtained when considering Standard Model fields to be living on the central brane. Solving the hierarchy problem requires stabilizing at least some of the radion modes.

II Dark Matter Limits

One of the motivations behind braneworld models is that the sequestering that occurs between matter on different branes may provide a natural explanation for the weakness of the coupling between normal matter and dark matter. Because of the different coupling factors of the metric to matter on different branes, there is a different Newton's constant for each brane, as well as different interaction strengths between matter on separate branes. As such, the Newton's constant becomes a Newton's matrix. In this section, we calculate the Newton's matrix measured by observers on different branes.

The Newton's matrix depends on the brane on which the observer resides, since the units in terms of which the Newton's constant is measured vary from one brane to another. As the above section constrains normal matter to live on the central brane, we calculate the Newton's matrix from the perspective of the central brane. Generalizing the arguments presented in the appendix of [78], for a theory of the form (3.4.1), we calculate the elements of the Newton's matrix to be

$$G_{\text{eff}}^{mn} = \frac{\kappa_4^2}{8\pi} e^{2\alpha_T} \left(1 + \frac{2}{\kappa_5^2} \gamma^{AB} \frac{\partial \alpha_m}{\Phi^A} \frac{\partial \alpha_n}{\Phi^B} \right), \tag{3.4.9}$$

where G_{eff}^{mn} measures the strength of the gravitational interaction between matter on brane n and matter on brane m. Note that for $m = n$, the quantity in the brackets is $1 + {}^n\alpha_0^2$.

When calculating the elements of (3.4.9), it is again convenient to write the quantities in terms of $\eta = \tanh(a/\mu)$. We also use $p(n)$ [Eq. (3.4.6)], and similarly define $q(m)$, in order to account for the missing term in the matter action sum in Eq. (3.3.4). We find

$$G_{\text{eff}}^{TT} = \frac{\kappa_4^2}{8\pi} e^{2\alpha_T} \left(1 + \frac{\eta^2}{3} \right), \tag{3.4.10a}$$

$$G_{\text{eff}}^{Tp} = \frac{\kappa_4^2}{8\pi} e^{2\alpha_T} \left(1 + \frac{1}{3} \right), \tag{3.4.10b}$$

$$\begin{aligned} G_{\text{eff}}^{pp} = \frac{\kappa_4^2}{8\pi} e^{2\alpha_T} \Bigg(1 + \frac{1}{3\eta^2} \Bigg[1+ \\ (1-\eta^2) \left\{ \sum_{j=1}^{p} \frac{\cot^2(\lambda_j)}{\prod_{k=1}^{j-1} \sin^2(\lambda_k)} + (1-\delta_{p,N-2}) \frac{\tan^2(\lambda_{p+1})}{\prod_{k=1}^{p} \sin^2(\lambda_k)} \right\} \Bigg] \Bigg), \end{aligned} \tag{3.4.10c}$$

$$G_{\text{eff}}^{pq} = \frac{\kappa_4^2}{8\pi} e^{2\alpha_T} \left(1 + \frac{1}{3\eta^2} \left[1 + (1-\eta^2) \left\{ \sum_{j=1}^{q} \frac{\cot^2(\lambda_j)}{\prod_{k=1}^{j-1} \sin^2(\lambda_k)} - \frac{1}{\prod_{k=1}^{q} \sin^2(\lambda_k)} \right\} \right] \right), \tag{3.4.10d}$$

where $m \neq n \neq T$, and $m < n$. In all cases, the "1" in the outermost brackets arises from graviton exchange, while the remaining terms come from the exchange of scalar quanta.

By considering the formation of the Sagittarius tidal streams, Kesden and Kamionkowski [79] have placed limits on the relative strengths of gravitational interaction between dark matter and normal matter. The constraint is roughly

$$\left| \frac{G_{M-DM}}{\sqrt{G_{M-M} G_{DM-DM}}} - 1 \right| \lesssim 0.02 \tag{3.4.11}$$

where "M" indicates matter, and "DM" indicates dark matter. If we assume that all the dark matter lives on branes other than the central brane, we can calculate the constraints on our model that this provides, finding that $\eta \gtrsim 0.8$. This disagrees with the constraint (3.4.5), which implies $\eta \lesssim 6 \times 10^{-3}$. Thus, this model is unable to explain dark matter by positing the existence of matter fields on other branes[1].

We next consider the possibility that some fraction of the dark matter lives on our (central) brane, and some fraction lives on other branes. We can then calculate the percentage of dark matter which must reside on the central brane in order to be compatible with the observational constraints (3.4.5) and (3.4.11). On average, a mass M of dark matter will be composed of a mass αM on our brane, say, and $(1-\alpha)M$ on other branes. The effective matter to dark matter coupling strengths will then be

$$G_{\text{eff}}^{MM} = G_{\text{eff}}^{TT} \tag{3.4.12a}$$

$$G_{\text{eff}}^{DD} = G_{\text{eff}}^{TT}\alpha^2 + G_{\text{eff}}^{nn}(1-\alpha)^2 + G_{\text{eff}}^{Tn}\alpha(1-\alpha) \tag{3.4.12b}$$

$$G_{\text{eff}}^{MD} = G_{\text{eff}}^{TT}\alpha + G_{\text{eff}}^{Tn}(1-\alpha). \tag{3.4.12c}$$

For simplicity, we use

$$G_{\text{eff}}^{nn} = G_{\text{eff}}^{mn} \sim \frac{\kappa_4^2}{8\pi} e^{2\alpha_T} \left(1 + \frac{1}{3\eta^2}\right) \tag{3.4.13}$$

as the "off-brane to off-brane" coupling strength. Combining values for $G_{\text{eff}}^{TT}, G_{\text{eff}}^{Tn}$ and G_{eff}^{nn} with Eqs. (3.4.12) in the constraint (3.4.11) and using $\eta^2 \sim 3.5 \times 10^{-5}$, we find $\alpha \gtrsim 0.998$, indicating that the vast majority of the dark matter must reside on our brane in this simplified model.

[1]Note, however, that if the radion fields are stabilized, then it is possible to circumvent this restriction. As such, we can only rule out braneworld models with no moduli stabilization as an explanation for the observed weak interaction strength between dark matter and normal matter.

3.5 Discussion

This completes our analysis of the observational constraints for a general uncompactified five-dimensional braneworld model with arbitrary numbers of branes and without a radion stabilization mechanism, in the low-energy four-dimensional regime. The parameter space of such models was restricted by excluding ghost modes, and the phenomenology of the resulting models was analyzed. For such models to be viable, there is only one brane upon which Standard Model fields may reside, and such a configuration was unable to provide any benefit for the hierarchy problem, nor a natural explanation for the weakness of the coupling between normal matter and dark matter by sequestration. The Kaluza-Klein modes in such a model behave very similarly to the original RS-II model. Our model was not found to be ruled out experimentally, although observational constraints on the change in the value of G_N between nucleosynthesis and today may do so.

The methodology discussed in these chapters is also applicable to orbifolded models. In Appendix B, we show that the low-energy theory for orbifolded models is very similar to that for the uncompactified model discussed here. In Appendix C, we discuss the spectrum of Kaluza-Klein modes in both orbifolded and uncompactified multibrane models.

Overall, we found that models with N branes are quantitatively very similar to the two-brane case. Furthermore, uncompactified and orbifolded models were also found to be very similar, giving rise to identical four-dimensional low-energy theories, after a scaling of parameters.

I Evalulation

Our approach to analyzing the five-dimensional model and obtaining a four-dimensional effective theory is straightforward and versatile. The general approach of a two-lengthscale expansion is applicable to actions involving different contributions, such as induced gravity on branes (for example, the DPG model [5]) and Gauss-Bonnet curvature terms in the bulk. However, to acquire the four-dimensional effective theory for such models would require performing the analysis of the previous chapter again, in particular, identifying the leading order contributions to the equations of motion.

Braneworld models such as the ones we have analyzed are often complemented by a radion stabilization mechanism. Radion stabilization is particularly useful in circumventing the observational constraints that we calculated here, as massive radion modes will be subject to Yukawa suppression and thus will have suppressed contributions to deviations in γ. A radion stabilization mechanism may be implemented in the model explicitly by including it in the action, and the new model analyzed in the two-lengthscale expansion. In the case where a bulk scalar field is used [45, 46], we expect interactions between the radion modes and the scalar field to give rise to nontrivial dynamics. On the other hand, if radion stabilization is implemented by hand, such as by giving masses to the ψ_n fields in Eq. (2.8.15) (corresponding to fixing the distance between successive branes), then our analysis will proceed unchanged, although our calculations of the observational constraints will not apply.

The approach of using a two-lengthscale expansion has been demonstrated to be a useful method for understanding the low-energy theory of braneworld models, as we have shown here in the case of simple N-brane models in a five-dimensional bulk. We hope that others find the method applicable to a broad range of models.

Part of the motivation for investigating these models was to evaluate if any possible explanations for dark energy could arise from this manner of construction. As all of the radion modes turn out to be massless scalar fields, they are unfortunately not useful for dark energy models. A possible modification to the models which might give rise to the desired behavior would be to more closely investigate detuned branes, which naturally give rise to effective four-dimensional cosmological constants, as well as potentials for the radion modes. However, the dynamics associated with such a detuning can easily violate the separation of lengthscales argument upon which this method is based, and so alternative analysis techniques would be required.

Having concentrated on analyzing a specific class of models for two chapters, we now change gears and look at dark energy models with a more general approach. Later, we will meet the two approaches in the middle.

Chapter 4

A Class of Effective Field Theory Models of Cosmic Acceleration

Contents

We now turn to a rather different approach to dark energy models. Instead of investigating individual models, here we construct an effective field theory model of dark energy, with the aim of being as generic and all-encompassing as possible. We pay close attention to the regime of validity of the effective field theory, and find that such an approach isn't as all-encompassing as we had hoped.

This chapter is based on work originally presented in [3].

4.1 Introduction

The accelerated expansion of the universe, to our current observations, appears to be progressing in a homogeneous and isotropic manner on the largest scales. Should this expansion be due to something other than a cosmological constant, then it can typically be attributed to an effective scalar mode, so that the expansion has no directional preference such as would be associated with modes of other spins.

If dark energy has a dynamical microphysical origin, then it would represent a modification to gravity on extreme infrared scales. However, it is important not to modify gravity on scales in which gravity has been stringently tested, namely solar system scales down to sub-millimetre scales. Gravity is eventually expected to differ from general relativity on length-scales smaller than this, at the Planck scale, if not before. However, deviations at small scales are unable to contribute to the expansion of the universe.

A famous theorem due to Weinberg [28] shows that the self-interactions of a Lorentz-invariant massless spin-two field are equivalent to general relativity in the low-energy limit, and so any modifications of gravity perforce require the addition of new degrees of freedom. It is therefore little surprise that a common feature of the majority of dark energy and modified gravity models is that in the low-energy limit, they are equivalent to general relativity coupled to one or more scalar fields, often called quintessence fields.

It is thus useful to try to construct very general low-energy effective quantum field theories of general relativity coupled to light scalar fields, in order to encompass broad classes of dark energy models. Considering dark energy models as quantum field theories is useful, even though the dynamics of dark energy is likely in a classical regime, because it facilitates discriminating against theories which are theoretically inconsistent or require fine tuning.

I Previous Work

A similar situation occurs in the study of models of inflation, where it is useful to construct generic theories using effective field theory. Cheung *et al.* [80] constructed a general effective field theory for gravity and a single inflaton field, for perturbations about a background FRW cosmology in unitary gauge. This work was later generalized in multiple directions

[81, 82] and has been very useful. An alternative approach to single field inflationary models was taken by Weinberg [83], who constructed an effective field theory to describe both the background cosmology and the perturbations. This theory consisted at leading order of a standard single field inflationary model with a potential, together with higher-order terms in a covariant derivative expansion up to four derivatives. More detailed discussions of this type of effective field theory were given by Burgess, Lee and Trott [84].

When one turns from inflationary effective field theories to quintessence effective field theories, the essential physics is very similar, but there are three important differences that arise:

- First, the hierarchy of scales is vastly more extreme in quintessence models. The Hubble parameter H is typically several orders of magnitude below the Planck scale $m_P \sim 10^{28}$ eV in inflationary models, whereas for quintessence models $H_0 \sim 10^{-33}$ eV is ~ 60 orders of magnitude below the Planck scale. Quintessence fields must have a mass that is smaller than or on the order of H_0. It is a well-known, generic challenge for quintessence models to ensure that loop effects do not give rise to a mass much larger than H_0. Because of the disparity of scales, this issue is more extreme for quintessence models than inflationary models.
- In most inflationary models, it is assumed that the dynamics of the Universe are dominated by gravity and the scalar field (at least until reheating). By contrast, for quintessence models in the regime of low redshifts relevant to observations, we know that cold dark matter gives an $O(1)$ contribution to the energy density. Therefore there are additional possible couplings and terms that must be included in an effective field theory.
- For any effective field theory, it is possible to pass outside the domain of validity of the theory even at energies E low compared to the theory's cutoff Λ, if the mode occupation numbers N are sufficiently large (see Section 4.5.II below for more details).

This corresponds to a breakdown of the classical derivative expansion. For quintessence theories, mode occupation numbers today can be as large as $N \sim (m_P/H_0)^2$ and it is possible to pass outside the domain of validity of the theory. By contrast in inflationary models, this is less likely to occur since mode occupation numbers for the perturbations are not large before modes exit the horizon. Thus, the effective field theory framework is less all-encompassing for quintessence models than for inflation models. This issue seems not to have been appreciated in the literature and we discuss it in Section 4.5.II below.

Several studies have been made of generic effective field theories of dark energy. Creminelli, D'Amico, Noreña and Vernizzi [85] constructed a the general effective theory of single-field quintessence for perturbations about an arbitrary FRW background, paralleling the similar construction for inflation [80]. Park, Watson and Zurek constructed an effective theory for describing both the background cosmology and the perturbations, following the approach of Weinberg [83] but generalizing it to include couplings to matter [86].

The two approaches to effective field theories of quintessence – specialization to perturbations about a specific background, and maintaining covariance and the ability to describe the dynamics of a variety of backgrounds – are complementary to one another. The dynamics of the cosmological background FRW solution can be addressed in the covariant approach of Weinberg, but not in the background specific approach of Creminelli *et al.*, which restricts attention to the dynamics of perturbations about a given, fixed background. On the other hand, a background specific approach can describe a larger set of dynamical theories for the perturbations than can a covariant derivative expansion[1].

[1] To see this, consider for example a term in the Lagrangian of the form $f(\phi)(\nabla\phi)^{2n}$, where ϕ is the quintessence field. Such a term would be omitted in the covariant derivative expansion for sufficiently large n. However, upon expanding this term using $\phi = \phi_0 + \delta\phi$, where ϕ_0 is the background solution, one finds terms $\sim (\nabla\phi_0)^{2n-2}(\nabla\delta\phi)^2$ which are included in the Creminelli approach of applying standard effective field theory methods to the perturbations.

II Approach

In this chapter, we revisit, generalize and correct slightly the covariant effective field theory analysis of Park, Watson and Zurek [86]. Following Weinberg and Park *et al.*, we restrict attention to theories where the only dynamical degrees of freedom are a graviton and a single scalar. We allow couplings to an arbitrary matter sector, but we assume the validity of the weak equivalence principle, motivated by the strong experimental evidence for this principle. We assume that the theory consists of a standard quintessence theory coupled to matter at leading order in a derivative expansion, with an action of the form

$$S[g_{\alpha\beta}, \phi, \psi_{\rm m}] = \int d^4x\sqrt{-g}\left\{\frac{m_P^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right\} + S_{\rm m}\left[e^{\alpha(\phi)}g_{\mu\nu}, \psi_{\rm m}\right]. \tag{4.1.1}$$

Here $\psi_{\rm m}$ denotes a set of matter fields, and m_P is the Planck mass. The factor $e^{\alpha(\phi)}$ in the matter action provides a leading-order non-minimal coupling of the quintessence field to matter, in a manner similar to Brans-Dicke models in the Einstein frame [87, 88].

Our analysis then consists of a series of steps:

1. We add to the action all possible terms involving the scalar field and metric, in a covariant derivative expansion up to four derivatives. We truncate the expansion at four derivatives, as this is sufficient to yield the leading corrections to the action (4.1.1). As described by Weinberg [83] there are ten possible terms, with coefficients that can be arbitrary functions of ϕ [see Eq. (4.2.3) below]. Section 4.5.I below describes one possible justification of this covariant derivative expansion from an effective field theory viewpoint, starting from a set of ultralight pseudo-Nambu-Goldstone bosons (pNGBs). It is likely that the same expansion can be obtained from other, more general starting points.

2. We allow for corrections to the coupling to matter by adding to the metric that appears in the matter action all possible terms involving the metric and ϕ allowed by the derivative expansion, that is, up to two derivatives. There are six such terms [see Eq. (4.2.4) below.]

We also add to the action terms involving the stress energy tensor $T_{\mu\nu}$ of the matter fields, up to the order allowed by the derivative expansion using $T_{\mu\nu} \sim m_P^2 G_{\mu\nu}$ [see Eq. (4.2.3) below]. Including such terms in the action seems poorly motivated, since *a priori* there is no reason to expect that the resulting theory would respect the weak equivalence principle. However, we show in Appendix D that the weak equivalence principle is actually satisfied, to the order we are working to in the derivative expansion. In addition, all the terms in the action involving $T_{\mu\nu}$ can be shown to have equivalent representations not involving the stress energy tensor, using field redefinitions (see Appendix D).

3. The various correction terms are not all independent because of the freedom to perform field redefinitions involving ϕ, $g_{\mu\nu}$ and the matter fields, again in a derivative expansion. In Section 4.3 we explore the space of such field redefinitions, finding eleven independent transformations and tabulating their effects on the coefficients in the action (see Table 1 below).

4. Several of the correction terms that are obtained from the derivative expansion are "higher-derivative" terms, by which we mean that they give contributions to the equations of motion which involve third- or higher-order time derivatives of the fields[2]. Normally, such higher-derivative terms give rise to additional degrees of freedom. However, if they are treated perturbatively (consistent with our derivative expansion) additional degrees of freedom do not arise. Specifically, one can perform a *reduction of order* procedure on the equations of motion [90, 91, 92], substituting the zeroth-order equations of motion into the higher derivative terms in the equations of motion to eliminate the higher

[2]The precise definition of higher-derivative that we use, which is covariant, is that an equation will be said not to contain any higher-derivative terms if there exists a choice of foliation of spacetime for which any third-order or higher-order derivatives contain at most two time derivatives. Theories which are higher-derivative in this sense are generically associated with instabilities (Ostragradski's theorem) [89], although the instabilities can be evaded in special cases, for example $f(R)$ gravity. For most of this work (except for the Chern-Simons term), a simpler definition of higher-derivative would be sufficient: a term in the action is "higher-derivative" if it gives rise to terms in the equation of motion that involve any third- or higher-order derivatives.

derivatives[3]. We actually use a slightly different but equivalent procedure of eliminating the higher derivative terms directly in the action using field redefinitions[4] (see Appendix E).

Weinberg [83] and Park *et al.* [86] use a slightly different method, consisting of substituting the leading order equations of motion directly into the higher derivative terms in the action. This method is not generally valid, but it is valid up to field redefinitions that do not involve higher derivatives, and so it suffices for the purpose of attempting to classify general theories of dark energy (see Appendix E).

5. Another issue that arises with respect to the higher derivative terms is the following. Is it really necessary to include such terms in an action when trying to write down the most general theory of gravity and a scalar field, in a derivative expansion? Weinberg [83] suggested that perhaps a more general class of theories is generated by including these terms and performing a reduction of order procedure on them, rather than by omitting them. However, since it is ultimately possible to obtain a theory that is perturbatively equivalent to the higher-derivative theory, and which has second-order equations of motion, it should be possible just to write down the action for this reduced theory. In other words, an equivalent class of theories should be obtained simply by omitting all the higher-derivative terms from the start. We show explicitly in Section 4.4 that this is the case for the class of theories considered here.

6. We fix the remaining field redefinition freedom by choosing a "gauge" in field space, thus fixing the action uniquely (see Section 4.4.II).

[3]This is more general than requiring the solutions of the equation of motion to be analytic in the expansion parameter, as advocated by Simon [93]; see Ref. [92].

[4]This procedure is counterintuitive since normally field redefinitions do not change the physical content of a theory; here however they do because the field redefinitions themselves involve higher derivatives.

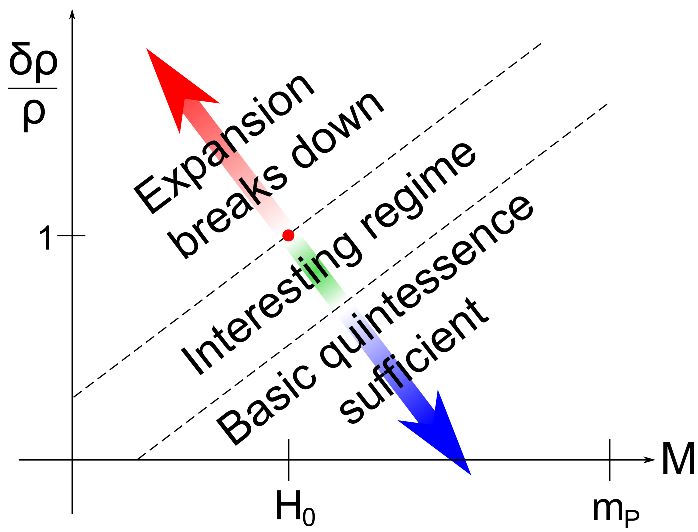

Figure 9: The parameter space of fractional density perturbation $\delta\rho/\rho$ for perturbations to the quintessence field, and cutoff scale M for the effective field theory, illustrating the constraint (4.1.3) on the domain of validity. Near the boundary of the domain of validity the higher derivative terms in the action are potentially observable, this is labeled the "interesting regime". Further away from the boundary the higher derivative terms are negligible and the theory reduces to a standard quintessence model with a matter coupling.

III Results and Implications

Our final action is [Eq. (4.4.5) below]

$$
\begin{aligned}
S = &\int d^4x\sqrt{-g}\left\{\frac{m_P^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right\} + S_{\rm m}[e^{\alpha(\phi)}g_{\alpha\beta}, \psi_{\rm m}] \\
&+ \epsilon\int d^4x\sqrt{-g}\Big\{a_1(\nabla\phi)^4 + b_2 T(\nabla\phi)^2 + c_1 G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\
&+ d_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) + d_4\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta} \\
&+ e_1 T^{\mu\nu}T_{\mu\nu} + e_2 T^2 + \ldots\Big\}.
\end{aligned}
\tag{4.1.2}
$$

Here the coefficients a_1, b_2 etc. of the next-to-leading order terms in the derivative expansion are arbitrary functions of ϕ, and the ellipsis $\ldots$ refers to higher-order terms with more than four derivatives. The corresponding equations of motion do not contain any higher derivative terms. This result generalizes that of Weinberg [83] to include couplings to matter.

We can summarize our key results as follows:

- The most general action contains nine free functions of ϕ: U, α, a_1, b_2, c_1, d_3, d_4, e_1, e_2, as compared to the four functions that are needed when matter is not present [83].

- There are a variety of different forms of the final theory that can be obtained using field redefinitions. In particular some of the matter-coupling terms in the action can be re-expressed as terms that involve only the quintessence field and metric. Specifically, the term $T(\nabla\phi)^2$ term could be eliminated in favor of $\Box\phi(\nabla\phi)^2$, the $(\nabla\phi)^4$ could be eliminated in favor of a term $T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, or the $G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ term could be eliminated in favor of a term $T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ (see Section 4.4.II).

- As mentioned above, one obtains the correct final action if one excludes throughout the calculation all higher-derivative terms.

- The final theory does contain terms involving the matter stress-energy tensor. Nevertheless, the weak equivalence principle is still satisfied (see Appendix D). It is possible to eliminate the stress-energy terms, but only if we allow higher derivative terms in the action (where it is assumed that the reduction of order procedure will be applied to these higher derivative terms). Thus, for a fully general theory, one must have either stress-energy terms or higher derivative terms; one cannot eliminate both (see Section 4.4.II).

- We can estimate how all the coefficients a_1 etc. scale with respect to a cutoff scale M for an effective field theory as follows (see Section 4.5.I). We assume that several ultralight scalar fields of mass $\sim H_0$ arise as pseudo-Nambu-Goldstone bosons from some high-energy theory [94, 95], and are described by a nonlinear sigma model at low energies. We then suppose that all but one of the these pNGB fields have masses M that are somewhat larger than $\sim H_0$, and integrate them out. This will give rise to a theory of the form discussed above for the single light scalar, where the higher derivative terms are suppressed by powers of M. The scalings for each of the coefficients in the action are summarized in Table 3. We find that the fractional corrections to the cosmological dynamics due to the higher derivative terms scale as H_0^2/M^2, as one would expect.

- Finally, we can use these scalings to estimate the domain of validity of the effective field theory (see Section 4.5.II). We find that cosmological perturbations with a density perturbation $\delta\rho$ in the quintessence field must have a fractional density perturbation that satisfies

$$\frac{\delta\rho}{\rho} \ll \frac{M^2}{H_0^2}. \tag{4.1.3}$$

Thus perturbations can become nonlinear, but only modestly so, if M is close to H_0. The parameter space of fractional density perturbation $\delta\rho/\rho$ and cutoff scale M is illustrated in Fig. 9. In addition there is the standard constraint for derivative expansions

$$E \ll M \tag{4.1.4}$$

where E^{-1} is the length-scale or time-scale for some process. We show in Fig. 10 the two constraints (4.1.3) and (4.1.4) on the two dimensional parameter space of energy E and mode occupation number N.

Finally, in Appendix F we compare our analysis to that of Park, Watson and Zurek [86], who perform a similar computation but in the Jordan frame rather than the Einstein frame (see also Ref. [96]). The main difference between our analysis and theirs is that they use a different method to estimate the scalings of the coefficients, and as a result their final action differs from ours, being parameterized by three free functions rather than nine.

4.2 Class of Theories Involving Gravity and a Scalar Field

As discussed in the introduction, our starting point is an action for a standard quintessence model with an arbitrary matter coupling, together with a perturbative correction which consists of a general derivative expansion up to four derivatives. The action is a functional of the Einstein-frame metric $g_{\alpha\beta}$, the quintessence field ϕ, and some matter fields which we

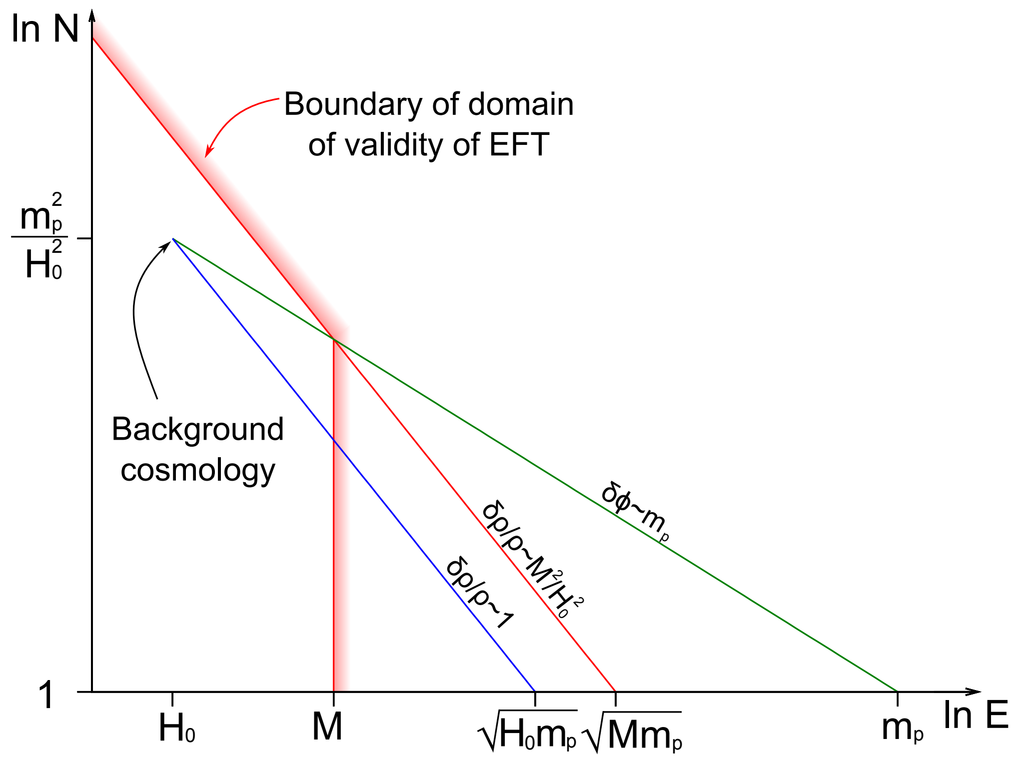

Figure 10: The domain of validity of the effective field theory in the two dimensional parameter space of energy E per quantum of a mode of the quintessence field, and mode occupation number N. The cutoff scale M must be larger than the Hubble parameter H_0 in order that the background cosmology lie within the domain of validity. Perturbation modes on length-scales that are small compared to H_0^{-1} but large compared to M^{-1} can be described, but only if the mode occupation number and fractional density perturbation are sufficiently small. See Section 4.5.II for details.

denote collectively by $\psi_{\rm m}$:

$$S[g_{\alpha\beta}, \phi, \psi_{\rm m}] = S_0[g_{\alpha\beta}, \phi] + \epsilon S_1[g_{\alpha\beta}, \phi, T_{\alpha\beta}(\psi_{\rm m})] + S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}] + O(\epsilon^2). \tag{4.2.1}$$

Here $S_{\rm m}$ is the action for the matter fields, and the quantity ϵ is a formal expansion parameter. We will see in Section 4.5.I below that ϵ can be identified as proportional to M^{-2}, where M is a cutoff scale or the mass of the lightest of the fields that have been integrated out to obtain the low-energy action. Equivalently, ϵ counts the number of derivatives in our derivative expansion, with ϵ^n corresponding to $2(n+1)$ derivatives. The notation in the second term indicates that the perturbative correction S_1 to the action can depend on the matter fields, but only through their stress energy tensor $T_{\alpha\beta}$ (as defined in the Preface). Explicitly we have

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{m_P^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right], \tag{4.2.2}$$

and [83, 86]

$$\begin{aligned} S_1 = \int d^4x\sqrt{-g}\Big\{ & a_1(\nabla\phi)^4 + a_2\Box\phi(\nabla\phi)^2 + a_3(\Box\phi)^2 + b_1 T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \\ & + b_2 T(\nabla\phi)^2 + b_3 T\Box\phi + b_4 T^{\mu\nu}\nabla_\mu\nabla_\nu\phi + b_5 R_{\mu\nu}T^{\mu\nu} \\ & + b_6 RT + b_7 T + c_1 G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + c_2 R(\nabla\phi)^2 + c_3 R\Box\phi \\ & + d_1 R^2 + d_2 R^{\mu\nu}R_{\mu\nu} + d_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) \\ & + d_4\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta} + e_1 T^{\mu\nu}T_{\mu\nu} + e_2 T^2\Big\}. \end{aligned} \tag{4.2.3}$$

Here $C_{\mu\nu\lambda\rho}$ is the Weyl tensor and $\epsilon^{\mu\nu\lambda\rho}$ is the antisymmetric tensor (our conventions for these are given in the Preface). There are additional terms with four derivatives that one can write down, but all such terms can be eliminated by integration by parts. Finally, the metric $\bar{g}_{\mu\nu}$ which appears in the matter action $S_{\rm m}$ in Eq. (4.2.1) is given by[5]

$$\begin{aligned} \bar{g}_{\mu\nu} = e^{\alpha}g_{\mu\nu} + \epsilon e^{\alpha}\big[& \beta_1\nabla_\mu\phi\nabla_\nu\phi + \beta_2(\nabla\phi)^2 g_{\mu\nu} + \beta_3\Box\phi g_{\mu\nu} \\ & + \beta_4\nabla_\mu\nabla_\nu\phi + \beta_5 R_{\mu\nu} + \beta_6 R g_{\mu\nu}\big] + O(\epsilon^2). \end{aligned} \tag{4.2.4}$$

All of the coefficients a_i, b_i, c_i, d_i, e_i, β_i and α are arbitrary functions of ϕ.

Let us briefly discuss each of the perturbative terms. The terms with coefficients a_i are corrections to the kinetic term of the scalar field. The b_i and β_i terms are couplings between the scalar field and the stress-energy tensor, or between curvature and the stress-energy tensor. The c_i terms are kinetic couplings between the scalar field and gravity. The d_i terms are quadratic curvature terms, which we have chosen to write as an R^2 term, an $R_{\mu\nu}R^{\mu\nu}$ term, and the Gauss-Bonnet term. Any constant piece of the coefficient d_3 is a topological

[5]We call this metric the Jordan frame metric, in an extension of the usual terminology which applies to the case when the relation (4.2.4) between the two metrics is just a conformal transformation.

term and may be omitted. The term d_4 is the gravitational Chern-Simons term, which may be excluded if one wishes to introduce parity as a symmetry of the theory, and again, any constant component of d_4 is topological and may be omitted. Finally, the e_i terms are quadratic in the stress-energy tensor.

Note that several of the terms in the action (4.2.3) are "higher derivative" terms, that is, they give rise to contributions to the equations of motion containing derivatives of order three or higher. The specific terms are those parameterized by the coefficients a_3, $b_3, \ldots, b_6$, c_2, c_3, d_1, d_2 and $\beta_3, \ldots, \beta_6$. As discussed in the introduction and in Appendix E, we will choose to define our theory by treating these terms perturbatively, which excludes the extra degrees of freedom and instabilities that are normally associated with higher derivative terms.

We also note that the theory (4.2.1) satisfies the weak equivalence principle, to linear order in ϵ, as we show in Appendix D. That is, objects with negligible self-gravity with different compositions all experience the same acceleration. It is not *a priori* obvious that the principle should be satisfied since, as we show in Appendix D, violations of the principle generically arise whenever the matter stress energy tensor appears explicitly in the gravitational action, as in Eq. (4.2.1).

4.3 Transformation Properties of the Action

The description of the theory provided by Eqs. (4.2.1) – (4.2.4) is very redundant, in part because of the freedom to perform field redefinitions. In this section we derive how the various coefficients in the action (4.2.1) are modified under various transformations. In the next section we will use these transformation laws to derive a canonical representation of the theory, involving only nine free functions.

I Expansion of the Matter Action

Consider first the perturbative terms parameterized by $\beta_1, \ldots, \beta_6$, in the definition (4.2.4) of the Jordan metric $\bar{g}_{\alpha\beta}$, which appears in the matter action $S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}]$. Using the definition (0.0.2) of the stress-energy tensor, we can eliminate these terms in favor of terms in the action involving $T_{\alpha\beta}$. Specifically we have from Eq. (0.0.2) that

$$S_{\rm m}[e^{\alpha}(g_{\mu\nu} + \delta g_{\mu\nu}), \psi_{\rm m}] = S_{\rm m}[e^{\alpha} g_{\mu\nu}, \psi_{\rm m}] + \frac{1}{2}\int d^4x \sqrt{-g} e^{2\alpha} T^{\mu\nu} \delta g_{\mu\nu} + O(\delta g^2). \tag{4.3.1}$$

Choosing

$$\delta g_{\mu\nu} = \epsilon[\tilde{\beta}_1 \nabla_\mu \phi \nabla_\nu \phi + \tilde{\beta}_2 (\nabla\phi)^2 g_{\mu\nu} + \tilde{\beta}_3 \Box\phi g_{\mu\nu} + \tilde{\beta}_4 \nabla_\mu \nabla_\nu \phi + \tilde{\beta}_5 R_{\mu\nu} + \tilde{\beta}_6 R g_{\mu\nu}] \tag{4.3.2}$$

then gives a transformation of the action (4.2.1) characterized by the following changes in the coefficients:

$$\begin{aligned}
\delta\beta_1 &= -\tilde{\beta}_1, & \delta b_1 &= \tfrac{1}{2} e^{2\alpha} \tilde{\beta}_1, \\
\delta\beta_2 &= -\tilde{\beta}_2, & \delta b_2 &= \tfrac{1}{2} e^{2\alpha} \tilde{\beta}_2, \\
\delta\beta_3 &= -\tilde{\beta}_3, & \delta b_3 &= \tfrac{1}{2} e^{2\alpha} \tilde{\beta}_3, \\
\delta\beta_4 &= -\tilde{\beta}_4, & \delta b_4 &= \tfrac{1}{2} e^{2\alpha} \tilde{\beta}_4, \\
\delta\beta_5 &= -\tilde{\beta}_5, & \delta b_5 &= \tfrac{1}{2} e^{2\alpha} \tilde{\beta}_5, \\
\delta\beta_6 &= -\tilde{\beta}_6, & \delta b_6 &= \tfrac{1}{2} e^{2\alpha} \tilde{\beta}_6.
\end{aligned} \tag{4.3.3}$$

Here the parameters $\tilde{\beta}_i$ can be arbitrary functions of ϕ. Similarly choosing $\delta g_{\mu\nu} = \epsilon \tilde{\alpha} g_{\mu\nu}$ gives a transformation characterized by

$$\delta\alpha = -\epsilon\tilde{\alpha}, \qquad \delta b_7 = \tfrac{1}{2} e^{2\alpha} \tilde{\alpha}. \tag{4.3.4}$$

II Field Redefinitions Involving just the Scalar Field

Consider a perturbative field redefinition of the form

$$\phi = \psi + \epsilon\gamma, \tag{4.3.5}$$

where the quantity γ can in general depend on any of the fields and their derivatives. To leading order in ϵ, the change in the action (4.2.1) is then proportional to the zeroth-order equation of motion (4.5.10b) for ϕ. Relabeling ψ as ϕ, the change induced in the action is

$$\delta S = \epsilon \int d^4x \sqrt{-g}\gamma \left[\Box\phi - U' + \frac{1}{2}e^{2\alpha}\alpha' T\right]. \tag{4.3.6}$$

There are three special cases that will be useful:

1. First, choose

$$\phi = \psi + \epsilon\sigma_1 T, \tag{4.3.7}$$

where σ_1 is an arbitrary function[6] of ψ, and T is the trace of the stress-energy tensor. Substituting this into Eq. (4.3.6) and comparing with the general action (4.2.3), we find the following transformation law for the coefficients:

$$\begin{aligned} \delta b_3 &= \sigma_1, \qquad \delta b_7 = -U'\sigma_1, \\ \delta e_2 &= \frac{1}{2}\alpha' e^{2\alpha}\sigma_1. \end{aligned} \tag{4.3.8}$$

2. Second, we use the field redefinition

$$\phi = \psi + \epsilon\sigma_2[\Box\psi + U'(\psi)]. \tag{4.3.9}$$

Here the second term in the square bracket is included in order to maintain canonical normalization of the scalar field, that is, to avoid generating terms in the action of the form $f(\phi)(\nabla\phi)^2$. The resulting transformation law is

$$\begin{aligned} \delta a_3 &= \sigma_2, \qquad & \delta b_3 &= \frac{1}{2}e^{2\alpha}\alpha'\sigma_2, \\ \delta b_7 &= \frac{1}{2}\alpha' e^{2\alpha}U'\sigma_2, \qquad & \delta U &= \epsilon(U')^2\sigma_2. \end{aligned} \tag{4.3.10}$$

[6]Because we are working to linear order in ϵ, it does not matter whether we take σ_1 to be a function of ϕ or of ψ.

3. Third, consider the field redefinition

$$\phi = \psi + \epsilon\sigma_3 - \epsilon\frac{1}{U'}\sigma_3'(\nabla\psi)^2, \tag{4.3.11}$$

where σ_3 is a function of ψ and again the particular combination of terms is chosen to maintain canonical normalization. Substituting into Eq. (4.3.6), performing some integrations by parts and comparing with Eq. (4.2.3) gives the transformation law

$$\begin{aligned} \delta a_2 &= -\frac{\sigma_3'}{U'}, & \delta b_2 &= -\frac{1}{2U'}e^{2\alpha}\alpha'\sigma_3', \\ \delta b_7 &= \frac{1}{2}e^{2\alpha}\alpha'\sigma_3, & \delta U &= \epsilon U'\sigma_3. \end{aligned} \tag{4.3.12}$$

Note that this transformation is not well defined in general in the limit $U' \to 0$, because of the factors of $1/U'$. However, it is well defined in the limit $U' \to 0$, $\sigma_3' \to 0$ with σ_3'/U' kept constant.

III Field Redefinitions Involving the Metric

We now consider a more general class of field redefinitions, where in addition to redefining the scalar field via Eq. (4.3.5), we also perturbatively redefine the metric via

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + \epsilon F_{\alpha\beta}. \tag{4.3.13}$$

Here the quantity $F_{\alpha\beta}$ can depend on ψ, $\hat{g}_{\alpha\beta}$, their derivatives and the stress energy tensor. The corresponding change in the action is proportional to the equation of motion (4.5.10a). Relabeling $\hat{g}_{\alpha\beta}$ as $g_{\alpha\beta}$ and ψ as ϕ, the total change in the action is

$$\begin{aligned} \delta S =& \frac{\epsilon}{2}\int d^4x\sqrt{-g}F_{\alpha\beta}\left[-m_P^2 G^{\alpha\beta} + \nabla^\alpha\phi\nabla^\beta\phi - \frac{1}{2}(\nabla\phi)^2 g^{\alpha\beta} - U g^{\alpha\beta} + e^{2\alpha}T^{\alpha\beta}\right] \\ &+ \epsilon\int d^4x\sqrt{-g}\gamma\left[\Box\phi - U' + \frac{1}{2}e^{2\alpha}\alpha' T\right]. \end{aligned} \tag{4.3.14}$$

Note that this formula includes the effect of the change in the Jordan frame metric (4.2.4) caused by the transformation (4.3.13). We now consider seven different transformations of this type:

Coeff.	Term	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
a_1	$(\nabla\phi)^4$						$\star$		$\star$	$\star$		
a_2	$\Box\phi(\nabla\phi)^2$			$\star$				$\star$		$\star$		
a_3 †	$(\Box\phi)^2$		$\star$									
b_1	$T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$								$\star$			$\star$
b_2	$T(\nabla\phi)^2$			$\star$			$\star$				$\star$	$\star$
b_3 †	$T\Box\phi$	$\star$	$\star$					$\star$				
b_4 †	$T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$									$\star$		
b_5 †	$R^{\mu\nu}T_{\mu\nu}$					$\star$						$\star$
b_6 †	RT				$\star$						$\star$	$\star$
b_7	T	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$
c_1	$G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$					$\star$			$\star$	$\star$		
c_2 †	$R(\nabla\phi)^2$				$\star$		$\star$					
c_3 †	$R\Box\phi$							$\star$				
d_1 †	R^2				$\star$	$\star$						
d_2 †	$R^{\mu\nu}R_{\mu\nu}$					$\star$						
d_3	Gauss-Bonnet											
d_4	Chern-Simons											
e_1	$T^{\mu\nu}T_{\mu\nu}$											$\star$
e_2	T^2	$\star$									$\star$	
	U (potential)		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$		

Table 1: This table shows which of the terms in our action (4.2.2) are affected by each of the eleven field redefinitions (4.3.7) – (4.3.29) that are parameterized by the functions $\sigma_1(\phi), \ldots, \sigma_{11}(\phi)$. The columns represent the redefinitions, and the rows represent terms. Daggers † in first column indicate "higher derivative" terms, that is, terms that give contributions to the equations of motion containing derivatives of higher than second-order. Stars $\star$ indicate that the coefficient of that row's term is altered by that column's field redefinition. We omit the coefficients α and $\beta_1, \ldots, \beta_6$ since those coefficients are degenerate with $b_1, \ldots, b_7$ by Eqs. (4.3.3) and (4.3.4).

4. The first case is a change to the metric proportional to $Rg_{\alpha\beta}$. In order to maintain canonical normalization of both the metric and the scalar field, that is, to avoid terms of the form $f(\phi)(\nabla\phi)^2$ and $f(\phi)R$, we need the following combination of terms in the field redefinition:

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} - 2\epsilon\sigma_4' \left(\frac{m_P^2}{U}\hat{R} + 4 \right) \hat{g}_{\alpha\beta}, \tag{4.3.15a}$$

$$\phi = \psi + 4\epsilon\sigma_4, \tag{4.3.15b}$$

for some function $\sigma_4(\psi)$. Substituting into Eq. (4.3.14), performing some integrations by parts and comparing with Eq. (4.2.3) we obtain for the transformation law

$$\begin{aligned}
\delta b_7 &= 2e^{2\alpha}\alpha'\sigma_4 - 4e^{2\alpha}\sigma_4', & \delta c_2 &= \frac{m_P^2}{U}\sigma_4', \\
\delta d_1 &= -\frac{m_P^4}{U}\sigma_4', & \delta b_6 &= -\frac{e^{2\alpha}}{U}m_P^2\sigma_4', \\
\delta U &= 4\epsilon\left[U'\sigma_4 - 4U\sigma_4'\right]. & &
\end{aligned} \tag{4.3.16}$$

5. Next consider changes to the metric proportional to $R_{\alpha\beta}$. In order to maintain canonical normalizations we use the following combination of terms in the field redefinition:

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta}(1 - 2\epsilon\sigma_5') - 2\epsilon\frac{m_P^2}{U}\sigma_5'\hat{R}_{\alpha\beta}, \tag{4.3.17a}$$

$$\phi = \psi + \epsilon\sigma_5, \tag{4.3.17b}$$

for some function $\sigma_5(\psi)$. This gives the transformation law

$$\begin{aligned}
\delta b_7 &= \frac{1}{2}e^{2\alpha}\alpha'\sigma_5 - e^{2\alpha}\sigma_5', & \delta c_1 &= -\frac{m_P^2}{U}\sigma_5', \\
\delta d_1 &= -\frac{m_P^4}{2U}\sigma_5', & \delta d_2 &= \frac{m_P^4}{U}\sigma_5', \\
\delta b_5 &= -\frac{m_P^2}{U}e^{2\alpha}\sigma_5', & \delta U &= \epsilon\left[U'\sigma_5 - 4U\sigma_5'\right].
\end{aligned} \tag{4.3.18}$$

6. The next case is a change to the metric proportional to $(\nabla\phi)^2 g_{\alpha\beta}$. To maintain canonical normalization of the scalar field, we need in addition a change to the scalar field, with

the combined transformation being

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} - 2\epsilon\frac{\sigma_6'}{U}(\hat{\nabla}\psi)^2\hat{g}_{\alpha\beta}, \tag{4.3.19a}$$

$$\phi = \psi + 4\epsilon\sigma_6, \tag{4.3.19b}$$

for some function σ_6. The resulting transformation law for the coefficients is

$$\begin{aligned} \delta a_1 &= \frac{\sigma_6'}{U}, & \delta b_2 &= -e^{2\alpha}\frac{\sigma_6'}{U}, \\ \delta b_7 &= 2e^{2\alpha}\alpha'\sigma_6, & \delta c_2 &= -\sigma_6'\frac{m_P^2}{U}, \\ \delta U &= 4\epsilon U'\sigma_6. \end{aligned} \tag{4.3.20}$$

7. Next consider changes to the metric proportional to $\Box\phi g_{\alpha\beta}$. The required form of field redefinition that preserves canonical normalization of ϕ is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_7\hat{\Box}\psi\hat{g}_{\alpha\beta}, \tag{4.3.21a}$$

$$\phi = \psi + 4\epsilon U\sigma_7, \tag{4.3.21b}$$

for some function σ_7. The coefficients in the action then change according to

$$\begin{aligned} \delta a_2 &= -\sigma_7, & \delta b_3 &= e^{2\alpha}\sigma_7, \\ \delta b_7 &= 2e^{2\alpha}\alpha' U\sigma_7, & \delta c_3 &= m_P^2\sigma_7, \\ \delta U &= 4\epsilon UU'\sigma_7. \end{aligned} \tag{4.3.22}$$

8. The fifth case is a change to the metric proportional to $\nabla_\alpha\phi\nabla_\beta\phi$. The required form of field redefinition that preserves canonical normalization of ϕ is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} - 2\epsilon\frac{\sigma_8'}{U}\hat{\nabla}_\alpha\psi\hat{\nabla}_\beta\psi, \tag{4.3.23a}$$

$$\phi = \psi + \epsilon\sigma_8, \tag{4.3.23b}$$

for some function σ_8. The coefficients in the action then change according to

$$\begin{aligned} \delta a_1 &= -\frac{\sigma_8'}{2U}, & \delta b_1 &= -e^{2\alpha}\frac{\sigma_8'}{U}, \\ \delta b_7 &= \frac{1}{2}e^{2\alpha}\alpha'\sigma_8, & \delta c_1 &= \frac{m_P^2}{U}\sigma_8', \\ \delta U &= \epsilon U'\sigma_8. \end{aligned} \tag{4.3.24}$$

9. Next consider a change in the metric proportional to $\nabla_\alpha \nabla_\beta \phi$. To preserve canonical normalization of ϕ we use the redefinitions

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_9 \hat{\nabla}_\alpha \hat{\nabla}_\beta \psi, \tag{4.3.25a}$$

$$\phi = \psi + \epsilon U \sigma_9, \tag{4.3.25b}$$

for some function σ_8. The coefficients in the action then change according to

$$\begin{aligned} \delta a_1 &= -\frac{1}{2}\sigma_9', & \delta a_2 &= -\sigma_9, \\ \delta b_4 &= e^{2\alpha}\sigma_9, & \delta b_7 &= \frac{1}{2}e^{2\alpha}\alpha' U \sigma_9, \\ \delta c_1 &= m_P^2 \sigma_9', & \delta U &= \epsilon U U' \sigma_9. \end{aligned} \tag{4.3.26}$$

10. A simple case is when the change in the metric is proportional to $T g_{\alpha\beta}$, for which no change to the scalar field is required. The redefinition is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_{10} T \hat{g}_{\alpha\beta}, \tag{4.3.27}$$

for some function σ_{10}. The transformation law for the coefficients is

$$\begin{aligned} \delta b_2 &= -\sigma_{10}, & \delta b_7 &= -4\sigma_{10} U, \\ \delta e_2 &= e^{2\alpha}\sigma_{10}, & \delta b_6 &= m_P^2 \sigma_{10}. \end{aligned} \tag{4.3.28}$$

11. Similarly, no transformation to the scalar is required for the case of a change in the metric proportional to $T_{\alpha\beta}$. The redefinition is

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_{11} T_{\alpha\beta}, \tag{4.3.29}$$

for some function σ_{11}, and the corresponding transformation law is

$$\begin{aligned} \delta b_1 &= \sigma_{11}, & \delta b_2 &= -\frac{1}{2}\sigma_{11}, \\ \delta b_7 &= -\sigma_{11} U, & \delta e_1 &= e^{2\alpha}\sigma_{11}, \\ \delta b_5 &= -m_P^2 \sigma_{11}, & \delta b_6 &= \frac{1}{2} m_P^2 \sigma_{11}. \end{aligned} \tag{4.3.30}$$

The eleven[7] field redefinitions (4.3.7) – (4.3.29) are summarized in Table 1, which shows which coefficients are modified by which transformations.

4.4 Canonical Form of Action

In this section, we derive our final, reduced action (4.1.2) from the starting action (4.2.1), using the transformation laws derived in Section 4.3. There is some freedom in which terms we choose to eliminate and which terms we choose to retain. We choose to eliminate all terms that give higher derivatives in the equations of motion, so that the final theory is not a "higher derivative" theory. However, even after this has been accomplished, there is still some freedom in how the final theory is represented. We discuss this further in Section 4.4.II below. The order of operations in the derivation is important, since we need to take care that terms which we have already set to zero are not reintroduced by subsequent transformations. Table 2 summarizes our calculations and their effects on the coefficients in the action at each stage in the computation.

I Derivation

The steps in the derivation are as follows:

1. *Elimination of Derivative Terms in the Jordan Frame Metric:* The transformation (4.3.3) can be used to eliminate all of the terms involving derivatives in the Jordan frame metric (4.2.4), which are parameterized by the coefficients $\beta_1, \ldots, \beta_6$. This changes the coefficients of the terms in the action that depend linearly on the stress energy tensor, namely $b_1, \ldots, b_6$. As discussed in Appendix D, these terms involving the stress-energy

[7]We could also consider a twelfth redefinition given by $g_{\alpha\beta} = \hat{g}_{\alpha\beta}(1 - 2\epsilon\sigma_{12}')$, $\phi = \psi + \epsilon\sigma_{12} - \epsilon m_P^2 \sigma_{12}' \hat{R}/U'$. However this redefinition is not independent of the first eleven; the same effect can be achieved by choosing $\sigma_1 = -e^{2\alpha}\sigma_{12}'/U'$, $\sigma_3 = -\sigma_{12}$, $\sigma_7 = \sigma_{12}'/U'$, $\sigma_{10} = e^{2\alpha}\alpha'\sigma_{12}'/(2U')$.

Step		1	2	3	3	4	5	6	6	7	Final
	Transformation	$\tilde{\beta}_j$	σ_4, σ_5	σ_9	σ_{10}, σ_{11}	σ_6, σ_7	σ_2, σ_3	σ_8	σ_1	$\tilde{\alpha}$	
Coeff.	Term in Action										
a_1	$(\nabla\phi)^4$			$\star$		$\star$		$\star$			$\checkmark$
a_2	$\Box\phi(\nabla\phi)^2$			$\star$		$\star$	$\to 0$				$\circ$
a_3 †	$(\Box\phi)^2$						$\to 0$				
b_1	$T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$	$\star$			$\star$			$\to 0$			$\circ$
b_2	$T(\nabla\phi)^2$	$\star$			$\star$	$\star$	$\star$				$\checkmark$
b_3 †	$T\Box\phi$	$\star$				$\star$	$\star$		$\to 0$		
b_4 †	$T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$	$\star$		$\to 0$							
b_5 †	$R^{\mu\nu}T_{\mu\nu}$	$\star$	$\star$		$\to 0$						
b_6 †	RT	$\star$	$\star$		$\to 0$						
b_7	T		$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\star$	$\to 0$	
c_1	$G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$		$\star$	$\star$				$\star$			$\checkmark$
c_2 †	$R(\nabla\phi)^2$		$\star$			$\to 0$					
c_3 †	$R\Box\phi$					$\to 0$					
d_1 †	R^2		$\to 0$								
d_2 †	$R^{\mu\nu}R_{\mu\nu}$		$\to 0$								
d_3	Gauss-Bonnet										$\checkmark$
d_4	Chern-Simons										$\checkmark$
e_1	$T^{\mu\nu}T_{\mu\nu}$				$\star$						$\checkmark$
e_2	T^2				$\star$				$\star$		$\checkmark$
	U (potential)		$\star$	$\star$		$\star$	$\star$	$\star$			$\checkmark$
Coeff.	Term in $\bar{g}_{\mu\nu}$										
β_1	$\nabla_\mu\phi\nabla_\nu\phi$	$\to 0$									
β_2	$(\nabla\phi)^2 g_{\mu\nu}$	$\to 0$									
β_3 †	$\Box\phi g_{\mu\nu}$	$\to 0$									
β_4 †	$\nabla_\mu\nabla_\nu\phi$	$\to 0$									
β_5 †	$R_{\mu\nu}$	$\to 0$									
β_6 †	$Rg_{\mu\nu}$	$\to 0$									
	α (conf. factor)									$\star$	$\checkmark$

Table 2: This table shows the progression of manipulations we make in this section. The second column on the left lists the various terms in the action (4.2.3), or in the Jordan-frame metric (4.2.4). The first column lists the corresponding coefficients; daggers † indicate higher derivative terms. The numbers in the first row along the top refer to the numbered steps in the derivation in Section 4.4.I. The second row shows which transformation functions are used in each step. In the table, a star $\star$ indicates that the corresponding row's term receives a contribution from the corresponding column's reduction process, while $\to 0$ indicates that the term has been eliminated. The check marks $\checkmark$ in the last column indicate the remaining terms that are non-zero in the final action (4.4.5). Finally, circles $\circ$ in the last column indicate terms that are nonzero in alternative forms of the final action obtained using the transformations (4.3.11) or (4.3.23), as discussed in Section 4.4.II.

tensor look like they might violate the weak equivalence principle, but in fact they do not.

2. *Elimination of Higher Derivative, Quadratic in Curvature Terms:* We next consider the terms in the action that are quadratic functions of curvature, whose coefficients are d_1, d_2, d_3 and d_4. The Chern-Simons term (d_4) and the Gauss-Bonnet term (d_3) give rise to well behaved equations of motion (in the sense that they not increase the number of degrees of freedom), so we do not attempt to eliminate these terms. By contrast, the terms proportional to the squares of the Ricci scalar and Ricci tensor, parameterized by d_1 and d_2, do increase the number of degrees of freedom. We can eliminate these terms by using the transformations (4.3.15) and (4.3.17), with parameters chosen to be

$$\sigma_4 = \int d\phi \, \frac{U}{m_P^4}(d_1 + d_2/2), \qquad \sigma_5 = -\int d\phi \, \frac{U}{m_P^4} d_2. \tag{4.4.1}$$

These transformations will then modify the coefficients b_5, b_6, b_7, c_1 and c_2, as well as the potential U (see Table 1).

3. *Elimination of some of the Linear Stress-Energy Terms:* We next turn to terms which depend linearly on the stress-energy tensor, parameterized by $b_1, \ldots, b_6$. First, we can eliminate the term $b_4 T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$ by using the transformation (4.3.25) with $\sigma_9 = -e^{-2\alpha}b_4$. This gives rise to changes in the coefficients a_1, a_2, b_7, c_1 as well as to the potential U. Second, we can eliminate the terms parameterized by b_5 and b_6 by using the transformations (4.3.27) and (4.3.29) with the parameters $\sigma_{10} = -(b_6 + b_5/2)/m_P^2$, $\sigma_{11} = b_5/m_P^2$. This changes the coefficients b_1, b_2, b_7, e_1 and e_2.

4. *Elimination of Kinetic Coupling of the Scalar to Curvature:* We next focus on the terms which kinetically couple the scalar field to gravity, namely $G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, $R(\nabla\phi)^2$ and $R\Box\phi$. The first of these does not produce higher derivative terms in the equation of motion, so we focus on the remaining two terms, which are parameterized by c_2 and c_3. These terms can be eliminated using the transformations (4.3.19) and (4.3.21), with the

parameters chosen to be

$$\sigma_6 = \int d\phi \, \frac{U}{m_P^2} c_2, \qquad \sigma_7 = -\frac{c_3}{m_P^2}. \tag{4.4.2}$$

These transformations then give rise to changes in the coefficients a_1, a_2, b_2, b_3, b_7 as well as to the potential U.

5. *Elimination of some of the Corrections to Scalar Field Kinetic Term:* Our action includes three corrections to the scalar kinetic term, parameterized by a_1, a_2 and a_3. Of these, only term a_3 contributes higher-order derivatives to the equations of motion. We eliminate this term, and also the term a_2, by using the transformations (4.3.9) and (4.3.11) with

$$\sigma_2 = -a_3, \qquad \sigma_3 = \int d\phi \, U' a_2. \tag{4.4.3}$$

This gives rise to corrections to the coefficients b_2, b_3 and b_7 and to the potential U.

6. *Elimination of some Kinetic Couplings of the Scalar to Stress-Energy:* We next turn to the term $b_1 T^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$. We can eliminate this using the transformation (4.3.23) with the parameter choice

$$\sigma_8 = \int d\phi \, b_1 U e^{-2\alpha}. \tag{4.4.4}$$

This gives rise to changes in the coefficients a_1, b_7, c_1 and U, from Table 1. We can also eliminate the term $b_3 T \Box \phi$ by using the transformation (4.3.7) with $\sigma_1 = -b_3$. This changes the coefficients e_2 and b_7.

7. *Elimination of Trace of Stress-Energy Tensor Term:* The last step is to re-express the term $b_7 T$ in terms of an $O(\epsilon)$ correction to the conformal factor e^α by using the transformation (4.3.4) with $\tilde{\alpha} = -2e^{-2\alpha} b_7$.

II Canonical Form of Action and Discussion

Applying the parameter specializations derived above to the action (4.2.1) we arrive at our final result:

$$
\begin{aligned}
S = &\int d^4x\sqrt{-g}\left\{\frac{m_P^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi)\right\} + S_{\rm m}[e^{\alpha(\phi)}g_{\alpha\beta}, \psi_{\rm m}] \\
&+ \epsilon \int d^4x\sqrt{-g}\Big\{a_1(\nabla\phi)^4 + b_2 T(\nabla\phi)^2 + c_1 G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + e_1 T^{\mu\nu}T_{\mu\nu} \\
&+ d_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) + d_4\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta} + e_2 T^2\Big\}.
\end{aligned} \tag{4.4.5}
$$

This action contains nine free functions of ϕ: $U, \alpha, a_1, b_2, c_1, d_3, d_4, e_1, e_2$. The corresponding equations of motion do not contain any higher derivative terms and are presented in Appendix G.

Our final result (4.4.5) can be re-expressed in a number of equivalent forms:

- First, the term $b_2 T(\nabla\phi)^2$ in the action can be eliminated in favor of a term proportional to $e^{2\alpha}\beta_2(\nabla\phi)^2 g_{\mu\nu}$ in the Jordan frame metric (4.2.4) using the transformation (4.3.3). As discussed in Appendix D the latter representation makes explicit that the weak equivalence principle is satisfied.

- The term $b_2 T(\nabla\phi)^2$ could also be eliminated in favor of a term $a_2\Box\phi(\nabla\phi)^2$, using the transformation (4.3.11) parameterized by σ_3, as long as $\alpha' \neq 0$[8]. The dynamics of a scalar quintessence field with kinetic terms of the latter type have recently been explored in detail in Ref. [97], who called the mixing of the scalar and metric kinetic terms in the equations of motion "kinetic braiding". The representation of this term as $a_2\Box\phi(\nabla\phi)^2$ has some advantages for cosmological analyses: in this representation the dynamics of the term are confined to the scalar sector, while in the b_2 representation they are coupled to matter.

[8] More precisely the criterion is that the zeroth-order term in the expansion in α' in powers of ϵ is nonzero. A nonzero α' that is proportional to ϵ would be insufficient to allow this transformation.

- The term $a_1(\nabla\phi)^4$ can be eliminated in favor of a term $b_1 T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, using the transformation (4.3.23) parameterized by σ_8.

- Alternatively, the term $c_1 G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ can be eliminated in favor of a term $b_1 T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$, using the transformation (4.3.23) parameterized by σ_8. Our result in this representation agrees with that of Weinberg [83] when all the matter terms are dropped. The c_1 representation has the advantage over the b_1 representation that the corrections are confined to the scalar sector and do not involve matter. The term $c_1 G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$ has interesting effects: it can give rise to a self-tuning cosmology as well as potentially support a Vainshtein screening mechanism [98].

- As discussed in Appendix D, it is possible to eliminate all the stress-energy terms from the action by applying field redefinitions. This yields a form of the theory in which the weak equivalence principle is manifest. However, the resulting action contains higher derivative terms, unlike all the representations discussed so far in this subsection. As discussed in the introduction and in Appendix E, to define the theory when higher derivative terms are present we use the reduction of order technique applied to the equations of motion.

- Finally, the result can be cast in the Jordan conformal frame by doing a conformal transformation, followed by some field redefinitions to simplify the answer. The result is similar in form to the Einstein frame action (4.4.5):

$$
\begin{aligned}
S[\tilde{g}_{\alpha\beta},\tilde{\phi},\psi_{\rm m}] = & \int d^4x\sqrt{-\tilde{g}}\left[\frac{1}{2}m_P^2 e^{-\alpha}\tilde{R} - \frac{1}{2}(\tilde{\nabla}\tilde{\phi})^2 - \tilde{U}(\tilde{\phi})\right] + S_{\rm m}[\tilde{g}_{\alpha\beta},\psi_{\rm m}] \\
& + \epsilon\int d^4x\sqrt{-g}\Big\{\tilde{a}_1(\tilde{\nabla}\tilde{\phi})^4 + \tilde{b}_2 T(\tilde{\nabla}\tilde{\phi})^2 + \tilde{c}_1\tilde{G}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} \\
& + \tilde{d}_3\left(\tilde{R}^2 - 4\tilde{R}^{\mu\nu}\tilde{R}_{\mu\nu} + \tilde{R}_{\mu\nu\sigma\rho}\tilde{R}^{\mu\nu\sigma\rho}\right) + \tilde{d}_4\epsilon^{\mu\nu\lambda\rho}\tilde{C}_{\mu\nu}{}^{\alpha\beta}\tilde{C}_{\lambda\rho\alpha\beta} \\
& + \tilde{e}_1 T^{\mu\nu}T_{\mu\nu} + \tilde{e}_2 T^2\Big\}.
\end{aligned}
\tag{4.4.6}
$$

Here $\tilde{g}_{\mu\nu} = e^{\alpha}g_{\mu\nu}$ and the field $\tilde{\phi}$ is a function of ϕ, where the function is chosen to give canonical normalization for $\tilde{\phi}$ in the Jordan frame action (4.4.6). All of the functions

$\tilde{U}$, $\tilde{a}_1$, etc. in this action differ from the corresponding functions in the Einstein frame representation (4.4.5), but can in principle be computed in terms of them. The Jordan frame result (4.4.6) can also be cast in a number of different forms using linearized field redefinitions, just as for the Einstein frame result (4.4.5). Note that the stress energy tensor we use is the same in both frames, and is defined in the Preface. The result (4.4.6) matches that found by Park *et al.* [86] (up to some minor adjustments, see Appendix F).

We note that the Chern-Simons term (d_4) gives rise to third-order derivatives in the equations of motion [see Eqs. (G.4) and (G.5) below]. However, with the choice of foliation[9] given by surfaces of constant ϕ, there are no third-order time derivatives, and so the Chern-Simons term is not a higher-derivative term according to our definition (see the discussion in Section 4.1.II above), and is not subject to the Ostrogradski instability. For further discussion of the Chern-Simons term in gravitational theories, see, e.g., Ref. [99]. As a parity-violating term, this term modifies the propagation speed of different polarizations of gravitons.

In the above derivation, we eliminated higher derivative terms using field redefinitions. As discussed in Appendix E, an alternative but equivalent procedure is to derive a form of the action which explicitly exhibits the extra degrees of freedom associated with the higher derivative terms, and then integrate out those degrees of freedom at tree level. This is shown explicitly for higher derivatives of the scalar field in Appendix E, and can also be shown explicitly for the terms d_1 and d_2 involving higher derivatives of the metric. A third, equivalent method is to perform a reduction of order procedure at the level of the equations of motion, as discussed in the introduction and in Appendix E.

The above derivation confirms the general argument made in the introduction that it should not be necessary to include higher derivative terms in the action. This is because the new terms that are generated when one eliminates the higher derivative terms should already be included in the derivative expansion. In the above derivation, if we eliminate

[9]This choice requires the assumption that $\nabla\phi$ is timelike everywhere, which will be true in cosmological applications when perturbations are sufficiently small.

from the start the higher derivative terms $(a_3, b_3, b_4, b_5, b_6, c_2, c_3, d_1, d_2)$, then we must also forbid all transformations that generate these terms, which includes all the transformations we have considered except Eqs. (4.3.3), (4.3.4), (4.3.11) and (4.3.23). The above derivation gets modified by dropping steps 2, 3, and 4, the portion of step 5 that sets a_3 to zero, and the portion of step 6 that sets b_3 to zero. The final result (4.4.5) is unchanged.

In a similar vein, the correct result can also be obtained by omitting from the initial action all terms involving the stress energy tensor, that is, the terms parameterized by b_1, $\ldots$, b_7 and e_1, e_2. If one follows all the steps of the derivation in Table 2, the same final result is obtained, and all the final coefficients are nonzero in general. This occurs because all the terms involving the stress energy tensor have alternative representations not involving it (although they do involve higher derivatives). Thus, from this point of view, it is not necessary to include in the action stress-energy terms.

However, it is *not* possible to do without both the higher derivative terms and the stress-energy terms. Suppose we throw out at the start all the higher derivative terms in both the action (4.2.1) and Jordan frame metric (4.2.4), and in addition omit all the stress-energy terms in the action. This would yield a version of the action (4.2.1) involving only the terms $a_1, a_2, \beta_1, \beta_2, c_1, d_3$ and d_4. Using the transformation (4.3.2) the terms β_1 and β_2 can be exchanged for b_1 and b_2, and the terms a_2 and b_1 can then be eliminated using the transformations parameterized by σ_3 and σ_8. This yields our final action (4.4.5) but without the terms e_1 and e_2, which in general arise from intergrating out heavy fields which are gravitationally coupled. Therefore, for a fully general theory, one can choose to eliminate higher derivative terms, or stress-energy terms, but not both.

III Extension to N scalar fields: Qui-N-tessence

The preceding analysis can be generalized straightforwardly to the case of N scalar fields, which we call "qui-N-tessence", an analog of multifield inflation [100, 81]. The zeroth-order action (4.2.2) gets replaced by a general nonlinear sigma model:

$$S_0 = \int d^4x\sqrt{-g}\left[\frac{m_P^2}{2}R - \frac{1}{2}q_{AB}(\phi^C)\nabla_\nu\phi^A\nabla_\mu\phi^B g^{\mu\nu} - U(\phi^C)\right], \tag{4.4.7}$$

where $\phi^A = (\phi^1, \ldots, \phi^N)$ are the N scalar fields and q_{AB} is a metric on the target space. In the remainder of the action, functions of ϕ are replaced by functions of ϕ^A. The first three terms of the second line of Eq. (4.4.5) are replaced by

$$\begin{aligned}&a_{1\,ABCD}\nabla_\mu\phi^A\nabla_\nu\phi^B\nabla_\lambda\phi^C\nabla_\sigma\phi^D g^{\mu\nu}g^{\lambda\sigma} + a_{2\,ABC}\nabla_\mu\nabla_\lambda\phi^A\nabla_\sigma\phi^B\nabla_\lambda\phi^C g^{\mu\nu}g^{\lambda\sigma}\\ &+ c_{1\,AB}G^{\mu\nu}\nabla_\mu\phi^A\nabla_\nu\phi^B.\end{aligned} \tag{4.4.8}$$

Thus the coefficients a_1, a_2 and c_1 become tensors on the target space of the indicated orders. Note that we must use the representation involving the coefficients $a_{1\,ABCD}, a_{2\,ABC}$ and not $b_{1\,AB}, b_{2\,AB}$ (we assume $\alpha_{,A} \neq 0$) since the latter are less general; the equivalence between the different representations discussed in Section 4.4.II does not generalize to the N field case. When $N \geq 4$ one could also add a term

$$a_{4\,ABCD}\nabla_\mu\phi^A\nabla_\nu\phi^B\nabla_\lambda\phi^C\nabla_\sigma\phi^D\epsilon^{\mu\nu\lambda\sigma}, \tag{4.4.9}$$

where $a_{4\,ABCD}$ is an arbitrary 4-form on the target space.

4.5 Order of Magnitude Estimates and Domain of Validity

In the previous sections, we started from the standard quintessence model with a matter coupling (4.2.2), and added arbitrary corrections involving the scalar field and metric in a derivative expansion up to four derivatives. We then exploited the field-redefinition freedom to eliminate all terms that give rise to additional degrees of freedom ("higher derivative terms"), and to reduce the set of operators in the action to the canonical and unique set given in our final action (4.4.5).

We now turn to estimating the scaling of the coefficients in the final action using effective field theory. We will then use these estimates to determine the domain of validity of the theory.

I Derivation of Scaling of Coefficients

We start by recalling the scenario of pseudo-Nambu-Goldstone Bosons [94, 95] discussed in the introduction that may give rise to the zeroth-order action (4.2.2). Suppose that at some high-energy scale M_* we spontaneously break a set of continuous global symmetries and thereby generate N massless Goldstone bosons $\phi^A = (\phi^1, \ldots, \phi^N)$. The theory then has N residual continuous symmetries. If we now suppose that these residual symmetries are explicitly broken at some much lower energy scale Λ, then a potential is generated that scales as $\Lambda^4 V(\phi^A/M_*)$, for some function V which is of order unity. In particular the mass of the pNGB fields scale as Λ^2/M_* and can be very light. For example, in axion models $M_* \sim 10^{12}$ GeV and $\Lambda \sim \Lambda_{\rm QCD} \sim 100$ MeV, giving an axion mass of order 10^{-5} eV.

The leading order action for the pNGB fields coupled to gravity at low energies will be that of a nonlinear sigma model,

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}m_P^2 R - \frac{1}{2}q_{AB}(\phi^C/M_*)\nabla_\nu\phi^A\nabla_\mu\phi^B g^{\mu\nu} - \Lambda^4 V(\phi^C/M_*)\right], \tag{4.5.1}$$

where q_{AB} is a metric on the target space which admits N Killing vectors (the residual symmetries). In the special case where q_{AB} is flat, these residual symmetries are shift symmetries $\phi^A \to \phi^A +$ constant. We now assume that these fields drive the cosmic acceleration, and in addition we assume that the kinetic and potential terms are of the same order, that is, we assume that slow roll parameters are only modestly small. It then follows from the action (4.5.1) that the scales of spontaneous and explicit symmetry breaking M_* and Λ must be of order[10]

$$M_* \sim m_P, \qquad \Lambda \sim \sqrt{H_0 m_P}, \tag{4.5.2}$$

where H_0 is the Hubble parameter, so that the quintessence fields have mass $\sim H_0$ and energy density $\sim m_P^2 H_0^2$. Defining the dimensionless fields $\varphi^A = \phi^A / m_P$ allows us to rewrite the action as

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} m_P^2 R - \frac{1}{2} m_P^2 q_{AB}(\varphi^C) \nabla_\nu \varphi^A \nabla_\mu \varphi^B g^{\mu\nu} - m_P^2 H_0^2 V(\varphi^C) \right]. \tag{4.5.3}$$

Consider now the stability of the theory (4.5.3) under loop corrections. The story is exactly the same here as in inflationary models [101, 94] (aside from couplings to matter, see below). Computing loop corrections starting from the action (4.5.3) does not lead to large corrections $\delta m \gg H_0$ to the mass of the quintessence fields, because in the limit where the explicit symmetry breaking scale $\Lambda = \sqrt{m_P H_0}$ goes to zero, the theory possesses exact symmetries which must be respected by the loop corrections. Hence the loop corrections to the potential must scale proportionally to $H_0^2 m_P^2$, as for the original potential. Thus the smallness of the mass of the quintessence field is natural in the sense of 't Hooft. However, this is not the entire story, since the form (4.5.3) of the low-energy theory imposes non trivial constraints on the physics at high energies, which must respect the residual symmetries. Indeed in general there is no guarantee that there exists a consistent high-energy theory with the low-energy limit (4.5.3). This question is beyond the scope of this work: we shall simply assume that a consistent UV theory can be found. See Ref. [102] for an example of an attempt to address this issue.

So far in the discussion we have neglected coupling to matter. If we assume the validity of the weak equivalence principle, the general leading order coupling of ϕ^C to matter will be of the form of a scalar-tensor theory, given by adding to the action (4.5.3) the term

$$S_{\rm m}\left[e^{\alpha(\phi^c/M_*)} g_{\mu\nu}, \psi_{\rm m}\right] = S_{\rm m}\left[e^{\alpha(\varphi^c)} g_{\mu\nu}, \psi_{\rm m}\right], \tag{4.5.4}$$

[10]The need to use the Hubble scale today in the symmetry breaking scale Λ is associated with the coincidence problem.

for some function α.

We now suppose that one or more of the pNGB fields has a mass $\sim M$ which is parametrically larger than H_0, and we integrate out these heavier fields, following the similar analysis of inflationary models by Burgess, Lee and Trott [84]. Integrating out the heavier fields gives rise to modifications to the target space metric and potential for the remaining light fields [that do not change the scalings shown in Eq. (4.5.3)], and also a set of correction terms to the leading order action (4.5.3). The leading, tree-level correction terms can be obtained simply by solving the classical equations of motion for the heavy fields in an adiabatic approximation and substituting back into the action. One finds that the induced correction terms have the form[11]

$$M^2 m_P^2 \sum_n \frac{c_n}{M^d} \mathcal{O}_n, \tag{4.5.5}$$

where the sum is over operators $\mathcal{O}_n$ involving d derivatives acting on k powers of the dimensionless fields φ and/or $g_{\mu\nu}$, and the coefficients c_n are of order unity (see Appendix H for details). In other words, each additional derivative is suppressed by a power of the mass M of the fields that have been integrated out (which we can think of as a cutoff scale), and the overall prefactor is such that the normal kinetic terms would be reproduced for the case $k = d = 2$.

Note that the rule (4.5.5) for how the coefficients of additional corrections to the action depend on the cutoff scale M differs from the usual rule of effective field theory, where an operator of dimension $D + 4$ has a coefficient $\sim M^{-D}$. The rule (4.5.5) instead gives a coefficient $\sim M^{-(d-2)} m_P^{-(k-2)}$, where d is the number of derivatives in the operator and k is the number of powers of (canonically normalized) fields, related to D by $D = d + k - 4$. The difference between the two rules arises from the fact that we are making nontrivial assumptions about the physics above the scale M, specifically that it is described by an

[11]These are the terms involving just the scalar field and metric. One also finds correction terms involving the matter stress energy tensor as long as $\alpha' \neq 0$, of the form indicated in Table 3.

Coefficient	Term in Action	Scaling
a_1	$(\nabla\phi)^4$	$\sim 1/(m_P^2 M^2)$
a_2	$\Box\phi(\nabla\phi)^2$	$\sim 1/(m_P M^2)$
a_3 †	$(\Box\phi)^2$	
b_1	$T^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$	$\sim 1/(m_P^2 M^2)$
b_2	$T(\nabla\phi)^2$	$\sim 1/(m_P^2 M^2)$
b_3 †	$T\Box\phi$	
b_4 †	$T^{\mu\nu}\nabla_\mu\nabla_\nu\phi$	
b_5 †	$R^{\mu\nu}T_{\mu\nu}$	
b_6 †	RT	
b_7	T	
c_1	$G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$	$\sim 1/M^2$
c_2 †	$R(\nabla\phi)^2$	
c_3 †	$R\Box\phi$	
d_1 †	R^2	
d_2 †	$R^{\mu\nu}R_{\mu\nu}$	
d_3	Gauss-Bonnet	$\sim m_P^2/M^2$
d_4	Chern-Simons	$\sim m_P^2/M^2$
e_1	$T^{\mu\nu}T_{\mu\nu}$	$\sim 1/(m_P^2 M^2)$
e_2	T^2	$\sim 1/(m_P^2 M^2)$

Table 3: This table gives the scalings of the various coefficients. The first column lists the coefficients, and the second column lists the corresponding terms in the action (4.2.3). Daggers in the first column indicate higher derivative terms. The third column gives our estimate of the scale of the coefficients, under the assumptions discussed in the text, for those coefficients that are nonzero in our final action (4.4.5), or in versions of that action obtained using the field redefinitions (4.3.11) or (4.3.23). The quantity M is the mass of the lightest field that is integrated out to produce our final action. In all cases, these scales for the coefficients correspond to fractional corrections to the leading order dynamics of order $\sim H_0^2/M^2$.

action of the pNGB form (4.5.3) [12]. If we were to allow arbitrary physics at energies above the scale M, then the coefficients would scale according to the standard rule.

We now specialize to the case of a single light field. The correction terms (4.5.5) have the form of a double power series, in number of derivatives and in powers of the fields. If we fix the number of derivatives and associated index structure, we can sum over all operators that differ only by powers of φ to obtain operators with prefactors that are functions of φ,

$$f(\varphi) = \sum c_k \varphi^k \tag{4.5.6}$$

with coefficients of order unity. We now write out all the resulting terms to leading order in $1/M^2$, imposing general covariance. The result is the theory (4.4.5) discussed in the previous section[13], but with additional information about the coefficients a_1, b_1 etc. Specifically we find that

$$a_1(\phi) = \frac{1}{m_P^2 M^2} \hat{a}_1(\phi/m_P), \tag{4.5.7}$$

where the function $\hat{a}_1$ is of order unity, i.e., the coefficients in its Taylor expansion are independent of m_P and M. The corresponding prefactors or overall scaling for the other coefficients are listed in Table 3.

Finally, we note that, as is well-known, Solar System tests of general relativity strongly constrain the coupling of ϕ to the matter sector[14]. If we define the dimensionless parameter

[12] More general interactions which are not of the form (4.5.3) can modify the scaling rule (4.5.5), even if they respect the residual (shift) symmetries. For example consider a scalar field ψ of mass m which couples to ϕ via a term $\psi(\nabla\phi)^2/m_*$ for some mass scale m_*. Integrating out this field gives a correction to the ϕ action $\sim (\nabla\phi)^4/(m^2 m_*^2)$ (see Appendix H). To keep such terms from invalidating the scaling rule we need to assume that $m m_* \gtrsim M m_P$, i.e. that any such fields are either sufficiently massive or sufficiently weakly coupled to the pNGB fields.

[13] The parity-violating Chern-Simons term is not generated in this way, since the fields we are integrating out do not violate parity. To obtain the Chern-Simons term with the scaling indicated in Table 3 would require integrating out some parity violating fields at the scale M which approximately respect the residual (shift) symmetries.

[14] Strictly speaking, Solar System tests lie outside the domain of validity of our effective field theory unless $M^{-1} \lesssim 1\,\mathrm{A.U.}$, which is very small compared to H_0^{-1}; see Section 4.5.II above.

$\lambda = m_P|\alpha'(\phi_0)|$, where ϕ_0 is the present day cosmological background value of ϕ, then the Solar System constraint is[15] $\lambda \lesssim 10^{-2}$ [77]. In addition the coupling of the scalar to the visible sector will generically give rise to large corrections to the quintessence potential via loop corrections [104, 105, 106, 107, 108]. For a fermion of mass m_f, the correction δm to the mass of the quintessence field will be of order

$$\frac{\delta m}{H_0} \sim \lambda \left(\frac{m_f^2}{H_0 m_P} \right). \tag{4.5.8}$$

If $\lambda \sim 1$ and $m_f \gg \sqrt{m_P H_0} \sim 10^{-3}$ eV, then $\delta m \gg H_0$, which is inconsistent if the quintessence field is to drive cosmic acceleration. This is a well-known naturalness problem for matter couplings in quintessence models, and it motivates setting[16] $\alpha = 0$.

II Domain of Validity of the Effective Field Theory

We now estimate the domain of validity for the theory (4.4.5) with the scalings given by Table 3, by requiring that the terms with higher derivatives be small compared to terms with fewer derivatives. If E is the energy involved in a given process, or equivalently E^{-1} is the corresponding time-scale or length-scale, then successive terms in the derivative expansion are suppressed by the ratio E/M, which yields the standard condition

$$E \ll M \tag{4.5.9}$$

for the domain of validity. As discussed in the introduction, M must be somewhat larger than H_0 in order to describe the background cosmology and observable perturbation modes. However if M is significantly larger than H_0 then the corrections due to the higher-order terms in Eq. (4.4.5) become negligible, and the theory reduces to a standard quintessence model with some matter coupling. Therefore, the interesting regime is when M is perhaps

[15]This constraint can be evaded in models where nonlinear effects in ϕ are important in the Solar System, such as Chameleon [103] and Galileon models [29, 30, 31].

[16]More precisely the condition is $\alpha' = 0$, i.e., $\alpha =$ constant, but the constant can be absorbed by a rescaling of all the dimensionful parameters in the matter action.

just one or two orders of magnitude larger than H_0, as indicated in Fig. 9. In particular, when the scale M is in this interesting regime, the theory is unable to describe gravitational effects in the Solar System and binary pulsars, which is a shortcoming of the effective field theory approach used here.

Consider now the background cosmological solution. The theory (4.2.1) to zeroth-order in ϵ (or equivalently $1/M^2$) has the equations of motion

$$m_P^2 G_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi - \frac{1}{2}(\nabla\phi)^2 g_{\alpha\beta} - U(\phi)g_{\alpha\beta} + e^{2\alpha(\phi)}T_{\alpha\beta}, \tag{4.5.10a}$$

$$\Box\phi = U'(\phi) - \frac{1}{2}\alpha' e^{2\alpha}T. \tag{4.5.10b}$$

For each of these two equations we assume that all of the terms are of the same order. For the matter terms this is this is a reasonable approximation, since $\Omega_\Lambda \sim 0.7$ and $\Omega_{\rm matter} \sim 0.3$. If the scalar potential term dominates over the kinetic term, then the following estimates need to be modified by including factors of slow roll parameters; we ignore these factors here since we expect them to be only modestly small. Similarly, our estimates assume that $m_P\alpha'$ is of order unity; some changes would be required if this quantity were very small. From these assumptions, and ignoring $O(1)$ functions of the scalar field, we have

$$m_P^2 R \sim (\nabla\phi)^2 \sim U \sim T \sim m_P\Box\phi \sim m_P U'(\phi) \sim H_0^2 m_P^2. \tag{4.5.11}$$

Inserting these estimates into the action (4.4.5) and using the scalings given in Table 3, we find that for each of the correction terms in the action, the fractional corrections to the leading order cosmological dynamics scale as H_0^2/M^2. The corrections therefore are of order unity at $M \sim H_0$, as we would expect, since at this scale the heavy fields which we have integrated out have the same mass scale as the light fields, and would be expected to give rise to $O(1)$ corrections to the dynamics. This gives a useful consistency check of the calculations underlying Table 3 discussed in the previous subsection.

In addition to the standard constraint (4.5.9), there are other constraints on the domain of validity which we now discuss. We focus attention on cosmological perturbations, for which $\phi(t, \mathbf{x}) = \phi_0(t) + \delta\phi(t, \mathbf{x})$, and consider the conditions under which the dynamics of the perturbation $\delta\phi$ can be described by the effective theory. Consider localized wavepacket modes $\delta\phi$, where the size of the wavepacket is of the same order as the wavelength, both $\sim E^{-1}$. For such modes we can characterize perturbations in terms of two parameters, the energy E and the number of quanta or mode occupation number N. The total energy of the wavepacket will be of order $NE \sim \int d^3x(\nabla\delta\phi)^2 \sim E^{-3}(E\delta\phi)^2$ which gives the estimate

$$\delta\phi \sim \sqrt{N}E. \tag{4.5.12}$$

The fractional density perturbation due to the wavepacket is of order

$$\frac{\delta\rho}{\rho} \sim \frac{(\nabla\delta\phi)^2}{H_0^2 m_P^2} \sim \frac{NE^4}{H_0^2 m_P^2}. \tag{4.5.13}$$

We now demand that the term $a_1(\nabla\delta\phi)^4$ in the action[17] be small compared to the leading order term $(\nabla\delta\phi)^2$. Using the scaling $a_1 \sim 1/(m_P^2 M^2)$ from Table 3 and combining with the estimate (4.5.13) of the fractional density perturbation then gives the constraint[18]

$$\frac{\delta\rho}{\rho} \ll \frac{M^2}{H_0^2}. \tag{4.5.14}$$

Thus, the theory can describe perturbations in the nonlinear regime, but the perturbations can only be modestly nonlinear if M is fairly close to H_0. In terms of the parameters E and N the constraint (4.5.14) is

$$NE^4 \ll M^2 m_P^2. \tag{4.5.15}$$

[17]Here we envisage computing an action for the perturbations by expanding the action (4.4.5) around the background cosmological solution, as in Ref. [84].

[18]In the previous subsection we showed that $a_1(\phi) = \hat{a}_1(\phi/m_P)/(M^2 m_P^2)$, where $\hat{a}_1$ is function for which all the Taylor expansion coefficients are of order unity. It follows that $\hat{a}_1 \sim 1$ for $\phi \sim m_P$. However the estimate (4.5.14) requires the stronger assumption $\hat{a}_1 \lesssim 1$ for $\phi \gg m_P$ which need not be valid. If we instead assume that $\hat{a}_1 \sim (\phi/m_P)^\alpha$ for $\phi \gg m_P$ then the constraint (4.5.15) gets replaced by $N(E/M)^\gamma \ll m_P^2/M^2$, where $\gamma = 2(4+\alpha)/(2+\alpha)$. This modifies the boundary of the domain of validity of the effective field theory shown in Fig. 10 by changing the slope of the tilted portion of the boundary. In the limit $\alpha \to \infty$ this portion of the boundary approaches the green curve $\delta\varphi \sim m_P$.

This gives a nontrivial constraint on the domain of validity of the theory in the regime $E \lesssim M$. The two dimensional parameter space (E, N) is illustrated in Fig. 10, which shows the constraints (4.5.9) and (4.5.14), the curves $\delta\rho/\rho \sim 1$ and $\delta\rho \sim M^2/H_0^2$, as well as the curve where $\delta\phi \sim m_P$.

Another potential constraint on the domain of the validity of the theory (4.4.5) with the scalings given by Table 3 is that the theory should be weakly coupled, i.e. the effects of loop corrections should be small. Using the power counting methods of Ref. [84] one can show that this is indeed true within the domain $H_0 \lesssim E \ll M$ of interest. Strong coupling can arise due to tri-linear couplings, as discussed in Section 2.2 of Ref. [84], but this only occurs for energies far below the Hubble scale H_0, and so is not relevant to cosmological applications of the theory.

We note that there are several well-known theories of cosmic acceleration that are not encompassed by our effective field theory. The form of our expansion requires that the dominant contribution to cosmic acceleration be the leading order scalar terms and not the higher-order terms, and so theories in which other mechanisms provide the acceleration cannot be described in our formalism. One example is provided by k-essence models in which terms in the action like $(\nabla\phi)^4$, $(\nabla\phi)^6$ ... are all equally important. In particular this is true for ghost condensate models [20]. Also there are many cosmic acceleration models that exploit the Vainshtein effect [109, 110, 111] to evade Solar System constraints on light fields with gravitational-strength couplings. The Vainshtein effect relies on nonlinear derivative terms in the scalar field action. Although our class of theories includes models that demonstrate the Vainshtein mechanism, the mechanism only operates outside the domain of validity of our approach, as we require the nonlinear derivative terms to be small. The chameleon mechanism [103, 112], on the other hand, does not require nonlinearities in the derivatives of the scalar field, and thus may be analyzed in our formalism, although the regime in which a screening mechanism would be required to evade fifth force experiments and solar system constraints will be in the regime of validity of our analysis only for large enough values of the cutoff M.

4.6 Discussion

In this chapter, we have investigated effective field theory models of cosmic acceleration involving a metric and a single scalar field. The set of theories we considered consists of a standard quintessence model with matter coupling, together with a general covariant derivative expansion, truncated at four derivatives. We showed that this class of theories can be obtained from a pNGB scenario, where one of the pNGB fields is lighter than all the others, and the heavier fields are integrated out. We showed that in constructing this class of theories, including higher derivative terms in the action, as suggested by Weinberg [83], does not give any increased generality. We also showed that complete generality requires one to include terms in the action that depend on the stress-energy tensor of the matter fields.

We now turn to a discussion of some of the advantages and shortcomings of the approach adopted here to describe models of dark energy. Some of the shortcomings are:

- By construction, our approach excludes theories where nonlinear kinetic terms in the action give an order unity contribution to the dynamics, such as k-essence, ghost condensates etc., since such theories do not arise from the pNGB construction used here, nor does their derivative expansion possess a small parameter. On the other hand, such theories are less natural than the class of theories considered here, from the point of view of loop corrections: they require very nontrivial physics at the scale $\sim H_0$, instead of at the scale $\sim \sqrt{H_0 m_P}$ required in the pNGB approach. The most general class of theories of this kind is that of Horndeski [113], which contains four free functions of ϕ and $(\nabla\phi)^2$ [26], and which is the most general class of theories of a metric and a scalar field for which the equations of motion are second-order. As discussed in the introduction, these theories are included in the alternative, background-dependent approach to effective field theories of quintessence of Creminelli *et al.* [85].

- Our class of theories will be observationally distinguishable from vanilla quintessence theories only if the cutoff M is near the Hubble scale H_0. In this regime, our framework cannot be used to analyze Solar System tests of general relativity, since they are outside the domain of validity of the effective field theory. Also, when the background cosmology is evolved backwards in time it passes outside the domain of validity at fairly low redshifts. (This is not a serious disadvantage since dark energy dominates only at low redshifts.)

- We have restricted attention to theories with a metric and a single scalar field, with the only symmetry being general covariance. Thus, our analysis does not include models with several scalar fields, vector fields etc. In addition, our analysis excludes an interesting class of models that one obtains by imposing that the action be invariant under $\phi \to f(\phi)$, where f is any monotonic function, as such a symmetry cannot be realized with our derivative expansion. This class of models includes Horava-Lifshitz gravity and has the same number of physical degrees of freedom as general relativity [26, 114]. It would be interesting to explore the most general dark energy models of this kind.

Some of the advantages of the approach used here are:

- Our class of theories is generic within the pNGB construction, which itself is a well motivated way to obtain the ultralight fields needed for cosmic acceleration. The theories are fairly simple and it should be straightforward to confront them with observational data.

- Our class of theories allow for a unified treatment of the cosmological background and perturbations, unlike the background-dependent approach of Ref. [85].

Finally, we list some possible directions in which the approach used here could be extended:

- It would be interesting to compute the relation between the nine free functions used in our theories to the free functions of the post-Friedmannian approach to parameterizing dark energy models [26].
- It would be interesting to explore the phenomenology of the various higher-order terms in our action, for the cosmological background evolution and perturbations. Many of the terms have already been explored in detail, see for example Refs. [97, 98].
- Either by using the post-Friedmannian approach, or more directly, it would be useful to compute the current observational constraints on the free functions in the action.
- An interesting open question is the extent to which our final action is generic. That is, is there a class of theories more general than nonlinear sigma model pNGB theories for which our action is obtained by integrating out some of the fields?

Chapter 5

Discussion and Conclusions

Contents

5.1 Combining Results

We now turn to one final calculation, demonstrating the ability of the EFT approach of Chapter 4 to include the braneworld low-energy theory described in Chapters 2 and 3. We start from the low-energy braneworld action (3.3.4), reproduced here.

$$S = \int d^4x\sqrt{-g}\left[\frac{R^{(4)}[g]}{2\kappa_4^2} - \frac{(\nabla a)^2}{2} - \frac{\mu^2}{2}\sinh^2\left(\frac{a}{\mu}\right)\sum_{n=1}^{N-2}\left\{\prod_{m=1}^{n-1}\sin^2(\lambda_m)\right\}(\nabla\lambda_n)^2\right]$$
$$+ {}^TS_m\left[\cosh^2\left(\frac{a}{\mu}\right)g_{ab}, {}^T\phi\right] + \sum_{\substack{n=0\\ n\neq T}}^{N-1} {}^nS_m\left[\sinh^2\left(\frac{a}{\mu}\right)\frac{f_n^2}{B_n'}g_{ab}, {}^n\phi\right] \tag{5.1.1}$$

From the analysis of the Eddington γ parameter, we know that $a/\mu \leq 0.05$. This implies that the only significant radion mode is the field a, and the remaining modes λ_n may be neglected. Furthermore, based on the result that the vast majority of matter must exist on our brane, we can assume that all matter fields live on the central brane. Using these assumptions, we can construct an approximate action

$$S = \int d^4x\sqrt{-g}\left[\frac{R^{(4)}[g]}{2\kappa_4^2} - \frac{(\nabla a)^2}{2}\right] + {}^T S_m\left[\cosh^2\left(\frac{a}{\mu}\right)g_{ab}, {}^T\phi\right]. \qquad (5.1.2)$$

In the EFT approach, we are then left with a simple correspondence. We make the identifications $a \equiv \phi$ and $\exp(\alpha(\phi)) \equiv \cosh^2\left(\frac{a}{\mu}\right)$, while all other functions in the EFT are vanishing. The factor μ is related to the four-dimensional gravitational scale by $\mu = \sqrt{6}/\kappa_4 = \sqrt{6}m_P^{(4)}$, and so the scaling of the conformal factor follows the form predicted by the EFT. While this theory obviously satisfies the requirement that the derivative expansion be valid, the model is unable to explain dark energy, as the scalar field is massless and thus cannot have the correct equation of state.

5.2 Summary of Results

Although our current understanding of gravity provides a remarkably accurate description of observations from terrestrial to cosmological scales, a number of theoretical problems remain unsolved. Chief amongst these is the issue of the accelerated expansion of the universe, attributed to an unknown energy density dubbed dark energy. This dissertation has described two separate approaches to theoretical investigations of dark energy.

In the first approach, described in Chapters 2 and 3, a class of extra-dimensional braneworld models were investigated. These models involved generalizations of the Randall-Sundrum models to include multiple branes in orbifolded and uncompactified configurations, without radion stabilization. The motivation for these generalizations was to investigate whether the addition of further branes on a variety of topological configurations is able to ameliorate the observational constraints that apply to the original RS-I and RS-II models, or lead to interesting new behavior.

A method to construct a four-dimensional low-energy description of such models was described in detail, and applied to the case of N branes in an uncompactified five-dimensional bulk as an example. The low-energy action of such a model was shown to be four-dimensional general relativity coupled to $N-1$ radion fields in a non-linear sigma model, as well as matter fields on the branes with conformal couplings to the radion modes. The requirement that the non-linear sigma model had no ghosts required that negative tension branes could only exist at orbifold fixed points (i.e., orientifolds). The subset of models with this condition have hyperbolic space as the target space. The Eddington PPN γ parameter was calculated, and it was found that it could be consistent with observations for only one brane, the equivalent of the Planck brane in the RS-I model. For essentially the same reasons as in that model, this implies that a potential solution to the hierarchy problem must involve radion stabilization. By comparing the gravitational coupling between matter on different branes, an estimate was made of whether dark matter could reside on another brane in this class of models.

From combining observational constraints on γ and dark-matter to normal-matter gravitational couplings, it was found that dark matter cannot reside wholly on another brane, at least without a radion stabilization mechanism. Qualitatively, models involving more than two branes were physically very similar to two-brane models. A single scalar mode dominated the dynamics of the system, and the effect of extra branes was found to be exponentially suppressed. As such, we found that the inclusion of multiple branes is unable to circumvent observational constraints on the original Randall-Sundrum models.

In the second approach, described in Chapter 3, we took an effective field theory approach to dark energy models. Considering single-field dark energy, we constructed a derivative expansion, where general relativity and normal quintessence with a non-minimal coupling are the leading order terms. This approach allowed us to provide a general description of dark energy models within the regime in which the derivative expansion holds. We constrained our approach by requiring that the perturbative terms did not lead to higher-order derivatives in the equations of motion, which introduce new degrees of freedom (typically ghost modes), and also by imposing the weak equivalence principle. We demonstrated how this construction could be arrived at by integrating out heavy modes in a non-linear sigma model of pseudo-Nambu-Goldstone bosons, and used the properties of this construction to derive the regime of validity of our effective field theory. Furthermore, we motivated the scaling of the different operators based on the pNGB construction. It is hoped that this construction will aid in establishing generic observational constraints on dark energy models. At least one collaboration is currently investigating this possibility.

5.3 Future Prospects

I Theoretical Prospects

The effective field theory approach taken in Chapter 4 is very general in the sense that it captures the leading-order effects of scalar field dark energy models. However, as has been previously noted, the regime of validity of the description is somewhat restrictive. Furthermore, the background behavior of dark energy models is somewhat degenerate, as even for standard minimally coupled quintessence, it is possible to choose a potential to yield any cosmological history $a(t)$. The inclusion of further free functions compounds this degeneracy. Therefore, it is of great interest to identify the perturbative behavior of dark energy models, whose influence on the growth of structure in the late universe will be fundamental in applying observational constraints to the parameter space.

In inflation, a very successful effective field theory of the perturbations in the inflaton field has been constructed by Cheung *et al.* [80], and applied to quintessence models by Creminelli *et al.* [85]. The background evolution of the universe must be specified as an input to the theory, but the theory can handle regimes in which our approach is invalid. A benefit of this approach is that there are fewer free functions present in the description. While this signals a degeneracy among the functions described in this work, it does give hope that the application of observational constraints will be more straightforward.

However, previous work has only considered minimally coupled quintessence fields. Work presented here motivates a number of possible couplings between quintessence fields and matter, and so it will be of use to extend the EFT of inflation work to describe various matter couplings. This work is currently in progress.

II Experimental Prospects

While our theoretical tools for probing dark energy are developing, it is also exciting to see a number of upcoming experiments that are designed to help yield information on the cosmological evolution. Currently underway and due to release data soon, the Planck mission will observe the CMB anisotropies to unprecedented accuracy. This will be of great use in describing cosmological parameters and understanding the spectrum of perturbations that seeded large scale structure. Looking towards the future, Stage IV experiments such as Euclid, the LSST, and WFIRST have been designed to undertake large imaging surveys of the sky, while experiments such as BigBOSS and Euclid will be making spectroscopic measurements of galaxies. The combined data sets from upcoming experiments will hopefully allow us to place stringent constraints on dark energy models, and ascertain whether or not dark energy is dynamical in our universe.

Appendices

Appendix A

Five-Dimensional Ricci Scalars and Exact Equations of Motion

Here we present the dimensionally reduced Ricci scalar and the exact equations of motion for the action (2.5.23). We include the order at which terms appear in terms of our scaling parameter, ϵ.

1 Dimensional Reduction of the Ricci Scalar

The constraint $\det \hat{\gamma} = -1$ may be enforced either at the level of the equations of motion, or by using a Lagrange multiplier.

If the constraint $\det \hat{\gamma} = -1$ is being enforced at the level of the equations of motion, then it is simplest to compute the equations of motion using the metric (2.5.4), and then perform a conformal transformation on the quantities in the equations of motion. In this metric, the five-dimensional Ricci scalar is given by

$$\begin{aligned} {}^{n}R^{(5)} = \epsilon^2 \left({}^{n}R^{(4)} - \frac{2\nabla^a \nabla_a {}^{n}\Phi}{{}^{n}\Phi} \right) - \frac{{}^{n}\gamma^{ab\,n}\gamma_{ab,yy}}{{}^{n}\Phi^2} + {}^{n}\gamma^{ab\,n}\gamma_{ab,y} \frac{{}^{n}\Phi_{,y}}{{}^{n}\Phi^3} \\ - \frac{1}{4{}^{n}\Phi^2} \left({}^{n}\gamma^{ab\,n}\gamma_{ab,y} \right)^2 + \frac{3}{4{}^{n}\Phi^2} {}^{n}\gamma^{ab\,n}\gamma_{ac,y}{}^{n}\gamma^{cd\,n}\gamma_{db,y}, \end{aligned} \tag{A.1}$$

where covariant derivatives and the four-dimensional Ricci scalar are those associated with ${}^{n}\gamma_{ab}$.

For the constraint $\det \hat{\gamma} = -1$ to be enforced at the level of the action, a Lagrange multiplier term must be added to the action

$$\Delta S = \sum_{n=0}^{N} \int_{\mathcal{R}_n} d^5 x_n \, {}^n\lambda(x^a, y) \left(\sqrt{-{}^n\hat{\gamma}} - 1 \right), \tag{A.2}$$

where ${}^n\lambda(x^a, y)$ are the Lagrange multiplier fields. Using the metric (2.6.5), the five-dimensional Ricci scalar is given by

$$\begin{aligned} {}^nR^{(5)} = \epsilon^2 e^{-{}^n\chi} \Bigg({}^nR^{(4)} - 3\nabla^a\nabla_a {}^n\chi - \frac{3}{2}(\nabla^{a\,n}\chi)(\nabla_a {}^n\chi) \\ - \frac{2\nabla^a\nabla_a {}^n\Phi}{{}^n\Phi} - \frac{2(\nabla^{a\,n}\chi)(\nabla_a {}^n\Phi)}{{}^n\Phi} \Bigg) \\ + \frac{1}{{}^n\Phi^2} \left(-\frac{1}{4} {}^n\hat{\gamma}^{ab\,n}\hat{\gamma}_{ac,y} {}^n\hat{\gamma}^{cd\,n}\hat{\gamma}_{db,y} - 5({}^n\chi_{,y})^2 - 4{}^n\chi_{,yy} + 4\frac{{}^n\Phi_{,y}}{{}^n\Phi} {}^n\chi_{,y} \right), \end{aligned} \tag{A.3}$$

where covariant derivatives and the four-dimensional Ricci scalar are those associated with ${}^n\hat{\gamma}_{ab}$. To obtain this form, we use the following two formulae that may be derived from the fact that $\det({}^n\hat{\gamma}_{ab}) = -1$:

$$ {}^n\hat{\gamma}^{ab} \; {}^n\hat{\gamma}_{ab,y} = 0, \tag{A.4}$$

$$ {}^n\hat{\gamma}^{ab} \; {}^n\hat{\gamma}_{ab,yy} = {}^n\hat{\gamma}^{ab} \; {}^n\hat{\gamma}_{bc,y} {}^n\hat{\gamma}^{cd} \; {}^n\hat{\gamma}_{da,y}. \tag{A.5}$$

The complete action (with ϵ scaling and Lagrange multipliers) is given by Eq. (2.7.1).

2 Varying the Action

We use ${}^n\hat{\gamma}_{ab}$ to compute covariant derivatives, the four-dimensional Ricci scalar ${}^nR^{(4)}$ and the four-dimensional Einstein tensor ${}^nG^{(4)}_{ab}$. Indices will also be raised and lowered using this metric.

Varying the action (2.7.1) with respect to ${}^n\Phi$, we find the bulk equation of motion

$$\begin{aligned} \epsilon^2 e^{-{}^n\chi} \left({}^nR^{(4)} - \frac{3}{2}(\nabla^{a\,n}\chi)(\nabla_a {}^n\chi) - 3\nabla^a\nabla_a {}^n\chi \right) - \frac{3}{{}^n\Phi^2} {}^n\chi_{,y}^2 \\ + \frac{1}{4{}^n\Phi^2} {}^n\hat{\gamma}^{ab\,n}\hat{\gamma}_{bc,y} {}^n\hat{\gamma}^{cd\,n}\hat{\gamma}_{da,y} - 2\kappa_5^2 \Lambda_n = 0. \end{aligned} \tag{A.6}$$

From combining the variations with respect to ${}^n\hat{\gamma}_{ab}$ and ${}^n\chi$ (after eliminating the Lagrange multiplier by tracing over the ${}^n\hat{\gamma}_{ab}$ equation of motion, or enforcing $\det\hat{\gamma} = -1$ on the equations of motion), we obtain a traceless tensor equation of motion in the bulk

$$\begin{aligned}
&\frac{1}{2}{}^n\Phi^2\epsilon^2 e^{-{}^n\chi}\Big(4{}^nG^{(4)}_{ab} + {}^n\hat{\gamma}_{ab}{}^nR^{(4)} + 2(\nabla_a{}^n\chi)(\nabla_b{}^n\chi) - \frac{1}{2}{}^n\hat{\gamma}_{ab}(\nabla^{c\,n}\chi)(\nabla_c{}^n\chi) \\
&\qquad - 4\nabla_a\nabla_b{}^n\chi + {}^n\hat{\gamma}_{ab}\nabla^c\nabla_c{}^n\chi\Big) \\
&+ \frac{3}{2}{}^n\Phi\epsilon^2 e^{-{}^n\chi}\left(-4\nabla_a\nabla_b{}^n\Phi + 4(\nabla_{(a}{}^n\Phi)(\nabla_{b)}{}^n\chi)\right. \\
&\qquad \left. + {}^n\hat{\gamma}_{ab}\nabla^c\nabla_c{}^n\Phi - {}^n\hat{\gamma}_{ab}(\nabla_c{}^n\Phi)(\nabla^{c\,n}\chi)\right) \\
&- {}^n\hat{\gamma}_{ab,yy} + \frac{{}^n\Phi_{,y}}{{}^n\Phi}{}^n\hat{\gamma}_{ab,y} - 2{}^n\chi_{,y}{}^n\hat{\gamma}_{ab,y} + {}^n\hat{\gamma}_{ac,y}{}^n\hat{\gamma}^{cd\,n}\hat{\gamma}_{db,y} = 0,
\end{aligned} \tag{A.7}$$

and a scalar equation of motion in the bulk

$$\begin{aligned}
&\frac{1}{2}{}^n\Phi^2\epsilon^2 e^{-{}^n\chi}\Big(-{}^nR^{(4)} + \frac{3}{2}(\nabla^{a\,n}\chi)(\nabla_a{}^n\chi) + 3\nabla^a\nabla_a{}^n\chi \\
&\qquad + \frac{5}{{}^n\Phi}\nabla^a\nabla_a{}^n\Phi + \frac{5}{{}^n\Phi}(\nabla_a{}^n\Phi)(\nabla^{a\,n}\chi)\Big) \\
&+ \frac{1}{4}{}^n\hat{\gamma}^{ab\,n}\hat{\gamma}_{ab,yy} + 3{}^n\chi_{,yy} + 3({}^n\chi_{,y})^2 - 3\frac{{}^n\Phi_{,y}}{{}^n\Phi}{}^n\chi_{,y} + 2{}^n\Phi^2\kappa_5^2\Lambda_n = 0.
\end{aligned} \tag{A.8}$$

These variations also give rise to the boundary conditions on the branes

$$\frac{1}{{}^n\Phi}{}^n\hat{\gamma}_{ab,y} - \frac{1}{{}^{n+1}\Phi}{}^{n+1}\hat{\gamma}_{ab,y} = 2\kappa_5^2\epsilon^2 e^{-{}^n\chi}\left({}^nT_{ab} - {}^n\hat{\gamma}_{ab}\frac{1}{4}{}^n\hat{\gamma}^{cd\,n}T_{cd}\right), \tag{A.9}$$

and

$$-\frac{3{}^n\chi_{,y}}{{}^n\Phi} + \frac{3{}^{n+1}\chi_{,y}}{{}^{n+1}\Phi} + 2\kappa_5^2\sigma_n = \frac{1}{2}\kappa_5^2\epsilon^2 e^{-{}^n\chi}\,{}^n\hat{\gamma}^{ab\,n}T_{ab}. \tag{A.10}$$

The four-dimensional stress energy tensors on the branes (${}^nT_{ab}$) are defined by Eq. (0.0.1), where factors of h are converted into factors of $\hat{\gamma}$ as appropriate.

Note that every factor of ϵ^2 is accompanied by a factor of $\exp(-{}^n\chi)$. Also note that the $O(1)$ terms in these equations are exactly our equations of motion (2.7.3) to (2.7.7).

Appendix B

Results on an Orbifold

In this appendix, we derive the four-dimensional low-energy action of an orbifolded N-brane model, and show that it is equivalent to the uncompactified model up to the rescaling of parameters.

We begin by describing the construction of the model, using the notation established in Chapter 2. Consider a model with N branes on an orbifold. The first and last branes are taken to be at the orbifold fixed points. The other $N-2$ branes lie between these two branes on one half of the orbifold, and are duplicated on the other half by the symmetry. These regions effectively lie on a circle, and so the coordinate describing the extra dimension will be periodic. To calculate the action for this model, we take there to be $2(N-1)$ regions and $2(N-1)$ branes. Let the first brane be labeled by $\mathcal{B}_0$, situated at $y=0$, where y is the coordinate describing the extra dimension. After gauge specializing, let there be $N-1$ branes located at $y=1,2,\ldots,N-1$. In between the branes, we have $N-1$ bulk regions. To account for the orbifolding, continue the extra dimension in the negative y direction, with another $N-1$ branes located at $y=-1,-2,\ldots,-N+1$, with the coordinates y and $-y$ identified. The y coordinate varies from $-N+1$ to $N-1$, and these endpoints are identified under periodic boundary conditions in y. The branes labeled $N-1$ and $-N+1$ are thus the

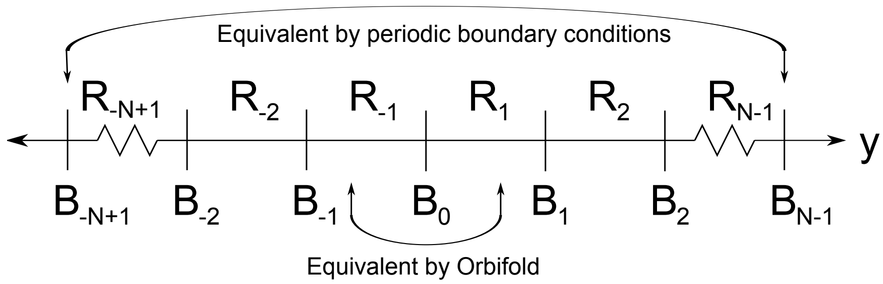

Figure 11: Diagram indicating how the branes are labeled in the construction of the orbifolded model. The orbifold symmetry identifies y with $-y$, and we impose the periodicity condition of identifying y with $y + 2N - 2$. To calculate the action, the model is broken up into $2(N-1)$ bulk regions $\mathcal{R}_n$, but regions $\mathcal{R}_n$ and $\mathcal{R}_{-n}$ coincide by the orbifold symmetry.

same brane. Labellings are described in Fig. 11. The action for this model is given by

$$\begin{aligned} S\left[g_{\Gamma\Sigma}, {}^n x^\Gamma, {}^n\phi\right] = &\left(\sum_{n=1}^{N-1} + \sum_{n=-N+1}^{-1}\right) \int_{\mathcal{R}_n} d^5x_n \sqrt{-{}^n g}\left(\frac{{}^nR^{(5)}}{2\kappa_5^2} - \Lambda_n\right) \\ &+ \sum_{n=-N+1}^{N-2} \frac{1}{\kappa_5^2} \int_{\mathcal{B}_n} d^4w_n \sqrt{-{}^n h}\left({}^nK^+ + {}^nK^-\right) \\ &- \sum_{n=-N+1}^{N-2} \sigma_n \int_{\mathcal{B}_n} d^4w_n \sqrt{-{}^n h} + {}^0S_m[{}^0h_{ab}, {}^0\phi] \\ &+ \frac{1}{2}\left(\sum_{n=1}^{N-1} + \sum_{n=-N+1}^{-1}\right) {}^nS_m[{}^nh_{ab}, {}^n\phi] \end{aligned} \tag{B.1}$$

The sums over branes which only run to $N-2$ are written so because the branes $-N+1$ and $N-1$ are the same brane. Note that the brane tensions at the orbifold fixed points are included once only, while the brane tensions on the other branes are doubly included. This is just a choice of how to describe the brane tensions in the orbifold. The choice of the factor of $1/2$ in the matter actions accounts for the doubling that occurs with the orbifolding.

The procedure described in Chapter 2 may now be followed for each region. We gauge specialize to the straight gauge, before separating length-scales in the action. Writing the metric in each region as

$$^n ds^2 = e^{{}^n\chi(x_n^c, y_n)}\, {}^n\hat{\gamma}_{ab}(x_n^c, y_n) dx_n^a dx_n^b + {}^n\Phi^2(x_n^c, y_n) dy_n^2 \tag{B.2}$$

with $\det(\hat{\gamma}) = -1$, we can find the equations of motion at lowest order in the separation of length-scales. The following equations and boundary conditions arise, corresponding to Eqs. (2.5.11), (2.5.12), (2.7.3), (2.7.4), (2.7.5), (2.7.6), and (2.7.7). Note that the equations in regions n and $-n$ are identical, as required by the orbifolding condition:

$$ {}^{n}\chi(w_n^c, n) = {}^{n+1}\chi(w_n^c, n) \tag{B.3} $$

$$ \frac{2}{3}\kappa_5^2\sigma_n = \left.\frac{{}^{n}\chi_{,y}}{{}^{n}\Phi}\right|_{y_n=n} - \left.\frac{{}^{n+1}\chi_{,y}}{{}^{n+1}\Phi}\right|_{y_{n+1}=n} \tag{B.4} $$

$$ {}^{n}\hat{\gamma}_{ab}(w_n^c, n) = {}^{n+1}\hat{\gamma}_{ab}(w_n^c, n) \tag{B.5} $$

$$ \frac{1}{{}^{n}\Phi}\,{}^{n}\hat{\gamma}_{ab,y}(w_n^c, n) = \frac{1}{{}^{n+1}\Phi}\,{}^{n+1}\hat{\gamma}_{ab,y}(w_n^c, n) \tag{B.6} $$

$$ 0 = \frac{1}{4}\,{}^{n}\hat{\gamma}^{ab}\ {}^{n}\hat{\gamma}_{bc,y}\,{}^{n}\hat{\gamma}^{cd}\ {}^{n}\hat{\gamma}_{da,y} - 3\ {}^{n}\chi_{,y}^2 - 2\kappa_5^2\,{}^{n}\Phi^2\Lambda_n \tag{B.7} $$

$$ {}^{n}\hat{\gamma}_{ad,yy} = {}^{n}\hat{\gamma}_{ab,y}\ {}^{n}\hat{\gamma}^{bc}\ {}^{n}\hat{\gamma}_{cd,y} - {}^{n}\hat{\gamma}_{ad,y}\left(2\ {}^{n}\chi_{,y} - \frac{{}^{n}\Phi_{,y}}{{}^{n}\Phi}\right) \tag{B.8} $$

$$ 0 = \frac{1}{12}\,{}^{n}\hat{\gamma}^{ab}\ {}^{n}\hat{\gamma}_{bc,y}\,{}^{n}\hat{\gamma}^{cd}\ {}^{n}\hat{\gamma}_{da,y} + {}^{n}\chi_{,y}^2 + {}^{n}\chi_{,yy} - \frac{{}^{n}\Phi_{,y}}{{}^{n}\Phi}\,{}^{n}\chi_{,y} + \frac{2}{3}\kappa_5^2\,{}^{n}\Phi^2\Lambda_n. \tag{B.9} $$

The boundary conditions at the first and last branes are

$$ 0 = \left.{}^{1}\gamma_{ab,y}\right|_{y_1=0^+} \tag{B.10} $$

$$ 0 = \left.{}^{N-1}\gamma_{ab,y}\right|_{y_{N-1}=(N-1)^-} \tag{B.11} $$

$$ -P_1\frac{1}{3}\kappa_5^2\sigma_0 = \left.\frac{{}^{1}\chi_{,y}}{{}^{1}\Phi}\right|_{y_1=0^+} \tag{B.12} $$

and

$$ P_{N-1}\frac{1}{3}\kappa_5^2\sigma_{N-1} = \left.\frac{{}^{N-1}\chi_{,y}}{{}^{N-1}\Phi}\right|_{y_{N-1}=(N-1)^-}. \tag{B.13} $$

Equation (B.8) should be solved first. The solution (in matrix notation and suppressing indices n) is

$$ \hat{\boldsymbol{\gamma}}(x^a, y) = \mathbf{A}(x^a)\exp\left(\mathbf{B}(x^a)\int^y \Phi(x^a, y')e^{-2\chi(x^a,y')}dy'\right) \tag{B.14} $$

where $\mathbf{A}(x^a)$ and $\mathbf{B}(x^a)$ are arbitrary 4×4 matrices such that $\hat{\boldsymbol{\gamma}}$ has the properties of a metric. Combining this with Eqs. (B.5) and (B.6), we see that $\mathbf{B}$ is independent of region.

The boundary conditions Eqs. (B.10) and (B.11) then imply that $\mathbf{B} = 0$ in all regions. Finally, the condition (B.5) then implies that $\mathbf{A}$ is independent of region, and so we can write ${}^n\hat{\gamma}_{ab}(x^c, y) = \hat{\gamma}_{ab}(x^c)$ for all regions.

The remaining equations of motion are then solved straightforwardly. Defining

$$k_n = \sqrt{\frac{-\kappa_5^2 \Lambda_n}{6}}, \tag{B.15}$$

we find

$${}^n\chi_{,y} = 2 P_n k_n {}^n\Phi \tag{B.16}$$

and the brane-tuning condition

$$k_n P_n - k_{n+1} P_{n+1} = \frac{1}{3}\kappa_5^2 \sigma_n. \tag{B.17}$$

For the first and last branes, this condition is

$$k_1 P_1 = -\frac{1}{6}\kappa_5^2 \sigma_0, \tag{B.18}$$

$$k_{N-1} P_{N-1} = \frac{1}{6}\kappa_5^2 \sigma_{N-1}. \tag{B.19}$$

The metric in each bulk region is

$${}^n ds^2 = e^{{}^n\chi(x^c,y)} \hat{\gamma}_{ab}(x^c) dx^a dx^b + \frac{{}^n\chi_{,y}^2(x^c,y)}{4k_n^2} dy^2. \tag{B.20}$$

Following our prescription, we now substitute this into the action (B.1) and integrate over the fifth dimension. The result is

$$\begin{aligned} S\left[\hat{\gamma}_{ab}, \Psi_n, {}^n\phi\right] = \int d^4x \sqrt{-\hat{\gamma}} \frac{1}{2\kappa_5^2} \Bigg[& \sum_{n=1}^{N-1} \left(\frac{e^{\chi_n}}{k_n P_n} - \frac{e^{\chi_{n-1}}}{k_n P_n} \right) R^{(4)} \\ & + \frac{3}{2} \sum_{n=1}^{N-1} \left(\frac{e^{\chi_n}}{k_n P_n} (\nabla\chi_n)^2 - \frac{e^{\chi_{n-1}}}{k_n P_n} (\nabla\chi_{n-1})^2 \right) \Bigg] \\ & + \sum_{n=0}^{N-1} {}^n S_m[e^{\chi_n}\hat{\gamma}_{ab}, {}^n\phi], \end{aligned} \tag{B.21}$$

where $\chi_n(x^a) = {}^n\chi(x^a, n)$.

We now make the following definitions.

$$A_n = \left|\frac{1}{k_n P_n} - \frac{1}{k_{n+1}P_{n+1}}\right| \tag{B.22}$$

$$A_0 = \left|-\frac{1}{k_1 P_1}\right| = \frac{1}{k_1} \tag{B.23}$$

$$A_{N-1} = \left|\frac{1}{k_{N-1}P_{N-1}}\right| = \frac{1}{k_{N-1}} \tag{B.24}$$

$$\epsilon_n = \mathrm{sgn}\left(\frac{1}{k_n P_n} - \frac{1}{k_{n+1}P_{n+1}}\right) \tag{B.25}$$

$$\epsilon_0 = \mathrm{sgn}(-P_1) = -P_1 \tag{B.26}$$

$$\epsilon_{N-1} = \mathrm{sgn}(P_{N-1}) = P_{N-1} \tag{B.27}$$

$$\Psi_n = \sqrt{A_n e^{\chi_n}} \tag{B.28}$$

With these definitions, the action is given by

$$\begin{aligned} S\left[\hat{\gamma}_{ab}, \Psi_n, {}^n\phi\right] = &\int d^4x \frac{\sqrt{-\hat{\gamma}}}{2\kappa_5^2}\left[R^{(4)}\left[\hat{\gamma}_{ab}\right]\left(\sum_{n=0}^{N-1}\epsilon_n\Psi_n^2\right) + 6\sum_{n=0}^{N-1}\epsilon_n(\nabla^a\Psi_n)(\nabla_a\Psi_n)\right] \\ &+ \sum_{n=0}^{N-1} {}^nS_m\left[\frac{\Psi_n^2}{A_n}\hat{\gamma}_{ab}, {}^n\phi\right]. \end{aligned} \tag{B.29}$$

This is identical to Eq. (2.8.9) above except for a factor of two multiplying $1/4\kappa_5^2$, which arises from integrating each region twice rather than once. Otherwise, only the definitions of ϵ_0, A_0, ϵ_{N-1} and A_{N-1} have changed, which corrects for the removal of the regions between the first and last branes and infinity in the bulk. Thus, the four-dimensional low-energy action for this model is the same as for the uncompactified case (2.9.17), although some parameters have been modified. A special case of the orbifolded model is the two-brane case, corresponding to the RS-I model (also see Section 2.4). In this case, the action (B.29) reduces to previously known four-dimensional actions [62].

Most of the analysis for the orbifolded scenario is identical to that for the orbifolded scenario. The only time when the orbifolded scenario requires a separate analysis is when removing ghost modes. In the orbifolded case, we again want all ϵ_n parameters to have the same sign except for one, which is opposite. Note that we now have $\epsilon_0 = \mathrm{sgn}(\sigma_0) = -P_1$ and

$\epsilon_{N-1} = \text{sgn}(\sigma_{N-1}) = P_{N-1}$. For the first and last branes, we may only choose whether ϵ is positive or negative, while for the intermediary branes, all of the previously discussed cases are possibilities.

For a single positive ϵ_n, we need one of the following configurations:

$$-, 5, \ldots, 5, (2 \text{ or } 6), 4, \ldots, 4, -,$$

$$+, 4, \ldots, 4, -,$$

$$-, 5, \ldots, 5, +.$$

For a single negative ϵ_n, the options are

$$-, 1, \ldots, 1, +,$$

$$+, 8, \ldots, 8, -,$$

$$+, 8, \ldots, 8, (3 \text{ or } 7), 1, \ldots, 1, +.$$

The analysis of each configuration proceeds exactly as in Section 3.2. We find that we must have a single positive ϵ_n, with all other ϵ_n negative. This implies that all branes must be positive tension, with the possible exception of the first and last branes, which may be negative. Again, the warp factor thus rises to a maximum and then falls again. If the first brane has the maximum warp factor, it has a positive tension, and similarly for the last brane. The four-dimensional low-energy action specialized to such a configuration is described by (3.3.4) above.

As the constraints on the Eddington γ factor and the dark matter limits arise only from this action, the constraints on this orbifolded model are identical to those in the uncompactified model.

In arriving at the four-dimensional low-energy action (B.29), we make the same approximations as for the uncompactified case, namely that the separation of length-scales is valid everywhere between the branes. However, we don't have any issues with the separation of length-scales breaking down towards infinity in the bulk, and nor do we need to invoke global hyperbolicity to constrain the behavior of the warp factor outside the collection of branes. Furthermore, the boundary conditions imposed by the orbifolding ensures that the degree of freedom $\mathbf{B}$ is projected out. In these regards, the orbifolded analysis is more robust than the uncompactified analysis.

Appendix C

Kaluza-Klein Modes

In this appendix, we venture away from the four-dimensional theory to discuss the Kaluza-Klein modes of our model. The methods and results here mimic the original RS-II model [7] closely.

Consider an uncompactified model with N branes (with brane tensions tuned) and no matter. The solution for the five-dimensional metric can be written as

$$ds^2 = e^{\chi(y)} \eta_{ab} dx^a dx^b + dy^2 \tag{C.1}$$

after appropriate gauge transformations, where $\chi_{,y} = 2k_n P_n$, and χ is continuous. Now consider metric fluctuations of the form

$$ds^2 = \left(e^{\chi(y)} \eta_{ab} + h_{ab}(x^c, y)\right) dx^a dx^b + dy^2. \tag{C.2}$$

Decomposing h_{ab} into Fourier modes $h_{ab}(x^c, y) = h_{ab}\psi(y)\exp(ip_c x^c)$, where p^c is a four-momentum with $p^2 = -m^2$, we find to first order in h

$$\left(-\frac{1}{2}m^2 e^{-\chi} - \frac{1}{2}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\left(\chi_{,y}\right)^2 + \frac{\chi_{,yy}}{2}\right)\psi = 0. \tag{C.3}$$

Our gauge choice is $h^a_a = \partial^a h_{ab} = 0$. Equation (C.3) is equivalent to Eq. (8) in [7]. As discussed there, the solutions to this equation are Bessel functions (although here, they must be defined piecewise because of the piecewise nature of χ). There is a massless graviton mode, which has been integrated to give the four-dimensional effective graviton in our low-energy theory (3.3.4), and a continuum of massive Kaluza-Klein graviton modes, which in this work were previously truncated.

As in the RS-II model, there is no mass gap. Note that there are no so-called “ultra-light” [49, 50, 68] modes present in this model, as such modes occur in a model where the mass spectrum is quantized. Although the presence of extra branes complicates the mathematics, the physical effect of the Kaluza-Klein modes in our model is essentially the same as in the RS-II model.

In an orbifolded model, the analysis of the Kaluza-Klein modes follows similarly, but the orbifolding condition implies that the mass spectrum is quantized, and we expect ultra-light modes to be present (see [68] and citations therein).

Appendix D

The Weak Equivalence Principle

In this appendix, we show that including terms in the action that depend explicitly on the matter stress energy tensor, as in Eq. (4.2.1) above, generically gives rise to violations of the weak equivalence principle. However, we also show that our specific model (4.2.1) does not, to linear order in ϵ. Since the parameter ϵ essentially counts the number of derivatives in our derivative expansion, it follows the weak equivalence principle is satisfied for our derivative expansion up to four derivatives.

1 Generic Violations of Weak Equivalence Principle when Stress-Energy Terms are Present in Action

Consider first an action principle of the general form

$$S[g_{\alpha\beta}, \phi, \psi_{\rm m}] = S_{\rm g}[g_{\alpha\beta}, \phi] + S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}]. \tag{D.1}$$

Here the first term is a gravitational action, depending only on the metric $g_{\alpha\beta}$ and the scalar field ϕ, and the second term is the matter action, in which all the matter fields $\psi_{\rm m}$ couple only to the Jordan metric $\bar{g}_{\alpha\beta}$ (some function of $g_{\alpha\beta}$ and ϕ), and not to $g_{\alpha\beta}$ and ϕ individually. By definition, any theory of this form obeys the weak equivalence principle. What this means is as follows. We define weakly self-gravitating bodies to be bodies for which we can neglect the perturbations they cause to $g_{\alpha\beta}$ and ϕ. From the form of the action (D.1), it follows that all weakly self-gravitating bodies will fall on geodesics of the metric $\bar{g}_{\alpha\beta}$, and hence will all fall on the same geodesics.

The action principle (4.2.1) we use in this work is not of the general form (D.1), because of the explicit appearance of terms involving the stress energy tensor in the gravitational action. Therefore one expects violation of the weak equivalence principle to arise. We now verify explicitly that this does occur in a specific example. We choose the following special case of the action (4.2.1), where the only perturbative term included is the term proportional to the trace of the stress energy tensor:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}m_P^2 R - \frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon f(\phi) T\right] + S_{\rm m}[\bar{g}_{\alpha\beta}, \psi_{\rm m}]. \tag{D.2}$$

We choose the matter field $\psi_{\rm m}$ to be a scalar field ψ with action

$$S_{\rm m} = -\int d^4x\sqrt{-\bar{g}}\left[\frac{1}{2}(\bar{\nabla}\psi)^2 + V(\psi)\right], \tag{D.3}$$

and we specialize the relation (4.2.4) between the two metrics to be the conformal transformation $\bar{g}_{\alpha\beta} = e^{\alpha(\phi)} g_{\alpha\beta}$. This gives $T = -e^{-\alpha}(\nabla\psi)^2 - 4V$ and the total action is therefore

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big[&\frac{1}{2}m_P^2 R - \frac{1}{2}(\nabla\phi)^2 - U(\phi) - \frac{1}{2}(e^{\alpha} + 2\epsilon e^{-\alpha} f)(\nabla\psi)^2 \\ &- (e^{2\alpha} + 4\epsilon f)V(\psi)\Big]. \end{aligned} \tag{D.4}$$

The kinetic term for ψ can be written as $\int d^4x\sqrt{-\hat{g}}(\hat{\nabla}\psi)^2$ where $\hat{g}_{\alpha\beta} = (e^{\alpha} + 2\epsilon e^{-\alpha} f) g_{\alpha\beta}$, and the potential term can be written as $\int d^4x\sqrt{-\tilde{g}}V(\psi)$, where $\tilde{g}_{\alpha\beta} = \sqrt{e^{2\alpha} + 4\epsilon f}\, g_{\alpha\beta}$. Therefore, objects whose stress energy is composed of different combinations of the kinetic term and the potential term will fall on different combinations of the metrics $\hat{g}_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$, violating the weak equivalence principle.

2 Validity of Weak Equivalence Principle to Linear Order

In the above analysis, we note that the metrics $\hat{g}_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ coincide to linear order in ϵ, so there is no violation to this order. We now show that, similarly, none of the stress-energydependent terms included in Eq. (4.2.1) violate the weak equivalence principle, to linear order in ϵ.

The key idea of the proof is to use the transformation laws derived in Section 4.3 above to rewrite the theory in the general form (D.1), which we know satisfies the weak equivalence principle. All of the terms in the action given by Eqs. (4.2.1) – (4.2.4) are of this form, except for the terms parameterized by the coefficients $b_1, \ldots, b_7$, e_1 and e_2. However, as we now show, we can use transformations to eliminate these terms in favor of the remaining terms which manifestly satisfy the principle.

Consider first the terms in the action (4.2.3) which depend linearly on the stress-energy tensor. We can eliminate the terms parameterized by $b_1, \ldots, b_6$ using the transformation (4.3.3) with $\tilde{\beta}_i = -2e^{-2\alpha} b_i$ for $1 \leq i \leq 6$. This generates contributions to the the terms parameterized by $\beta_1, \ldots, \beta_6$ in the definition (4.2.4) of the Jordan metric. Similarly, by using the transformation (4.3.4) with $\tilde{\alpha} = -2e^{-2\alpha} b_7$, we can eliminate the term parameterized by b_7 in favor of an $O(\epsilon)$ correction to the function α in Eq. (4.2.4).

We now turn to the terms in the action (4.2.3) which depend quadratically on the stress-energy tensor, namely the terms parameterized by e_1 and e_2. For e_1 we use the transformation (4.3.29) with $\sigma_{11} = -e^{-2\alpha} e_1$, and for e_2 we use the transformation (4.3.27) with $\sigma_{10} = -e^{-2\alpha} e_2$. These transformations generates new contributions to the linear stress-energy terms parameterized by b_1, b_2, b_5, b_6 and b_7 (see Table 1), but we have already shown that all of those terms satisfy the weak equivalence principle.

To summarize, we have shown that our model (4.2.1) satisfies the weak equivalence principle despite the explicit appearance of stress energy terms in the action. Of course, there can be violations of the strong equivalence principle in models of this kind, which can even be of order unity [115]. In addition, the weak equivalence principle will generically be violated by quantum loop corrections, although this is a small effect [116].

3 Potential Ambiguity in Definition of Weak Equivalence Principle

We next discuss a potential ambiguity that arises in the definition of the weak equivalence principle. In the definition one restricts attention to bodies whose gravitational fields, as measured by the perturbations they produce to the metric $g_{\mu\nu}$ and scalar field ϕ, can be neglected. However, consider for example the field redefinition (4.3.27), where the metric transforms according to

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + 2\epsilon\sigma_{10} T \hat{g}_{\alpha\beta}. \tag{D.5}$$

It is possible for the perturbation $\delta\hat{g}_{\alpha\beta}$ generated by the body to be negligible, but the perturbation $\delta g_{\alpha\beta}$ to be non-negligible, because of the appearance of the stress-energy term in Eq. (D.5). If this occurs then the weak equivalence principle could be valid for one choice of variables, but not valid for the other choice.

To assess this ambiguity, we now make some order of magnitude estimates. Consider a body of mass $\sim M_b$ and size $\sim R$. Then in general relativity the size of the metric perturbation due to the body is of order $\delta\hat{g}_{\alpha\beta} \sim M_b/(m_P^2 R)$. Suppose now that $\sigma_{10} \sim 1/(m_P^2 M^2)$, as indicated by Eq. (4.3.28) and Table 3. Then the contribution to the metric perturbation $\delta g_{\alpha\beta}$ from the second term in Eq. (D.5) will be of order $M_b/(R^3 m_P^2 M^2)$, which will be much larger than $\delta\hat{g}_{\alpha\beta}$ whenever $R \ll M^{-1}$. Therefore the ambiguity could in principle arise.

However, in the models considered in this work, the ambiguity does not occur. This is because the condition $R \ll M^{-1}$ is excluded by the condition (4.5.9) for the validity of the effective field theory.

Appendix E

Equivalence Between Field Redefinitions, Integrating Out New Degrees of Freedom, and Reduction of Order

The action (4.2.1) we start with in Chapter 4 contains several higher-derivative terms, that is, terms which gives contributions to the equations of motion which involve third-order and fourth-order derivatives of the fields. As discussed in the introduction, the theory with these higher-derivative terms contains additional degrees of freedom compared to our zeroth-order action (4.2.2), which contains a single graviton and scalar. Our goal in this work is to describe a general class of theories containing just one tensor and one scalar degree of freedom, so we wish to exclude these additional degrees of freedom[1].

Therefore, as discussed in the introduction, we define the theory we wish to consider, associated with our action (4.2.1), to be that obtained from the following series of steps:

1. Vary the action to obtain the equations of motion, which will contain third-order and fourth-order derivative terms which are proportional to ϵ.
2. Perform a *reduction of order* procedure on the equations of motion [90, 91, 92]. That is,

[1] Higher derivative terms are also generically associated with instabilities [89], although this can be evaded in special cases, for example R^2 terms.

substitute the zeroth-order in ϵ equations of motion into the higher derivative terms in order to obtain equations that contain only second-order and lower order derivatives, which are equivalent to the original equations up to correction terms of $O(\epsilon^2)$ which we neglect.

3. Optionally, one can then derive the action principle that gives the reduced-order equations of motion.

In this appendix, we show that this procedure is equivalent to the computational procedure we use in Chapter 4, in which we apply perturbative field redefinitions directly to the action in order to obtain an action with no higher-derivative terms. We also show that it is equivalent to integrating out at tree level the extra degrees of freedom that are associated with the higher derivative terms.

We note that the analyses of general quintessence models by Weinberg [83] and Park *et al.* [86] used a different method of eliminating higher derivative terms. They performed a reduction of order procedure directly at the level of the action, that is, they substituted the zeroth-order equations of motion directly into the higher-derivative terms in the action, to obtain an action with no higher-derivative terms. We will show that this method is not in general correct; it does not agree with the theory obtained by applying the reduction of order method to the equations of motion[2]. However, it differs from the correct result only by field redefinitions (that do not involve higher derivatives), and so for the purpose of attempting to classify general theories of quintessence, Weinberg's method is adequate.

[2]The reason is that substituting the zeroth-order equations of motion into the action gives an action which is correct off-shell to $O(\epsilon^0)$ and on-shell to $O(\epsilon)$, but it needs to be valid off-shell to $O(\epsilon)$.

1 Reduction of Order Method

We start by considering the case of just a scalar field; a more general argument valid for scalar and tensor fields will be given below. Consider a general action of the form

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon F[\phi, (\nabla\phi)^2, \Box\phi]\right\}, \tag{E.1}$$

where F is an arbitrary function. We introduce the notation $K = (\nabla\phi)^2$ and $L = \Box\phi$. We first show that applying the reduction of order procedure to the equations of motion (steps 1 – 3 above) give rise to a theory of the form (E.1) but with $F(\phi, K, L)$ replaced by another function $\hat{F}(\phi, K, L)$, given by

$$\hat{F}(\phi, K, L) = F[\phi, K, U'(\phi)] + [L - U'(\phi)]F_{,L}[\phi, K, U'(\phi)]. \tag{E.2}$$

To see this, we vary the action (E.1) to obtain the equation of motion

$$\Box\phi - U'(\phi) + \epsilon F_{,\phi} - 2\epsilon\nabla_\alpha(F_{,K}\nabla^\alpha\phi) + \epsilon\Box F_{,L} = 0. \tag{E.3}$$

We now make the field redefinition

$$\psi = \phi + \epsilon F_{,L}[\phi, (\nabla\phi)^2, \Box\phi]. \tag{E.4}$$

Rewriting the equation of motion (E.3) in terms of ψ yields

$$\Box\psi - U'(\psi) + \epsilon U''(\psi)F_{,L} + \epsilon F_{,\phi} - 2\epsilon\nabla_\alpha(F_{,K}\nabla^\alpha\psi) = O(\epsilon^2), \tag{E.5}$$

where the arguments of $F_{,\phi}$, $F_{,L}$ and $F_{,K}$ are now $[\psi, (\nabla\psi)^2, \Box\psi]$.

We now apply the reduction of order procedure to the equation of motion given by Eqs. (E.4) and (E.5), that is, we substitute in the zeroth-order equation of motion $\Box\psi = U'(\psi)$. The field redefinition (E.4) gets replaced by the following field redefinition which does not involve higher derivatives:

$$\psi = \phi + \epsilon F_{,L}[\phi, (\nabla\phi)^2, U'(\phi)] + O(\epsilon^2). \tag{E.6}$$

The equation of motion (E.5) is unchanged, except that the arguments of $F_{,\phi}, F_{,L}$ and $F_{,K}$ are now $[\psi, (\nabla\psi)^2, U'(\psi)]$. This equation of motion can be obtained from the action

$$S = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}(\nabla\psi)^2 - U(\psi) + \epsilon F[\psi, (\nabla\psi)^2, U'(\psi)]\right\}. \tag{E.7}$$

Finally we rewrite this action in terms of ϕ using the change of variable (E.6). The result is

$$\begin{aligned} S = \int d^4x\sqrt{-g}\Big\{ &-\frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon F[\phi, (\nabla\phi)^2, U'(\phi)] \\ &+ \epsilon[\Box\phi - U'(\phi)]F_{,L}[\phi, (\nabla\phi)^2, U'(\phi)]\Big\}. \end{aligned} \tag{E.8}$$

Note that although this action contains second-order derivatives, the corresponding equations of motion contain derivatives only up to second order, that is, the theory is no longer a "higher derivative" theory [97]. The final, reduced-order action (E.8) is of the form (E.2) claimed above.

The final result (E.8) shows explicitly that the method of reducing order directly in the action used in Refs. [83, 86] is not correct. Applying this procedure to the action (E.1) would yield the first three terms in the action (E.8), but not the fourth term.

2 Method of Integrating Out the Additional Fields

We next show that the same result (E.8) can be obtained by integrating out the new degrees of freedom that are associated with the higher derivative terms. Starting from the action (E.1), we introduce an auxiliary scalar field ψ and consider the action

$$\begin{aligned} S[\phi,\psi] = \int d^4x\sqrt{-g}\Big\{ &-\frac{1}{2}(\nabla\phi)^2 - U(\phi) + \epsilon F[\phi, (\nabla\phi)^2, \psi] \\ &+ \epsilon(\Box\phi - \psi)F_{,L}[\phi, (\nabla\psi)^2, \psi].\Big\}, \end{aligned} \tag{E.9}$$

The equation of motion for ψ from this action is $\psi = \Box\phi$, assuming $F_{,LL} \neq 0$, and substituting this back into the action (E.9) yields the action (E.1). Thus the two actions are equivalent classically.

We now proceed to integrate out the field ψ, at tree level, i.e., classically. The equation of motion for ϕ is $\psi = U'(\phi) + O(\epsilon)$, and substituting this back the action (E.9) gives the same result (E.8) as was obtained from the reduction of order method.

3 Field Redefinition Method

We next turn to a discussion of the method we use to eliminate higher derivative terms in Section 4.3, using perturbative field redefinitions. That method is not generally applicable, but when it can be used, it is equivalent to the method of reduction of order (steps 1-3 above), as we now show. We start with an action of the form (E.1), with the function F chosen to be of the form

$$F(\phi, K, L) = g(\phi, K) + [L - U'(\phi)]h(\phi, K, L), \tag{E.10}$$

for some functions g and h. This is the most general form of F for which the field redefinition method can be used to eliminate the higher derivatives, and is sufficiently general to encompass the cases used in the work presented here. First, we apply the reduction of order method. Inserting the formula (E.10) into Eq. (E.2) shows that the reduced-order action is characterized by the function $\hat{F}$ given by

$$\hat{F}(\phi, K, L) = g(\phi, K) + [L - U'(\phi)]h[\phi, K, U'(\phi)]. \tag{E.11}$$

However, the same result is obtained by starting with the action given by Eqs. (E.1) and (E.10) and performing the field redefinition

$$\phi \to \phi + \epsilon h[\phi, (\nabla\phi)^2, U'(\phi)] - \epsilon h[\phi, (\nabla\phi)^2, \Box\phi]. \tag{E.12}$$

This shows the reduction of order and field redefinition methods are equivalent.

We now give a more general and abstract argument for the equivalence, valid for any field content. Suppose we have a theory containing higher-derivative terms in the action, proportional to ϵ. Suppose that we can find a linearized field redefinition, involving higher derivatives, that has the effect of eliminating all higher derivative terms from the action. We can then consider this process in reverse: starting from a theory which is not higher derivative, by making a linearized field redefinition we obtain another theory which has higher derivative terms, proportional to ϵ. However, the change in the action induced by the field redefinition must be proportional to the equations of motion. Hence, these higher derivative terms will be eliminated by applying Weinberg's method of substituting the zeroth-order equations of motion into the $O(\epsilon)$ terms in the action. As we have discussed, Weinberg's procedure is valid up to a field redefinition of the type (E.6) which does not change the differential order.

Appendix F

Comparison with Previous Work

In this appendix we compare our analysis and results in Chapter 4 to those of Park, Watson and Zurek [86], who perform a similar computation with similar motivation, but obtain a somewhat different final result [Eq. (1) of their paper]. The main differences that arise are:

- They work throughout in the Jordan frame, whereas we work in the Einstein frame. This is a minor difference which only affects the appearance of the computations and results, since it is always possible to translate from one frame to another.
- As discussed in the introduction and in Appendix E, they use Weinberg's method of eliminating the higher derivative terms, consisting of substituting the zeroth-order equations of motion into the higher derivative terms in the action, whereas we use the field redefinition method. The two methods are not equivalent for a given specific theory with specific coefficients, but are equivalent for the purpose of determining a general class of theories.
- After eliminating higher derivative terms, their result is an action [Eq. (5) of their paper] that contains eleven functions of the scalar field, whereas our corresponding result (4.4.6) has only nine free functions. However, this is a minor difference: their function $Z(\phi)$ can be eliminated by redefining the scalar field to attain canonical normalization, and their function $f(\phi)$ can be eliminated by the transformation used in step 7 in Section 4.4.I above.

- Another minor difference is that in their analysis they have in their action a Weyl squared term $\propto C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$, which is unaffected by any of the transformation they make to the action. This Weyl squared term gives rise to higher derivative terms in the equation of motion that are associated with ghost-like additional degrees of freedom [117]. In our analysis the Weyl squared term is replaced by the Gauss-Bonnet term, which is not a higher derivative term, because it would be a topological term if it were not for the ϕ-dependent prefactor.

- Aside from the above minor differences, our result (4.4.5) is equivalent to the result given in Eq. (5) of their paper. Two major differences arise subsequently in the estimates of the scalings for the coefficients of the operators in the Lagrangian.

  First, Park *et al.* use the standard effective theory scaling rule wherein an operator of dimension $4+n$ has a coefficient $\sim \Lambda^{-n}$, where Λ is the cutoff. As discussed in Section 4.5.I above, this corresponds to placing no restrictions on the theory that applies above the cutoff scale Λ. By contrast, our approach does place restrictions on the physics at scales above Λ, and yields the modified scaling rule (4.5.5). As a consequence, our cutoff Λ (which we denote by M in our work) can be taken all the way down to the Hubble scale $H_0 \sim 10^{-33}$ eV, whereas their cutoff must be larger than $\sim \sqrt{H_0 m_P} \sim 10^{-3}$ eV.

  Second, Park *et al.* actually assume separate cutoffs for the gravitational, matter and scalar sectors of the theory, and estimate how each of their coefficients scale as functions of these three cutoffs. We do not understand completely their method of derivation of these scalings, but we do note that some of their scaling estimates are inconsistent with how the coefficients transform into one another under field redefinitions as discussed in Section 4.3 above. They then proceed to drop some terms which their scalings indicate are subdominant, and arrive at a final action [Eq. (1) in their paper] which differs from ours, being parameterized by three free functions rather than nine.

Appendix G

Equations of Motion for Reduced Theory

In this appendix we compute the equations of motion for the final action in Chapter 4, given by Eq. (4.4.5), with the e_1 and e_2 terms omitted. We start by using a transformation of the form (4.3.2) with $\tilde{\beta}_2 = -2e^{-2\alpha}b_2$. This yields the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{m_p^2}{2}R - \frac{1}{2}(\nabla\phi)^2 - U(\phi) + a_1(\nabla\phi)^4 + c_1 G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \right.$$
$$\left. + d_3\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) + d_4\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta}\right\}$$
$$+ S_{\rm m}\left[e^{\alpha(\phi)}g_{\mu\nu}\left(1+\beta(\nabla\phi)^2\right), \psi_{\rm m}\right]. \tag{G.1}$$

Here we have defined $\beta = 2e^{-2\alpha}b_2$; this was denoted β_2 in Chapter 4. We have also set $\epsilon = 1$ for simplicity. The representation (G.1) is more convenient than (4.4.5) for computing the equations of motion since it avoids varying of the stress-energy tensor.

Next, we vary the matter action in Eq. (G.1) using the definition (0.0.2) of the stress energy tensor $T_{\mu\nu}$ and the definition (4.2.4) of the Jordan metric $\bar{g}_{\mu\nu}$. This yields

$$\delta S_m = -\frac{1}{2}\int d^4x\sqrt{-g}e^{2\alpha}\left\{\delta g^{\mu\nu}\left[T_{\mu\nu} + 2T_{\mu\nu}\beta(\nabla\phi)^2 - \beta T\nabla_\mu\phi\nabla_\nu\phi\right]\right.$$
$$\left. + \delta\phi\left[-\alpha' T + 2\alpha'\beta T(\nabla\phi)^2 + \beta' T(\nabla\phi)^2 + 2\beta\nabla_\mu T\nabla^\mu\phi + 2\beta T\Box\phi\right]\right\}. \tag{G.2}$$

Combining this with the variation of the gravitational action gives the equations of motion

$$
\begin{aligned}
\Box\phi = U'(\phi) - \frac{1}{2}e^{2\alpha}\alpha' T + 4a_1\left[(\nabla\phi)^2\Box\phi + 2\nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi\right] + 3a_1'(\nabla\phi)^4 \\
+ c_1' G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 2c_1 G^{\mu\nu}\nabla_\mu\nabla_\nu\phi - d_3'\left(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho}\right) \\
- d_4'\epsilon^{\mu\nu\lambda\rho}C_{\mu\nu}{}^{\alpha\beta}C_{\lambda\rho\alpha\beta} \\
+ \frac{1}{2}e^{2\alpha}\left[2\alpha'\beta T(\nabla\phi)^2 + \beta' T(\nabla\phi)^2 + 2\beta\nabla_\mu T\nabla^\mu\phi + 2\beta T\Box\phi\right],
\end{aligned}
\tag{G.3}
$$

and

$$
\begin{aligned}
m_p^2 G_{\mu\nu} = e^{2\alpha}T_{\mu\nu} + \nabla_\mu\phi\nabla_\nu\phi - \left[\frac{1}{2}(\nabla\phi)^2 + U(\phi)\right]g_{\mu\nu} - 4a_1(\nabla\phi)^2\nabla_\mu\phi\nabla_\nu\phi \\
+ a_1(\nabla\phi)^4 g_{\mu\nu} + g_{\mu\nu}c_1 G^{\sigma\lambda}\nabla_\sigma\phi\nabla_\lambda\phi - 4c_1 R_{\sigma(\mu}\nabla_{\nu)}\phi\nabla^\sigma\phi + c_1 R_{\mu\nu}(\nabla\phi)^2 \\
+ c_1 R\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\nabla_\sigma\nabla_\lambda(c_1\nabla^\sigma\phi\nabla^\lambda\phi) + g_{\mu\nu}\Box[c_1(\nabla\phi)^2] \\
+ 2\nabla_\lambda\nabla_{(\mu}(c_1\nabla_{\nu)}\phi\nabla^\lambda\phi) - \nabla_\mu\nabla_\nu[c_1(\nabla\phi)^2] - \Box(c_1\nabla_\mu\phi\nabla_\nu\phi) \\
+ 2R\nabla_\mu\nabla_\nu d_3 - 2g_{\mu\nu}R\Box d_3 + 4R_{\mu\nu}\Box d_3 - 8R^\sigma{}_{(\mu}\nabla_{\nu)}\nabla_\sigma d_3 \\
+ 4g_{\mu\nu}R_{\sigma\rho}\nabla^\sigma\nabla^\rho d_3 + 4R_{\rho\mu\nu\sigma}\nabla^\rho\nabla^\sigma d_3 + 16C_{\mu\nu} + 2e^{2\alpha}T_{\mu\nu}\beta(\nabla\phi)^2 \\
- e^{2\alpha}\beta T\nabla_\mu\phi\nabla_\nu\phi.
\end{aligned}
\tag{G.4}
$$

Here the tensor $C_{\mu\nu}$ comes from the Chern-Simons term, and is defined by

$$
C^{\mu\nu} = (\nabla_\sigma d_4)\epsilon^{\sigma\lambda\rho(\mu}\nabla_\rho R^{\nu)}{}_\lambda + (\nabla_\sigma\nabla_\lambda d_4){}^\star R^{\lambda(\mu\nu)\sigma}
\tag{G.5}
$$

where ${}^\star R^{\mu\nu\sigma\lambda} = \epsilon^{\sigma\lambda\rho\tau}R^{\mu\nu}{}_{\rho\tau}/2$. Note that the zeroth-order terms involving the stress-energy tensor depend implicitly on β through the expression for the Jordan metric given in Eq. (G.1).

The terms involving c_1 are written in the most compact manner we could find. Although it looks unlikely, the higher-order derivatives in these terms do cancel; the full expansion of these terms is

$$
\begin{aligned}
&2c_1 g_{\mu\nu} R^{\sigma\lambda}\nabla_\sigma\phi\nabla_\lambda\phi - \frac{1}{2}c_1 g_{\mu\nu} R(\nabla\phi)^2 - 4c_1 R_{\sigma(\mu}\nabla_{\nu)}\phi\nabla^\sigma\phi + c_1 R_{\mu\nu}(\nabla\phi)^2 \\
&+ g_{\mu\nu}\left[c_1'\nabla_\sigma\phi\nabla_\lambda\phi\nabla^\sigma\nabla^\lambda\phi + c_1\nabla_\sigma\nabla_\lambda\phi\nabla^\sigma\nabla^\lambda\phi - c_1'(\nabla\phi)^2\Box\phi - c_1(\Box\phi)^2\right] \\
&- 2c_1\nabla_\sigma\nabla_\mu\phi\nabla^\sigma\nabla_\nu\phi - 2c_1'\nabla_\sigma\phi\nabla_{(\mu}\phi\nabla_{\nu)}\nabla^\sigma\phi + c_1'\nabla_\mu\nabla_\nu\phi(\nabla\phi)^2 + c_1'\nabla_\mu\phi\nabla_\nu\phi\Box\phi \\
&+ 2c_1\nabla_\mu\nabla_\nu\phi\Box\phi + 2c_1\nabla^\lambda\phi\nabla^\sigma\phi R_{\sigma\mu\nu\lambda} + c_1 R\nabla_\mu\phi\nabla_\nu\phi. \qquad \text{(G.6)}
\end{aligned}
$$

Appendix H

Scaling of Coefficients Obtained by Integrating Out Pseudo-Nambu-Goldstone Fields

In this appendix we give some more details of the derivation discussed in Section 4.5.I of the scaling of the coefficients of the operators in the Lagrangian. We divide the pNGB fields Φ^A into two groups, a set χ^a with mass $\sim H_0$ and a set ψ^Γ with mass $\sim M$, where $M \gg H_0$:

$$\Phi^A = (\chi^a, \psi^\Gamma). \tag{H.1}$$

We assume an action for these fields of the form

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}q_{AB}(\Phi^A)\nabla_\mu\Phi^A\nabla_\nu\Phi^B g^{\mu\nu} - H_0^2 V\left(\chi^a, \frac{M}{H_0}\psi^\Gamma\right)\right\}. \tag{H.2}$$

This is the same as the action (4.5.3) of Section 4.5.I above, except that an extra factor has been inserted into the potential to make the ψ^Γ fields have mass $\sim M$ rather than $\sim H_0$, and we have specialized to units where $m_P = 1$. We assume that the target space coordinates have been chosen so that the potential is minimized at $\psi^\Gamma = 0$, i.e.

$$V_{,\Gamma} = 0 \tag{H.3}$$

at $\psi^\Gamma = 0$.

We now want to let M become large and integrate out the fields ψ^Γ at tree level. This can be done by using Feynman diagrams and using power counting[1], as in Ref. [84]. Alternatively and more simply, it can be done by writing out the equations of motion for the fields ψ^Γ and invoking an adiabatic approximation. At zeroth order in $1/M$, the theory obtained for the fields χ^a is a nonlinear sigma model where the potential is just the potential of the action (H.2) evaluated on the surface $\psi^\Gamma = 0$, and the target space metric is just the metric induced on the surface from the metric q_{AB}.

To obtain the higher-order corrections we can proceed as follows. The equation of motion for the fields ψ^Γ is

$$\begin{aligned}\Box\psi^\Sigma + \Gamma^\Sigma_{ab}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b + \Gamma^\Sigma_{\Theta\Upsilon}\vec{\nabla}\psi^\Theta \cdot \vec{\nabla}\psi^\Upsilon + 2\Gamma^\Sigma_{a\Theta}\vec{\nabla}\chi^a \cdot \vec{\nabla}\psi^\Theta \\ = H_0^2 q^{\Sigma a}V_{,a} + H_0 M q^{\Sigma\Theta}V_{,\Theta}.\end{aligned} \tag{H.4}$$

Here the connection coefficients are those of the target space metric q_{AB}. We next expand this equation to linear order in ψ^Γ and use the condition (H.3) to obtain

$$\begin{aligned}\Box\psi^\Sigma + \left[\Gamma^\Sigma_{ab,\Theta}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b - H_0^2 q^{\Sigma a}{}_{,\Theta}V_{,a} - M^2 q^{\Sigma\Upsilon}V_{,\Upsilon\Theta}\right]\psi^\Theta \\ +2\Gamma^\Sigma_{a\Theta}\vec{\nabla}\chi^a \cdot \vec{\nabla}\psi^\Theta = -\Gamma^\Sigma_{ab}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b + H_0^2 q^{\Sigma a}V_{,a},\end{aligned} \tag{H.5}$$

where all the metric coefficients, connection coefficients and their derivatives are evaluated at $\psi^\Gamma = 0$. Now in the large M or adiabatic limit, the dominant term on the left hand side will be the term proportional to M^2, and dropping the other terms gives a simple algebraic equation for the leading order contribution to ψ^Γ:

$$\left[q^{\Sigma\Upsilon}V_{,\Upsilon\Theta}\right]\psi^\Theta = \frac{1}{M^2}\left[\Gamma^\Sigma_{ab}\vec{\nabla}\chi^a \cdot \vec{\nabla}\chi^b - H_0^2 q^{\Sigma a}V_{,a}\right]. \tag{H.6}$$

Substituting the solution given by Eq. (H.6) into the action (H.2) gives the required, $O(1/M^2)$ corrections to the action. The first term on the right hand side of Eq. (H.6) will give nonlinear

[1]We note that Burgess *et al.* [84] write down a scaling rule in their Eqs. (2.3) and (2.5) which is identical to our scaling rule (4.5.5) except that it is suppressed by an overall factor of M^2/m_P^2 for $d > 2$, where d is the number of derivatives. They say in their footnote 2 that this rule comes from integrating out a pNGB field of mass M. However we find that the detailed power counting calculations given in the second example in their Section 2.2 actually yield our scaling rule rather than theirs.

corrections to the kinetic energy. (We assume that the second fundamental form or extrinsic curvature of the surface $\psi^\Gamma = 0$ is nonzero, otherwise these corrections would vanish.)

As a simple example, consider the theory

$$\mathcal{L} = -\frac{1}{2}(\nabla\chi)^2 - \frac{1}{2}(\nabla\psi)^2 - \frac{1}{2}M^2\psi^2 + \psi(\nabla\chi)^2/m_P. \tag{H.7}$$

The equation of motion for ψ is $\Box\psi - M^2\psi = (\nabla\chi)^2/m_P$ with leading order solution $\psi = -(\nabla\chi)^2/(m_P M^2)$. The corresponding corrections to the action for χ scale as $(\nabla\chi)^4/(m_P^2 M^2)$, in agreement with Eq. (4.5.5). The scaling (4.5.5) of other operators can be derived similarly.

References

[1] Jolyon K. Bloomfield and Eanna E. Flanagan. A Four-dimensional description of five-dimensional N-brane models. *Physical Review D*, 82:124013, 2010. arXiv:1004.2477, doi:10.1103/PhysRevD.82.124013.

[2] Jolyon K. Bloomfield and Eanna E. Flanagan. Gravitational Interactions in a General Multibrane Model. *Physical Review D*, 84:104016, 2011. arXiv:1009.4686, doi:10.1103/PhysRevD.84.104016.

[3] Jolyon K. Bloomfield and Eanna E. Flanagan. A Class of Effective Field Theory Models of Cosmic Acceleration. 2011. arXiv:1112.0303.

[4] Nima Arkani-Hamed, Savas Dimopoulos, and G.R. Dvali. The Hierarchy problem and new dimensions at a millimeter. *Physics Letters B*, 429:263–272, 1998. arXiv:hep-ph/9803315, doi:10.1016/S0370-2693(98)00466-3.

[5] G.R. Dvali, Gregory Gabadadze, and Massimo Porrati. 4-D gravity on a brane in 5-D Minkowski space. *Physics Letters B*, 485:208–214, 2000. arXiv:hep-th/0005016, doi:10.1016/S0370-2693(00)00669-9.

[6] Lisa Randall and Raman Sundrum. A Large mass hierarchy from a small extra dimension. *Physical Review Letters*, 83:3370–3373, 1999. arXiv:hep-ph/9905221, doi:10.1103/PhysRevLett.83.3370.

[7] Lisa Randall and Raman Sundrum. An Alternative to compactification. *Physical Review Letters*, 83:4690–4693, 1999. arXiv:hep-th/9906064, doi:10.1103/PhysRevLett.83.4690.

[8] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman, September 1973.

[9] S. Perlmutter et al. Measurements of Omega and Lambda from 42 high redshift supernovae. *The Astrophysical Journal*, 517:565–586, 1999. `arXiv:astro-ph/9812133`, `doi:10.1086/307221`.

[10] Adam G. Riess et al. Observational evidence from supernovae for an accelerating universe and a cosmological constant. *The Astronomical Journal*, 116:1009–1038, 1998. `arXiv:astro-ph/9805201`, `doi:10.1086/300499`.

[11] Scott Dodelson. *Modern Cosmology*. Academic Press, 2003.

[12] WMAP. Cosmological Parameter Estimation from WMAP7 Data, 2012. URL: `http://lambda.gsfc.nasa.gov/product/map/dr4/parameters.cfm`.

[13] N. Suzuki, D. Rubin, C. Lidman, G. Aldering, R. Amanullah, et al. The Hubble Space Telescope Cluster Supernova Survey: V. Improving the Dark Energy Constraints Above z>1 and Building an Early-Type-Hosted Supernova Sample. *Astrophysical Journal*, 746:85, 2012. `arXiv:1105.3470`, `doi:10.1088/0004-637X/746/1/85`.

[14] Saul Perlmutter. Reprinted with permission from “Supernovae, Dark Energy, and the Accelerating Universe”. Copyright 2003, American Institute of Physics. *Physics Today*, pages 53–60, April 2003. URL: `http://supernova.lbl.gov/PhysicsTodayArticle.pdf`.

[15] Malcolm Hicken, W. Michael Wood-Vasey, Stephane Blondin, Peter Challis, Saurabh Jha, et al. Improved Dark Energy Constraints from 100 New CfA Supernova Type Ia Light Curves. *Astrophysical Journal*, 700:1097–1140, 2009. `arXiv:0901.4804`, `doi:10.1088/0004-637X/700/2/1097`.

[16] Andreas Albrecht, Gary Bernstein, Robert Cahn, Wendy L. Freedman, Jacqueline Hewitt, et al. Report of the Dark Energy Task Force. 2006. `arXiv:astro-ph/0609591`.

[17] Chris Blake, Eyal Kazin, Florian Beutler, Tamara Davis, David Parkinson, et al. The WiggleZ Dark Energy Survey: mapping the distance-redshift relation with baryon

acoustic oscillations. *Monthly Notices of the Royal Astronomical Society*, 418:1707–1724, 2011. arXiv:1108.2635, doi:10.1111/j.1365-2966.2011.19592.x.

[18] C. Armendariz-Picon, Viatcheslav F. Mukhanov, and Paul J. Steinhardt. A Dynamical solution to the problem of a small cosmological constant and late time cosmic acceleration. *Physical Review Letters*, 85:4438–4441, 2000. arXiv:astro-ph/0004134, doi:10.1103/PhysRevLett.85.4438.

[19] Takeshi Chiba, Takahiro Okabe, and Masahide Yamaguchi. Kinetically driven quintessence. *Physical Review D*, 62:023511, 2000. arXiv:astro-ph/9912463, doi:10.1103/PhysRevD.62.023511.

[20] Nima Arkani-Hamed, Hsin-Chia Cheng, Markus A. Luty, and Shinji Mukohyama. Ghost condensation and a consistent infrared modification of gravity. *Journal of High Energy Physics*, 0405:074, 2004. arXiv:hep-th/0312099, doi:10.1088/1126-6708/2004/05/074.

[21] P.J.E. Peebles and Bharat Ratra. The Cosmological constant and dark energy. *Reviews of Modern Physics*, 75:559–606, 2003. arXiv:astro-ph/0207347, doi:10.1103/RevModPhys.75.559.

[22] Stefan Nobbenhuis. Categorizing different approaches to the cosmological constant problem. *Foundations of Physics*, 36:613–680, 2006. arXiv:gr-qc/0411093, doi:10.1007/s10701-005-9042-8.

[23] Edmund J. Copeland, M. Sami, and Shinji Tsujikawa. Dynamics of dark energy. *International Journal of Modern Physics*, D15:1753–1936, 2006. arXiv:hep-th/0603057, doi:10.1142/S021827180600942X.

[24] Robert R. Caldwell and Marc Kamionkowski. The Physics of Cosmic Acceleration. *Annual Review of Nuclear and Particle Science*, 59:397–429, 2009. arXiv:0903.0866, doi:10.1146/annurev-nucl-010709-151330.

[25] Alessandra Silvestri and Mark Trodden. Approaches to Understanding Cosmic Acceleration. *Reports on Progress in Physics*, 72:096901, 2009. arXiv:0904.0024, doi:10.1088/0034-4885/72/9/096901.

[26] Timothy Clifton, Pedro G. Ferreira, Antonio Padilla, and Constantinos Skordis. Modified Gravity and Cosmology. *Physics Reports*, 513:1–189, 2012. `arXiv:1106.2476`, `doi:10.1016/j.physrep.2012.01.001`.

[27] Luca Amendola and Shinji Tsujikawa. *Dark Energy: Theory and Observations*. Cambridge University Press, 2010.

[28] Steven Weinberg. Photons and Gravitons in Perturbation Theory: Derivation of Maxwell's and Einstein's Equations. *Physical Review*, 138:B988–B1002, May 1965. `doi:10.1103/PhysRev.138.B988`.

[29] Alberto Nicolis, Riccardo Rattazzi, and Enrico Trincherini. The Galileon as a local modification of gravity. *Physical Review D*, 79:064036, 2009. `arXiv:0811.2197`, `doi:10.1103/PhysRevD.79.064036`.

[30] C. Deffayet, Gilles Esposito-Farese, and A. Vikman. Covariant Galileon. *Physical Review D*, 79:084003, 2009. `arXiv:0901.1314`, `doi:10.1103/PhysRevD.79.084003`.

[31] C. Deffayet, S. Deser, and G. Esposito-Farese. Arbitrary p-form Galileons. *Physical Review D*, 82:061501, 2010. `arXiv:1007.5278`, `doi:10.1103/PhysRevD.82.061501`.

[32] H. van Dam and M. Veltman. Massive and mass-less Yang-Mills and gravitational fields. *Nuclear Physics B*, 22(2):397 – 411, 1970. `doi:10.1016/0550-3213(70)90416-5`.

[33] Vladimir E. Zakharov. Linearized gravitation theory and the graviton mass. *JETP Letters*, 12:312, 1970.

[34] David G. Boulware and S. Deser. Can Gravitation Have a Finite Range? *Physical Review D*, 6:3368–3382, Dec 1972. `doi:10.1103/PhysRevD.6.3368`.

[35] Claudia de Rham, Gregory Gabadadze, and Andrew J. Tolley. Resummation of Massive Gravity. *Physical Review Letters*, 106:231101, 2011. `arXiv:1011.1232`, `doi:10.1103/PhysRevLett.106.231101`.

[36] Serguei Chatrchyan et al. Observation of a new boson at a mass of 125 GeV with

the CMS experiment at the LHC. *Physics Letters B*, 2012. arXiv:1207.7235, doi:10.1016/j.physletb.2012.08.021.

[37] Georges Aad et al. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Physics Letters B*, 2012. arXiv:1207.7214.

[38] Theodor Kaluza. Zum Unitätsproblem in der Physik. *Sitzungsber. Preuss. Akad. Wiss. Berlin. (Math. Phys.)*, 1921:966–972, 1921.

[39] Oskar Klein. Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift für Physik A*, 37:895–906, 1926.

[40] Philippe Brax and Carsten van de Bruck. Cosmology and brane worlds: A Review. *Classical and Quantum Gravity*, 20:R201–R232, 2003. arXiv:hep-th/0303095, doi:10.1088/0264-9381/20/9/202.

[41] V.A. Rubakov. Large and infinite extra dimensions: An Introduction. *Physics-Uspekhi*, 44:871–893, 2001. arXiv:hep-ph/0104152, doi:10.1070/PU2001v044n09ABEH001000.

[42] Roy Maartens and Kazuya Koyama. Brane-World Gravity. *Living Reviews in Relativity*, 13(5), 2010. URL: http://www.livingreviews.org/lrr-2010-5.

[43] Ignatios Antoniadis, Nima Arkani-Hamed, Savas Dimopoulos, and G.R. Dvali. New dimensions at a millimeter to a Fermi and superstrings at a TeV. *Physics Letters B*, 436:257–263, 1998. arXiv:hep-ph/9804398, doi:10.1016/S0370-2693(98)00860-0.

[44] Walter D. Goldberger and Mark B. Wise. Bulk fields in the Randall-Sundrum compactification scenario. *Physical Review D*, 60:107505, 1999. arXiv:hep-ph/9907218, doi:10.1103/PhysRevD.60.107505.

[45] Walter D. Goldberger and Mark B. Wise. Modulus stabilization with bulk fields. *Physical Review Letters*, 83:4922–4925, 1999. arXiv:hep-ph/9907447, doi:10.1103/PhysRevLett.83.4922.

[46] Walter D. Goldberger and Mark B. Wise. Phenomenology of a stabilized modulus. *Physics Letters B*, 475:275–279, 2000. arXiv:hep-ph/9911457, doi:10.1016/S0370-2693(00)00099-X.

[47] Freddy P. Zen, Arianto, Bobby Eka Gunara, and Hishamuddin Zainuddin. The Low energy effective equations of motion for multibrane world gravity. 2005. arXiv:hep-th/0511257.

[48] Ludovica Cotta-Ramusino. Low energy effective theory for brane world models. Master's thesis, University of Portsmouth, 2004. URL: http://www.icg.port.ac.uk/theses/cotta-ramusino.pdf.

[49] Ian I. Kogan, Stavros Mouslopoulos, Antonios Papazoglou, Graham G. Ross, and Jose Santiago. A Three three-brane universe: New phenomenology for the new millennium? *Nuclear Physics B*, 584:313–328, 2000. arXiv:hep-ph/9912552, doi:10.1016/S0550-3213(00)00241-8.

[50] Ian I. Kogan, Stavros Mouslopoulos, Antonios Papazoglou, and Graham G. Ross. Multi-brane worlds and modification of gravity at large scales. *Nuclear Physics B*, 595:225–249, 2001. arXiv:hep-th/0006030, doi:10.1016/S0550-3213(00)00664-7.

[51] Ian I. Kogan, Stavros Mouslopoulos, Antonios Papazoglou, and Luigi Pilo. Radion in multibrane world. *Nuclear Physics B*, 625:179–197, 2002. arXiv:hep-th/0105255, doi:10.1016/S0550-3213(02)00009-3.

[52] Eanna Flanagan, Nicholas T. Jones, Horace Stoica, S.H. Henry Tye, and Ira Wasserman. A Brane world perspective on the cosmological constant and the hierarchy problems. *Physical Review D*, 64:045007, 2001. arXiv:hep-th/0012129, doi:10.1103/PhysRevD.64.045007.

[53] Csaba Csaki. TASI lectures on extra dimensions and branes. pages 605–698, 2004. arXiv:hep-ph/0404096.

[54] Abdel Perez-Lorenzana. An Introduction to the Brane World. 2004. arXiv:hep-ph/0406279.

[55] Jaume Garriga and Takahiro Tanaka. Gravity in the brane world. *Physical Review Letters*, 84:2778–2781, 2000. arXiv:hep-th/9911055, doi:10.1103/PhysRevLett.84.2778.

[56] Hideaki Kudoh and Takahiro Tanaka. Second order perturbations in the Randall-Sundrum infinite brane world model. *Physical Review D*, 64:084022, 2001. arXiv:hep-th/0104049, doi:10.1103/PhysRevD.64.084022.

[57] Hideaki Kudoh and Takahiro Tanaka. Second order perturbations in the radius stabilized Randall-Sundrum two branes model. *Physical Review D*, 65:104034, 2002. arXiv:hep-th/0112013, doi:10.1103/PhysRevD.65.104034.

[58] Marcela S. Carena, Joseph D. Lykken, and Minjoon Park. The Interval approach to braneworld gravity. *Physical Review D*, 72:084017, 2005. arXiv:hep-ph/0506305, doi:10.1103/PhysRevD.72.084017.

[59] Petter Callin and Finn Ravndal. Lagrangian formalism of gravity in the Randall-Sundrum model. *Physical Review D*, 72:064026, 2005. arXiv:hep-ph/0412109, doi:10.1103/PhysRevD.72.064026.

[60] Tetsuya Shiromizu, Kei-ichi Maeda, and Misao Sasaki. The Einstein equation on the 3-brane world. *Physical Review D*, 62:024012, 2000. arXiv:gr-qc/9910076, doi:10.1103/PhysRevD.62.024012.

[61] Sugumi Kanno and Jiro Soda. Brane world effective action at low-energies and AdS / CFT. *Physical Review D*, 66:043526, 2002. arXiv:hep-th/0205188, doi:10.1103/PhysRevD.66.043526.

[62] Sugumi Kanno and Jiro Soda. Radion and holographic brane gravity. *Physical Review D*, 66:083506, 2002. arXiv:hep-th/0207029, doi:10.1103/PhysRevD.66.083506.

[63] Jiro Soda and Sugumi Kanno. Holographic view of cosmological perturbations. *Astrophysics and Space Science*, 283:639–644, 2003. arXiv:gr-qc/0209086, doi:10.1023/A:1022519910504.

[64] Frederico Arroja. *On the four-dimensional effective theories in brane-worlds*. PhD thesis, University of Portsmouth, 2008. arXiv:0812.1431.

[65] Toby Wiseman. Strong brane gravity and the radion at low-energies. *Classical and Quantum Gravity*, 19:3083–3106, 2002. arXiv:hep-th/0201127, doi:10.1088/0264-9381/19/11/322.

[66] Takeshi Chiba. Scalar tensor gravity in two 3-brane system. *Physical Review D*, 62:021502, 2000. arXiv:gr-qc/0001029, doi:10.1103/PhysRevD.62.021502.

[67] Sugumi Kanno and Jiro Soda. On the validity of a factorizable metric ansatz in string cosmology. *Physical Review D*, 71:044031, 2005. arXiv:hep-th/0410061, doi:10.1103/PhysRevD.71.044031.

[68] Thibault Damour and Ian I. Kogan. Effective Lagrangians and universality classes of nonlinear bigravity. *Physical Review D*, 66:104024, 2002. arXiv:hep-th/0206042, doi:10.1103/PhysRevD.66.104024.

[69] Rachel Bean, Eanna E. Flanagan, Istvan Laszlo, and Mark Trodden. Constraining Interactions in Cosmology's Dark Sector. *Physical Review D*, 78:123514, 2008. arXiv:0808.1105, doi:10.1103/PhysRevD.78.123514.

[70] G.W. Gibbons and S.W. Hawking. Action Integrals and Partition Functions in Quantum Gravity. *Physical Review D*, 15:2752–2756, 1977. doi:10.1103/PhysRevD.15.2752.

[71] W. Israel. Singular hypersurfaces and thin shells in general relativity. *Nuovo Cimento B*, 44S10:1, 1966. doi:10.1007/BF02710419,10.1007/BF02712210.

[72] Andreas Karch and Lisa Randall. Locally localized gravity. *Journal of High Energy Physics*, 2001(05):008, 2001. arXiv:hep-th/0011156, doi:10.1088/1126-6708/2001/05/008.

[73] R. Gregory and R. Laflamme. Black strings and p-branes are unstable. *Physical Review Letters*, 70:2837–2840, 1993. arXiv:hep-th/9301052, doi:10.1103/PhysRevLett.70.2837.

[74] A. Chamblin, S.W. Hawking, and H.S. Reall. Brane world black holes. *Physical Review D*, 61:065007, 2000. `arXiv:hep-th/9909205`, `doi:10.1103/PhysRevD.61.065007`.

[75] Pau Figueras, Toby Wiseman, and Toby Wiseman. Gravity and large black holes in Randall-Sundrum II braneworlds. *Physical Review Letters*, 107:081101, 2011. `arXiv:1105.2558`, `doi:10.1103/PhysRevLett.107.081101`.

[76] Eanna E. Flanagan. Higher order gravity theories and scalar tensor theories. *Classical and Quantum Gravity*, 21:417–426, 2003. `arXiv:gr-qc/0309015`, `doi:10.1088/0264-9381/21/2/006`.

[77] Clifford M. Will. The Confrontation between General Relativity and Experiment. *Living Reviews in Relativity*, 9(3), 2006. URL: `http://www.livingreviews.org/lrr-2006-3`.

[78] Rachel Bean, Eanna E. Flanagan, and Mark Trodden. Adiabatic instability in coupled dark energy-dark matter models. *Physical Review D*, 78:023009, 2008. `arXiv:0709.1128`, `doi:10.1103/PhysRevD.78.023009`.

[79] Michael Kesden and Marc Kamionkowski. Galilean Equivalence for Galactic Dark Matter. *Physical Review Letters*, 97:131303, 2006. `arXiv:astro-ph/0606566`, `doi:10.1103/PhysRevLett.97.131303`.

[80] Clifford Cheung, Paolo Creminelli, A. Liam Fitzpatrick, Jared Kaplan, and Leonardo Senatore. The Effective Field Theory of Inflation. *Journal of High Energy Physics*, 0803:014, 2008. `arXiv:0709.0293`, `doi:10.1088/1126-6708/2008/03/014`.

[81] Leonardo Senatore and Matias Zaldarriaga. The Effective Field Theory of Multifield Inflation. *Journal of High Energy Physics*, 1204:024, 2012. `arXiv:1009.2093`, `doi:10.1007/JHEP04(2012)024`.

[82] Matteo Fasiello. *Effective Field Theory for Inflation*. PhD thesis, Universita degli Studi di Milano Bicocca, 2011. `arXiv:1106.2189`.

[83] Steven Weinberg. Effective Field Theory for Inflation. *Physical Review D*, 77:123541, 2008. `arXiv:0804.4291`, `doi:10.1103/PhysRevD.77.123541`.

[84] C.P. Burgess, Hyun Min Lee, and Michael Trott. Power-counting and the Validity of the Classical Approximation During Inflation. *Journal of High Energy Physics*, 0909:103, 2009. `arXiv:0902.4465`, `doi:10.1088/1126-6708/2009/09/103`.

[85] Paolo Creminelli, Guido D'Amico, Jorge Norena, and Filippo Vernizzi. The Effective Theory of Quintessence: the w<-1 Side Unveiled. *Journal of Cosmology and Astroparticle Physics*, 0902:018, 2009. `arXiv:0811.0827`, `doi:10.1088/1475-7516/2009/02/018`.

[86] Minjoon Park, Kathryn M. Zurek, and Scott Watson. A Unified Approach to Cosmic Acceleration. *Physical Review D*, 81:124008, 2010. `arXiv:1003.1722`, `doi:10.1103/PhysRevD.81.124008`.

[87] C. H. Brans and R. H. Dicke. Mach's Principle and a Relativistic Theory of Gravitation. *Physical Review*, 124:925–935, November 1961. `doi:10.1103/PhysRev.124.925`.

[88] C. H. Brans. Mach's Principle and a Relativistic Theory of Gravitation. II. *Physical Review*, 125:2194–2201, March 1962. `doi:10.1103/PhysRev.125.2194`.

[89] Richard P. Woodard. Avoiding dark energy with 1/r modifications of gravity. *Lecture Notes in Physics*, 720:403–433, 2007. `arXiv:astro-ph/0601672`, `doi:10.1007/978-3-540-71013-4_14`.

[90] L. D. Landau and E. M. Lifshitz. *The classical theory of fields*. Pergamon Press, 1971.

[91] Leonard Parker and Jonathan Z. Simon. Einstein equation with quantum corrections reduced to second order. *Physical Review D*, 47:1339–1355, 1993. `arXiv:gr-qc/9211002`, `doi:10.1103/PhysRevD.47.1339`.

[92] Eanna E. Flanagan and Robert M. Wald. Does back reaction enforce the averaged null energy condition in semiclassical gravity? *Physical Review D*, 54:6233–6283, 1996. `arXiv:gr-qc/9602052`, `doi:10.1103/PhysRevD.54.6233`.

[93] Jonathan Z. Simon. Higher-derivative Lagrangians, nonlocality, problems, and solutions. *Physical Review D*, 41:3720, 1990. `doi:10.1103/PhysRevD.41.3720`.

[94] Joshua A. Frieman, Christopher T. Hill, Albert Stebbins, and Ioav Waga. Cosmology with ultralight pseudo Nambu-Goldstone bosons. *Physical Review Letters*, 75:2077–2080, 1995. arXiv:astro-ph/9505060, doi:10.1103/PhysRevLett.75.2077.

[95] Nima Arkani-Hamed, Hsin-Chia Cheng, Paolo Creminelli, and Lisa Randall. Pseudo-natural inflation. *Journal of Cosmology and Astroparticle Physics*, 0307:003, 2003. arXiv:hep-th/0302034, doi:10.1088/1475-7516/2003/07/003.

[96] Raul Jimenez, P. Talavera, and Licia Verde. An effective theory of accelerated expansion. 2011. arXiv:1107.2542.

[97] Cedric Deffayet, Oriol Pujolas, Ignacy Sawicki, and Alexander Vikman. Imperfect Dark Energy from Kinetic Gravity Braiding. *Journal of Cosmology and Astroparticle Physics*, 1010:026, 2010. arXiv:1008.0048, doi:10.1088/1475-7516/2010/10/026.

[98] Christos Charmousis, Edmund J. Copeland, Antonio Padilla, and Paul M. Saffin. General second order scalar-tensor theory, self tuning, and the Fab Four. *Physical Review Letters*, 108:051101, 2012. arXiv:1106.2000, doi:10.1103/PhysRevLett.108.051101.

[99] Stephon Alexander and Nicolas Yunes. Chern-Simons Modified General Relativity. *Physics Reports*, 480:1–55, 2009. arXiv:0907.2562, doi:10.1016/j.physrep.2009.07.002.

[100] David Wands. Multiple Field Inflation. In Martin Lemoine, Jerome Martin, and Patrick Peter, editors, *Inflationary Cosmology*, volume 738 of *Lecture Notes in Physics*, pages 275–304. Springer Berlin / Heidelberg, 2007.

[101] Daniel Baumann and Liam McAllister. Advances in Inflation in String Theory. *Annual Review of Nuclear and Particle Science*, 59:67–94, 2009. arXiv:0901.0265, doi:10.1146/annurev.nucl.010909.083524.

[102] Sudhakar Panda, Yoske Sumitomo, and Sandip P. Trivedi. Axions as Quintessence in String Theory. *Physical Review D*, 83:083506, 2011. arXiv:1011.5877, doi:10.1103/PhysRevD.83.083506.

[103] Justin Khoury and Amanda Weltman. Chameleon fields: Awaiting surprises for tests of gravity in space. *Physical Review Letters*, 93:171104, 2004. `arXiv:astro-ph/0309300`, `doi:10.1103/PhysRevLett.93.171104`.

[104] Philippe Brax and Jerome Martin. The Robustness of quintessence. *Physical Review D*, 61:103502, 2000. `arXiv:astro-ph/9912046`, `doi:10.1103/PhysRevD.61.103502`.

[105] Michael Doran and Jörg Jäckel. Loop corrections to scalar quintessence potentials. *Physical Review D*, 66:043519, Aug 2002. `arXiv:astro-ph/0205206`, `doi:10.1103/PhysRevD.66.043519`.

[106] Massimo Pietroni. Dark energy condensation. *Physical Review D*, 72:043535, 2005. `arXiv:astro-ph/0505615`, `doi:10.1103/PhysRevD.72.043535`.

[107] Mathias Garny. Quantum corrections in quintessence models. *Physical Review D*, 74:043009, 2006. `arXiv:hep-ph/0606120`, `doi:10.1103/PhysRevD.74.043009`.

[108] Alexandre Arbey and Farvah Mahmoudi. One-loop quantum corrections to cosmological scalar field potentials. *Physical Review D*, 75:063513, 2007. `arXiv:hep-th/0703053`, `doi:10.1103/PhysRevD.75.063513`.

[109] A.I. Vainshtein. To the problem of nonvanishing gravitation mass. *Physics Letters B*, 39:393–394, 1972. `doi:10.1016/0370-2693(72)90147-5`.

[110] Markus A. Luty, Massimo Porrati, and Riccardo Rattazzi. Strong interactions and stability in the DGP model. *Journal of High Energy Physics*, 0309:029, 2003. `arXiv:hep-th/0303116`, `doi:10.1088/1126-6708/2003/09/029`.

[111] Alberto Nicolis and Riccardo Rattazzi. Classical and quantum consistency of the DGP model. *Journal of High Energy Physics*, 0406:059, 2004. `arXiv:hep-th/0404159`, `doi:10.1088/1126-6708/2004/06/059`.

[112] Justin Khoury and Amanda Weltman. Chameleon cosmology. *Physical Review D*, 69:044026, 2004. `arXiv:astro-ph/0309411`, `doi:10.1103/PhysRevD.69.044026`.

[113] Gregory Walter Horndeski. Second-order scalar-tensor field equations in a four-dimensional space. *International Journal of Theoretical Physics*, 10(6):363–384, September 1974. doi:10.1007/BF01807638.

[114] Diego Blas, Oriol Pujolas, and Sergey Sibiryakov. Models of non-relativistic quantum gravity: The Good, the bad and the healthy. *Journal of High Energy Physics*, 1104:018, 2011. arXiv:1007.3503, doi:10.1007/JHEP04(2011)018.

[115] Lam Hui, Alberto Nicolis, and Christopher Stubbs. Equivalence Principle Implications of Modified Gravity Models. *Physical Review D*, 80:104002, 2009. arXiv:0905.2966, doi:10.1103/PhysRevD.80.104002.

[116] Cristian Armendariz-Picon and Riccardo Penco. Quantum Equivalence Principle Violations in Scalar-Tensor Theories. *Physical Review D*, 85:044052, 2012. arXiv:1108.6028, doi:10.1103/PhysRevD.85.044052.

[117] Takeshi Chiba. Generalized gravity and ghost. *Journal of Cosmology and Astroparticle Physics*, 0503:008, 2005. arXiv:gr-qc/0502070, doi:10.1088/1475-7516/2005/03/008.