Identities inside the Gluon and the Graviton Scattering Amplitudes— A Proof of BCJ conjecture

The duality between the color/kinematic factors and the duality between gluon and graviton scattering amplitude via Heterotic string theory

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Motivation

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$$A_{1234} + A_{2134} + A_{1324} = 0.$$ photon decoupling theorem.

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BCJ conjecture

M-gluon tree amplitude in pure YM theory is

$$\mathcal{A}_M^{\text{YM}} = \sum_i^{(2M-5)!!} \frac{c_i n_i}{P_i}.$$ c_i color factor. n_i kinematic factors. P_i poles.

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1. If three color factors satisfy (Jacobi) $c_i + c_j + c_k = 0$, then the corresponding $n_i + n_j + n_k = 0$.
2. M-graviton tree amplitude in Einstein theory is

$$A_M^{\text{Grav}} = \sum_i^{(2M-5)!!} \frac{n_i n_i}{P_i}.\ \text{same } n_i \text{ and } P_i$$

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Although the BCJ conjecture-1 seems simple, it was not noticed until recently when people are working on loop amplitude. The direct proof with Feynman rules soon became too complicated. BCJ conjecture-2 is almost impossible to prove just by Feynman rules.

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Recall that in heterotic string theory, the color index is represented by discrete momentum in root lattice. The whole Lie-algebra structure can be understood as the interaction of strings with discrete momentum.

And heterotic string theory contains graviton!

That is a hint for BCJ conjecture-2...

Our strategy: Heterotic string theory + KLT relation

Heterotic string theory is closed string theory, within it

$$\begin{aligned} \text{Gluon} &= \text{color sector} \times \text{vector sector} \\ \text{Graviton} &= \text{vector sector} \times \text{vector sector} \end{aligned}$$

KLT relation, (H.Kawai, D.C.Lewellen and H.Tye), shown that

$$\text{closed amplitude} \propto (\text{left open amplitude}) \times (\text{right open amplitude})$$

- Open amplitudes, by contour integral argument, would satisfy the same kind of identities, no matter they are left/right, vector/color. BCJ conjecture-1 is proven.
- When left sector: color → vector, the c_i are replaced by n_i's, so KLT relation gives,

$$A^{\text{YM}} = \sum_i \frac{c_i n_i}{P_i} \to A^{\text{Grav}} = \sum_i \frac{n_i n_i}{P_i}$$

Outline

- Introduction
- (Physics 651) BCJ conjecture in the view point of field theory.
- Review of the heterotic string theory, in the low energy limit
- Proof of BCJ conjecture-1: 4-point example
- Proof of BCJ conjecture-1: general case
- Graviton scattering amplitude and other amplitudes
- Summary

4-gluon example

Scattering amplitude for four gluons, (k_1, a_1, ζ_1), (k_2, a_2, ζ_2), (k_3, a_3, ζ_3) and (k_4, a_4, ζ_4) is easily obtained by Feynman rules,

$$\mathcal{A}_4^{\mathrm{YM}} = \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}$$

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$$\mathcal{A}_4^{\mathrm{YM}} = \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}$$

where the 4-point vertex contribution is absorb into s, t and u channels. $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$, $c_t = f^{a_2 a_3 b} f^{b a_1 a_4}$ and $c_u = f^{a_3 a_1 b} f^{b a_2 a_4}$.

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$$\begin{aligned} n_s \quad = \quad & i[(\zeta_1 \cdot \zeta_2)(k_2 - k_1) - (2k_2 \cdot \zeta_1)\zeta_2 + (2k_1 \cdot \zeta_2)\zeta_1] \\ & \times[(\zeta_3 \cdot \zeta_4)(k_4 - k_3) - (2k_4 \cdot \zeta_3)\zeta_4 + (2k_3 \cdot \zeta_4)\zeta_3] \\ & -i[(\zeta_1 \cdot \zeta_3)(\zeta_2 \cdot \zeta_4) - (\zeta_1 \cdot \zeta_4)(\zeta_2 \cdot \zeta_3)]s \\ & n_t = ..., n_u = ... \end{aligned}$$

4-gluon scattering example

It is easy to see that, by Jacobi identity,

$$c_s + c_t + c_u = f^{a_1 a_2 b} f^{b a_3 a_4} + f^{a_2 a_3 b} f^{b a_1 a_4} + f^{a_3 a_1 b} f^{b a_2 a_4} = 0$$

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However, it is amazing that the kinematic factors satisfy the same identity as the color factors,

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Why do the color factors and the kinematic factor satisfy the same kind of identity?

5-gluon scattering example

More complicated, 15 channels

$$A_5^{\text{YM}} = \frac{c_1 n_1}{s_{12}s_{45}} + \frac{c_2 n_2}{s_{15}s_{23}} + \frac{c_3 n_3}{s_{12}s_{34}} + \frac{c_4 n_4}{s_{23}s_{45}} + \frac{c_5 n_5}{s_{15}s_{34}} + \frac{c_6 n_6}{s_{14}s_{25}} + \frac{c_7 n_7}{s_{14}s_{23}} + \frac{c_8 n_8}{s_{34}s_{25}} + \frac{c_9 n_9}{s_{13}s_{25}} + \frac{c_{10} n_{10}}{s_{13}s_{24}} + \frac{c_{11} n_{11}}{s_{15}s_{24}} + \frac{c_{12} n_{12}}{s_{12}s_{35}} + \frac{c_{13} n_{13}}{s_{24}s_{35}} + \frac{c_{14} n_{14}}{s_{14}s_{35}} + \frac{c_{15} n_{15}}{s_{13}s_{45}}$$

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Still, the color factors and the kinematic factors satisfy the same identities,

$$\begin{aligned}
c_4 + c_{15} - c_1 &= 0, & n_4 + n_{15} - n_1 &= 0 \\
c_4 + c_7 - c_2 &= 0, & n_4 + n_7 - n_2 &= 0 \\
c_8 + c_9 - c_6 &= 0, & n_8 + n_9 - n_6 &= 0 \\
c_3 + c_8 - c_5 &= 0, & n_3 + n_8 - n_5 &= 0 \\
&\dots
\end{aligned}$$

10 identities for c_i's, and 10 same identities for n_i's.

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The field theory amplitude calculation seems obscure so we may turn to string theory (and take its low energy limit).

Why heterotic string theory?

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Massless spectrum in Heterotic string theory

As a closed string theory,

$$\text{State} = \text{left-moving sector} \times \text{right-moving sector}$$

Massless left-moving sector

1. Vector sector. $i\xi_\mu \partial X^\mu e^{ik_\nu X^\nu}$
2. Color sector. $e^{ik_\nu X^\nu + iK_I X^I}$ or $i\zeta_I \partial X^I e^{ik_\nu X^\nu}$. K, discrete momentum, ζ^I, Cartan Lie algebra.

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$$\begin{aligned}
\text{Gluon} &= \text{color sector} \times \text{vector sector} \\
\text{Graviton} &= \text{vector sector} \times \text{vector sector}\big|_{\xi_\mu \zeta_\nu \to \epsilon_{\mu\nu}} \\
\text{Gluino} &= \text{color sector} \times \text{spinor sector} \\
\text{Gravitino} &= \text{vector sector} \times \text{spinor sector}
\end{aligned}$$

Color sector

We look at the color sector more carefully. The Lie algebra of G can be decomposed into the Cartan sub-algebra and the root. Simplest example,

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where ζ is an element in Cartan sub-algebra. For gluon with the color index $\in$ as a root, the vertex operator is

$$e^{ik_\nu X^\nu + iK_I X^I}$$

. where K is a root in the root lattice, which is the momentum space of the extra dimensions.

KLT

KLT relation, by H.Kawai, D.C.Lewellen and H.Tye,

$$\begin{aligned} & \text{closed string amplitude} \\ = \; & \sum \text{left open string amplitude} \times \text{right open string amplitude} \end{aligned}$$

So we will first calculate the left-moving open string amplitude and right-moving open string amplitude separately. In this calculation, we find that the analytic property of the left-moving open amplitude will give the Jacobi identity

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So we will first calculate the left-moving open string amplitude and right-moving open string amplitude separately. In this calculation, we find that the analytic property of the left-moving open amplitude will give the Jacobi identity while the same kind of analytic property of the right-moving amplitude will give the BCJ dual identities.

Left-moving open amplitude

We have 3 partial amplitudes (different vertex orderings),

$$\mathbf{A}^{L(c)}_{2134} = co(2134)\int_{-\infty}^{0} dx_2\ (-x_2)^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2)$$

$$\mathbf{A}^{L(c)}_{1234} = co(1234)\int_{0}^{1} dx_2\ x_2^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2)$$

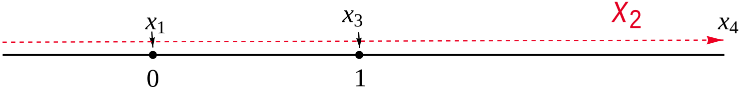
$$\mathbf{A}^{L(c)}_{1324} = co(1324)\int_{1}^{\infty} dx_2\ x_2^{\frac{\alpha'}{2}k_1\cdot k_2+2\alpha' K_1\cdot K_2}(x_2-1)^{\frac{\alpha'}{2}k_2\cdot k_3+2\alpha' K_2\cdot K_3} f(x_2)$$

where $co(1234)$ and etc are the product of co-cycles, which can only be ± 1. $f(x)$ contains the possible polarization in lattice, i.e., color index in Cartan sub-algebra. The three amplitude are related via analytic continuation!

Analytic continuation

The integral in $\mathbf{A}^{L(c)}_{1234}$ can be continued to a contour integral which equals zero,

$$\int_0^1 dx_2\; x_2^{\cdots}(1-x_2)^{\cdots} f(x_2) \rightarrow \int_{-\infty}^{\infty} dx_2\; x_2^{\cdots}(1-x_2)^{\cdots} f(x_2) = 0$$



$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathbf{A}^{L(c)}_{2134} + \mathbf{A}^{L(c)}_{1234} + e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathbf{A}^{L(c)}_{1324} = 0.$$

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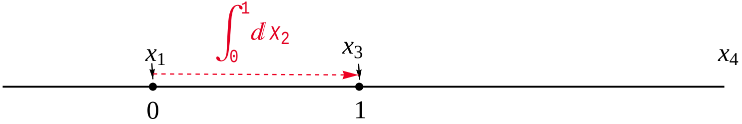

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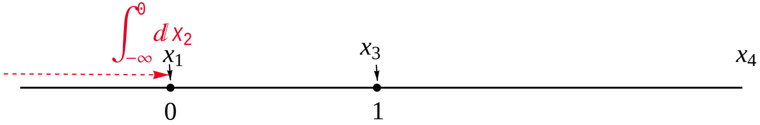
$$\int_0^1 dx_2\ x_2^{\cdots}(1-x_2)^{\cdots} f(x_2) \rightarrow \int_{-\infty}^{\infty} dx_2\ x_2^{\cdots}(1-x_2)^{\cdots} f(x_2) = 0$$



$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathbf{A}^{L(c)}_{2134} + \mathbf{A}^{L(c)}_{1234} + e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathbf{A}^{L(c)}_{1324} = 0.$$

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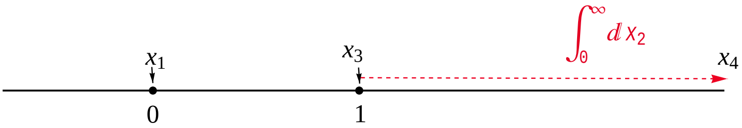

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Low energy limit

In the low energy limit, i.e., the zero slope limit only the massless state (gluon, graviton, etc.) survived so we get the field theory,

$$\lim_{\alpha' \to 0} \mathbf{A}^{L(c)}_{1234} \to A^{L(c)}_{1234}$$

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The contour integral identity is reduced to

$$A^{L(c)}_{2134} + A^{L(c)}_{1234} + A^{L(c)}_{1324} = 0, \text{ real part}$$
$$sA^{L(c)}_{2134} = tA^{L(c)}_{1324}, \text{ imaginary part}$$

where $s = -(k_1 + k_2)^2$, $u = -(k_1 + k_3)^2$ and $t = -(k_1 + k_4)^2$.

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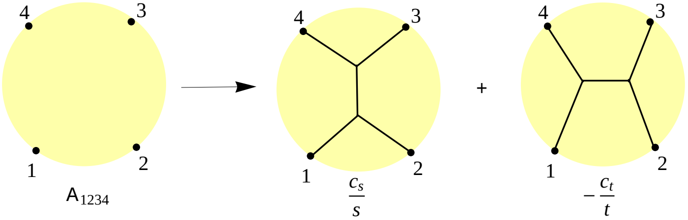
where $s = -(k_1 + k_2)^2$, $u = -(k_1 + k_3)^2$ and $t = -(k_1 + k_4)^2$. The meaning of this identity is not clear, so we look at it more carefully by the channel decomposition.

Channels

One string amplitude, in the low energy limit, will decompose into several



channels,

$$A^{L(c)}_{2134} = -\frac{\tilde{c}_s}{s} + \frac{c_u}{u}, A^{L(c)}_{1234} = \frac{c_s}{s} - \frac{\tilde{c}_t}{t}, A^{L(c)}_{1324} = -\frac{\tilde{c}_u}{u} + \frac{c_t}{t}.$$

Plug into the contour integral identities, we will get the result,

$$A^{L(c)}_{2134} + A^{L(c)}_{1234} + A^{L(c)}_{1324} = 0, \text{ real part}$$
$$sA^{L(c)}_{2134} = tA^{L(c)}_{1324}, \text{ imaginary part}$$

Jacobi identity

We have

$$\tilde{c}_s = c_s, \quad \tilde{c}_u = c_u, \quad \tilde{c}_t = c_t.$$

and,

$$c_s + c_t + c_u = 0.$$

The direct calculation shows that $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$, $c_t = f^{a_2 a_3 b} f^{b a_1 a_4}$ and $c_u = f^{a_3 a_1 b} f^{b a_2 a_4}$.
The contour integral for the left-moving color sector just gives the Jacobi identity, while the same method, applied on the right-moving vector sector will give the non-trivial identities $n_s + n_t + n_u = 0$.

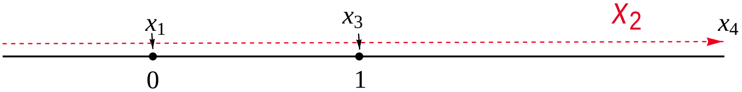
Right-moving amplitude

$$\mathbf{A}_{1234}^{R(v)} = \int_0^1 dx_2 \; x_2^{\frac{\alpha'}{2}k_1\cdot k_2}(1-x_2)^{\frac{\alpha'}{2}k_2\cdot k_3}\bar{f}(x_2), \text{etc.}$$

$$\bar{f}(x_2) = \exp\left(\frac{\alpha'}{2}\sum_{i>j}\frac{\zeta_i\cdot\zeta_j}{(x_i-x_j)^2} - \frac{\alpha'}{2}\sum_{i\neq j}\frac{\zeta_i\cdot k_j}{x_i-x_j}\right)\Bigg|_{\text{multiple-linear}}.$$

The contour integral in x_2 gives,

$$e^{i\pi(\frac{\alpha'}{2}k_1\cdot k_2)}\mathbf{A}_{2134}^{R(v)} + \mathbf{A}_{1234}^{R(v)} + e^{-i\pi(\frac{\alpha'}{2}k_2\cdot k_3)}\mathbf{A}_{1324}^{R(v)} = 0.$$



kinematic identity

$$A^{R(v)}_{2134} = -\frac{n_s}{s} + \frac{n_u}{u}, \; A^{R(v)}_{1234} = \frac{n_s}{s} - \frac{n_t}{t}, \; A^{R(v)}_{1324} = -\frac{n_u}{u} + \frac{n_t}{t}.$$

Unlike the c_i's, the definition of n_s, n_t and n_u is not unique because we can move the contact terms between each other, $n'_s = n_s + cs$, $n'_t = n_t + ct$, $n'_u = n_u + cu$.

kinematic identity

$$A_{2134}^{R(v)} = -\frac{n_s}{s} + \frac{n_u}{u},\ A_{1234}^{R(v)} = \frac{n_s}{s} - \frac{n_t}{t},\ A_{1324}^{R(v)} = -\frac{n_u}{u} + \frac{n_t}{t}.$$

Unlike the c_i's, the definition of n_s, n_t and n_u is not unique because we can move the contact terms between each other, $n'_s = n_s + cs$, $n'_t = n_t + ct$, $n'_u = n_u + cu$.

In the low-energy limit, the imaginary part of the contour integral identity,

$$sA_{2134}^{R(v)} = tA_{1324}^{R(v)}$$

gives,

$$n_s + n_t + n_u = 0,$$

This identity is invariant under the contact term rearrangement,

$$n'_s + n'_t + n'_u = n_s + n_t + n_u + c(s + t + u) = 0$$

4-gluon amplitude

KLT,

$$\mathcal{A}^{\text{het}}_{\text{4-gluon}} \propto \sin\left(\pi \frac{\alpha'}{2} k_2 \cdot k_3\right) \cdot \mathbf{A}^{L(c)}_{1234} \mathbf{A}^{R(v)}_{1324}.$$

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in the low energy limit

$$\begin{aligned}
\mathcal{A}_{\text{4-gluon}} &\propto t\left(\frac{c_s}{s} - \frac{c_t}{t}\right)\left(-\frac{n_u}{u} + \frac{n_t}{t}\right) \\
&= \left(\left(-\frac{c_s n_u}{s} - \frac{c_s n_t}{s}\right) + \left(-\frac{c_s n_u}{u} - \frac{c_t n_u}{u}\right) + \frac{c_t n_t}{t} \right) \\
&= \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t},
\end{aligned}$$

so we get the deserved result with the identities $c_s + c_t + c_u = 0$ and $n_s + n_t + n_u = 0$. The duality between the two identities comes from the same contour integral identity.

M-gluon

This method can be used for arbitary M-gluon tree scattering amplitude. Now there are $(2M-5)!!$ channels, so $(2M-5)!!$ c_i's and n_i's.

$$\mathcal{A}_M^{\rm YM} = \sum_i \frac{c_i n_i}{P_i}$$

New feature We have to integrate over $M-2$ variables, there are many different ways to do contour integral so there are many open string identities.

M-gluon

New feature: One contour integral argument gives $\binom{M-1}{3}$ color (kinematic identities). For instance, if we consider the continuation of the x_2 integral in $A^{L(c)}_{12345}$,

$$-\frac{c_3+c_8-c_5}{s_{34}}-\frac{c_4-c_2+c_7}{s_{23}}+\frac{c_4+c_{15}-c_1}{s_{45}}+\frac{c_8+c_9-c_6}{s_{25}}=0,$$

whose residues are,

$$c_3+c_8-c_5=0,\ c_4-c_2+c_7=0,\ c_4+c_{15}-c_1=0,\ c_8+c_9-c_6=0.$$

By detailed combinatorics, we proved that for arbitary M, the contour integral identities will give all the color identities between c_i's.

The subtlety in n_i's

It seems that as the $M = 4$ case, all the analysis on the color sectors can be directly applied on the vector sector. However, there is a subtlety since n_i contains the contact terms, for example,

$$-\frac{n_3 + n_8 - n_5}{s_{34}} - \frac{n_4 - n_2 + n_7}{s_{23}} + \frac{n_4 + n_{15} - n_1}{s_{45}} + \frac{n_8 + n_9 - n_6}{s_{25}} = 0,$$

n_3, n_8 and n_5 may contain contact terms which are proportional to s_{34} and not residues. By general channel choice, the sum, $n_3 + n_8 - n_5$ always vanishes except that contact terms. (4-point case does not have this subtlety.)

We think that (still working in progress),

- there exist a way to rearrange the contact terms in n_i's such that $n_i + n_j + n_k$ exactly vanish.
- such a way is not unique and actually these choices form a subspace with the dimension $(M-2)! - (M-3)!$.

When the existence of the rearrangement is found, then as the 4-point case, the dual kinematic identities are dual to the Jacobi identities

Graviton amplitude and other amplitudes

Turn to the M-graviton amplitude,

$$\text{Graviton} = \text{vector sector} \times \text{vector sector}$$

Now the left-mover is also vector section. We can repeat all what we did in the gluon scattering case just with some label changing

$$A^{L(c)} \rightarrow A^{R(v)}, c_i \rightarrow n_i.$$

Because we know that the gluon heterotic string amplitude, in the low energy limit, would finally reduce into,

$$\mathcal{A}_M^{\text{YM}} = \sum_i \frac{c_i n_i}{P_i}$$

so the graviton heterotic string amplitude, in the low energy limit, would finally reduce into,

$$\mathcal{A}_M^{\text{grav}} = \sum_i \frac{n_i n_i}{P_i}.$$

So the BCJ conjecture on graviton amplitude is also proven.

4-graviton example

When KLT relation is used on color sector $\times$ vector sector, we have.

$$\begin{aligned} \mathcal{A}_{\text{4-gluon}} &\propto t(\frac{c_s}{s} - \frac{c_t}{t})(-\frac{n_u}{u} + \frac{n_t}{t}) \\ &= \frac{c_s n_s}{s} + \frac{c_u n_u}{u} + \frac{c_t n_t}{t}, \end{aligned}$$

On the other hand, When KLT relation is used on vector sector $\times$ vector sector, we have,

$$\begin{aligned} \mathcal{A}_{\text{4-graviton}} &\propto t(\frac{n_s}{s} - \frac{n_t}{t})(-\frac{n_u}{u} + \frac{n_t}{t}) \\ &= \frac{n_s n_s}{s} + \frac{n_u n_u}{u} + \frac{n_t n_t}{t}, \end{aligned}$$

which is the 4-graviton tree amplitude. The calculation is totally identical except $c_i \longrightarrow n_i$.

Summary

1. Up to the subtlety of the contact terms, we prove BCJ conjecture via heterotic string theory and the dualities between color/kinematic identities and also gluon/graviton are natural.
2. When BCJ conjecture is proven, the calculation of graviton amplitude is dramatically simplified.

Summary

1. Up to the subtlety of the contact terms, we prove BCJ conjecture via heterotic string theory and the dualities between color/kinematic identities and also gluon/graviton are natural.
2. When BCJ conjecture is proven, the calculation of graviton amplitude is dramatically simplified.

Further directions,

1. KLT relation, applied in heterotic string theory, seems to give a duality between the gauge amplitude and gravity amplitude, but different from AdS/CFT. Does this relation illustrate the gauge and gravity in different regime?
2. The loop amplitude is related to the tree amplitude via unitarity relations. So the BCJ conjecture would be generalized to the loop amplitude case.