# Renormalized CFT/ Effective AdS

JiJi Fan

Princeton University

Work in progress

# Outline

- Motivation and goal
- Prescription
- Examples
- Conclusion and open questions

AdS/CFT correspondence: Maldacena conjecture `97

N=4 conformal SYM
$g_s = g^2_{YM}$

↔

Type IIB string theory on $AdS_5 \times S^5$
$R^4 = 4\pi g_s N\, l^4_s$

Phenomenological works motivated by the duality:

Randall-Sundrum models...

AdS/QCD

"bottom-up" AdS/CMT...

- Rules of thumb for model-building

| | | |
|---|---|---|
| Bulk of AdS | $\leftrightarrow$ | CFT |
| Coordinate ($z$) along AdS | $\leftrightarrow$ | Energy scale in CFT |
| Appearance of UV brane | $\leftrightarrow$ | CFT has a cutoff |
| Appearance of IR brane | $\leftrightarrow$ | conformal symmetry broken spontaneously by CFT |
| KK modes localized on IR brane | $\leftrightarrow$ | composites of CFT |
| Modes on the UV brane | $\leftrightarrow$ | Elementary fields coupled to CFT |
| Gauge fields in bulk | $\leftrightarrow$ | CFT has a global symmetry |
| Bulk gauge symmetry broken on UV brane | $\leftrightarrow$ | Global symmetry not gauged |
| Bulk gauge symmetry unbroken on UV brane | $\leftrightarrow$ | Global symmetry weakly gauged |
| Higgs on IR brane | $\leftrightarrow$ | CFT becoming strong produces composite Higgs |
| Bulk gauge symmetry broken on IR brane by BC's | $\leftrightarrow$ | Strong dynamics that breaks CFT also breaks gauge symmetry |

Copied from Csaki, Hubisz and Meade 05’

But unlike the original N=4/Type IIB duality,

the “phenomenological” duality

- Finite N
- Non-supersymmetric

## The "phenomenological" duality

- Finite N — Higher order corrections
- Non-supersymmetric — Instability (won't consider further)
  e.g.: Horowitz-Orgera-Polchinski instablity `07;

- Goal: assuming existence of an effective AdS field theory/CFT correspondence, understand how bulk interactions renormalize CFT 2-point correlator beyond the leading order in N.

  More specifically, calculate anomalous dimensions of single- and double-trace operators arising from bulk interactions. E.g.: scalar single-trace operator:

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2 + \gamma}$$

- Aside: definition of single- and double-trace operators

  Given a matrix-valued field Φ(N × N hermitian matrix),

  Single-trace operators: $\mathrm{Tr}\Phi^n$

  Double-trace operators: $(\mathrm{Tr}\Phi^n)^2$

- Example:

  Scalar operators

  Single-trace operators: $\mathrm{Tr}F^2$ $\psi\bar{\psi}$

  Double-trace operators: $(\mathrm{Tr}F^2)^2$ $(\psi\bar{\psi})^2$

More on the effective AdS/CFT correspondence

# Conjecture: (Heemskerk, Penedones, Polchinski and Sully `09; Fitzpatrick, Katz, Poland, Simmons-Duffin `10)

Sufficient conditions for CFT to have a local bulk dual

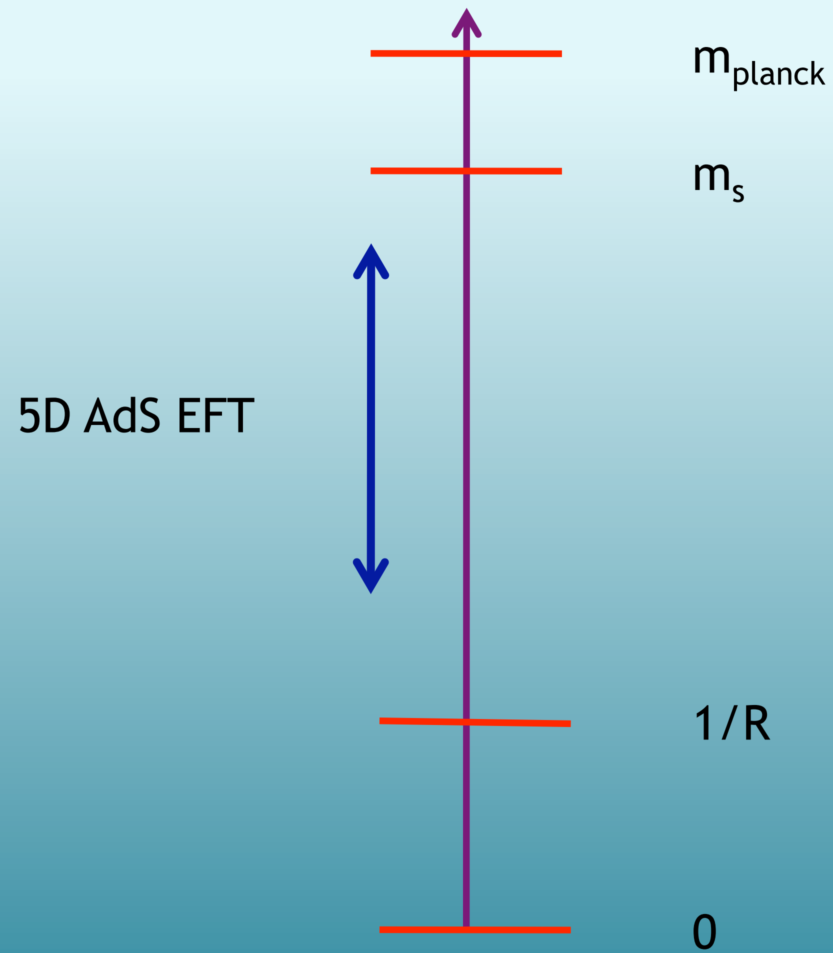
- a mass gap: all single-trace operators of spin greater than 2 have parametrically large dimension.

More specifically, $m_{KK} \sim 1/R$; R: AdS radius

String states $m_s \sim 1/l_s \sim \lambda^{1/4}/R$;

- a small parameter such as 1/N

$M_5 \sim N^{2/3}/R$



$m_{planck}$
$m_s$
5D AdS EFT
1/R
0

- Setup

AdS$_{d+1}$: metric $ds^2 = \frac{z_0^2 + \sum_{i=1}^{d} dx_i^2}{z_0^2}$

scalar field theory with contact interactions

CFT: at the bottom of the spectrum, only one single-trace scalar operator $\mathcal{O}(x)$

Double-trace operator from OPE O×O

$$\mathcal{O}_{n,l}(\mathbf{x}) \equiv \mathcal{O}(\overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}^\mu)^n (\overleftrightarrow{\partial}_{\nu_1} \overleftrightarrow{\partial}_{\nu_2} \cdots \overleftrightarrow{\partial}_{\nu_l})\mathcal{O}(\mathbf{x})$$

n: twist

l: spin

# Boundary action

To make precise sense of the AdS/CFT, needs to introduce a radial regulator ε: $z_0 \geq \varepsilon > 0$

$$S_\epsilon = S_\epsilon^{local} + S_\epsilon^{non-local}$$

$$S = S_\epsilon + \int d^d x \int_{z_0 > \epsilon} dz_0 \mathcal{L}_{bulk}$$

## $S_\epsilon^{local}$

- Consist of all (high-dimensional) local counter-terms to restore conformal invariance.

- Requiring the total action S is independent of ε, holographic

  RGE of the boundary local operators at classical level:

$$\partial_\epsilon S_\epsilon^{local} = -\int_{z=\epsilon} d^d x(\Pi\partial_z\phi - \sqrt{-g}\mathcal{L}) = -\int d^d x\mathcal{H}$$

Lewandowski, May and Sundrum `02…;

Heemskerk and Polchinski `10;

Faulkner, Liu and Rangamani `10.

$$S_\epsilon^{non-local}$$

- Encodes the correlators of the dual field theory.
- The ansatz:

Boundary value of bulk field

$$\langle exp \int_\epsilon \phi_0 \mathcal{O} \rangle_{CFT} = Z_\epsilon(\phi_0)$$

Generating function of CFT

Boundary partition function

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \epsilon^{2(d-\Delta)} \langle \phi_0(x)\phi_0(0)\rangle^{1PI}_{nonlocal}$$

Normalization

$$S_\epsilon^{non-local}$$ Continued

- Example: free AdS scalar theory with Dirichlet b.c: (mixed momentum-position rep.)

$$\begin{aligned} S_\epsilon &= \frac{1}{2}\int d^dk d^dk' \delta(\vec{k}+\vec{k'})\epsilon^{-d+1}\phi_0(\epsilon,\vec{k})\partial_{z_0}\phi(z_0,\vec{k'})|_{z_0=\epsilon} \\ &= \frac{1}{2}\int d^dk d^dk' \delta(\vec{k}+\vec{k'})\epsilon^{-d+1}\phi_0(\vec{k})(\boxed{\partial_{z_0}K(z_0,\vec{k'})|_{z_0=\epsilon}}\phi_0(\vec{k'})) \end{aligned}$$

$$\underbrace{\partial_{z_0}K(z_0,\vec{k'})|_{z_0=\epsilon}}_{\langle \mathcal{O}(\vec{k})\mathcal{O}(\vec{k'})\rangle}$$

$$\begin{aligned} \phi(z_0,\vec{k}) &= K(z_0,\vec{k})\phi_0(\vec{k}) \\ K(z_0,\vec{k}) &= \left(\frac{z_0}{\epsilon}\right)^{d/2}\frac{K_\nu(kz_0)}{K_\nu(k\epsilon)} \end{aligned}$$ Bulk to boundary propagator

# Prescription

- After turning on bulk interactions, CFT gets renormalized

$$\mathcal{O}_R(x) = Z\mathcal{O}(x)$$

- Calculate correction to the two-point $\langle \mathcal{O}(x)\mathcal{O}(0)\rangle^{(1)}$

from AdS : correction to the 1PI boundary action

$$\epsilon^{2(d-\Delta)}\langle \phi_0(x)\phi_0(0)\cdots\rangle^{1PI}_{non-local}$$

↑ Insertions of bulk interaction

The renormalized two-point functions are **Finite** after taking the regulator away

$$\epsilon \longrightarrow 0$$

$$\langle \mathcal{O}_R(x)\mathcal{O}_R(0)\rangle = Z^2\langle \mathcal{O}(x)\mathcal{O}(0)\rangle$$

Z factor absorbs the ε dependence, more specifically, log ε dependence

$$\boxed{\gamma \equiv -\epsilon\frac{\partial \log Z}{\partial \epsilon}}$$

# A toy example: mass perturbation

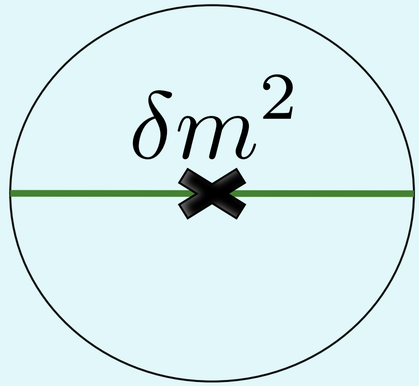
$$\delta m^2 \phi^2$$

Exact solution:

$$\Delta_{\text{exact}} = \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2 + \delta m^2}$$
$$= \Delta_0 + \frac{\delta m^2}{2\nu} + \mathcal{O}((\delta m^2)^2)$$

$$\nu = \sqrt{\frac{d^2}{4} + m^2}$$

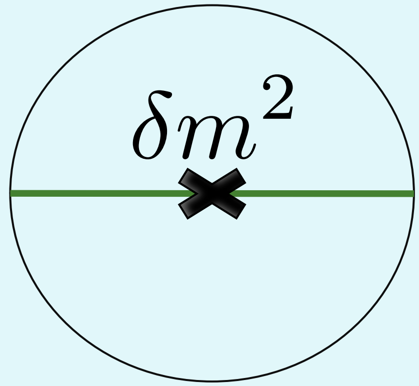
$$\delta m^2 \phi^2$$




A bit of information beforehand:
Main ingredient for calculating
anomolous dimensions
from contact interactions

$$\begin{aligned}
&\langle \phi_0(\mathbf{x})\phi_0(\mathbf{0}) \int \frac{dz_0 d^d z}{z_0^{d+1}} \frac{1}{2}\delta m^2 \phi(z_0,\mathbf{z})^2 \rangle^{1PI}_{nonlocal} \\
&= \delta m^2 \int_\epsilon \frac{dz_0}{z_0} \frac{1}{\epsilon^d} \int \frac{d^d k}{(2\pi)^d} \left( \frac{K_\nu(kz_0)}{K_\nu(k\epsilon)} \right)^2 e^{-i\mathbf{k}\cdot\mathbf{x}} \\
&= \cdots - 2\delta m^2 \int_\epsilon \frac{dz_0}{z_0} \int \frac{d^d k}{(2\pi)^d} \epsilon^{2\nu-d} \left( \frac{k}{2} \right)^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} e^{-i\mathbf{k}\cdot\mathbf{x}} + \cdots \\
&= \cdots + \epsilon^{2\nu-d} \log\epsilon \frac{2\delta m^2}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta-d/2)} |\mathbf{x}|^{-2\Delta} + \cdots ,
\end{aligned}$$

$\delta m^2 \phi^2$




A bit of information beforehand:
Main ingredient for calculating anomalous dimensions from contact interactions

$$
\begin{aligned}
&\langle \phi_0(x)\phi_0(0) \int \frac{dz_0 d^d z}{z_0^{d+1}} \frac{1}{2}\delta m^2 \phi(z_0,z)^2 \rangle^{1PI}_{nonlocal} \\
= \quad &\cdots + \epsilon^{2\nu-d} \log\epsilon \frac{2\delta m^2}{\pi^{d/2}} \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} |x|^{-2\Delta} + \cdots \\
Z^2 - 1 \quad &= \quad -\frac{\delta m^2}{\nu} \log\epsilon + \text{finite terms} \\
\gamma \quad &= \quad -\epsilon \frac{\partial \log Z}{\partial \epsilon} = \frac{\delta m^2}{2\nu}
\end{aligned}
$$

Compared to the exact solution:

$$
\begin{aligned}
\Delta_{\text{exact}} &= \frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 + m^2 + \delta m^2} \\
&= \Delta_0 + \frac{\delta m^2}{2\nu} + \mathcal{O}((\delta m^2)^2)
\end{aligned}
$$

- On the CFT side,

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{1}{|x|^{2(\Delta+\gamma)}} = \frac{1}{|x|^2}(1-2\gamma\log(|x|\Lambda))$$

$$Z^2 - 1 = 2\gamma \log \Lambda \qquad \gamma = \frac{\partial \log Z}{\partial \log \Lambda}$$

UV/IR duality $\qquad \Lambda \leftrightarrow \frac{1}{\epsilon}$

$$\boxed{\gamma \equiv -\epsilon \frac{\partial \log Z}{\partial \epsilon}}$$

Example 2: mass perturbation again
-- toy model of integrating out heavy states

$$\mathcal{V} = -\frac{1}{2}m_1^2\phi^2 - \frac{1}{2}m_2^2\chi^2 - \delta m^2\phi\chi, \quad \delta m^2 \ll m_1^2 \neq m_2^2$$

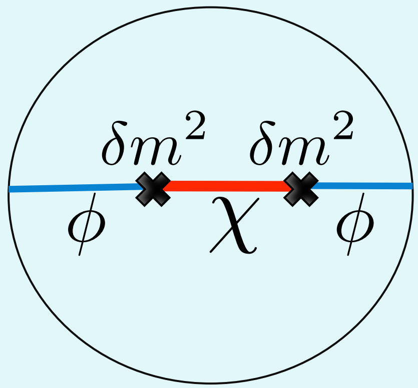
Again one could obtain the exact solution:

Diagonalize mass matrix $\begin{pmatrix} m_1^2 & \delta m^2 \\ \delta m^2 & m_2^2 \end{pmatrix}$

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2}$$

$$\gamma_1 = \frac{(\delta m^2)^2}{m_1^2 - m_2^2}\frac{1}{2\nu_1}$$

## Example 2: continued



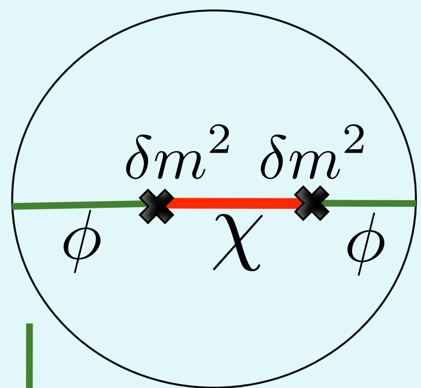

$$\begin{aligned}\langle \phi_0(x)\phi_0(0)\rangle^{(1)} &= \int \frac{dz_0}{z_0^{d+1}}\frac{dw_0}{w_0^{d+1}}\int \frac{d^dk}{(2\pi)^d}e^{-ik\cdot x}K(z_0,k)G(z_0,w_0;k)K(w_0,k) \\ &= 2(\delta m^2)^2\int_\epsilon \frac{dz_0}{z_0}\frac{1}{\epsilon^d}\frac{K_{\nu_1}(kz_0)}{K_{\nu_1}(k\epsilon)}K_{\nu_2}(kz_0)\int_\epsilon^{z_0}\frac{dw_0}{w_0}I_{\nu_2}(kw_0)\frac{K_{\nu_1}(kw_0)}{K_{\nu_1}(k\epsilon)}\end{aligned}$$

$$\gamma_1 = \frac{(\delta m^2)^2}{m_1^2 - m_2^2}\frac{1}{2\nu_1}$$

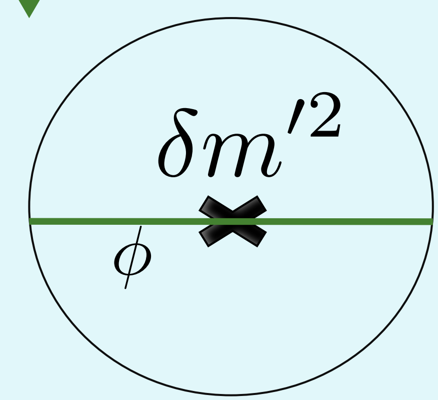
## Example 2:Continue

$$\mathcal{V} = -\frac{1}{2}m_1^2\phi^2 - \frac{1}{2}m_2^2\chi^2 - \delta m^2\phi\chi, \quad \delta m^2 \ll m_1^2 \ll m_2^2$$

$$\gamma_1 = \frac{(\delta m^2)^2}{m_1^2 - m_2^2}\frac{1}{2\nu_1}$$



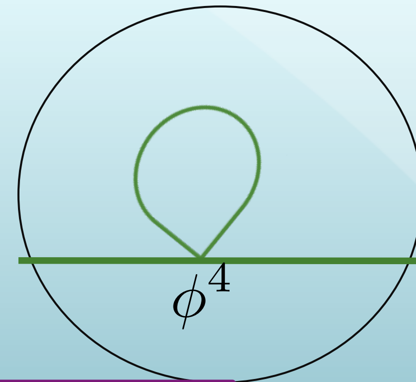

$$\delta m'^2 = -\frac{(\delta m^2)^2}{m_2^2}$$




$$\gamma_1 = \frac{\delta m'^2}{2\nu_1}$$

Contact interaction $\phi^4$

- Single-trace operator




$$\epsilon^{-d}\int_\epsilon \frac{dz_0}{z_0}\int \frac{d^dk}{(2\pi)^d}\left(\frac{K_\nu(kz_0)}{K_\nu(k\epsilon)}\right)^2 e^{-ik\cdot x}\,\frac{\mu}{2}\int \frac{d^dp}{(2\pi)^d}G(z_0,z_0;p)$$

$$\delta m^2$$

Momentum loop integration is divergent

$$\int_0^{\alpha/z_0}\frac{d^dp}{(2\pi)^d}G(z_0,z_0;p)$$

Position-dependent cutoff

$$\gamma = \frac{\delta m^2}{2\nu}$$

$$\delta m^2 = \int_0^{\alpha/z_0}\frac{d^dp}{(2\pi)^d}G(z_0,z_0;p)$$

- Power countings of the divergences:

Flat space: $$\int (d^{d+1}p)\frac{1}{p^2} \quad \sim \quad \Lambda^{d-1}$$

Warped: $$\int (d^d p) G(z_0, z_0; p) \quad \sim \quad \int (d^d p) K_\nu(p z_0) I_\nu(p z_0) \sim \int (d^d p)\frac{1}{p} \sim \Lambda^{d-1}$$

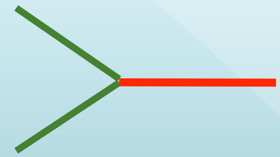
- Could also regulate the momentum loop with Pauli-Villas

  Final answer is scheme-dependent

# Double-trace operators

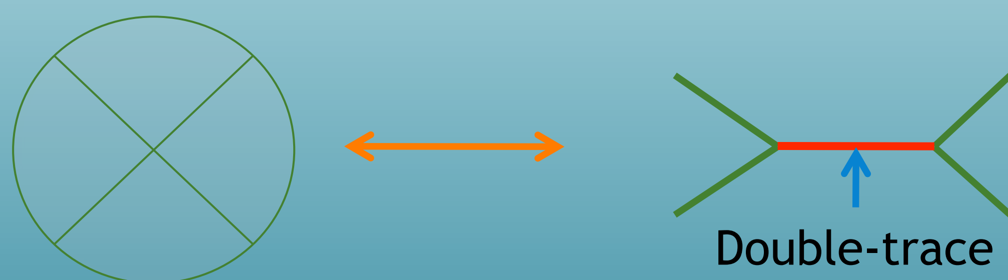
$$\mathcal{O}_{n,l}(\mathbf{x}) \equiv \mathcal{O}(\overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}^\mu)^n(\overleftrightarrow{\partial}_{\nu_1}\overleftrightarrow{\partial}_{\nu_2}\cdots\overleftrightarrow{\partial}_{\nu_l})\mathcal{O}(\mathbf{x})$$

$$\mathcal{O}(x)\mathcal{O}(0) = \sum c \;\; \mathcal{O}_{n,l}$$



Why should we care about these composite CFT operators?

- They encode information about scattering in AdS




- Anomalous dimension of double-trace operators and OPE coefficients, the two sets of data, contain all the dynamical information of CFT at $\mathcal{O}(1/N^2)$

More recently,

- Anomolous dimension of double-trace gives important information about bulk locality

  Heemskerk, Penedones, Polchinski and Sully `09;
  Fitzpatrick, Katz, Poland, Simmons-Duffin `10

- RGE of boundary local operators ⇔ multitrace-trace flow (classical level)

  Heemskerk and Polchinski `10; Faulkner, Liu and Rangamani `10

- Possible phenomelogical applications of double-trace deformation

- Turning on $\phi^4$

$$\mathcal{O}_n(x) \equiv \mathcal{O}(\overleftrightarrow{\partial}_\mu \overleftrightarrow{\partial}^\mu)^n \mathcal{O}(x)$$

Would be renormalized

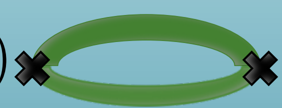
$$\Delta_n = 2\Delta + 2n + \gamma(n)$$

↑ Single-trace operator dim.

Assume $\phi_0^2$ sources the double-trace, calculate

$$\langle \phi_0^2(x)\phi_0^2(0) \int \frac{dz_0 d^d z}{z_0^{d+1}} \frac{\mu}{4!} \phi(z_0, z)^4 \rangle^{1PI}_{non-local}$$

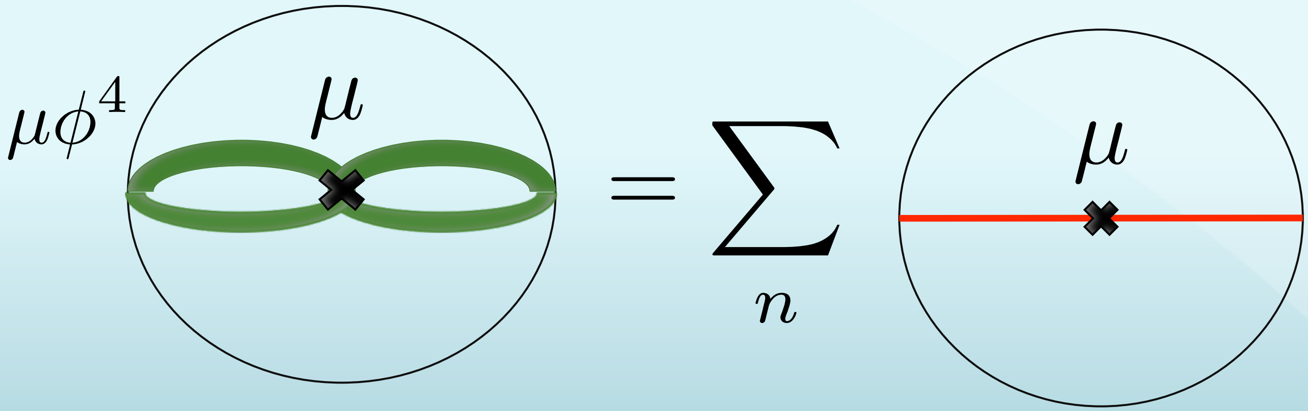
Trick: Instead of using the single-particle propagator, use the two-particle propagator

$\phi^2(x)$  $\phi^2(0) = \sum_n \frac{1}{N_n^2}$ 

 Single-particle propagator: $G_\Delta(x, 0)$

 Two-particles' propagator: $G_{\Delta_n}(x, 0)$

Analog of partial-wave decomposition




$$d = 2 \quad \gamma(n) = \frac{\mu}{8\pi}\frac{1}{2\Delta + 2n - 1},$$

$$d = 4 \quad \gamma(n) = \frac{\mu}{16\pi^2}\frac{(n+1)(\Delta + n - 1)(2\Delta + n - 3)}{(2\Delta + 2n - 1)(2\Delta + 2n - 3)}$$

Agrees with Heemskerk, Penedones, Polchinski and Sully `09;
Fitzpatrick, Katz, Poland, Simmons-Duffin `10

Santa Barbara group: calculate 4-point function and project onto each individual two-particle partial wave;
Boston group: in global AdS, calculate perturbations of the dilatation operator using the old-fashioned perturbation theory.

- Physical interpretation

$$d = 2 \quad \gamma(n) = \frac{\mu}{8\pi}\frac{1}{2\Delta + 2n - 1},$$

$$d = 4 \quad \gamma(n) = \frac{\mu}{16\pi^2}\frac{(n+1)(\Delta+n-1)(2\Delta+n-3)}{(2\Delta+2n-1)(2\Delta+2n-3)}$$

$$n \gg 1 \quad \frac{\phi^4}{\Lambda^{d-3}} \quad \gamma(n) \sim n^{d-3}$$

- Physical interpretation

$$\frac{\phi^4}{\Lambda^{d-3}} \quad \gamma(n) \sim n^{d-3}$$

$$\frac{\mathcal{O}}{\Lambda^p} \quad \mathcal{A} \sim E^p$$

Eg: Euler-Heisenberg Lagrangain

$F^4 \quad \mathcal{A} \sim E^4$

$$\boxed{\begin{array}{ccc} \gamma(n) & \leftrightarrow & \mathcal{A} \\ n & \leftrightarrow & E \end{array}}$$

# Three cases

| AdS UV b.c | IR b.c | CFT |
|---|---|---|
| Dirichlet b.c | Regular at $z_0 \rightarrow \infty$ | Standard quantization $\Delta > \frac{d}{2}$ |
| Mixed Neuman/ Dirichlet b.c | Regular at $z_0 \rightarrow \infty$ | CFT w/ a double-trace deformation: $CFT^{UV} \rightarrow CFT^{IR}$ |
| Dirichlet b.c | $z_0 = z_{IR}$ | Spontaneous breaking of CFT |

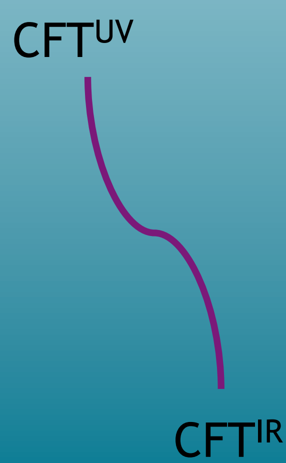
# CFT with double-trace deformation

- CFT: $\mathcal{L}_{CFT} + \tilde{f}\mathcal{O}^2$
- Dual to mixed boundary condition: Witten 01';

$$f\phi(k) + \epsilon\partial_{z_0}\phi(k)\big|_{z_0=\epsilon} = 0$$

$$UV: f = 0 \quad \Delta_- = d/2 - \nu \quad \Delta[\mathcal{O}^2] < d \quad \text{relevant perturbation}$$

$$IR: f \to \infty \quad \Delta_+ = d/2 + \nu$$




Some possible pheno applications:
(unsuccessful) attempt to explain QCD confinement
D.B. Kaplan, Lee, Son, Stephanov 09;
Non-susy theory w/ natural light scalar:
Strassler 03;
Split SUSY: Sundrum 09;

- Repeat calculation with the new boundary conditions and corresponding propagators

For instance, for the toy example of mass perturbation,

$$\text{UV} \quad f \to 0 \quad \gamma = -\frac{\delta m^2}{2\nu},$$

$$\text{IR} \quad f \to \infty \quad \gamma = \frac{\delta m^2}{2\nu},$$

$$\begin{aligned} \Delta_- &= \frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 + m^2 + \delta m^2} \\ &= \Delta_0 - \frac{\delta m^2}{2\nu} + \mathcal{O}((\delta m^2)^2) \end{aligned}$$

# Spontaneously broken CFT

- Supposing the existence of a Wilsonian scheme, the renormalization of CFT should not be sensitive to the interior boundary condition.
- Impose an IR cutoff surface with Dirichlet b.c., for mass perturbations, the answers do not change!

# Spontaneously broken CFT (continued)

- Impose an IR cutoff surface with Dirichlet b.c., for mass perturbations, the answers do not change!

$$K(z_0, \vec{k}) = \left(\frac{z_0}{\epsilon}\right)^{d/2} \frac{K_\nu(kz_0) + aI_\nu(kz_0)}{K_\nu(k\epsilon) + aI_\nu(k\epsilon)}$$

$$\langle \phi_0(x)\phi_0(0) \int \frac{dz_0 d^d z}{z_0^{d+1}} \frac{1}{2} \delta m^2 \phi(z_0, z)^2 \rangle^{1PI}_{nonlocal}$$

$$= \cdots + \epsilon^{2\nu-d} \log \epsilon \frac{2\delta m^2}{\pi^{d/2}} \left( \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} + 2a \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2 + 1)\Gamma(\nu)\Gamma(-\nu)} \right) |\vec{x}|^{-2\Delta}$$

$$\langle \mathcal{O}(x)\mathcal{O}(0) \rangle^{(0)} = \frac{2\nu}{\pi^{d/2}} \left( \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2)} + 2a \frac{\Gamma(\Delta)}{\Gamma(\Delta - d/2 + 1)\Gamma(\nu)\Gamma(-\nu)} \right) |\vec{x}|^{-2\Delta}$$

$$\gamma = -\epsilon \frac{\partial \log Z}{\partial \epsilon} = \frac{\delta m^2}{2\nu}$$

- For contact interactions, however,
- Single-trace:

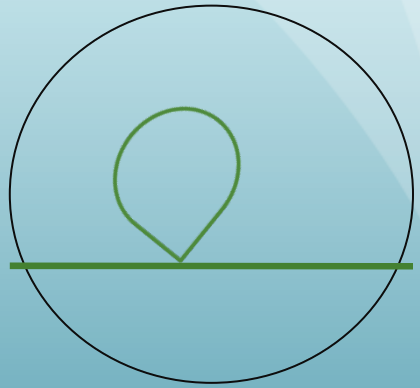
  the loop momentum integration is dependent of the particular choice of the bulk propagator;

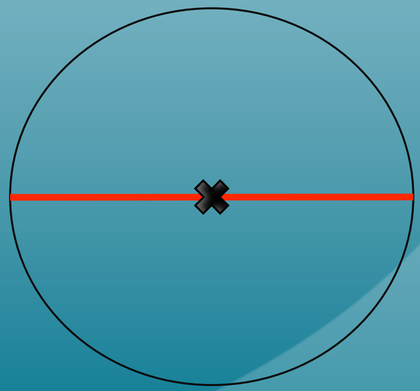
  UV divergences: determined by

  short-distance physics; unaffected by the

  IR boundary;

  Finite correction: sensitive to the interior b.c..





- Double-trace: similar to mass perturbation,

independent of IR condition

# Conclusion

- I present a simple prescription to calculate the anomalous dimensions of CFT operators from bulk interactions.
- The key ingredient is to use the radial position as the regulator.

# Open questions

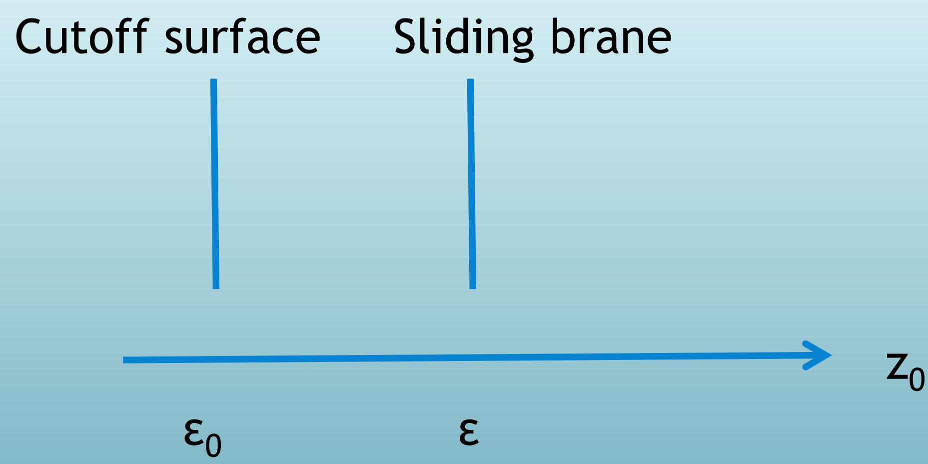
- RGE of local boundary operators beyond leading order in N:

  Lewandowski `04; However, part of his answer does not look Wilsonian-like as it depends on the IR b.c.'s;

- Understand correction to non-RG quantity, e.g., OPE coefficients
- Theory with gauge interactions and fermions.

Thank you!

# Boundary action




$$S_\epsilon = S_\epsilon^{local} + S_\epsilon^{non-local}$$

$$S = S_\epsilon + \int d^d x \int_{z_0 > \epsilon} dz_0 \mathcal{L}_{bulk}$$

$$\psi(kz_0) \equiv z_0^{d/2} K_\nu(kz_0),$$
$$K(\mathbf{k}, z_0) = \frac{\psi(kz_0)}{\tilde{f}\psi(k\epsilon) + \partial\psi(k\epsilon)\cdot\vec{n}}.$$

$$\langle\mathcal{O}(k)\mathcal{O}(k')\rangle_f^{(0)} = -\epsilon^{-d}\delta^d(k+k')\frac{\psi(k\epsilon)}{\tilde{f}\psi(k\epsilon) + \partial\psi(k\epsilon)\cdot\vec{n}}$$
$$= -\epsilon^{-d}\delta^d(k+k')\frac{1}{-f\epsilon^{2\nu}\left(2\pi^{d/2}\frac{\Gamma(1-\nu)}{\Gamma(\Delta_-)}\right) + \epsilon^{2\nu}(\frac{k}{2})^{2\nu}\frac{\Gamma(-\nu)}{\Gamma(\nu)}(2\nu)},$$

$$\langle\mathcal{O}(k)\mathcal{O}(k')\rangle_f^{(1)} = -\epsilon^{-d}\delta^d(k+k')\delta m^2 \int_\epsilon \frac{dz_0}{z_0}\left(\frac{\psi(kz_0)}{\tilde{f}\psi(kz_0) + \partial\psi(kz_0)\cdot\vec{n}}\right)^2$$
$$= -\epsilon^{-d}\delta^d(k+k')\delta m^2 \int_\epsilon \frac{dz_0}{z_0}\left(\frac{2^{\nu-1}\Gamma(\nu)(kz_0)^{-\nu} + 2^{-\nu-1}\Gamma(-\nu)(kz_0)^{\nu}}{-f\epsilon^\nu\pi^{d/2}2^\nu\frac{\Gamma(1-\nu)\Gamma(\nu)}{\Gamma(\Delta_-)}k^{-\nu} - 2^{-\nu}\Gamma(1-\nu)(k\epsilon)^\nu}\right)^2$$