Resonant Tunneling in Quantum Field Theory

Dan Wohns

Cornell University
work in progress with S.-H. Henry Tye

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Motivation

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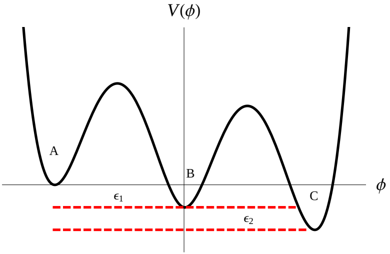
- Relevant to study of potentials with many minima
- Generic existence of resonance effects

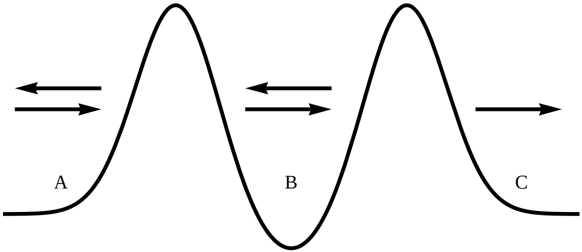
Results

Results

- Show existence of resonant tunneling in QFT
- New effect “catalytic tunneling” can enhance single-barrier tunneling

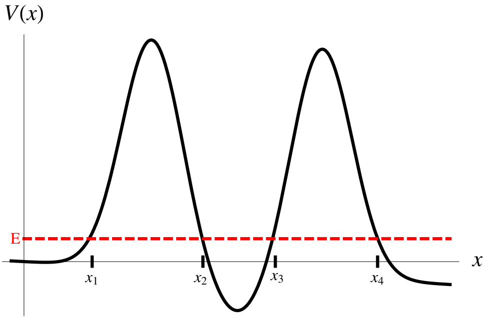


General Argument



- Tunneling rate for single-barrier tunneling is $\Gamma_{A\to B} = Ae^{-S}$
- Tunneling probability for single-barrier tunneling is $T_{A\to B} = Ke^{-S}$

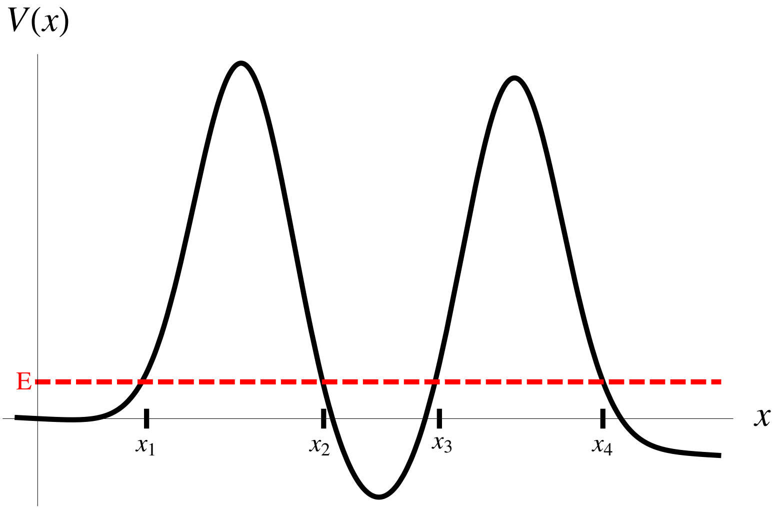
General Argument



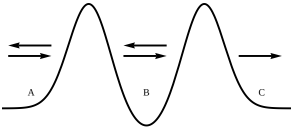
- Naive WKB analysis $T_{A\to C} \approx T_{A\to B} T_{B\to C}$ is incorrect
- Instead $t_{A\to C} = t_{A\to B} + t_{B\to C}$ or equivalently $T_{A\to C} = \frac{T_{A\to B} T_{B\to C}}{T_{B\to C} + T_{B\to C}}$.
- Enhancement due to resonant tunneling effect.

Outline

Potential



WKB Approximation



- Expand a general wavefunction $\Psi(x) = e^{if(x)/\hbar}$ in powers of $\hbar$
- $\psi_{L,R}(x) \approx \frac{1}{\sqrt{k(x)}} \exp\left(\pm i \int dx k(x) \right)$ in classically allowed region, where $k(x) = \sqrt{\frac{2m}{\hbar^2}(E - V(x))}$
- $\psi_{\pm}(x) \approx \frac{1}{\sqrt{\kappa(x)}} \exp\left(\pm \int dx \kappa(x) \right)$ in the classically forbidden region, where $\kappa(x) = \sqrt{\frac{2m}{\hbar^2}(V(x) - E)}$

Matching Conditions

- Complete solution $\psi(x) = \alpha_L \psi_L(x) + \alpha_R \psi_R(x)$ in vacuum A
- $\psi(x) = \alpha_+ \psi_+(x) + \alpha_- \psi_-(x)$ in the classically forbidden region
- $\psi(x) = \beta_L \psi_L(x) + \beta_R \psi_R(x)$ in vacuum B
- $\begin{pmatrix} \alpha_R \\ \alpha_L \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Theta + \Theta^{-1} & i(\Theta - \Theta^{-1}) \\ -i(\Theta - \Theta^{-1}) & \Theta + \Theta^{-1} \end{pmatrix} \begin{pmatrix} \beta_R \\ \beta_L \end{pmatrix}$
- $\Theta \simeq 2 \exp\left(\frac{1}{\hbar} \int_{x_1}^{x_2} dx \sqrt{2m(V(x) - E)} \right)$
- Tunneling probability $T_{A \to B} = |\frac{\beta_R}{\alpha_R}|^2 = 4 \left(\Theta + \frac{1}{\Theta} \right)^{-2} \simeq \frac{4}{\Theta^2}$

Double-Barrier Tunneling

- Same method of analysis
- $T_{A\to C} = 4\left(\left(\Theta\Phi + \frac{1}{\Theta\Phi}\right)^2 \cos^2 W + \left(\frac{\Theta}{\Phi} + \frac{\Phi}{\Theta}\right)^2 \sin^2 W\right)^{-1}$
- $\Phi \simeq 2\exp\left(\frac{1}{\hbar}\int_{x_3}^{x_4} dx\sqrt{2m\left(V(x) - E\right)}\right)$
- $W = \frac{1}{\hbar}\int_{x_2}^{x_3} dx\sqrt{2m(E - V(x))}$
- If B has zero width $T_{A\to C} \simeq 4\Theta^{-2}\Phi^{-2} = T_{A\to B}T_{B\to C}/4$
- If $W = (n_B + 1/2)\pi$ then $T_{A\to C} = \frac{4}{(\Theta/\Phi+\Phi/\Theta)^2}$

Tunneling Probability versus Energy

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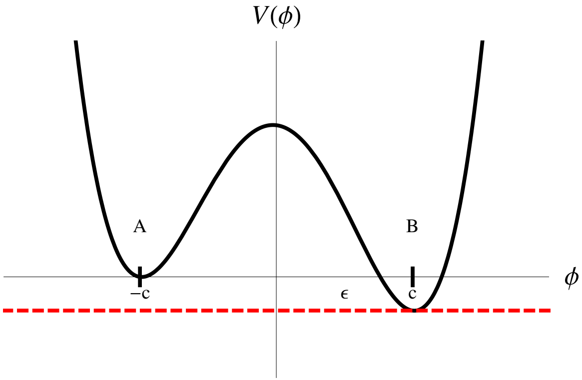
[1] Copeland, Padilla, Saffin

Potential



- $V(\phi) = \frac{1}{4}g(\phi^2 - c^2)^2 - B(\phi + c)$

Thin-Wall Approximation

- Tunneling rate per unit volume [2] is $\Gamma/V = A\exp(-S_E/\hbar)$
- Euclidean EOM is $\left(\frac{\partial^2}{\partial\tau^2} + \nabla^2\right)\phi = V'(\phi)$
- Assume O(4) symmetry
- Solution to Euclidean EOM is $\phi_{DW}(\tau, x, R) = -c\tanh\left(\frac{\mu}{2}(r - R)\right)$
- Inverse thickness of domain wall $\mu = \sqrt{2gc^2}$

[2] Coleman

Euclidean Action

- $S_E = \int d\tau d^3x \left[\frac{1}{2} \left(\frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right]$
- $S_E = -\frac{1}{2} \pi^2 R^4 \epsilon + 2\pi^2 R^3 S_1$
- Domain-wall tension is $S_1 = \int_{-c}^{c} d\phi \sqrt{2V(\phi)} = \frac{2}{3} \mu c$
- $\frac{dS_E}{dR} = 0$ implies $\mathcal{E} = -\frac{4}{3} \pi R^3 \epsilon + 4\pi R^2 S_1 = 0$
- $R = \lambda_c \equiv 3S_1/\epsilon$
- Euclidean action is $S_E = \frac{\pi^2}{2} S_1 \lambda_c^3 = \frac{27\pi^2}{2} \frac{S_1^4}{\epsilon^3}$
- Bubble grows if $d\mathcal{E}/dR < 0$ or $R > 2\lambda_c/3$

Functional Schrödinger Method

Basic Idea

Infinite-dimensional QFT $\longrightarrow$ one-dimensional QM problem

- In semiclassical limit, the vacuum tunneling rate is dominated by a discrete set of classical paths [3] [4] [5]
- Equivalent to Euclidean instanton method for single-barrier tunneling
- Easily generalizes to multiple-barrier tunneling

[3] Bender, Banks, Wu
[4] Gervais, Sakita
[5] Bitar, Chang

Functional Schrödinger Equation

- $H = \int d^3x \left(\frac{\dot{\phi}^2}{2} + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right)$
- Quantize using $[\dot{\phi}(x), \phi(x')] = i\hbar\delta^3(x - x')$
- $H = \int d^3x \left(-\frac{\hbar^2}{2} \left(\frac{\delta}{\delta\phi(x)} \right)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \right)$
- Make ansatz $\Psi(\phi) = A \exp(-\frac{i}{\hbar} S(\phi))$
- $H\Psi(\phi(x)) = E\Psi(\phi(x))$

Semiclassical Expansion

- $S(\phi) = S_{(0)}(\phi) + \hbar S_{(1)}(\phi) + \ldots$
- $\int d^3x \left[\frac{1}{2} \left(\frac{\delta S_{(0)}(\phi)}{\delta \phi} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right] = E$
- $\int d^3x \left[-i \frac{\delta^2 S_{(0)}(\phi)}{\delta \phi^2} + 2 \frac{\delta S_{(0)}(\phi)}{\delta \phi} \frac{\delta S_{(1)}(\phi)}{\delta \phi} \right] = 0$
- Ignore higher-order terms

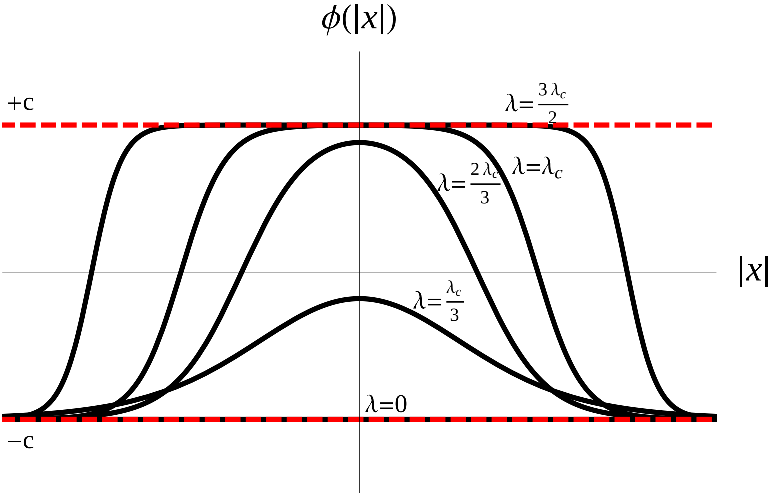
Goal

Determine value of $S_{(0)}$ that gives dominant contribution to the tunneling probability

Most Probable Escape Paths

- $\phi_0(x, \lambda)$
- Trajectory in the configuration space of $\phi(x)$ parameterized by λ
- Give dominant contribution to tunneling probability
- Variation of $S_{(0)}$ vanishes perpendicular to the MPEP
- Variation of $S_{(0)}$ is nonvanishing along the MPEP

Most Probable Escape Paths



Determining $S_{(0)}$

- Effective tunneling potential
  $U(\lambda) = U(\phi(x,\lambda)) = \int d^3x \left(\frac{1}{2}(\nabla\phi(x,\lambda))^2 + V(\phi(x,\lambda))\right)$
- Path length $(ds)^2 = \int d^3x (d\phi(x))^2 =$
  $(d\lambda)^2 \int d^3x \left(\frac{\partial\phi(x,\lambda)}{\partial\lambda}\right)^2 = (d\lambda)^2 m(\phi(x,\lambda))$

Zeroth-Order Solution (Classically Forbidden Region)

$$S_{(0)} = i\int_0^s ds' \sqrt{2[U(\phi(x,s')) - E]} =$$
$$i\int_{\lambda_1}^{\lambda_2} d\lambda \left(\frac{ds}{d\lambda}\right)\sqrt{2[U(\phi(x,\lambda)) - E]}$$

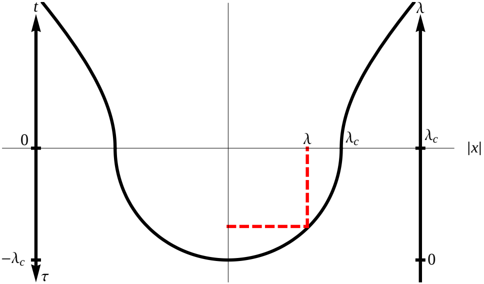
Determing the MPEP

Euler-Lagrange Equation

$$\frac{\partial^2\phi(x,\tau)}{\partial\tau^2} + \nabla^2\phi(x,\tau) - \frac{\partial V(\phi(x,\tau))}{\partial\phi} = 0$$

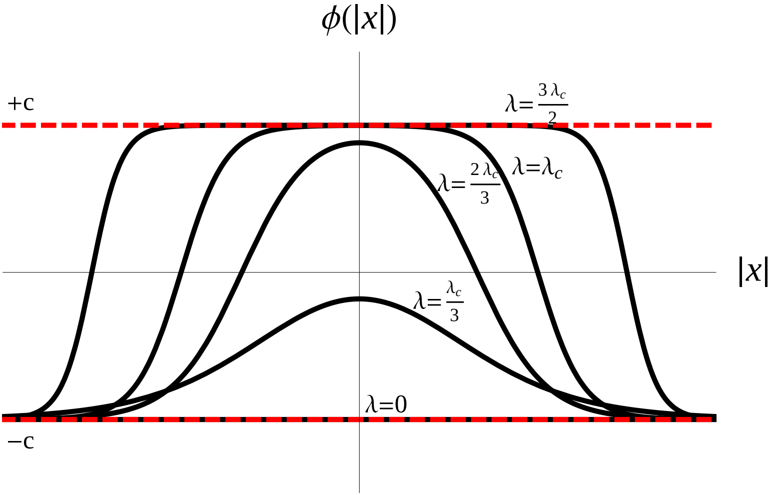
- $\frac{ds}{d\tau} = \sqrt{2[U(\phi(x,\tau)) - E]}$
- τ plays role of Euclidean time
- In thin-wall approximation solution is $\phi_0(x,\tau) = -c\tanh\left(\frac{\mu}{2}(r-\lambda_c)\right)$
- Euler-Lagrange equation same as Euclidean instanton method

Parameterization of MPEP



- Reparameterize solution in terms of $\lambda = \sqrt{\lambda_c^2 - \tau^2}$
- MPEP is $\phi_0(x, \lambda) \approx -c \tanh\left(\frac{\mu}{2}(|x| - \lambda)\frac{\lambda}{\lambda_c}\right)$

Parameterization of MPEP



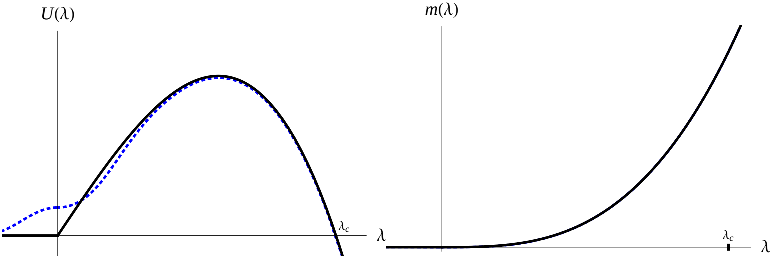
WKB Wavefunction

WKB Wavefunction

$$\Psi(\phi(x,\lambda)) = Ae^{iS_0/\hbar} = A\exp\left(-\frac{1}{\hbar}\left[\int_0^{\lambda_c} d\lambda\sqrt{2m(\lambda)[U(\lambda)-E]}\right]\right)$$

- Position-dependent mass $m(\lambda) \equiv \int d^3x \left(\frac{\partial\phi_0(x,\lambda)}{\partial\lambda}\right)^2$
- Effective tunneling potential $U(\phi(x,\lambda)) = \int d^3x \left(\frac{1}{2}(\nabla\phi(x,\lambda))^2 + V(\phi(x,\lambda))\right)$
- Now have one-dimensional QM problem

Equivalence of FSM with EIM

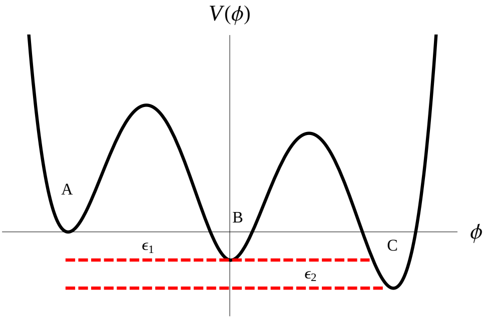


- Integrate spatially to get $U(\lambda) \approx \frac{2\pi S_1}{\lambda_c}\lambda(\lambda_c^2 - \lambda^2)$
- Position dependent mass $m(\lambda) \approx 4\pi S_1 \frac{\lambda^3}{\lambda_c}$
- Amplitude is $\exp\left(\frac{27\pi^2}{4}\frac{S_1^4}{\epsilon^3}\right) = \exp\left(S_E/2\right)$

Advantages of FSM

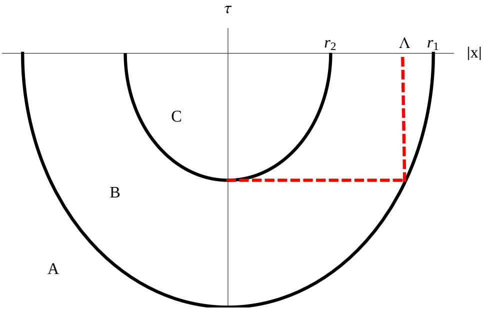
- Same arguments lead to $S_{(0)}(\phi(x,\lambda)) = \int d\lambda \sqrt{2m((\phi(x,\lambda))[E - U((\phi(x,\lambda))]}$ and Lorentzian EOM in classically allowed regions
- If we choose $\lambda = \sqrt{\lambda_c^2 + t^2}$ as parameter, MPEP takes form $\phi_0(x,\lambda) = -c \tanh\left(\frac{\mu}{2}(|x| - \lambda)\frac{\lambda}{\lambda_c}\right) = -c \tanh\left(\frac{\mu}{2}\frac{(|x|-\lambda)}{\sqrt{1-\dot{\lambda}^2}}\right)$
- Single real parameter describes entire system

Potential



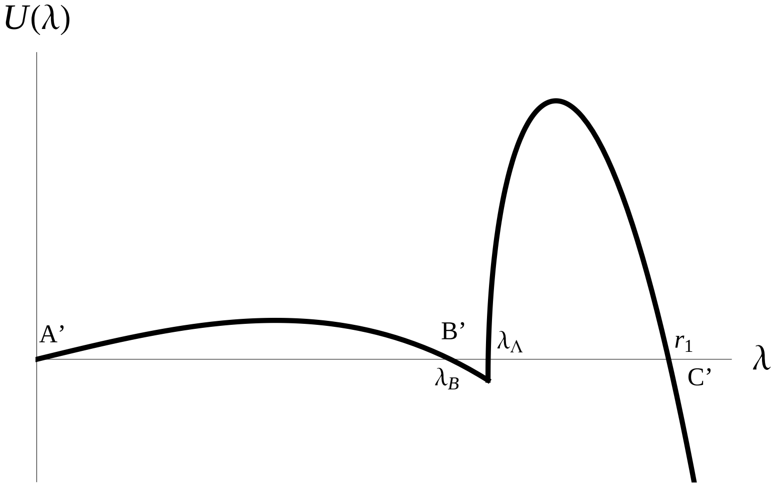
- $V(\phi) = \begin{cases} \frac{1}{4} g_1((\phi + c_1)^2 - c_1^2)^2 - B_1\phi - 2B_1c_1 & \phi < 0 \\ \frac{1}{4} g_2((\phi - c_2)^2 - c_2^2)^2 - B_2\phi - 2B_1c_1 & \phi > 0 \end{cases}$

MPEP

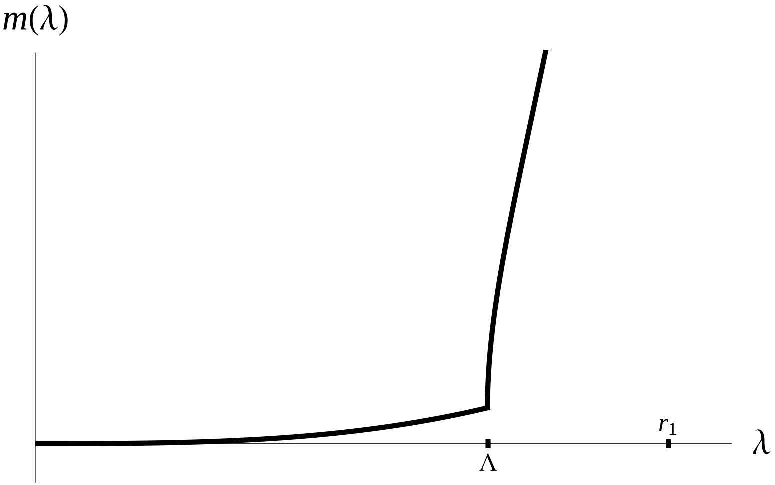


- $\phi_0(|x|,\lambda) = -c_1 \tanh\left(\frac{\mu_1}{2}\frac{\lambda}{r_1}(|x|-\lambda)\right) - \Theta\left(\frac{\lambda}{\Lambda} - 1\right) c_2 \tanh\left(\frac{\mu_2}{2}\frac{\lambda'}{r_2}(|x|-\lambda')\right) + c_2 - c_1$ solves both Euclidean and Lorentzian equation of motion

Effective Tunneling Potential



Position-Dependent Mass



Consistency Conditions

- We require zero total energy at nucleation
- $\mathcal{E}_{(2)} = 4\pi(S_1^{(1)} - \frac{1}{3}r_1\epsilon_1)r_1^2 + 4\pi(S_1^{(2)} - \frac{1}{3}r_2\epsilon_2)r_2^2 = 0$
- We do not demand that the energy of each bubble vanishes individually
- Equivalent to condition that action is stationary
- Must also ensure existence of a classically allowed region $U(\lambda) < 0$ for $\Lambda > \lambda > \lambda_B$
- Also require the existence of a second classically forbidden region

Resonant Tunneling or Catalytic Tunneling

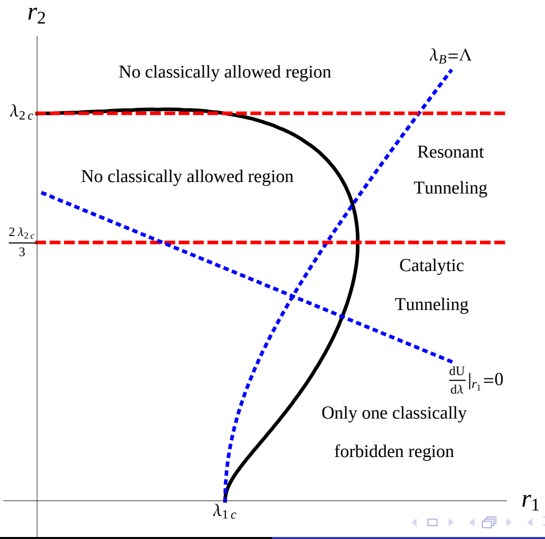
Resonant Tunneling

If the inside bubble is large enough $\lambda_{2c} > r_2 > 2\lambda_{2c}/3$ tunneling from A to C will complete.

Catalytic Tunneling

- If the inside bubble is too small $0 < r_2 < 2\lambda_{2c}/3$, inside bubble will collapse
- Effect will be tunneling from A to B
- Tunneling rate is exponentially enhanced by presence of C

Consistency Conditions



Resonant Tunneling in QFT

- $T_{A\to C} = 4\left(\left(\Theta\Phi + \frac{1}{\Theta\Phi}\right)^2 \cos^2 W + \left(\frac{\Theta}{\Phi} + \frac{\Phi}{\Theta}\right)^2 \sin^2 W\right)^{-1}$
- $W = \int_{\lambda_2}^{\lambda_3} d\lambda \sqrt{2m(\lambda)(-U(\lambda))}$
- $W = \frac{S_1^{(1)}\lambda_\Lambda}{\lambda_B}\sqrt{\lambda_\Lambda^2 - \lambda_B^2} - S_1^{(1)}\lambda_B \log\left[\frac{\lambda_\Lambda + \sqrt{\lambda_\Lambda^2 - \lambda_B^2}}{\lambda_B}\right]$
- Resonance condition is $W = (n + \frac{1}{2})\pi$

Probability of Hitting Resonance

- Treat tunneling probability as function of λ_Λ
- Expand around resonance at $\lambda_\Lambda = \lambda_R$ of width Γ_{λ_Λ}
- $\Gamma_{\lambda_\Lambda} = \frac{2}{\Theta\Phi(\frac{\partial W}{\partial \lambda_\Lambda})}\left(\frac{\Theta}{\Phi} + \frac{\Phi}{\Theta}\right)$
- Separation between resonances $\Delta\lambda \simeq \frac{\pi}{(\frac{\partial W}{\partial \lambda_\Lambda})}$
- Probability of hitting resonance $P(A \to C) = \frac{\Gamma_\Lambda}{\Delta\Lambda} \simeq \frac{2}{\pi\Theta\Phi} = \frac{1}{2\pi}(T_{A\to B} + T_{B\to C})$ is the larger of two decay probabilities
- Average tunneling probability $< T_{A\to C} >= P(A \to C) T_{A\to C} \sim \frac{T_{A\to B} T_{B\to C}}{T_{A\to B} + T_{B\to C}}$ given by smaller of two tunneling probabilities

Conclusions and Future Directions

- Used functional Schrödinger method to show how resonant tunneling takes place in QFT with a single scalar
- Showed existence of catalytic tunneling
- Expect resonance tunneling phenomena to persist outside of thin-wall approximation
- Interesting to study the more general case, especially including gravity