Finite temperature QFT: A dual path integral representation

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ABSTRACT:

Constraints in the imaginary time path integral

$$\updownarrow$$

Periodicity conditions corresponding to the Matsubara formalism for Thermal Field Theory

$$\updownarrow$$

Alternative Path Integral representation

TOPICS:

- **Introduction**
- **Outline of the method**
- **Scalar Fields**
- **Fermi Fields**
- **Conclusions**

Introduction

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In this context:

Fields are used to impose the constraints

$\Updownarrow$

integrating out the original variables

$\Updownarrow$

effective model

Where the dynamical fields live on the constrained surface.

Here, we extend that kind of approach to QFT at finite temperature $(T > 0)$, in order to deal with the periodicity constraints in the imaginary time.

⋆ First a glimpse of the usual QFT at $T > 0$:

T. Matsubara (1955)

H. Ezawa, Y. Tomonaga and H. Umezawa (1957)

This original approach now called Matsubara (or 'imaginary-time') formalism has been very successful (see, for instance, Kapusta, Finite-Temperature Field Theory, Cambridge University Press, Cambridge (1989)),

both in High Energy and Condensed Matter Physics.

A fundamental property introduced by this formalism is the imaginary-time periodicity (antiperiodicity) conditions for the bosonic (fermionic) field configurations in the path integral. Which can be seen at the level of the partition function

$$\mathcal{Z}(\beta) \;=\; \mathrm{Tr}\big(e^{-\beta\hat{H}}\big) \;=\; \int dq\ \langle q|e^{-\beta H}|q\rangle \;=\; \int dq\ \langle q,-i\beta|q,0\rangle. \tag{1}$$

The standard path integral construction for the transition amplitude between different times may be applied, to obtain the partition function in the Matsubara formalism:

$$\mathcal{Z}(\beta) \;=\; \int_{\mathrm{q}(0)=\mathrm{q}(\beta)} \mathcal{D}p\,\mathcal{D}q\; e^{\int_0^\beta d\tau\left[ip\dot{q}\,-\,H(p,q)\right]}\;, \tag{2}$$

For a bosonic field theory in $d+1$ spacetime dimensions

$\Updownarrow$

field paths are periodic in the imaginary time
canonical momentum ones are unrestricted

$\Updownarrow$

Hamiltonian quadratic in the canonical momentum

$\Updownarrow$ *Integrating*

model where the dynamical field is defined on $S^1 \times R^d$,
(radius of S^1) $\propto \beta = 1/T$

$\Updownarrow$

in Fourier space, frequencies become the usual Matsubara ones.

A characteristic feature of the Matsubara formalism (shared with the real-time formulation) is that the introduction of a time dependence for the fields seems to be unavoidable, even if one limits oneself to the calculation of time independent objects.

- We construct a representation where **only static fields** are involved, an **alternative way of dealing with $T > 0$ QFT** calculations.
- Inspired by the **constrained functional integral approach** used in the Casimir effect (Kardar et al).
- The periodicity conditions are met by **Lagrange multipliers** (d-dimensional when the field lives in $d + 1$ dimensions).
- Integrating the original fields leaves a functional of the d-dimensional Lagrange multipliers.)

The method

0.1 The periodicity constraint

We start from the phase-space path integral of $\mathcal{Z}_0$, the (zero temperature) vacuum persistence amplitude:

$$\mathcal{Z}_0 \;=\; \int \mathcal{D}p\,\mathcal{D}q\; e^{-\mathcal{S}[q(\tau),p(\tau)]}\;, \tag{3}$$

where $\mathcal{S}$ is the first-order action, $\mathcal{S} = \int_{-\infty}^{+\infty} d\tau\,\mathcal{L}$, with $\mathcal{L} = -ip\dot{q} + H(p,q)$, and H denotes the Hamiltonian, assumed to be of the form: $H(p,q) = T(p) + V(q)$.

$\mathcal{Z}_0$ is the limit of an imaginary-time transition amplitude,

$$\mathcal{Z}_0 = \lim_{T\to+\infty} \langle q_0, -iT|q_0, iT\rangle$$
$$= \lim_{T\to+\infty} \sum_n |\langle q_0|n\rangle|^2 e^{-2TE_n} = \lim_{T\to+\infty} |\langle q_0|0\rangle|^2 e^{-2TE_0} \quad (4)$$

- $|n\rangle$ are the eigenstates of $\hat{H}$,
- $\hat{H}|n\rangle = E_n|n\rangle$, and q_0, the asymptotic value for q_0 at $T \to \pm\infty$ (usually, $q_0 \equiv 0$).
- E_0 is the energy of $|0\rangle$, the ground state.

Next we obtain an alternative expression for $\mathcal{Z}(\beta)$.
Starting from $\mathcal{Z}_0$, and imposing the appropriate constraints on the paths.

We first introduce decompositions of the identity at the imaginary times corresponding to $\tau = 0$ and $\tau = \beta$, so that we may write:

$$\mathcal{Z}_0 \;=\; \lim_{T\to\infty} \int dq_2 dq_1 \, \langle q_0, -iT | q_2, -i\beta\rangle \, \langle q_2, -i\beta | q_1, 0\rangle \, \langle q_1, 0 | q_0, iT\rangle \;, \tag{5}$$

or, in a path integral representation,

$$\begin{aligned} \mathcal{Z}_0 \;&=\; \lim_{T\to\infty} \int dq_2 dq_1 \int_{q(\beta)=q_2}^{q(T)=q_0} \mathcal{D}p\,\mathcal{D}q \; e^{-\int_\beta^T d\tau \mathcal{L}} \\ &\times \int_{q(0)=q_1}^{q(\beta)=q_2} \mathcal{D}p\,\mathcal{D}q \; e^{-\int_0^\beta d\tau \mathcal{L}} \int_{q(-T)=q_0}^{q(0)=q_1} \mathcal{D}p\,\mathcal{D}q \; e^{-\int_{-T}^0 d\tau \mathcal{L}} \end{aligned} \tag{6}$$

In short, one obtains a thermal partition function by imposing periodicity constraints for both phase space variables.

Indeed, let us introduce an object $\mathcal{Z}_s(\beta)$ that results from imposing those constraints on the $\mathcal{Z}_0$ path integral, and extracting a $\mathcal{Z}_0$ factor:

$$\mathcal{Z}_s(\beta) \;\equiv\; \frac{\int \mathcal{D}p\,\mathcal{D}q\;\delta\big(q(\beta)-q(0)\big)\,\delta\big(p(\beta)-p(0)\big)\,e^{-\mathcal{S}}}{\int \mathcal{D}p\,\mathcal{D}q\,e^{-\mathcal{S}}}\,. \qquad (7)$$

Then, the use of the superposition principle yields:

$$\int \mathcal{D}p\,\mathcal{D}q\;\delta\big(q(\beta)-q(0)\big)\delta\big(p(\beta)-p(0)\big)\;e^{-\mathcal{S}} =$$

$$\lim_{T\to\infty}\int dp_1 dq_1\,\Big[\langle q_0,-iT|p_1,-i\beta\rangle\,\langle p_1,-i\beta|q_1,-i\beta\rangle\,\langle q_1,-i\beta|q_1,0\rangle$$

$$\times\,\langle q_1,0|p_1,0\rangle\,\langle p_1,0|q_0,iT\rangle\Big]$$

or

$$\int \mathcal{D}p\,\mathcal{D}q\;\delta\big(q(\beta)-q(0)\big)\delta\big(p(\beta)-p(0)\big)e^{-\mathcal{S}} \;=\; \lim_{T\to\infty}\; e^{-E_0(2T-\beta)}$$

$$\times \int \frac{dp_1 dq_1}{2\pi}\;\langle q_0|0\rangle\langle 0|p_1\rangle\;\langle q_1,-i\beta|q_1,0\rangle\;\langle p_1|0\rangle\langle 0|q_0\rangle$$

$$= \lim_{T\to\infty}\; e^{-E_0(2T-\beta)}\;|\langle q_0|0\rangle|^2\;\int dq_1\;\langle q_1,-i\beta|q_1,0\rangle.$$

$$= \mathcal{Z}_0\;\times\;e^{\beta E_0}\,\mathcal{Z}(\beta)\;=\;\mathcal{Z}_0\;\times\;\mathrm{Tr}\big[e^{-\beta(\hat{H}-E_0)}\big]\,. \qquad (8)$$

Then we conclude that

$$\mathcal{Z}_s(\beta)\;=\;\mathrm{Tr}\big[e^{-\beta\,:\hat{H}:}\big] \qquad (9)$$

where $:\hat{H}:$ denotes the normal-ordered Hamiltonian operator, i.e.:

$$:\hat{H}:\,\equiv\;\hat{H}\;-\;E_0\,. \qquad (10)$$

- So, by imposing periodicity on both phase space variables, and discarding β-independent factors (since they would be canceled by the normalization constant) we obtain $\mathcal{Z}_s(\beta)$.
- the partition function corresponding to the original Hamiltonian, the ground state energy redefined to zero.
- The subtraction of the vacuum energy is usually irrelevant (except in some known situations), as it is wiped out when taking derivatives of the free energy to calculate physical quantities.

Periodicity constraints for both variables is not in contradiction with the usual representation, (2), where they only apply to q, since they corresponds to different sets of paths. These constraints get rid of the unwelcome factors coming from paths which are outside of the $[0, \beta]$ interval (which are absent from the standard approach).

Summarizing, we have shown that a way to extract the partition function from the $T = 0$ partition function $\mathcal{Z}_0$, is to impose periodicity constraints for both the coordinate and its canonical momentum, a procedure that yields a $\mathcal{Z}_0$ factor times the thermal partition function, $\mathcal{Z}_s(\beta)$.

0.2 Rephrasing with auxiliary fields

The two δ-functions require the introduction of two auxiliary fields, ξ_1 and ξ_2.

Using $Q_1 \equiv q$ and $Q_2 \equiv p$, we have

$$\prod_{a=1}^{2} \left\{ \delta\left[Q_a(\beta) - Q_a(0) \right] \right\} = \int \frac{d^2\xi}{(2\pi)^2}\, e^{i \sum_{a=1}^{2} \xi_a \left[Q_a(\beta) - Q_a(0) \right]} \, . \quad (11)$$

Using this representation for the constraints we have:

$$\begin{aligned} \mathcal{Z}_s(\beta) &= \mathcal{N}^{-1} \int_{-\infty}^{\infty} \frac{d\xi_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\xi_2}{2\pi} \int \mathcal{D}Q \\ &\times e^{-\mathcal{S}(Q) + i \int_{-\infty}^{\infty} d\tau j_a(\tau) Q_a(\tau)} \, , \end{aligned} \quad (12)$$

where $\mathcal{N} \equiv \mathcal{Z}_0$, and we have introduced the notation:

$$j_a(\tau) \equiv \xi_a \left[\delta(\tau - \beta) - \delta(\tau)\right] . \tag{13}$$

The phase-space measure has been written in terms of Q:

$$\mathcal{D}Q \equiv \prod_{-\infty<\tau<\infty} \frac{dq(\tau)dp(\tau)}{2\pi} . \tag{14}$$

For the particular case of a harmonic oscillator with unit mass and frequency ω, we have

$$\mathcal{S}(Q) \;=\; \mathcal{S}_0(Q) \;=\; \frac{1}{2}\int_{-\infty}^{+\infty} d\tau\, Q_a(\tau)\widehat{\mathcal{K}}_{ab}Q_a(\tau) \;, \tag{15}$$

where the 2×2 operator matrix $\widehat{\mathcal{K}}$, given by:

$$\widehat{\mathcal{K}} \; = \; \begin{pmatrix} \omega^2 & i\frac{d}{d\tau} \\ -i\frac{d}{d\tau} & 1 \end{pmatrix} \, . \tag{16}$$

Thus the integral over Q is a Gaussian; it may therefore be written as follows:

$$\mathcal{Z}_s(\beta) \; = \; 2\pi \, \mathcal{N}^{-1} \left(\det \widehat{\mathcal{K}} \right)^{-\frac{1}{2}} \int \frac{d^2\xi}{(2\pi)^2} \, e^{-\frac{1}{2}\xi_a M_{ab} \xi_b} \, , \tag{17}$$

with

$$M \; \equiv \; \Omega(0_+) + \Omega(0_-) \, - \, \Omega(\beta) \, - \, \Omega(-\beta) \, , \tag{18}$$

where:

$$\widehat{\mathcal{K}}_{ac} \Omega_{cb}(\tau) \; = \; \delta_{ab} \, \delta(\tau). \tag{19}$$

Such that:

$$\Omega(\tau) \equiv \begin{pmatrix} \frac{1}{2\omega} & \frac{i}{2}\mathrm{sgn}(\tau) \\ -\frac{i}{2}\mathrm{sgn}(\tau) & \frac{\omega}{2} \end{pmatrix} e^{-\omega|\tau|} \tag{20}$$

($\mathrm{sgn} \equiv$ sign function).

Equation (20) can be used in (18), to see that:

$$M = \begin{pmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{pmatrix} (n_B(\omega) + 1)^{-1}, \tag{21}$$

where

$$n_B(\omega) \equiv (e^{\beta\omega} - 1)^{-1} \tag{22}$$

is the Bose-Einstein distribution function (the zero of energy set at the ground state).

Finally, note that $\mathcal{N}$ cancels the $\left(\det\widehat{\mathcal{K}}\right)^{-\frac{1}{2}}$ factor, and thus we arrive to a sort of 'dual' description for the partition function, as an integral over the ξ_a variables:

$$\mathcal{Z}_s(\beta) \;=\; \int \frac{d^2\xi}{2\pi}\, e^{-\frac{\omega^{-1}\,\xi_1^2 + \omega\,\xi_2^2}{2[n_B(\omega)+1]}} \;. \tag{23}$$

This integral is over two real variables ξ_a, which are 0-dimensional fields, one dimension less than the $0+1$ dimensional original theory.

Evaluating the partition function in the classical (high-temperature) limit we can see that $\mathcal{Z}_s(\beta)$ becomes:

$$\mathcal{Z}_s(\beta) \simeq \int \frac{d^2\xi}{2\pi} e^{-\beta H(\xi_1,\xi_2)} \quad (\beta << 1) \,, \tag{24}$$

where:

$$H(\xi_1,\xi_2) \equiv \frac{1}{2}\left(\xi_1^2 + \omega^2\xi_2^2\right) . \tag{25}$$

Such that (24) corresponds exactly to the classical partition function for a harmonic oscillator, when the identifications: $\xi_1 = p$ (classical momentum), and $\xi_2 = q$ (classical coordinate) are made

$$\mathcal{Z}_s(\beta) \simeq \int \frac{dpdq}{2\pi} e^{-\beta \frac{1}{2}\left(p^2 + \omega^2 q^2\right)} \quad (\beta << 1) \,. \tag{26}$$

On the other hand, had the exact form of the integral been kept (no approximation), we could still have written an expression similar to the classical partition function, albeit with an 'effective Hamiltonian' $H_{eff}(\xi_1, \xi_2)$:

$$\mathcal{Z}_s(\beta) \; = \; \int \frac{d^2\xi}{2\pi} \, e^{-\beta H_{eff}(\xi_1,\xi_2)} \; , \tag{27}$$

where:

$$H_{eff}(\xi_1, \xi_2) \; \equiv \; \frac{1}{2\beta} \left(n_B(\omega) + 1 \right)^{-1} \left(\omega^{-1} \, \xi_1^2 \, + \, \omega \, \xi_2^2 \right) . \tag{28}$$

This shows that the quantum partition function may also be written as a classical one, by using a β-dependent Hamiltonian, which tends to its classical counterpart in the high-temperature limit.

By integrating out the auxiliary fields in the (exact) expression for the partition function (23), we obtain:

$$\mathcal{Z}_s(\beta) \;=\; n_B(\omega) + 1 \;=\; \frac{1}{1 - e^{-\beta\omega}} \tag{29}$$

which is the usual result.

0.3 Interacting theories

When the action $\mathcal{S}$ is not quadratic, we may still give a formal expression for the alternative representation. Indeed, denoting by $\mathcal{Z}(J)$ the zero-temperature generating functional of correlation functions of the canonical variables:

$$\mathcal{Z}(J) \;=\; \int \mathcal{D}Q \, e^{-\mathcal{S}(Q)+\int_{-\infty}^{\infty} d\tau \, J_a(\tau) Q_a(\tau)} \tag{30}$$

and by $\mathcal{W}(J)$ the corresponding functional for connected ones, we see that

$$\mathcal{Z}_s(\beta) \;=\; [\mathcal{Z}(0)]^{-1} \int \frac{d^2\xi}{(2\pi)^2} \, \exp\{\mathcal{W}[i\, j(\tau)]\} \,, \tag{31}$$

where, with our normalization conventions, $\mathcal{Z}(0) = \mathcal{Z}_0$ (the vacuum functional for the interacting case).

Thus, a possible way to derive the effective Hamiltonian in the interacting case is to obtain first $\mathcal{W}[J]$, and then to replace the (arbitrary) source $J(\tau)$ by $i[j(\tau)]$, where $j(\tau)$ is the function of the auxiliary field defined in (13). Of course, $\mathcal{W}$ cannot be obtained exactly, except in very special cases. Otherwise, a suitable perturbative expansion can be used. In any case, $\mathcal{W}$ can be functionally expanded in powers of the source $J(\tau)$:

$$\mathcal{W}[J] = \mathcal{W}[0] + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\tau_1,\ldots,\tau_n} \mathcal{W}_{ab}^{(n)}(\tau_1,\ldots,\tau_n) J_{a_1}(\tau_1) \ldots J_{a_n}(\tau_n) \tag{32}$$

where each coefficient $\mathcal{W}^{(n)}$ is the n-point connected correlation function. The expansion above yields an expansion for H_{eff} in powers of the auxiliary fields. Not necessarily a perturbative expansion.

Indeed, the strength of each term is controlled by $\mathcal{W}^{(n)}$, which could even be exact (non-perturbative) in a coupling constant. To fix ideas, let us see what happens when one keeps only up to the $n = 4$ term, assuming also that there is $\mathcal{Q}_a \to -\mathcal{Q}_a$ symmetry in $\mathcal{S}$. Then, we first see that the $\mathcal{W}[0]$ is cancelled by the $\mathcal{N}$ factor, and on the other hand we obtain

$$\mathcal{Z}_s(\beta) \;=\; \int \frac{d^2\xi}{(2\pi)^2}\, e^{-\beta H_{eff}(\xi_1,\xi_2)} \;, \tag{33}$$

where

$$\begin{aligned} H_{eff} \quad &= \frac{1}{2\beta}\int_{\tau_1,\tau_2} \mathcal{W}^{(2)}_{a_1 a_2}(\tau_1,\tau_2) j_{a_1}(\tau_1) j_{a_2}(\tau_2) \\ &- \quad \frac{1}{4!\beta}\int_{\tau_1,\tau_2} \mathcal{W}^{(2)}_{a_1 a_2 a_3 a_4}(\tau_1,\tau_2,\tau_3,\tau_4) j_{a_1}(\tau_1) j_{a_2}(\tau_2) j_{a_3}(\tau_3) j_{a_4}(\tau_4) \\ &+ \quad \ldots \end{aligned} \tag{34}$$

Using the explicit form of $j_a(\tau)$ in terms of the auxiliary fields, we see that:

$$H_{eff} = H_{eff}^{(2)} + H_{eff}^{(4)} + \dots \qquad (35)$$

where

$$\begin{aligned} H_{eff}^{(2)} &= \frac{1}{2}\mathcal{M}_{ab}^{(2)}\xi_a\,\xi_b \\ H_{eff}^{(4)} &= \frac{1}{4!}\mathcal{M}_{abcd}^{(4)}\,\xi_a\,\xi_b\,\xi_c\,\xi_b \\ \dots &= \dots \\ H_{eff}^{(2k)} &= \frac{1}{(2k)!}\mathcal{M}_{a_1\dots a_{2k}}^{(2k)}\,\xi_{a_1}\dots\xi_{a_{2k}}\ , \end{aligned} \qquad (36)$$

where the explicit forms of the coefficients $\mathcal{M}^{(2k)}$ in terms of $\mathcal{W}^{(2k)}$ may be found, after some algebra.

For example $\mathcal{M}^{(2)}$ is a diagonal matrix:

$$\mathcal{M}^{(2)} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \tag{37}$$

where

$$c_a = \frac{1}{\beta} \int \frac{d\nu}{\pi} \left(1 - e^{-i\nu\beta}\right) \tilde{\mathcal{W}}_{aa}(\nu) \tag{38}$$

(where the tilde denotes Fourier transform). c_1 plays the role of an effective coefficient for the kinetic term ($\propto p^2$) in the effective Hamiltonian, while c_2 does introduce an effective quadratic potential. Note that they will, in general, depend on β, ω, and on any additional coupling constant the system may have. For the harmonic oscillator case we have the rather simple form:

$$
\begin{aligned}
c_1 &= \frac{1}{\omega(n_B(\omega)+1)} \\
c_2 &= \frac{\omega}{n_B(\omega)+1}\,.
\end{aligned}
\tag{39}
$$

The quartic term involves $\mathcal{M}^{(4)}$, which may be written in terms of the connected 4-point function:

$$
\mathcal{M}^{(4)}_{abcd} = \frac{1}{\beta}\Big[-\mathcal{W}^{(4)}_{abcd}(0,0,0,0) + 4\,\mathcal{W}^{(4)}_{abcd}(\beta,\beta,\beta,0) - 6\,\mathcal{W}^{(4)}_{abcd}(\beta,\beta,0,0) + 4\,\mathcal{W}^{(4)}_{abcd}(\beta,0,0,0) - \mathcal{W}^{(4)}_{abcd}(0,0,0,0)\Big]_{sym}
\tag{40}
$$

where the sym suffix denotes symmetrization under simultaneous interchange of time arguments and discrete indices.

Scalar field

The extension of the harmonic oscillator results to the QFT of a real scalar field φ in $d+1$ (Euclidean) dimensions is quite straightforward. Let $\varphi(x) = \varphi(\tau, \mathbf{x})$ where $x = (\tau, \mathbf{x}) \in \mathbb{R}^{(d+1)}$, $\tau \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^{(d)}$.

0.4 Free partition function

The free Euclidean action in terms of the phase-space variables $\mathcal{S}_0$, is in this case given by:

$$\mathcal{S}_0 \; = \; \int d^{d+1}x \left[-\, i\pi\partial_\tau\varphi + \mathcal{H}_0(\pi, \varphi) \right] , \tag{41}$$

with

$$\mathcal{H}_0(\pi,\varphi) \;\equiv\; \frac{1}{2}\Big[\pi^2 \;+\; |\nabla\varphi|^2 + m^2\varphi^2\Big]\;. \tag{42}$$

We then have to implement the periodic boundary conditions both for $\varphi(\tau,\mathbf{x})$ and its canonical momentum $\pi(\tau,\mathbf{x})$

$$\varphi\,(\beta,\mathbf{x}) \;=\; \varphi\,(0,\mathbf{x}) \;\;,\quad \pi\,(\beta,\mathbf{x}) \;=\; \pi\,(0,\mathbf{x}) \;\;,\;\; \forall\,\mathbf{x}\,\in\mathbb{R}^{(d)}\;, \tag{43}$$

which requires the introduction of two time-independent Lagrange multiplier fields: $\xi_a(\mathbf{x})$, $a = 1,\,2$. Defining a two-component field $\Phi = (\Phi_a)$, $a = 1,\,2$, such that $\Phi_1 = \varphi$ and $\Phi_2 = \pi$, an analogous procedure to the one followed for the harmonic oscillator yields, for the free partition function $\mathcal{Z}_0(\beta)$:

$$\mathcal{Z}_0(\beta) \;=\; \mathcal{N}^{-1}\int \mathcal{D}\xi \int \mathcal{D}\Phi\; e^{-\frac{1}{2}\int d^{d+1}x\,\Phi_a\hat{\mathcal{K}}_{ab}\Phi_b + i\int d^{d+1}x j_a\Phi_a}\;, \tag{44}$$

where $j_a(x) \equiv \xi_a(\mathbf{x})\big[\delta(\tau-\beta) - \delta(\tau)\big]$ and:

$$\widehat{\mathcal{K}} \;=\; \begin{pmatrix} \hat{h}^2 & i\frac{\partial}{\partial\tau} \\ -i\frac{\partial}{\partial\tau} & 1 \end{pmatrix} , \qquad (45)$$

where we have introduced $\hat{h} \equiv \sqrt{-\nabla^2 + m^2}$, the first-quantized energy operator for massive scalar particles. Performing the integral over Φ, yields the partition function in terms of the Lagrange multipliers:

$$\mathcal{Z}_0(\beta) \;=\; \int \mathcal{D}\xi \, e^{-\frac{1}{2}\int d^dx \int d^dy \, \xi_a(\mathbf{x}) \, \langle \mathbf{x}|\hat{M}_{ab}|\mathbf{y}\rangle \, \xi_b(\mathbf{y})} , \qquad (46)$$

with $\hat{M} \equiv \hat{\Omega}(0_+) + \hat{\Omega}(0_-) - \hat{\Omega}(\beta) - \hat{\Omega}(-\beta)$ and

$$\hat{\Omega}(\tau) \equiv \begin{pmatrix} \frac{1}{2}\hat{h}^{-1} & \frac{i}{2}\mathrm{sgn}(\tau) \\ -\frac{i}{2}\mathrm{sgn}(\tau) & \frac{1}{2}\hat{h} \end{pmatrix} e^{-\hat{h}|\tau|} \, . \tag{47}$$

Then,

$$\hat{M} \equiv \begin{pmatrix} \hat{h}^{-1} & 0 \\ 0 & \hat{h} \end{pmatrix} (\hat{n}_B + 1)^{-1} \, , \tag{48}$$

where

$$\hat{n}_B \equiv \frac{1}{e^{\beta\hat{h}} - 1} \, . \tag{49}$$

So, we see that:

$$\mathcal{Z}_0(\beta) = \tag{50}$$

$$\int \mathcal{D}\xi \, \exp\Big\{ - \frac{1}{2} \int d^d x \int d^d y \big[\xi_1(\mathbf{x}) \, \langle \mathbf{x} | \hat{h}^{-1} \, (\hat{n}_B + 1)^{-1} | \mathbf{y} \rangle \, \xi_1(\mathbf{y})$$

$$+ \xi_2(\mathbf{x}) \, \langle \mathbf{x} | \hat{h} \, (\hat{n}_B + 1)^{-1} | \mathbf{y} \rangle \, \xi_2(\mathbf{y}) \big] \Big\} \, . \tag{51}$$

Which gives:

$$\mathcal{Z}_0(\beta) \;=\; \det \big(\hat{n}_B + 1 \big) \tag{52}$$

and can be evaluated in the basis of eigenstates of momentum to yield:

$$\mathcal{Z}_0(\beta) \;=\; \prod_{\mathbf{k}} \big[n_B(E_{\mathbf{k}}) + 1 \big] \tag{53}$$

where $E_{\mathbf{k}} \equiv \sqrt{\mathbf{k}^2 + m^2}$.

The free-energy density, $F_0(\beta)$, is:

$$F_0(\beta) \;=\; \frac{1}{\beta} \int \frac{d^d k}{(2\pi)^d} \, \ln\left(1 \,-\, e^{-\beta E_{\mathbf{k}}}\right) \,. \tag{54}$$

In the classical, high-temperature limit, the path integral for the partition function becomes:

$$\mathcal{Z}_0(\beta) \;\simeq\; \int \mathcal{D}\xi \, e^{-\beta \, H(\xi)} \,, \tag{55}$$

where:

$$H(\xi) \;=\; \frac{1}{2} \int d^d x \left[\xi_1^2(\mathbf{x}) \,+\, |\nabla \xi_2(\mathbf{x})|^2 \,+\, m^2 \, \xi_2^2(\mathbf{x})\right] \,. \tag{56}$$

This is, again, the usual classical expression for the partition function, with the Lagrange multipliers playing the role of phase space variables.

0.5 Self-interacting real scalar field

When field self-interactions are included, instead of the free action S_0, we must consider instead

$$S \;=\; S_0 + S_I \;, \tag{57}$$

where the free action S_0, has already been defined in (41), while S_I is a self-interaction term. We shall assume it to be of the type:

$$S_I \;=\; \int d^{d+1}x \, V(\varphi) \;, \tag{58}$$

$V(\varphi)$ being an even polynomial in φ, with only one (trivial) minimum. Proceeding along similar lines to the ones followed for the free field case in the preceding section, the partition function for the interacting system can be written in the form:

$$\mathcal{Z}(\beta) \;=\; \mathcal{N} \int \mathcal{D}\xi \int \mathcal{D}\Phi \; e^{-S(\Phi) + i \int d^{d+1}x j_a \Phi_a} \;, \tag{59}$$

where Φ, as well as the 'current' j_a have already been defined for the free case, in the previous subsection. The constant $\mathcal{N}$ is introduced to satisfy $\mathcal{Z}(\infty) = 1$. On the other hand, since the fields are assumed to tend to zero at infinity, $\beta \to \infty$ implies that the term involving j vanishes in this limit. This means that

$$\mathcal{N}^{-1} \;=\; \int \mathcal{D}\xi \int \mathcal{D}\Phi \; e^{-S(\Phi)} \;. \tag{60}$$

There are many different paths one could follow from now on in order to evaluate the partition function. We choose to adopt a procedure that makes contact with quantities defined for QFT at $T = 0$, in such a way that the $T \neq 0$ theory is built 'on top of it'.

Indeed, recalling the definition of the generating functional for connected correlation functions, $\mathcal{W}$, we may write:

$$\mathcal{N} \int \mathcal{D}\Phi \; \exp\left[-S(\Phi) + i \int d^{d+1}x j_a \Phi_a\right] \equiv e^{-\mathcal{W}(j)}, \qquad (61)$$

so that the partition function $\mathcal{Z}(\beta)$ becomes:

$$\mathcal{Z}(\beta) = \int \mathcal{D}\xi \, e^{-\mathcal{W}(j)} . \qquad (62)$$

We use the small j to denote the 2-component current which is a function of the Lagrange multipliers, as defined in (13). A capital J shall be reserved to denote a completely arbitrary 2-component source, so that:

$$\mathcal{N} \int \mathcal{D}\Phi \, exp\left[-S(\Phi) + i \int d^{d+1}x J_a \Phi_a\right] \equiv e^{-\mathcal{W}(J)} . \qquad (63)$$

Defining $H_{eff}(\xi)$, the 'effective Hamiltonian' for ξ, by means of the expression:

$$H_{eff}(\xi) \;=\; \frac{1}{\beta}\,\mathcal{W}(j)\;, \tag{64}$$

we see that the partition function is given by:

$$\mathcal{Z}(\beta) \;=\; \int \mathcal{D}\xi \;\exp\left[-\beta H_{eff}(\xi)\right]\;. \tag{65}$$

This yields the path integral for the quantum partition function as a classical-looking functional integral, involving an effective Hamiltonian which takes into account all the $T = 0$ quantum effects.

Dirac field

The final example we consider is a massive Dirac field in $d+1$ spacetime dimensions. The procedure will be essentially the same as for the real scalar field, once the relevant kinematical differences are taken into account. The action S_0^f for the free case is given by $S_0^f = \int d^{d+1}x \bar{\psi}(\not\partial + m)\psi$ where $\not\partial = \gamma_\mu \partial_\mu$, $\gamma_\mu^\dagger = \gamma_\mu$ and $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

We then impose antiperiodic conditions for both fields:

$$\psi(\beta, \mathbf{x}) = -\psi(0, \mathbf{x}) \;,\;\; \bar{\psi}(\beta, \mathbf{x}) = -\bar{\psi}(0, \mathbf{x}) \tag{66}$$

as constraints on the Grassmann fields.

Those conditions lead to the introduction of the two $\delta-$functions:

$$\begin{aligned} \mathcal{Z}_0^f(\beta) &= \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \delta\big(\psi(\beta,\mathbf{x}) + \psi(0,\mathbf{x})\big)\, \delta\big(\bar{\psi}(\beta,\mathbf{x}) + \bar{\psi}(0,\mathbf{x})\big) \\ &\times \exp\Big[- S_0^f(\bar{\psi},\psi)\Big] \,. \end{aligned} \tag{67}$$

Those auxiliary fields, denoted by $\chi(\mathbf{x})$ and $\bar{\chi}(\mathbf{x})$ must be time-independent Grassmann spinors. The resulting expression for $\mathcal{Z}_0^f(\beta)$ is then

$$\mathcal{Z}_0^f(\beta) = \mathcal{N}^{-1} \int \mathcal{D}\chi \mathcal{D}\bar{\chi} \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-S_0^f(\bar{\psi},\psi) + i \int d^{d+1}x\, (\bar{\eta}\psi + \bar{\psi}\eta)}, \tag{68}$$

where η and $\bar{\eta}$ are (Grassmann) sources depending on χ and $\bar{\chi}$ through the relations:

$$\eta(x) = \chi(\mathbf{x})\big[\delta(\tau - \beta) + \delta(\tau)\big] \,, \quad \bar{\eta}(x) = \bar{\chi}(\mathbf{x})\big[\delta(\tau - \beta) + \delta(\tau)\big] \,. \tag{69}$$

Integrating out $\psi, \bar{\psi}$, we arrive to:

$$\mathcal{Z}_0^f(\beta) \;=\; \int \mathcal{D}\chi \mathcal{D}\bar{\chi}\; \exp\Big[-\beta H_{eff}\big(\bar{\chi}, \chi\big)\Big] \tag{70}$$

where

$$H_{eff}\big(\bar{\chi}, \chi\big) = \int d^d x \int d^d y\, \bar{\chi}(\mathbf{x}) H^{(2)}\big(\mathbf{x}, \mathbf{y}\big) \chi(\mathbf{y}) \tag{71}$$

with:

$$\begin{aligned} H^{(2)}\big(\mathbf{x}, \mathbf{y}\big) \;&=\; \langle \mathbf{x}, 0|(\not{\partial} + m)^{-1}|\mathbf{y}, 0\rangle + \langle \mathbf{x}, \beta|(\not{\partial} + m)^{-1}|\mathbf{y}, \beta\rangle \\ &\quad + \langle \mathbf{x}, 0|(\not{\partial} + m)^{-1}|\mathbf{y}, \beta\rangle + \langle \mathbf{x}, \beta|(\not{\partial} + m)^{-1}|\mathbf{y}, 0\rangle \\ &=\; \frac{1}{\beta}\Big[2\, S_f\big(0, \mathbf{x} - \mathbf{y}\big) + S_f\big(\beta, \mathbf{x} - \mathbf{y}\big) + S_f\big(-\beta, \mathbf{x} - \mathbf{y}\big)\Big]. \end{aligned} \tag{72}$$

On the last line, S_f, denotes the Dirac propagator. It is possible to show that

$$H\left(\mathbf{x}, \mathbf{y}\right) = \frac{1}{\beta} \langle \mathbf{x} | \hat{u} (1 - \hat{n}_F)^{-1} | \mathbf{y} \rangle \tag{73}$$

where $\hat{n}_F \equiv \left(1 + e^{\beta \hat{n}}\right)^{-1}$ is the Fermi-Dirac distribution function, written in terms of $\hat{h}$, the energy operator (defined identically to its real scalar field counterpart); $\hat{u}$ is a unitary operator, defined as

$$\hat{u} \equiv \frac{\hat{h}_D}{\hat{h}} , \;\; \hat{h}_D \equiv \gamma \cdot \nabla + m \, . \tag{74}$$

Then we verify that:

$$\mathcal{Z}_0^f(\beta) = \det \hat{u} \, \det{}^{-1} \left[(1 - \hat{n}_F) \, \mathbf{I} \right] , \tag{75}$$

($\mathbf{I} \equiv$ identity matrix in the representation of Dirac's algebra)

$$\mathcal{Z}_0^f(\beta) \;=\; \left\{ \prod_{\vec{p}} \left[1 + e^{-\beta E(\vec{p})} \right] \right\}^{r_d} \qquad (76)$$

with $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$ and $r_d \equiv$ dimension of the representation (we have used the fact that $\det \hat{u} = 1$).

Again, the procedure has produced the right result for the partition function, with a normal-ordered Hamiltonian.

1 Conclusions

- We have shown that, by introducing the periodicity conditions as constraints for the paths in the Euclidean path integral for the $T = 0$ vacuum functional, one can obtain a representation for a thermal partition function.
- These constraints should be applied on fields and canonical momenta, and when they are represented by means of auxiliary fields, they lead to an alternative, 'dual' representation for the corresponding thermal observable.
- The resulting representation for the partition function may be thought of as a dimensionally reduced path integral over phase space, similar to the one of a classical thermal field theory, with the auxiliary fields playing the role of canonical

variables, but with an effective Hamiltonian, H_{eff}, which reduces to the classical one in the corresponding (high-temperature) limit.

- The effective Hamiltonian can be constructed by assuming the knowledge of the corresponding $T = 0$ generating functional of connected correlation functions. If this knowledge is perturbative, one recovers the perturbative expansion for the thermal partition function.

- We believe that the most important applications of this formalism are to be found in the case of having non-perturbative information about the $T = 0$ correlation functions: here, it is possible to incorporate that knowledge into the formalism, and to compute thermal corrections from it.

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Thank You!