CFT/AdS

Ami Katz

Boston University

w/ L. Fitzpatrick, D. Poland, & D. Simmons-Duffin.

arXiv:1007.2412

Outline

1. Introduction - Global AdS: a tool for describing perturbative modification of the dilatation operator.
2. "CFT perturbative unitarity" implies that non-renormalizable AdS interactions are small.
3. Large dimension operators & unitarization of an "effective conformal theory."
4. Large dimension double-trace operators and the flat space limit of AdS.
5. Conclusions.

What CFTs give good AdS?

Important for:

1. Understanding warped Extra-Dim models in UV:
   a) Is SUSY important for good AdS?
   b) What from the p.o.v. of the CFT suppresses bulk non-renormalizable terms?

2. Duals of condensed matter systems.

Introduction

Global-AdS is just a convenient math tool to describe a CFT with (I. Heemskerk, J. Penedones, J. Polchinski, J. Sully) :

1. A parameter like N of SU(N):

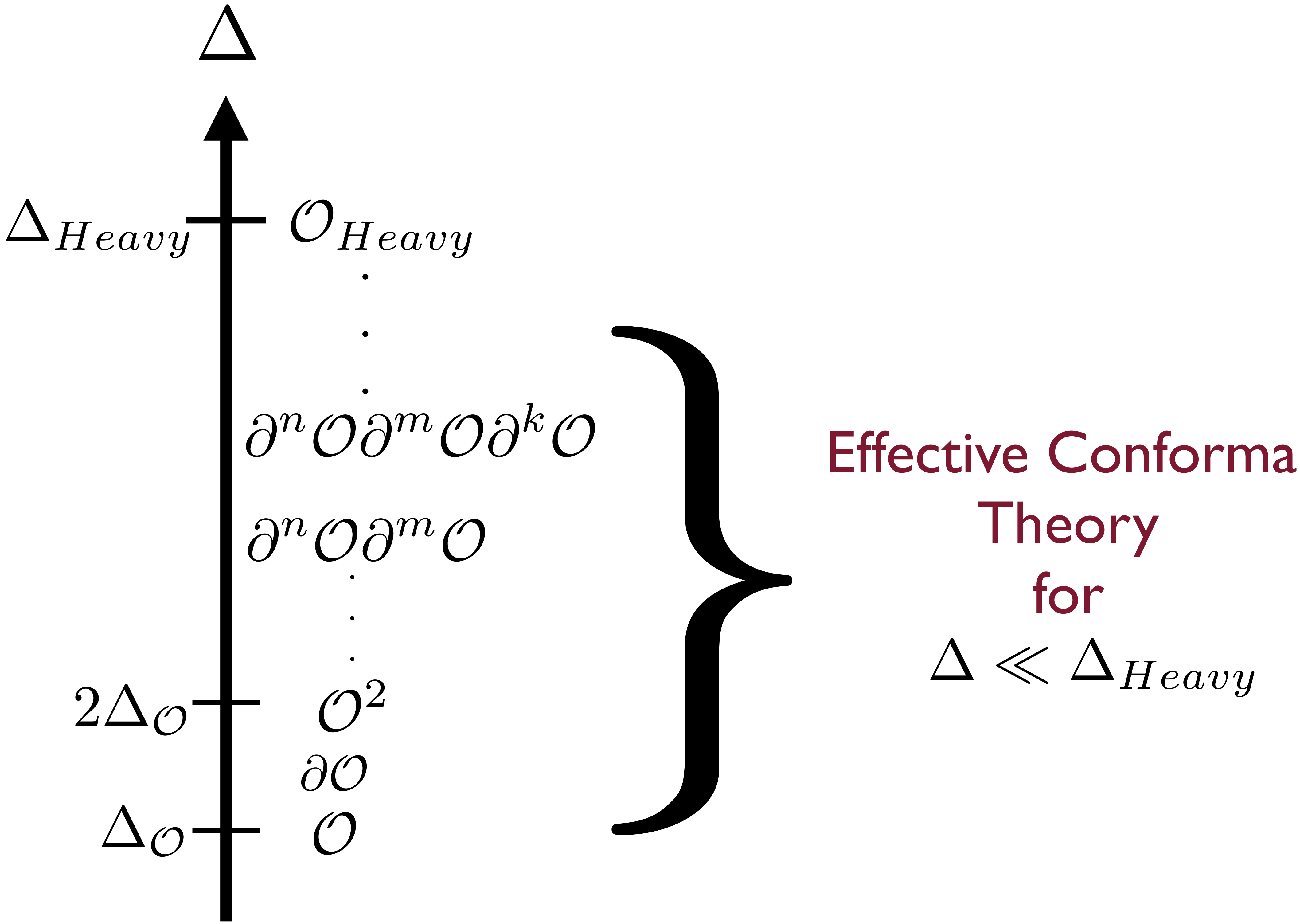
$$\langle O(x)O(0)\rangle = \frac{1}{x^{2\Delta}}, \quad \langle O(x)O(y)O(z)O(w)\rangle_{conn} = \frac{1}{N^2}F(x,y,z,w)$$

Dilatation Op: $(Ex : \mathcal{O} = tr(F^2))$

$$\mathcal{O}_{n,l}(x) = \mathcal{O}(\partial^2)^n \partial_{\{\mu_1}\ldots\partial_{\mu_l\}}\mathcal{O}(x)$$

$$\Delta_{n,l} = 2\Delta_{\mathcal{O}} + 2n + l + \mathcal{O}\left(\frac{1}{N^2}\right)$$

2. A hierarchy in the dimension of operators:



A comparison

A perturbative, Lorentz-Inv. theory of particles: $H = H_0 + V$	A perturbative, Conformal-Inv. theory of operators: $D = D_0 + V$

1. For $E < M_{Heavy}$, H describes

a few particles (say one type).

2. E-states of H_0:

$$H_0|\overrightarrow{p}\rangle = \sqrt{p^2 + m^2}|\overrightarrow{p}\rangle$$

$$H_0|\overrightarrow{p}_1, \overrightarrow{p}_2\rangle = \left(\sqrt{p_1^2 + m^2} + \sqrt{p_2^2 + m^2}\right)|\overrightarrow{p}_1, \overrightarrow{p}_2\rangle$$

$$\vdots$$

A perturbative, Lorentz-Inv. theory of particles: $H = H_0 + V$	A perturbative, Conformal-Inv. theory of operators: $D = D_0 + V$
3. V will mix states with different particle number:	

$$a^\dagger_{\overrightarrow{p}} |0\rangle \equiv |\overrightarrow{p}\rangle$$

$$V\left(\{a^\dagger_{\overrightarrow{p}}, a_{\overrightarrow{p}}\}\right)$$

A perturbative, Lorentz-Inv. theory of particles: $H = H_0 + V$	A perturbative, Conformal-Inv. theory of operators: $D = D_0 + V$

4. Making Lorentz Inv manifest:

$$\{a^\dagger_{\overrightarrow{p}}, a_{\overrightarrow{p}}\} \longrightarrow \phi(\overrightarrow{x}, t)$$

a) $U(\Lambda)\phi(x)U(\Lambda)^\dagger = \phi(\Lambda \cdot x)$

b) $[\phi(x), \phi(y)] = 0, (x-y)^2 < 0$

$$V = \int d^{d-1}x \mathcal{V}(\overrightarrow{x}, t)$$

$$\mathcal{V}(x) = \lambda\phi^4(x) + \frac{1}{\Lambda^4}(\partial\phi)^4 + \cdots$$

$\mathcal{V}(x)$ satisfies a) & b):

$$[\overrightarrow{K}, H] = i\overrightarrow{P}$$

$$H_0 \to H_0 + V \longrightarrow \overrightarrow{K}^{(0)} \to \overrightarrow{K}^{(0)} + \overrightarrow{K}^{(1)}$$

a): $\overrightarrow{K}^{(1)} = \int d^{d-1}x \; \overrightarrow{x}\mathcal{V}(\overrightarrow{x}, t)$

b): $[\overrightarrow{K}^{(1)}, V] = 0$

A comparison

A perturbative, Lorentz-Inv. theory of particles: $H = H_0 + V$	A perturbative, Conformal-Inv. theory of operators: $D = D_0 + V$
1. For $E < M_{Heavy}$, H describes a few particles (say one type).	1. For $\Delta < \Delta_{Heavy}$, D describes a few single-trace Ops (say one type).
2. E-states of H_0:	2. E-states of D_0 (in radial quantization):
$H_0\|\overrightarrow{p}\rangle = \sqrt{p^2+m^2}\|\overrightarrow{p}\rangle$	$(\partial^2)^n\partial_{\{\mu_1}\cdots\partial_{\mu_l\}}\mathcal{O} \longleftrightarrow \|\Delta_{\mathcal{O}}; n, l, J\rangle_1$
$H_0\|\overrightarrow{p}_1, \overrightarrow{p}_2\rangle = \left(\sqrt{p_1^2+m^2}+\sqrt{p_2^2+m^2}\right)\|\overrightarrow{p}_1, \overrightarrow{p}_2\rangle$	$D_0\|\Delta_{\mathcal{O}}; n, l, J\rangle_1 = (\Delta_{\mathcal{O}}+2n+l)\|\Delta_{\mathcal{O}}; n, l, J\rangle_1$
	$D_0\|\Delta_{\mathcal{O}}; n, l, J\rangle_1\|\Delta_{\mathcal{O}}; n', l', J'\rangle_1 = ((\Delta_{\mathcal{O}}+2n+l)+((\Delta_{\mathcal{O}}+2n'+l'))\|\Delta_{\mathcal{O}}; n, l, J\rangle_1\|\Delta_{\mathcal{O}}; n', l', J'\rangle_1$
⋮	⋮

A perturbative, Lorentz-Inv. theory of particles: $H = H_0 + V$	A perturbative, Conformal-Inv. theory of operators: $D = D_0 + V$
3. V will mix states with different particle number:	3. At O(1/N) V will mix operators with different number of traces:
$a^\dagger_{\overrightarrow{p}}\|0\rangle \equiv \|\overrightarrow{p}\rangle$	$a^\dagger_{n,l,J}\|0\rangle = \|\Delta_{\mathcal{O}}; n, l, J\rangle_1$
↓	↓
$V\left(\{a^\dagger_{\overrightarrow{p}}, a_{\overrightarrow{p}}\}\right)$	$V\left(\{a^\dagger_{n,l,J}, a_{n,l,J}\}\right)$

A perturbative, Lorentz-Inv. theory of particles: $H = H_0 + V$

4. Making Lorentz Inv manifest:

$$\{a^\dagger_{\vec{p}}, a_{\vec{p}}\} \longrightarrow \phi(\vec{x}, t)$$

a) $U(\Lambda)\phi(x)U(\Lambda)^\dagger = \phi(\Lambda \cdot x)$

b) $[\phi(x), \phi(y)] = 0, (x-y)^2 < 0$

$$V = \int d^{d-1}x \mathcal{V}(\vec{x}, t)$$

$$\mathcal{V}(x) = \lambda\phi^4(x) + \frac{1}{\Lambda^4}(\partial\phi)^4 + \cdots$$

$\mathcal{V}(x)$ satisfies a) & b):

$$[\vec{K}, H] = i\vec{P}$$

$$H_0 \to H_0 + V \longrightarrow \vec{K}^{(0)} \to \vec{K}^{(0)} + \vec{K}^{(1)}$$

a): $\vec{K}^{(1)} = \int d^{d-1}x \; \vec{x}\mathcal{V}(\vec{x}, t)$

b): $[\vec{K}^{(1)}, V] = 0$

A perturbative, Conformal-Inv. theory of operators: $D = D_0 + V$

4. Making Conformal Inv manifest:

$$\{a^\dagger_{n,l,J}, a_{n,l,J}\} \longrightarrow \phi\left(\vec{\theta}, \rho, t\right)$$

(t is global AdS time ~ dilatation)

a) $U(\vec{\alpha}_{conf})\phi(x)U(\vec{\alpha}_{conf})^\dagger = \phi\left(\vec{\theta'}, \rho', t'\right)$

b) $[\phi(\vec{x}, t), \phi(\vec{y}, t)] = 0, \; \vec{x} \neq \vec{y}$

$$V = \int \sqrt{g} d^d x \; \mathcal{V}(\vec{x}, t)$$

$$\mathcal{V}(x) = \lambda\phi^4(x) + \frac{1}{\Lambda^4}(\partial\phi)^4 + \cdots$$

$\mathcal{V}(x)$ satisfies a) & b):

$$[D, K_\mu] = -K_\mu$$

$$D_0 \to D_0 + V \longrightarrow K_\mu^{(0)} \to K_\mu^{(0)} + K_\mu^{(1)}$$

a): $K_\mu^{(1)} = \int \sqrt{g} d^d x \; f(\vec{x}, t)_\mu \mathcal{V}(\vec{x}, t)$

b): $[V, K_\mu^{(1)}] = 0$

A perturbative, Lorentz-Inv. theory of particles: $H = H_0 + V$

5. When is V small?

When perturbative unitarity isn't violated.

For $E < M_{Heavy}$ cross-sections are well behaved:

$$\mathcal{V}(x) = \lambda\phi^4(x) + \frac{1}{\Lambda^4}(\partial\phi)^4 + \cdots$$

$$\lambda \lesssim (4\pi)^2, \;\; \Lambda \gtrsim M_{Heavy}$$

A hierarchy in the spectrum ensures

that non-renorm terms are small.

A perturbative, Conformal-Inv. theory of operators: $D = D_0 + V$

5. When is V small?

AdS: Good behavior for local bulk scattering

at $E_{d+1} < M_{Heavy}$:

$$\mathcal{V}(x) = \lambda\phi^4(x) + \frac{1}{\Lambda^4}(\partial\phi)^4 + \cdots$$

$$\lambda \lesssim (4\pi)^2, \;\; \Lambda \gtrsim M_{Heavy}$$

CFT: ??? $(M_{Heavy} \leftrightarrow \Delta_{Heavy})$

A hierarchy in the dimensions ensures

that non-renorm bulk terms are small ???

CFT: ???

$$\langle J^a_\mu(p) J^b_\nu(q) J^c_\rho(k)\rangle$$

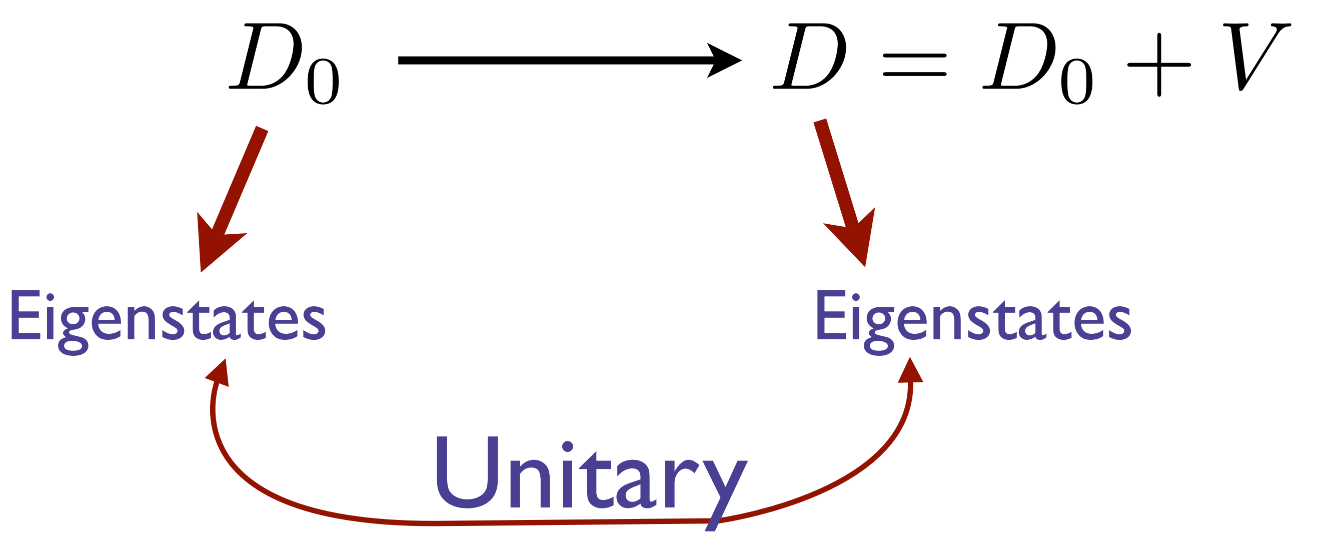
$$\sim f^{abc}(\eta_{\mu\nu}(p-q)_\rho + \cdots) F_1(p,q)$$ ← tr(F^2)

$$+ f^{abc}(p_\rho q_\mu k_\nu + \cdots) F_2(p,q)$$ ← tr(F^3)

↓

Suppressed for $\Delta_{Heavy} \gg 1$

CFT perturbative unitarity



$$|A\rangle = (\delta_{AB} + T_{AB})|B\rangle^{(0)}$$

$$|\mathcal{R}e(T_{AA})| = \frac{1}{2}\sum_{B}\frac{|V_{AB}|^2}{(E_A - E_B)^2} < 2$$

Anomalous dimensions of double-trace operators

Primary: $\mathcal{O}_{n,l}(x) \equiv \mathcal{O}(\overleftrightarrow{\partial}_\nu \overleftrightarrow{\partial}^\nu)^n \overleftrightarrow{\partial}_{\mu_1} ... \overleftrightarrow{\partial}_{\mu_l} \mathcal{O}(x) - \text{traces}$

$$\Delta_{n,l} = 2\Delta + 2n + l + \gamma(n,l)$$ ← Pert-Thy in V

Degenerate P.T?

$$[D^{(0)}, K^{(1)}] + K^{(1)} = [K^{(0)}, V] \longrightarrow K^{(1)}_{AB} \sim \frac{V_{AB}}{E_A - E_B}$$

Smoothness of $K^{(1)}_{AB}$ requires that $V_{AB} = 0$ for $E_A = E_B$

$$\mathcal{O}_{n,l}(0)|0\rangle = |n,l\rangle_2$$

$$\gamma(n,l) = {}_2\langle n,l|V|n,l\rangle_2 + \sum_\alpha \frac{|\langle\alpha|V|n,l\rangle_2|^2}{E_{n,l} - E_\alpha} + \ldots$$

$$\frac{|V_{n,l;n+1,l}|^2}{4} < \sum_B \frac{|V_{n,l;B}|^2}{(E_{n,l} - E_B)^2} < 4$$

$(|B\rangle = |n+1, l\rangle_2)$

$(V_{n,l;B} \equiv {}_2\langle n, l|V|B\rangle)$

$$|V_{n,l;n+1,l}| < 4$$

$$V_{n,l;n+1,l} = V_{n,l;n,l} + O(\frac{1}{n}), \ n \gg 1$$

Perturbative Unitarity Bound:

$$|\gamma(n,l)| < 4, \ n \gg 1$$

Anomalous dimensions from global AdS

$$ds^2 = \frac{1}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho\, d\Omega^2\right)$$

$D = -i\partial_t$: global energy is dimension.

$\gamma(n,l) =$ the energy shift of a 2-particle primary state, $|n,l\rangle_2$, due to the interaction V.

Example: $$V = \frac{\mu^{3-d}}{4!}\int d^d x\sqrt{-g}\phi^4(x)$$

$$\phi(x) = \sum_{n,l,J}\psi_{nlJ}(x)a_{nlJ} + \psi^*_{nlJ}(x)a^\dagger_{nlJ}$$

$$\gamma(n,0) = \frac{\mu^{3-d}}{4}\int d\Omega\int_0^{\pi/2} d\rho\frac{\sin^{d-1}\rho}{\cos^{d+1}\rho}\,{}_2\langle n,0|\phi^2(x)|0\rangle\underline{\langle 0|\phi^2(x)|n,0\rangle_2}$$

primary wave-fcn ?

Primary scalar of dimension $2\Delta + 2n$:

$$\langle 0|\phi^2(x)|n,0\rangle_2 = \frac{1}{N_n}(e^{it}\cos\rho)^{2\Delta+2n}$$

$$\gamma(n,0) = \frac{\mu^{3-d}\pi^{d/2}}{4(N_n)^2}\frac{\Gamma(2\Delta+2n-\frac{d}{2})}{\Gamma(2\Delta+2n)}$$

$$\gamma(n,0) \xrightarrow{n\gg 1} \left(\frac{\mu}{n}\right)^{3-d}$$

Dimensional analysis with $n \sim E$!

$\gamma(n,0)$ for $\frac{1}{\Lambda^{d+1}}(\nabla\phi)^4$: $\nabla^2\langle 0|\phi^2(x)|n,0\rangle_2,\ \nabla_\mu\nabla_\nu\langle 0|\phi(x)^2|n,0\rangle_2$

$$\longrightarrow \gamma(n,0) \xrightarrow{n\gg 1} \left(\frac{n}{\Lambda}\right)^{d+1}$$

In general - for an interaction of the form: $\frac{1}{\Lambda^p}(\nabla^2)^{\frac{p-d+3}{2}}\phi^4$

$$\gamma(n,0) \xrightarrow{n\gg 1} \left(\frac{n}{\Lambda}\right)^p$$

Perturbative Unitarity:

$$|\gamma(n,l)| < 4, \;\; n \gg 1$$

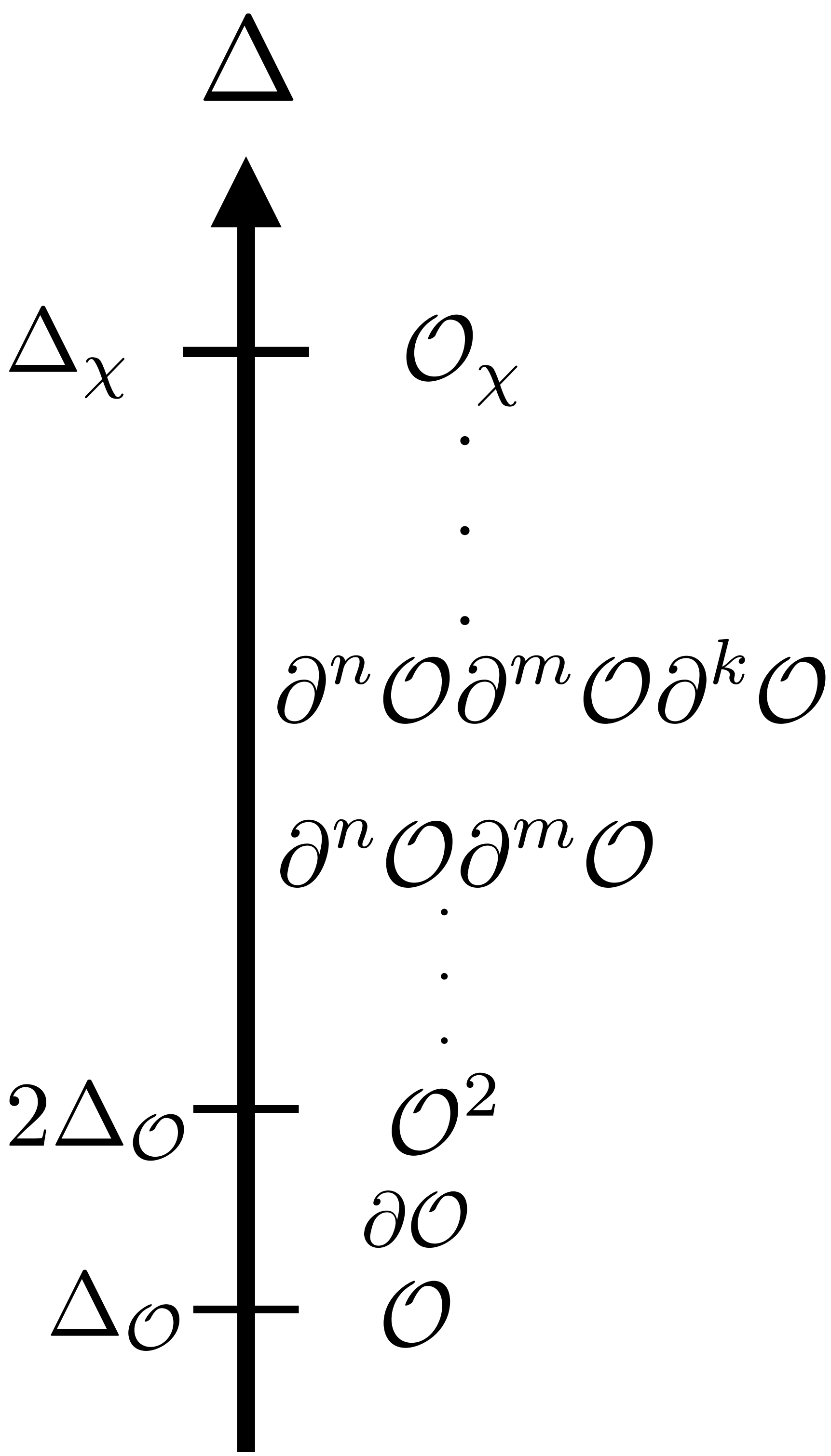
1. If no single-trace operators appear for $\Delta < \Delta_{Heavy}$ besides $\mathcal{O}$.
2. Perturbative unitarity obeyed for $\Delta < \Delta_{Heavy}$.

$$\Lambda \gtrsim \Delta_{Heavy}$$

All non-renormalizable terms suppressed by hierarchy!

P.T. : AdS/CFT no different than Mink/LIT!

Perturbative unitarization of Eff. Conf. Thy



$$\Delta_\chi \gg \Delta$$

$$V = \frac{\mu^{\frac{5-d}{2}}}{2}\int d^dx\sqrt{-g}\phi^2(x)\chi(x),\ \ m_\chi \gg m_\phi$$

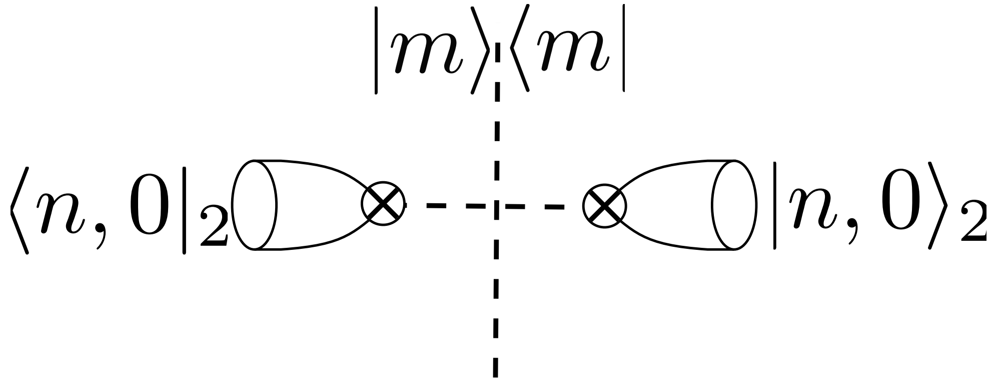
For low dimension double-trace ops:

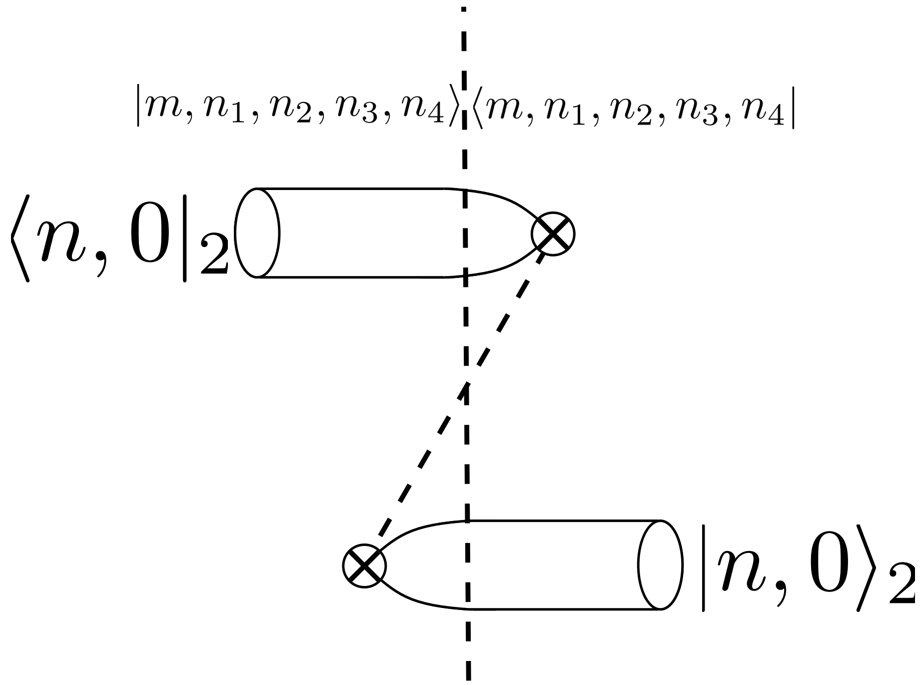
$$V_{eff} \sim \frac{\mu^{5-d}}{m_\chi^2}\int d^dx\sqrt{-g}\phi(x)^4 + \ldots$$

Does the $\mathcal{O}_\chi$ unitarize the growth in the anomalous dimensions?

$$\gamma(n,0) \quad = \quad \sum_{\alpha} \frac{|\langle \alpha | V | n, 0\rangle_2|^2}{E_n - E_\alpha}$$

S-channel contribution:

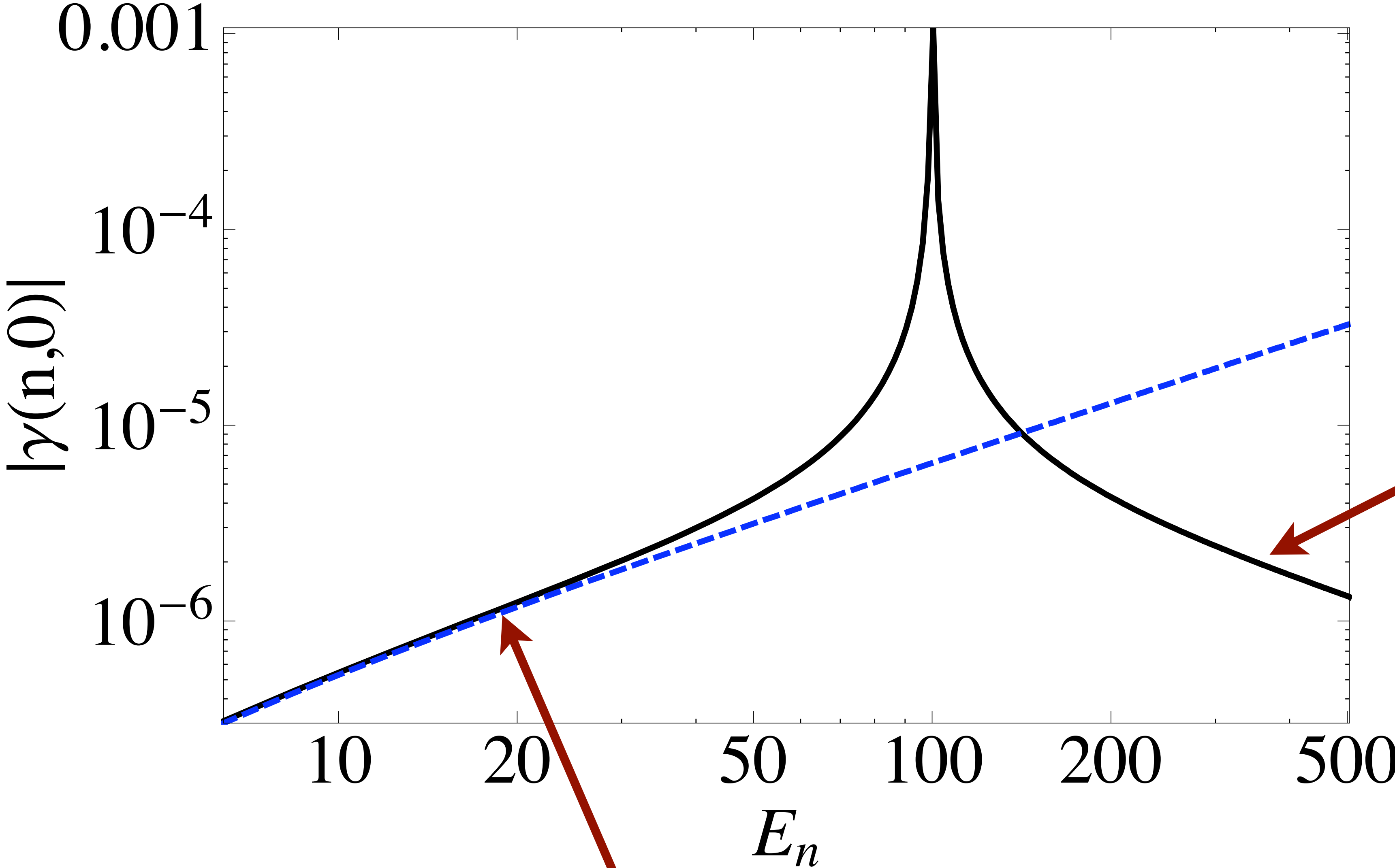




$$\gamma(n,0) \quad = \quad \sum_{m=0}^{\infty} |\langle \chi; m, 0 | V | n, 0\rangle_2|^2 \left(\frac{2E_m^\chi}{E_n^2 - E_m^{\chi 2}} \right)$$

$$E_n = 2\Delta + 2n, \;\; E_m^\chi = \Delta_\chi + 2m$$

AdS5: $\mathcal{V} = \sqrt{\mu}\phi^2\chi, \ \Delta = 2.2, \ \Delta_\chi = 100.1$



$\sim \dfrac{\mu}{n}$

$\sim n$ for $\dfrac{1}{\Lambda}\phi^4$

Flat-space limit of AdS

For $n \gg 1$ we seem to reproduce features similar to flat-space-why?

$$ds^2 = \frac{1}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho\, d\Omega^2\right)$$

Primary scalar wavefcn:

$$\langle 0|\phi^2(x)|n,0\rangle_2 = \frac{1}{N_n}(e^{it}\cos\rho)^{2\Delta+2n}$$

As $n \to \infty$, the wavefcn is increasingly concentrated at $\rho \approx 0$!

$$ds^2 \approx -dt^2 + d\rho^2 + \rho^2 d\Omega^2$$

$$|n,0\rangle_2 \underset{n\to\infty}{\longrightarrow} |\overrightarrow{P} = 0, E_n, l = 0\rangle_{partialwave}$$

Emergence of momentum conservation

Special conformal generators:

$$K_\mu = -R\frac{\partial}{\partial x^\mu} + ix_\mu\frac{\partial}{\partial t} + it\frac{\partial}{\partial x^\mu} + O(t/R, x/R)$$

Primary: $K_\mu|\psi\rangle_2 = 0$

$$\longrightarrow \langle 0|\phi(x_1)\phi(x_2)|\psi\rangle_2 \sim e^{ip\cdot(x_1-x_2)-\frac{E(x_1+x_2)^2}{4R}}, \; x_i \ll R$$

$$|n, lJ\rangle_2 = \frac{|2p|^{\frac{d-2}{2}}}{(2\pi)^d\sqrt{2RE}}\int d\hat{p}\, Y_{lJ}(\hat{p})\int d^dq\, \frac{e^{-\frac{Rq^2}{E}}}{N(E)}\, |\vec{p}+\vec{q}\rangle|-\vec{p}+\vec{q}\rangle \qquad (n \gg 1, l),$$

$\delta^d(\vec{q})$ with fuzziness $\delta q \sim \sqrt{E/R}$

For matrix elements what matters is $\dfrac{\delta q}{E} \to 0$

$$\phi^2\chi: \langle\chi; \overrightarrow{p}|V|\overrightarrow{P}=0, E, l=0\rangle_2 \sim \delta^d(\overrightarrow{p})$$ (flat-space)

$$\langle\chi; m, 0|V|n, 0\rangle_2 \to \sim \exp\left(\frac{-m(m+\Delta_\chi)-\Delta-\frac{\Delta_\chi}{2}}{n}\right)$$ (AdS)

(fuzziness $\delta q \sim \sqrt{E/R}$)

$$\gamma(n,l) \quad = \quad {}_2\langle n,l|V|n,l\rangle_2 + \sum_\alpha \frac{|\langle\alpha|V|n,l\rangle_2|^2}{E_{n,l}-E_\alpha} + \ldots$$

$$|n,l,J\rangle_2 \to |\overrightarrow{P}=0, E_n, l, J\rangle_{PW}, \;\; n\to\infty$$

↓

$$\gamma(n,l) \to {}_{PW}\langle E_n, l, J|T|E_n, l, J\rangle_{PW}$$

$$\mathcal{M}(s,t,u)^{d+1}_{flat\ space} \approx \frac{(4\pi)^d}{vol(S^{d-1})} \frac{E_n}{(E_n^2 - 4\Delta^2)^{\frac{d-2}{2}}} \sum_l [\gamma(n,l)]_{n \gg l} P_l^{(d)}(\cos(\theta))$$

$$E_n = 2(n+\Delta),\ p_n^2 = (E_n/2)^2 - \Delta^2 = n(n+2\Delta)$$

$$s = (E_n)^2,\ t = -2p_n^2(1-\cos(\theta)),\ u = -2p_n^2(1+\cos(\theta))$$

(Large n, Δ limit)

Examples

ϕ^4 :

$$\gamma(n,0) \;=\; \mu^{3-d}\frac{\mathrm{vol}(S^{d-1})}{8(2\pi)^d}\left(\frac{[n(n+2\Delta)]^{\frac{d-2}{2}}}{\Delta+n}\right) \sim \frac{p^{d-2}}{E}$$

$(\nabla\phi)^4,\ d=2$:

$$6\pi\mu^3\gamma(n,0) \quad \xrightarrow{n,\Delta\gg 1} \quad \frac{7n^4+28n^3\Delta+36n^2\Delta^2+16n\Delta^3+3\Delta^4}{8(\Delta+n)} \rightarrow \frac{3E^4+2E^2p^2+2p^4}{8E},$$

$$6\pi\mu^3\gamma(n,2) \quad \xrightarrow{n,\Delta\gg 1} \quad \frac{n^2(n+2\Delta)^2}{16(\Delta+n)} \rightarrow \frac{p^4}{16E}.$$

$$\mathcal{M}_{flat\ \ space} = \frac{\mu^3}{3}(3E^4+2E^2p^2+2p^4)P_0^{(2)}(\cos(\theta)) + \frac{\mu^3}{3}p^4P_2^{(2)}(\cos(\theta)).$$

$$\sim s^2+t^2+u^2$$

$(\nabla^2)^L \phi^4$:

d=2:

$$\gamma(n,L) = \pi \frac{\Gamma(n+L+1)\Gamma(2\Delta+n+L-1)\Gamma(\Delta+n-\frac{1}{2})\Gamma(\Delta+n+L)}{4\Gamma(1+n)\Gamma(\Delta+n)\Gamma(\Delta+n+L+\frac{1}{2})\Gamma(2\Delta+n-1)}$$

$$\xrightarrow{n,\Delta\gg 1} \quad \frac{\pi}{4}\frac{[n(n+2\Delta)]^L}{\Delta+n} \to \frac{\pi}{4}\frac{p^{2L}}{E}$$

d=4:

$$\gamma(n,L) \overset{n,\Delta\gg 1}{\propto} \frac{[n(n+2\Delta)]^{L+1}}{\Delta+n} \to \frac{p^{2(L+1)}}{E}$$

s-channel exchange:

$$\gamma(n,0) = \sum_{m=0}^{\infty} |\langle\chi;m,0|V|n,0\rangle_2|^2 \left(\frac{2E_m^\chi}{E_n^2 - E_m^{\chi 2}}\right)$$

$$\approx \frac{\mu^3}{(8\pi)E_n(E_n^2-\Delta_\chi^2)}$$

Momentum conservation from CFT

$$|\vec{P}=0,E,l=0\rangle_{PW} \sim \int d\Omega\ d^d q\ \delta(\vec{q})|\vec{p}+\vec{q}\rangle|\vec{p}-\vec{q}\rangle$$

Energy is shared equally

How does this happen in the CFT?

2D Ex, holomorphic only: $\mathcal{O}_h(z),\ \mathcal{O}_{2h+n} \sim \mathcal{O}_h \partial^n \mathcal{O}_h$

From 3pt fcn $\langle \mathcal{O}_{2h+n}\mathcal{O}_h\mathcal{O}_h\rangle$: $|2h+n\rangle = \sum_m c_m |h;m\rangle|h;n-m\rangle$

$$c_m \sim \binom{n}{m} \longrightarrow e^{-\frac{(m-\frac{n}{2})^2}{n}},\ n \gg 1$$

"Entropy of derivatives": $\mathcal{O}_{2h+n} \sim \partial^{\frac{n}{2}}\mathcal{O}_h\partial^{\frac{n}{2}}\mathcal{O}_h$

Conclusion

1. AdS is just a clever tool to deal with a large N CFT with a hierarchy in operator dimensions.
2. The CFT has a notion of perturbative unitarity which controls the size of bulk non-renorm terms.
3. Flat space S-matrix (to leading order) can be obtained directly from anomalous dimensions of double trace operators.