# Betatron-function measurement in lattices with $90°$ sections[*].

U. Wienands, SLAC, Stanford, CA 94309, USA;
M.E. Biagini, INFN, LFN Frascati, Italy

*Abstract*

Lattice functions derived from betatron phase-advance measurements have been used successfully at many $e^+$–$e^-$ facilities in the world, including at the PEP-II High Energy Ring. For the Low energy Ring of PEP-II, however, extraction of meaningful beta functions is hampered by the $90°$ phase advance/cell in the arcs, which causes a singularity in the expressions for beta. An algorithm has been developed calculating beta functions based on $\beta$ and $\alpha$ at the beginning of an arc and tracking the Twiss parameters through the arc while matching the observed phase advance/cell. Stability of the algorithm is improved by doing the same calculation "backward" as well as forward and averaging the result. The algorithm allows estimating beta functions at bad BPMs in many cases. The paper presents the algorithm used as well as examples of use in PEP.

## 1 INTRODUCTION

The ability to measure the envelope ($\beta-$) functions in the High Energy Ring (HER) of PEP-II has been an important diagnostic during beam commissioning and machine operation and tuning. Beating of the $\beta$ functions has been identified and corrected successfully using this diagnostic. The method uses betatron phase-advance measurements taken using the single-turn capability of the PEP BPM system[1], exciting the horizontal or vertical eigenmode of the beam using the tune-measurement facility.[2] The HER has $60°$ phase advance/cell through most of its sections and accurate results are obtained with a typical oscillation amplitude of $0.5\ldots1$ mm. The data analysis follows closely the formulae used at CERN.[3]

The same diagnostics exist for the Low Energy Ring (LER), except that the LER has $90°$ cells in the arcs and therefore the evaluation of the $\beta$ functions fails, not surprisingly. Limited information can still be gained from just comparing the measured and the model phase advance, but this is non-intuitive and in practice of limited value.

We have therefore developed an algorithm that will calculate meaningful $\beta$ functions throughout the LER arcs based on the $\beta$ and $\alpha$ functions measured in the straight sections (where the phase advance/cell $\mu \neq 90°$) and the measured phase advance throughout the arcs. Implicit in this algorithm is the assumption that the lattice optics is "locally correct". One could, of course, use the BPM amplitudes to derive the beta functions but the amplitudes depend on the calibration of the BPMs to be correct, which is not necessarily true. We rather use the BPM amplitudes as a cross check for consistency, on the assumption that

the BPM calibrations within the arc sections are closely matched since they are identical units in identical chamber cross sections, although they may have an overall error and may deviate from section to section.

## 2 BEAM OPTICS

### 2.1 Non-$90°$/cell optics

The transfer through a ring section is given by the TRANSPORT matrix $M$ describes the optical structure of the machine. $M$ can be expressed in terms of the (design-) lattice functions:

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\mu_{12} + \alpha_1\sin\mu_{12}) & \sqrt{\beta_1\beta_2}\sin\mu_{12} \\ -\frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\mu_{12} + \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\mu_{12} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\mu_{12} - \alpha_2\sin\mu_{12}) \end{pmatrix}, \quad (1)$$

where the indices 1 and 2 refer to the beginning and the end of the section described by $M$. $\mu_{12}$ is the betatron phase advance from point 1 to point 2. Note that the Twiss functions $\beta$ and $\alpha$ are not assumed to be at their matched values. We can write down the phase advance $\mu$ expressed in terms of the lattice functions at location 1 only:

$$\tan\mu_{12} = \frac{m_{12}}{\beta_1 m_{11} - \alpha_1 m_{12}}. \quad (2)$$

In a measurement scenario we now take $\mu$ from betatron-phase measurements. Since $M = (m_{ij})$ is known (the machine model presumably correctly describing the machine lattice) we can express $\beta$ and $\alpha$ in terms of the measured phase advance $\mu_m$. If we have two independent measurements, say $\mu_{12,m}$ and $\mu_{23,m}$ we can write two equations like (2), one for each section:

$$\tan\mu_{23} = \frac{n_{12}}{\beta_2 n_{11} - \alpha_2 n_{12}}, \quad (3)$$

$$\tan\mu_{12} = \frac{-m_{12}^*}{\beta_2 m_{11}^* - \alpha_2 m_{12}^*}, \quad (4)$$

where $m_{ij}^*$ is element *ij* of the inverse matrix $M_{12}^{-1}$, *i.e.* of $M_{21}$. $M_{21}$ is found from $M_{12}$ using Eq. (1) by exchanging $\beta_1, \alpha_1$ with $\beta_2, \alpha_2$ and $\mu_{12}$ by $\mu_{21} = -\mu_{12}$. $n_{ij}$ are the elements of $M_{23}$.

The (model–) phases $\mu_{12}$ and $\mu_{23}$ can be replaced by the measured phases $\mu_{12,m}$ and $\mu_{23,m}$ and we can solve the

resulting two equations for $\beta_2$ and $\alpha_2$:

$$\beta_{2,m}\frac{n_{11}}{n_{12}} - \alpha_{2,m} = \frac{1}{\tan\mu_{23,m}} \quad (5)$$

$$\beta_{2,m}\frac{m^*_{11}}{m^*_{12}} - \alpha_{2,m} = \frac{-1}{\tan\mu_{12,m}}. \quad (6)$$

and, thus,

$$\beta_{2,m} = \frac{\left(\frac{1}{\tan\mu_{23,m}} - \frac{1}{-\tan\mu_{12,m}}\right)}{\left(\frac{n_{11}}{n_{12}} - \frac{m^*_{11}}{m^*_{12}}\right)} \quad (7)$$

and

$$\alpha_{2,m} = \frac{\left(\frac{m^*_{11}}{m^*_{12}\tan\mu_{23,m}} + \frac{n_{11}}{n_{12}\tan\mu_{12,m}}\right)}{\left(\frac{n_{11}}{n_{12}} - \frac{m^*_{11}}{m^*_{12}}\right)}, \quad (8)$$

Using Eq. (1) we find the matrix elements in terms of the model-Twiss parameters:

$$\frac{n_{11}}{n_{12}} = \frac{1}{\beta_2}\left(\cot\mu_{23} + \alpha_2\right), \quad (9)$$

$$\frac{m^*_{11}}{m^*_{12}} = \frac{-1}{\beta_2}\left(\cot\mu_{12} - \alpha_2\right), \quad (10)$$

and get finally

$$\beta_{2,m} = \frac{\cot\mu_{23,m} + \cot\mu_{12,m}}{\cot\mu_{23} + \cot\mu_{12}}\beta_2, \quad (11)$$

and

$$\alpha_{2,m} = \frac{\cot\mu_{23,m} + \cot\mu_{12,m}}{\cot\mu_{23} + \cot\mu_{12}}\alpha_2 \quad (12)$$
$$- \frac{\cot\mu_{23,m}\cot\mu_{12} - \cot\mu_{12,m}\cot\mu_{23}}{\cot\mu_{12} + \cot\mu_{23}}.$$

This, of course, is what was worked out at CERN by P. Castro-Garcia to measure the lattice functions in LEP[3] and what is also used at CESR and at the PEP-II High Energy Ring (HER), although it is written here for the specific case calculating $\beta$, $\alpha$ at the middle one of three BPMs, see Fig. 1. It works quite well as long as the phase advances $\mu$ between BPMs are suitable.

Figure 1: Location of BPM phase measurements
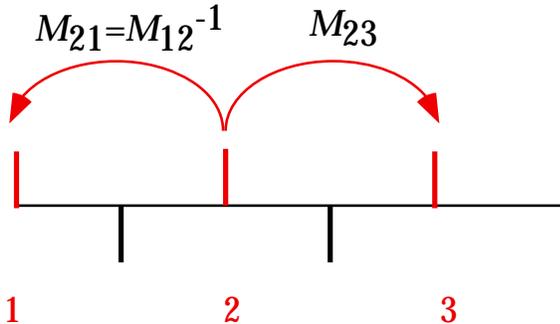


## 3 $\beta$ FUNCTIONS IN 90° SECTIONS

Problems arise when the denominator in the above equations becomes zero. This happens when $\mu$, the model phase advance/BPM for the section, is $(2n+1)\pi/2$, but also when $|\mu_{12} + \mu_{23}|$ is $n\pi$ ($n = 0, 1, \ldots$). In fact, similar problems arise in all regions where the phase advances between BPMs 1 and 3 are too close to 0 or $\pi$, e.g. the interaction region. We empirically set the following criterion:

$$|\mu_{12} + \mu_{23}| \geq 0.1, \quad \text{and} \quad |\mu_{12} + \mu_{23} - \pi| \geq 0.1 \quad (13)$$

for BPM #2 to be at a phase unsuitable for applying Eqs. (11) and (12). In the practical implementation, the threshold phase difference (0.1 radians) is implemented as a user-changeable constant.

For a short section with the above condition (1 pair of cells, say), the problem can be circumvented by switching to a different set of BPMs. This approach is taken successfully in the IR of the HER.[4] to reduce the scatter on the measured $\beta$ functions. In extended regions the approach breaks down due to lack of BPMs at suitable phases.

The phase measurements in these 90° sections are valid and, under the assumption of locally correct modelling as above, we can propagate the Twiss function values found in the sections with non-$\pi/2$ phase advance through the arcs for the LER, using the measured phase advances. This is done by observing that the $m_{12}$ matrix element relates $\beta$ and the phase advance $\mu$:

$$m_{12} = \sqrt{\beta_1\beta_2}\sin\mu_{12} = \sqrt{\beta_{1,m}\beta_{2,m}}\sin\mu_{12,m}, \quad (14)$$

and therefore

$$\beta_{2,m} = \frac{\beta_1}{\beta_{1,m}}\frac{\sin^2\mu_{12}}{\sin^2\mu_{12,m}}\beta_2 \quad (15)$$

at the next BPM (#2) is determined by the measurement at location 1. We solve Eq. (6) for $\alpha_{2,m}$ and get

$$\alpha_{2,m} = \cot\mu_{12,m} + \beta_{2,m}\frac{m^*_{11}}{m^*_{12}}$$
$$= \cot\mu_{12,m} - \frac{\beta_{2,m}}{\beta_2}(\cot\mu_{12} - \alpha_2). \quad (16)$$

Since we measure $\beta$ and $\alpha$ at BPM #1, the phase advance to BPM #2 is not really needed but can be derived from the measurements at location 1 as well. We rewrite Eq. (2) for the transformation from loc. 2 backwards to loc. 1 in terms of the measured Twiss parameters at location 1:

$$\tan\mu_{12,m} = \frac{m_{12}}{\beta_{1,m}m_{11} - \alpha_{1,m}m_{12}} \quad (17)$$

from which we can calculate the phase advance

$$\mu_{12,m} = \text{arccot}\left(\frac{\beta_{1,m}}{\beta_1}\left(\cot\mu_{12} + \alpha_1\right) - \alpha_{1,m}\right) \quad (18)$$

and thus we know $\beta_{2,m}$ as well. This is useful in the PEP-II context since most BPMs in the PEP rings are single-view, either $x$ or $y$, only.

Figure 2: Forward and backward tracking calculation of $\beta_x$ for the LER

blk:forward, red:backward
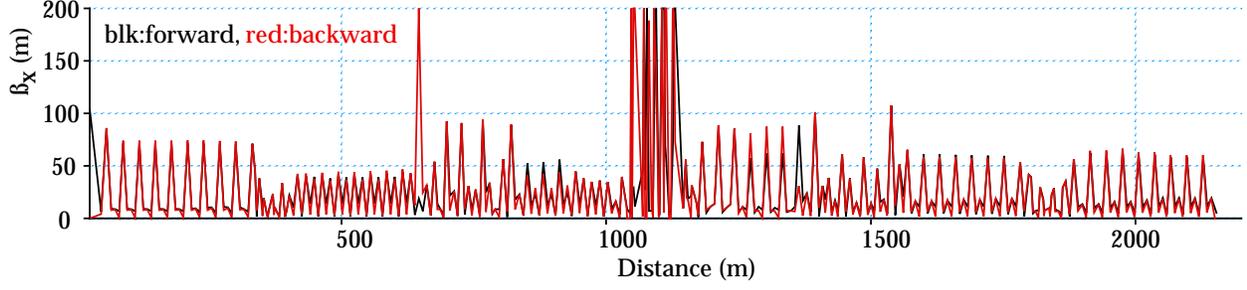
Figure 3: Average measured $\beta_x$ (top) and normalized amplitude (bottom)

ß measured, average
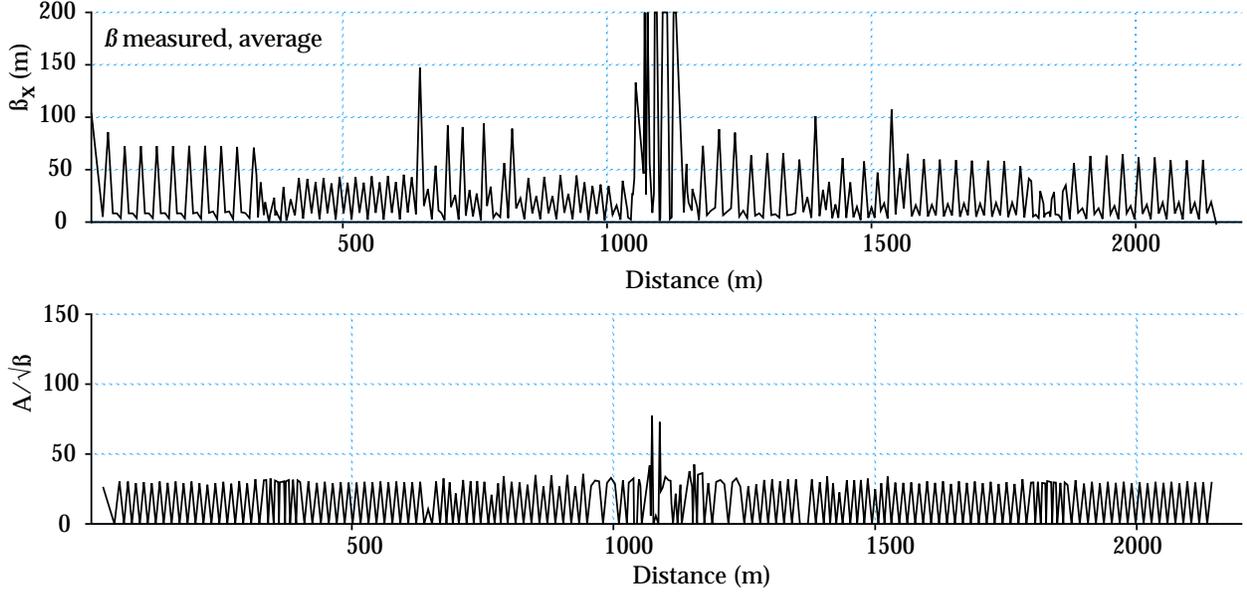
## 4 FORWARD & BACKWARD CALCULATION

With these added expressions we can calculate lattice functions at every location in the ring, regardless of phase advance. In fact, this can be done in either direction, beam-following (forward) as well as backward. The two solutions do not necessarily agree due to noise and other errors in the measurements. Straight-forward averaging of the two solutions should reduce the error, but in practise one is often faced with a "good" solution and a "bad" one spoiled due to a bad reading, it being not obvious which one is "good." We get around this to a certain extent by weighting the two solutions in the averaging by the deviation of the amplitude normalized to $\beta_m$ from 1.

The overall averaged normalized amplitude is calculated by

$$A_{norm} = \frac{\sum_{BPMs} \frac{A_i}{\sqrt{\beta_i}} p_i}{\sum_{BPMs} p_i}, \qquad (19)$$

where the $\beta_i$ are the model values. The weights are then

$$p_i = \frac{1}{(a_i/\sqrt{\beta_{i,m}} - A_{norm})^2} \qquad (20)$$

In cases where the amplitude is not available (bad BPM or BPM reading only in the other plane) the straight average is taken. There may be some cases where the $\beta$ values calculated in either forward or backward direction are not meaningful (*e.g.* negative due to an error in BPM phase) in which case only the valid reading is used.

## 5 MEASUREMENTS

The above algorithm was programmed in *MathPad*[5] and run on a number of previously saved data sets. Fig. 2 shows the comparison of forward and backward tracking of the lattice functions through the LER lattice. While in good general agreement the calculated lattice functions deviate from each other to a degree in certain regions. In Fig. 3 the weighted average is shown on the top, on the bottom the quantity $A_{norm,i}$ is shown for the average. Good consistency is obtained throughout the whole lattice, with some deviations in region 2, the interaction region where the local solenoid compensation introduces non-negligible coupling. This provides confidence in the derived $\beta$ functions as well as the gain-calibration of the BPMs.

The $\beta$ beating apparent in the example was observed when the working point in the LER was moved to 0.53 in

Figure 4: $\beta_x$ in the LER at $\nu_x \approx 0.63$.
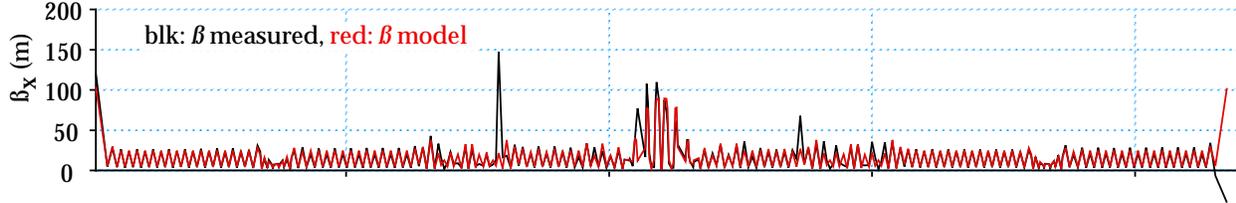


Figure 5: Measured LER $\beta_x$ and design model with initial conditions adjusted to fit the measurements between 10 and 300 m.
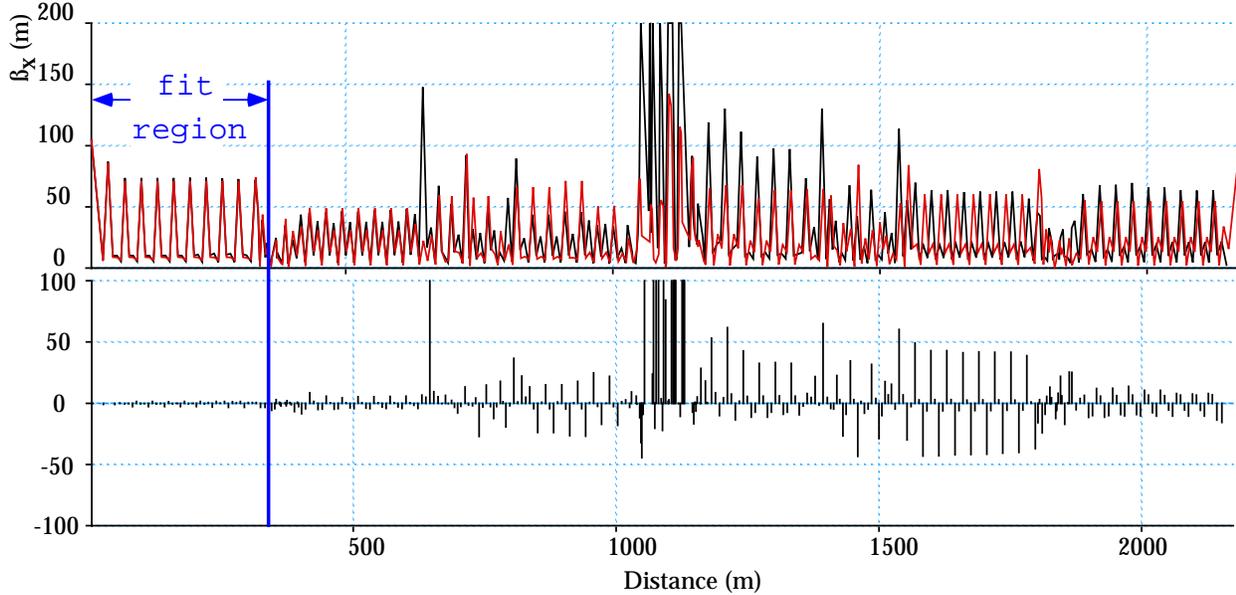


$x$. At its usual working point ($\approx 0.63$), the lattice is much less sensitive to focusing errors and the lattice functions are in fact quite regular, as shown in Fig. 4. In both figures isolated extrema in $\beta$ ("spikes") are visible. These have been traced back to faulty BPMs.

## 6  COMPARISON WITH THE LATTICE MODEL

Since the model-lattice functions simply encode the optical properties of the lattice, one can use them to "track" initial Twiss-parameter values through the machine model. Varying the initial values of $\alpha$ and $\beta$ one can fit the model values of a region to the measured values and then propagate through the remainder of the lattice, thus possibly localizing points where the optics deviates from the intended behavior. The result, using the design-lattice model of the LER, is shown in Fig. 5. Deviations are developing at the tune section (phase trombones) and across the interaction region. The phase trombones are set to different values in the machine, therefore deviations are not surprising. In the interaction region these measurements have lead to the uncovering of an inconsistency between the magnet settings and the model, which is still being analyzed.

## 7  FURTHER DEVELOPMENT

The present algorithm, while already being instrumental in shaking down the $\beta$ beating in the LER, can be further enhanced. Work is underway to implement the algorithm online.[4] Work is also underway to include correct treatment of measurement uncertainties and make the algorithm more robust against erroneous BPM readings. A more fundamental extension will include the determination of elements of the coupling matrix $C$, the information for which is present in the acquired data as the response of the off-plane BPMs is also recorded.

## 8  REFERENCES

[1] S.R. Smith et al. Proc. $17^{th}$ Part. Accel. Conf., Vancouver, BC, Canada, p. 2122 (1997).

[2] A. Fisher et al., Proc. $10^{th}$ Beam Inst. Workshop, BNL, May 2002, to be published, and SLAC-PUB-9215.

[3] P. Castro-Garcia, SL/Note 92-63 (BI), CERN, Geneva, CH, 1992.

[4] M. Zelazny, private communications.

[5] M. Widholm, Univ. of New Hampshire, Durham, NH, 1993–2000.