Longitudinal BBU Threshold Current in Recirculating Linacs

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Abstract

The total current that can be accelerated in an Energy Recovery Linear Accelerator (ERL) can be limited by the longitudinal recirculating beam-breakup instability (BBU), as well as the transverse BBU instability. In this report we discuss a simple model of the longitudinal BBU instability that can be solved analytically. More complicated cases are analyzed by means of numerical simulation. It should be noted that the discussion in this report applies to all recirculation linacs and the ERL is a special case with recirculation phase π. The theory and simulation are compared and they agree with each other very well in the simple model case. Similar to the transverse case, the influence of the higher order modes (HOM) frequency spread among different RF cavities is also discussed. Moreover a theory of the longitudinal HOM power growth is presented and the analytical result agrees well with the simulation.

1 Longitudinal Beam Breakup Instability

When an electron bunch passes through an RF cavity, it can excite both transverse and longitudinal HOMs. Previous papers have described how the recirculation of electron bunches can cause the exponential growth of tansverse HOM power leading to transverse BBU instability [1, 2, 3, 4]. Similar to the transverse case, longitudinal HOMs excited by electron bunches can lead to the longitudinal BBU instability [5]. Such longitudinal HOMs can disturb the beam motion in the longitudinal direction and a potential feedback loop can be formed if electrons return to the same RF cavity with a certain phase. The time of flight term r_{56} plays an important role in the longitudinal BBU, which is similar to the role of T_{12} element in the transverse BBU instability. The main effect of a longitudinal HOM on an electron bunch is changing the bunch energy. Such a deviation from the design energy is translated into a change of arrival time through the r_{56} term. Below the threshold current, the longitudinal HOM power is driven only by the current generated by the unperturbed sequence of bunches. Above the threshold, the beam current is modulated by the longitudinal HOMs with frequency matching the HOM frequency. Thus the longitudinal HOMs can be enhanced by this modulated beam current, which in turn leads to an even larger current modulation. Eventually such self-enhancement leads to the longitudinal BBU instability.

It should be noted that, unlike the transverse phase space motion, which is mainly determined by the linear optics, the longitudinal phase space motion is quite nonlinear due to the sinusoidal RF potential and the nonlinear time of flight term. Thus the longitudinal BBU instability saturates at a large but limited HOM power. It is this saturation behavior that distinguishes the longitudinal BBU from its transverse counterpart.

In the following discussion, the derivation in [5] is repeated in a more detailed form and the numerical simualtion is done by a code mainly derived from "bi" code written by Ivan Bazarov. Consider a sequence of bunches with bunching frequency $\omega_b = 2\pi/t_b$. Disregarding the finite bunch length, the current is of the form

$$I(t) = I_0 t_b \sum_{n=-\infty}^{+\infty} \delta(t - nt_b) \tag{1}$$

It's useful to find the Fourier decomposition of the current. Notice that the current is a periodic function with period $T = t_b$, so the Fourier decomposition coefficient I_n can be calculated as

$$I_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt e^{-in\omega_b t} I_0 t_b \sum_{n=-\infty}^{+\infty} \delta(t - nt_b) = I_0 \tag{2}$$

So the Fourier decomposition of the current is of the form

$$I(t) = I_0 \sum_{n=-\infty}^{+\infty} e^{i\omega_b nt} \tag{3}$$

This result implies that a uniform sequence of point bunches produces a signal at all harmonics of the bunching frequency.

When the instability starts, the arrival time of the bunches on their second turn has a small amplitude modulation at a frequency $\nu\omega_b$. So the modulated current can be written as

$$I(t) = I_0 t_b \sum_{n=-\infty}^{+\infty} \delta\left(t - t_r - nt_b - \Delta t \sin(\nu\omega_b n t_b + \phi)\right) \tag{4}$$

where t_r is the return time of a bunch, Δt is the small amplitude of the perturbation and ϕ is an arbitrary phase of the perturbation. Similarly we can get the Fourier transform of this function instead of the Fourier decomposition

$$\begin{aligned}\tilde{I}(\omega) &= \frac{I_0 t_b}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dt e^{i\omega t} \delta\left(t - t_r - nt_b - \Delta t \sin(\nu\omega_b n t_b + \phi)\right) \\ &= \frac{I_0 t_b}{\sqrt{2\pi}} \sum_{n=-\infty}^{+\infty} e^{i\omega t_r} e^{i\omega n t_b} e^{i\omega\Delta t \sin(\nu\omega_b n t_b + \phi)}\end{aligned}$$

Using the identity

$$e^{ix\sin y} = \sum_{\mu=-\infty}^{+\infty} J_\mu(x) e^{i\mu y} \tag{5}$$

we can write the Fourier transform of the current in the form of

$$\begin{aligned}\tilde{I}(\omega) &= \frac{I_0 t_b}{\sqrt{2\pi}} \sum_{\mu=-\infty}^{+\infty} e^{i(\omega t_r + \mu\phi)} J_\mu(\omega\Delta t) \sum_{n=-\infty}^{+\infty} e^{i(\omega + \mu\nu\omega_b) n t_b} \\ &= \frac{I_0 t_b}{\sqrt{2\pi}} \sum_{\mu=-\infty}^{+\infty} e^{i(\omega t_r + \mu\phi)} J_\mu(\omega\Delta t) \omega_b \sum_{n=-\infty}^{+\infty} \delta(\omega + \mu\nu\omega_b - n\omega_b)\end{aligned} \tag{6}$$

1.1 The Wake Potential in a RF Cavity

From the general theory of wake fields in a cavity, the wake potential at time t is given by

$$\begin{aligned}V(t) &= \int_{-\infty}^{+\infty} dt' I(t') W(t - t') \\ W(t - t') &= 0, \quad t - t' < 0\end{aligned} \tag{7}$$

where $I(t')$ is the current at t' and $W(t - t')$ is the wake function.

The longitudinal impedance $\tilde{Z}(\omega)$ is defined as the Fourier transform of the wake function $W(t)$.

$$\tilde{Z}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{i\omega t} W(t) \tag{8}$$

So the Fourier transform of the longitudinal wake potential can be written as

$$\tilde{V}(\omega) = \tilde{I}(\omega)\tilde{Z}(\omega) \tag{9}$$

Since $\tilde{Z}(\omega)$ is independent of the current and only related to the shape of the cavity, we can send a single bunch through the cavity and calculate the longitudinal impedance and then use this impedance to calculate wake potentials under other currents.

Considering a charge q passing through a cavity at $t = 0$, hence the current is

$$I(t) = q\delta(t) \tag{10}$$

So the Fourier transform of this current is in the form of

$$\tilde{I}(\omega) = \frac{q}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{i\omega t} \delta(t) = \frac{q}{\sqrt{2\pi}} \tag{11}$$

The HOM voltage $V(t)$ induced by the traversal of this charge through the cavity is given by

$$V(t) = \frac{q\omega_\lambda}{2} \left(\frac{R}{Q}\right)_\lambda \cos\omega_\lambda t e^{-\frac{\omega_\lambda}{2Q_\lambda} t} \quad (t > 0) \tag{12}$$

where ω_λ is the resonant HOM frequency, $(R/Q)_\lambda$ is the shunt impedance and Q_λ is the quality factor.

The Fourier transform of this HOM voltage is given by

$$\tilde{V}(\omega) = \frac{i}{\sqrt{2\pi}} \left(\frac{R}{Q}\right)_\lambda \frac{q\omega_\lambda}{4} \left\{ \frac{1}{\omega + \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} + \frac{1}{\omega - \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} \right\} \tag{13}$$

The longitudinal impedance $Z(\omega)$ can be calculated by Eq. (9):

$$\tilde{Z}(\omega) = \frac{\tilde{V}(\omega)}{\tilde{I}(\omega)}$$

Thus we have

$$\tilde{Z}(\omega) = i \left(\frac{R}{Q}\right)_\lambda \frac{\omega_\lambda}{4} \left\{ \frac{1}{\omega + \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} + \frac{1}{\omega - \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} \right\} \tag{14}$$

1.2 Voltage Induced by a Modulated Current

Let there be an excitation $V(t)$ of a HOM of the cavity at frequency $\nu\omega_b$. As the bunches pass through the cavity their energy can be modulated at the HOM frequency.

Define the slip factor η by the relation

$$\Delta T_f = \eta t_r \frac{\Delta E}{E} \tag{15}$$

where ΔT_f is the time offset of a bunch of energy offset $\Delta E/E$ for an on-energy recirculation time t_r and first pass energy (at the cavity) of E.

The energy modulation varying as $\sin(\nu\omega_b t + \phi)$ induced on the first pass will through η cause a modulation of the arrival time of bunch m for the second pass of the form

$$t_m = mt_b + t_r + \Delta t \sin(\nu\omega_b m t_b + \phi) \tag{16}$$

This modulation can generate the current

$$\tilde{I}(\omega) = \frac{I_0 t_b}{\sqrt{2\pi}} \sum_{\mu=-\infty}^{+\infty} e^{i(\omega t_r + \mu\phi)} J_\mu(\omega\Delta t)\omega_b \sum_{n=-\infty}^{+\infty} \delta(\omega + \mu\nu\omega_b - n\omega_b) \tag{17}$$

This current can induce a HOM voltage given by

$$\begin{aligned} V(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega e^{-i\omega t} \tilde{I}(\omega)\tilde{Z}(\omega) \\ &= i \left(\frac{R}{Q}\right)_\lambda \frac{I_0\omega_\lambda}{4} \sum_{\mu=-\infty}^{+\infty} \sum_n J_\mu((n-\mu\nu)\omega_b\Delta t) e^{i((n-\mu\nu)\omega_b(t_r - t) + \mu\phi)} \\ &\quad \cdot \left\{ \frac{1}{(n-\mu\nu)\omega_b + \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} + \frac{1}{(n-\mu\nu)\omega_b - \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} \right\} \end{aligned} \tag{18}$$

1.3 Analysis of Longitudinal Multipass BBU

In the limit of a small coherent modulation of the bunching frequency, $\omega\Delta t$ is a small quantity. Thus only the low order terms of it are important. So the J_0 and J_1 terms of the expansion will dominate because they contains the constant and the linear terms. The J_0 term to the lowest order is independent of the amplitude of the modulation and describes simple energy loss to the HOM. Since it does not provide feedback with respect to the modulation amplitude, it can not at this level of approximation contribute to a possible instability [5].

$$\begin{aligned} J_n(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+n)!} \left(\frac{x}{2}\right)^{2m+n} \\ J_0(x) &= 1 - \frac{x^2}{4} + \frac{x^4}{64} + \cdots \\ J_1(x) &= \frac{x}{2} - \frac{x^3}{16} + \frac{x^5}{384} + \cdots \\ J_{-n}(x) &= (-1)^n J_n(x) \end{aligned} \tag{19}$$

The $J_{\pm 1}$ to the lowest order is a linear function of the ampulitude of the modulation. Therefore the $J_{\pm 1}$ terms on the other hand do provide such a feedback mechanism.

If a narrow resonance is presumed, only one particular term in $V(t)$ such that $|n^* - \nu|\omega_b \approx \omega_\lambda$ will dominate.

Define the tuning angle ψ_n of the HOM

$$\tan\psi_n = \frac{\omega_\lambda - (n-\nu)\omega_b}{\omega_\lambda / 2Q_\lambda} \tag{20}$$

Thus we can simplify the formula of $V(t)$ in terms of ψ_n.

$$\begin{aligned} \frac{1}{(n-\nu)\omega_b - \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} &= \frac{2Q_\lambda}{i\omega_\lambda}\frac{1}{1 + i\tan\psi_n} = \frac{2Q_\lambda}{i\omega_\lambda} e^{-i\psi_n}\cos\psi_n \\ \frac{1}{(n-\nu)\omega_b - \omega_\lambda - i\frac{\omega_\lambda}{2Q_\lambda}} &= \frac{2Q_\lambda}{i\omega_\lambda}\frac{1}{-1 + i\tan\psi_n} = -\frac{2Q_\lambda}{i\omega_\lambda} e^{i\psi_n}\cos\psi_n \end{aligned} \tag{21}$$

$V(t)$ at bunch-crossing times mt_b is given by

$$\begin{aligned} V(mt_b) = i\left(\frac{R}{Q}\right)_\lambda \frac{I_0\omega_\lambda}{4} \sum_{\mu=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} J_\mu((n-\mu\nu)\omega_b\Delta t) e^{i((n-\mu\nu)\omega_b(t_r - mt_b) + \mu\phi)} \\ \cdot \left\{ \frac{1}{(n-\mu\nu)\omega_b + \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} + \frac{1}{(n-\mu\nu)\omega_b - \omega_\lambda + i\frac{\omega_\lambda}{2Q_\lambda}} \right\} \end{aligned} \tag{22}$$

When $|(n-\mu\nu)\omega_b\Delta t| << 1$, the main contribution is from J_1 and J_{-1} terms. For a narrow resonace, only $|n^* - \nu|\omega_b \approx \omega_\lambda$ contributes to the HOM voltage. Thus by keeping the J_1 term, we have

$$V^{J_1}(mt_b) = \left(\frac{R}{Q}\right)_\lambda \frac{Q_\lambda I_0}{2} J_1((n^* - \nu)\omega_b\Delta t) e^{i((n^*-\nu)\omega_b(t_r - mt_b) + \phi - \psi_{n^*})} \cos\psi_{n^*} \tag{23}$$

For the J_{-1} term, we have

$$\begin{aligned} V^{J_{-1}}(mt_b) &= \left(\frac{R}{Q}\right)_\lambda \frac{Q_\lambda I_0}{2} J_1((n^* - \nu)\omega_b\Delta t) e^{-i((n^*-\nu)\omega_b(t_r - mt_b) + \phi - \psi_{n^*})} \cos\psi_{n^*} \\ &= V^{J_1}(mt_b)^* \end{aligned} \tag{24}$$

Therefore we can get

$$\begin{aligned} V(mt_b) &= V^{J_1}(mt_b) + V^{J_{-1}}(mt_b) \\ &= \left(\frac{R}{Q}\right)_\lambda Q_\lambda I_0 J_1((n^* - \nu)\omega_b\Delta t) \cos\{(n^*-\nu)\omega_b(t_r - mt_b) + \phi - \psi_n\} \cos\psi_{n^*} \end{aligned} \tag{25}$$

When $x \ll 1, \quad J_1(x) \approx \frac{x}{2}$.

$$V(mt_b) = \left(\frac{R}{Q}\right)_\lambda \frac{Q_\lambda I_0}{2}(n^* - \nu)\omega_b \Delta t \cos\{(n^* - \nu)\omega_b(t_r - mt_b) + \phi - \psi_{n^*}\} \cos\psi_{n^*} \tag{26}$$

Therefore the induced time delay is

$$\begin{aligned} \Delta T_f &= t_r \left(\frac{R}{Q}\right)_\lambda \frac{\eta Q_\lambda I_0}{2E_0}(n^* - \nu)\omega_b \Delta t \sin(\nu\omega_b mt_b + \varphi) \cos\psi_{n^*} \\ \varphi &= \frac{\pi}{2} + (n^* - \nu)\omega_b t_r + \phi - \psi_{n^*} \end{aligned} \tag{27}$$

Thus the induced time delay has the same frequency as the initial perturbation $\Delta T_i = \Delta t \sin(\nu\omega_b mt_b + \phi)$, but with a phase shift. The feedback coefficient is defined as the absolute value of the ratio of ΔT_f and ΔT_i.

$$\begin{aligned} r &= \left|\frac{\Delta T_f}{\Delta T_i}\right| \\ &= \left|t_r \cos\psi_{n^*} \left(\frac{R}{Q}\right)_\lambda \frac{\eta Q_\lambda I_0}{2E_0}(n^* - \nu)\omega_b\right| \\ r &< 1, \quad \text{Stable;} \\ r &= 1, \quad \text{Threshold;} \\ r &> 1, \quad \text{Unstable, BBU} \end{aligned} \tag{28}$$

For the growth of instability, the induced time delay should have the same phase as the initial perturbation. Such phase conformity is essential for building up the longitudinal HOM power. In order to find a stable oscillation at the threshold current, we need $r = 1$ and $\varphi = \phi$, which indicates.

$$\begin{aligned} & t_r \cos\psi_{n^*} \left(\frac{R}{Q}\right)_\lambda \frac{\eta Q_\lambda I_0}{2E_0}(n^* - \nu)\omega_b = 1 \\ & \frac{\pi}{2} + (n^* - \nu)\omega_b t_r + \phi - \psi_{n^*} = \phi + 2p\pi \quad p \text{ is an arbitrary integer.} \end{aligned} \tag{29}$$

Define $\omega = (n^* - \nu)\omega_b$, then we have

$$\begin{aligned} \cos\psi_{n^*} \left(\frac{R}{Q}\right)_\lambda \frac{\eta Q_\lambda I_0}{2E_0}\omega t_r &= 1 \\ \omega t_r + \frac{\pi}{2} &= \psi_{n^*} - 2p\pi \end{aligned} \tag{30}$$

These two equations indicate that if the system is self-sustained, I_0 and ω are constrained by these equations. Recall the definition of ψ_{n^*}

$$\tan\psi_{n^*} = \frac{\omega_\lambda - (n^* - \nu)\omega_b}{\omega_\lambda / 2Q_\lambda} \tag{31}$$

and define

$$\begin{aligned} \epsilon_r &= \frac{\omega_\lambda}{2Q_\lambda} t_r \\ \delta_\omega &= \frac{\omega - \omega_\lambda}{\omega_\lambda}, \end{aligned} \tag{32}$$

we have

$$\begin{aligned} \tan(\omega t_r + \frac{\pi}{2}) &= \tan(\psi_{n^*} - 2p\pi) \\ \frac{1}{\tan\{(1 - \delta_\omega)\omega_\lambda t_r\}} &= \frac{\delta_\omega}{\epsilon_r}\omega_\lambda t_r \end{aligned} \tag{33}$$

For a narrow resonance $\delta_\omega \ll 1$ we only keep linear terms of δ_ω, thus we have

$$1 - \delta_\omega \omega_\lambda t_r \tan\omega_\lambda t_r = \frac{\delta_\omega}{\epsilon_r}\omega t_r \tan\omega_\lambda t_r \quad . \tag{34}$$

Table 1: The four dominant longitudinal HOMs in the 7-cell ERL cavity

	f_λ(**GHz**)	Q_λ	$(R/Q)_\lambda[\Omega]$
1	3.85763	13728	31
2	2.45658	1778.8	134.5
3	5.93396	27887	5.99
4	3.85758	40172	2.94

Therefore δ_ω can be solved as

$$\delta_\omega = \frac{\epsilon_r}{(1+\epsilon_r)\omega_\lambda t_r \tan \omega_\lambda t_r} \tag{35}$$

Since

$$|\cos\psi_{n^*}| = \frac{1}{\sqrt{1+\tan^2\psi_{n^*}}} = \frac{1}{\sqrt{1+\left(\frac{\delta_\omega}{\epsilon_r}\omega_\lambda t_r\right)^2}} \approx 1 - \frac{1}{2}\left(\frac{\delta_\omega}{\epsilon_r}\omega_\lambda t_r\right)^2 + \cdots \tag{36}$$

the solution of I_0 is

$$I_0 = \frac{1}{1+\delta_\omega}\frac{2E_0}{\eta t_r \omega_\lambda \left(\frac{R}{Q}\right)_\lambda Q_\lambda} \tag{37}$$

The worst case is $|(n^*-\nu)\omega_b| = \omega_\lambda$, thus $\delta_\omega = 0$ and

$$I_{\text{th}} = \frac{2E_0}{\eta t_r \omega_\lambda \left(\frac{R}{Q}\right)_\lambda Q_\lambda} \tag{38}$$

In the simulation that will be discussed in the next section we have $f_\lambda = 3.85763\text{GHz}$, $t_r = 4.738328773 \times 10^{-6}\text{s}$ and $Q_\lambda = 13728$ thus we can calculate ϵ_r and δ_ω according to Eq. (32) and Eq. (35).

$$\epsilon_r = 4.183\,, \qquad \delta_\omega = 1.376 \times 10^{-6}$$

It is clear that if δ_ω is very small compared to 1, it hardly has any contribution in the formula of the threshold current. In theory δ_ω could become a large number if $\tan\omega_\lambda t_r$ is very small. But for a real resonance, the frequency ω does not deviate from the HOM frequency ω_λ more than ω_b itself. Thus it is safe to assume that $\delta < 1$ and the threshold current we get from Eq. (38) does not deviate from the result of Eq. (37) by more than $I_0/2$.

1.4 The Comparison between Theory and Simulation

In the simulation of longitudinal BBU, we are using BMAD standard. That is

$$\begin{cases} z = -\beta c \Delta t \\ P_z = \dfrac{\Delta E}{E_0} \end{cases}$$

$$\begin{aligned} &\text{if} \quad z \sim r_{56} P_z, \quad \Longrightarrow \Delta t \sim \frac{r_{56}}{\beta c}\frac{\Delta E}{E_0} \sim \eta t_r \frac{\Delta E}{E_0} \\ &\text{thus} \quad \frac{r_{56}}{\beta c} = \eta t_r \end{aligned} \tag{39}$$

In the code we have $f_\lambda = 3.85763\text{GHz}$, $(R/Q)_\lambda = 31\Omega$, $Q_\lambda = 13728$, which are obtained from cavity simulation, and $r_{56}/\beta c E_0 = 2.249 \times 10^{-18}$, which is taken from ERL lattice simulation. Thus we can get the theoretical value of the threshold current as

$$I_{\text{th}} = \frac{2E_0}{\eta t_r \omega_\lambda (R/Q)_\lambda Q_\lambda} = \frac{2}{\frac{r_{56}}{\beta c E_0} \cdot 2\pi f_\lambda (R/Q)_\lambda Q_\lambda} \approx 8.6 \times 10^4\text{mA} \tag{40}$$

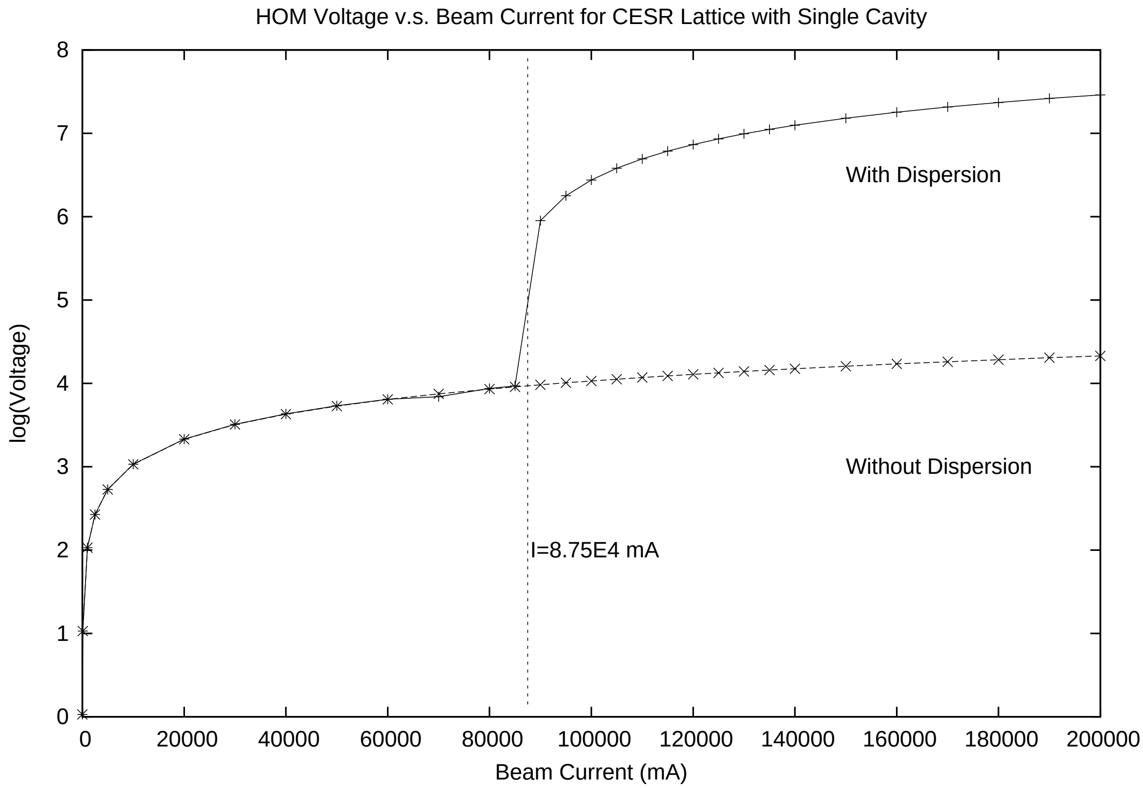

Figure 1: The HOM voltage with different beam currents in a single-cavity linac

In the simulation we use different beam currents and track the bunch motion until the HOM voltage in the cavity becomes stable. At the end of simulation the HOM voltage is recorded. The simulation result is shown in Fig. 1. We can see that the threshold current is about 8.6×10^4mA, just as the theory predicts.

For a full CESR lattice simulation, the results of up to four modes in each cavity are shown in Fig. 2. We can see from the simulation that the mode with $f_1 = 3.85763$GHz is dominant in this case. When we add the other two modes with significantly different frequencies, the threshold current is not affected. When the fourth mode with $f_4 = 3.85758$GHz is added, the threshold current decreases by about 50mA because f_4 is very close to f_1, equivalent to increasing the (R/Q) of the dominant mode.

In the simulation for CESR lattice we can observe the changing behavior of the average power in one cavity with respect to the tracking time. When the beam current is as low as 100mA, the equilibrium can be quickly established and the HOM voltage is stablized at about 110eV as Fig. 3 shows. When we increase the beam current, it takes longer to establish the equilibrium. But before the instability occurs, the equilibrium HOM voltage is always proportional to the beam current. For example,

$$\frac{U_{600}}{U_{100}} = \frac{668.2\text{eV}}{111.3\text{eV}} \approx 6 = \frac{600\text{mA}}{100\text{mA}} \tag{41}$$

The instability occurs at about 800mA, amid an extremely long relaxation time and a sudden jump of the equilibrium HOM voltage. As we can see in Fig. 3

$$\frac{U_{900}}{U_{100}} = \frac{197611\text{eV}}{111.3\text{eV}} \approx 1775 \gg \frac{900\text{mA}}{100\text{mA}} \tag{42}$$

1.5 Simulation Results for the New CESR Lattice

In the new CESR lattice the threshold current is significantly reduced to 82 mA because of the large r_{56} of the CESR ring($r_{56} = -1.055 \times 10^3$cm), as is shown in Fig. 4. This raises a critical issue in the lattice design. In order to suppress longitudinal BBU, we must build a return loop with small r_{56}. In order to study the influence of the

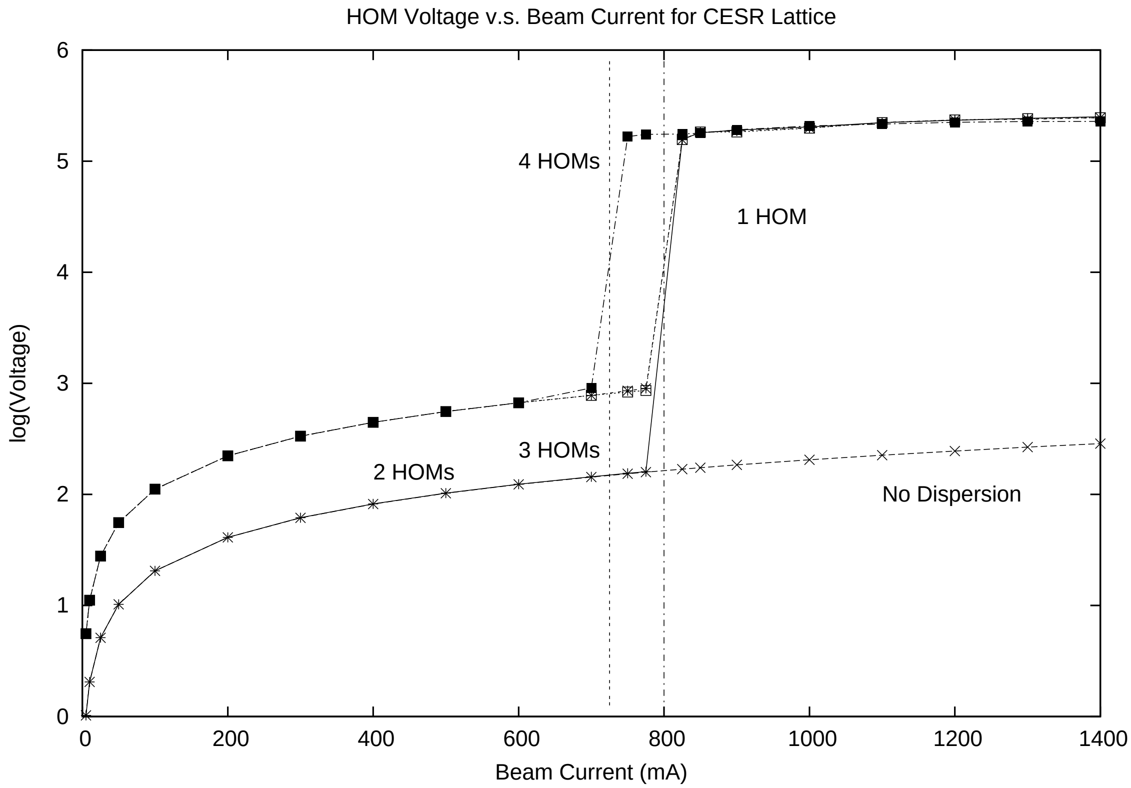

Figure 2: The HOM voltage with different beam currents in the CESR lattice

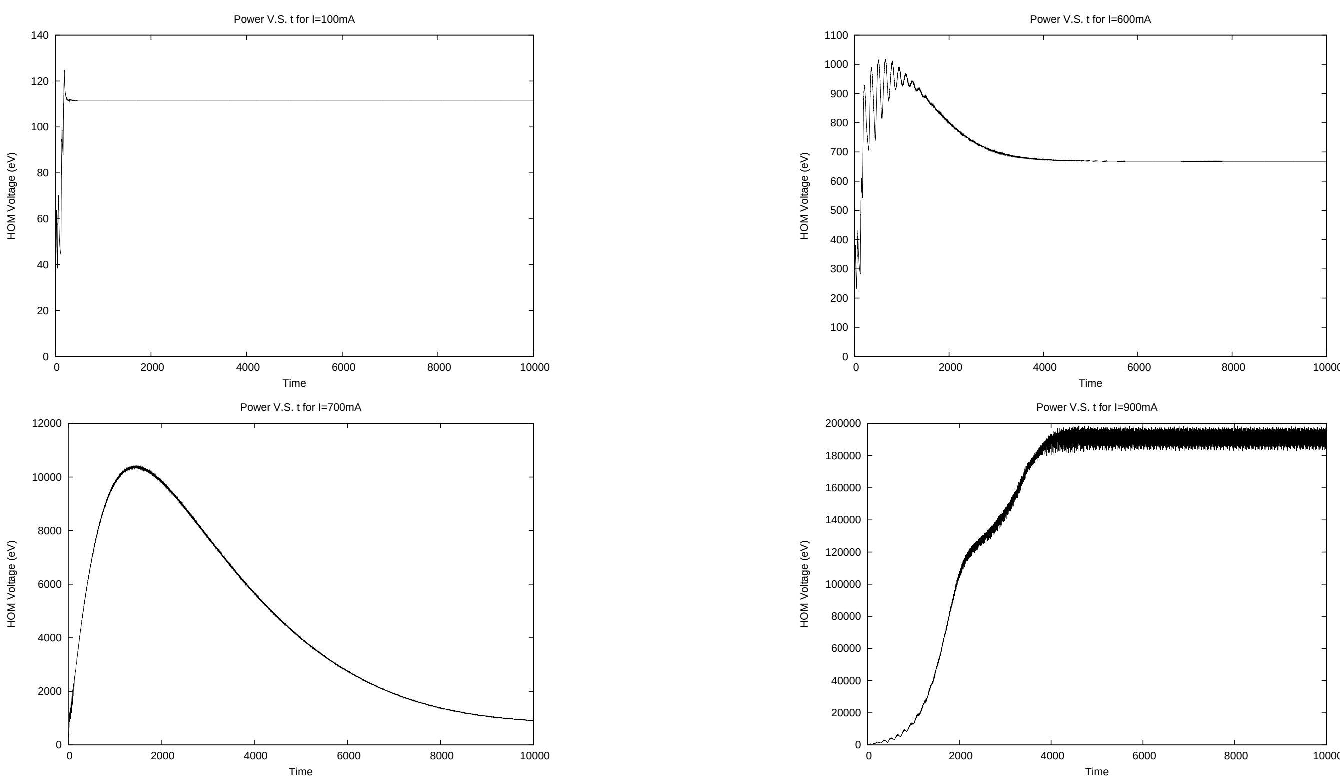

Figure 3: The power in the cavity v.s. Tracking time with I=100, 600, 700, 900mA

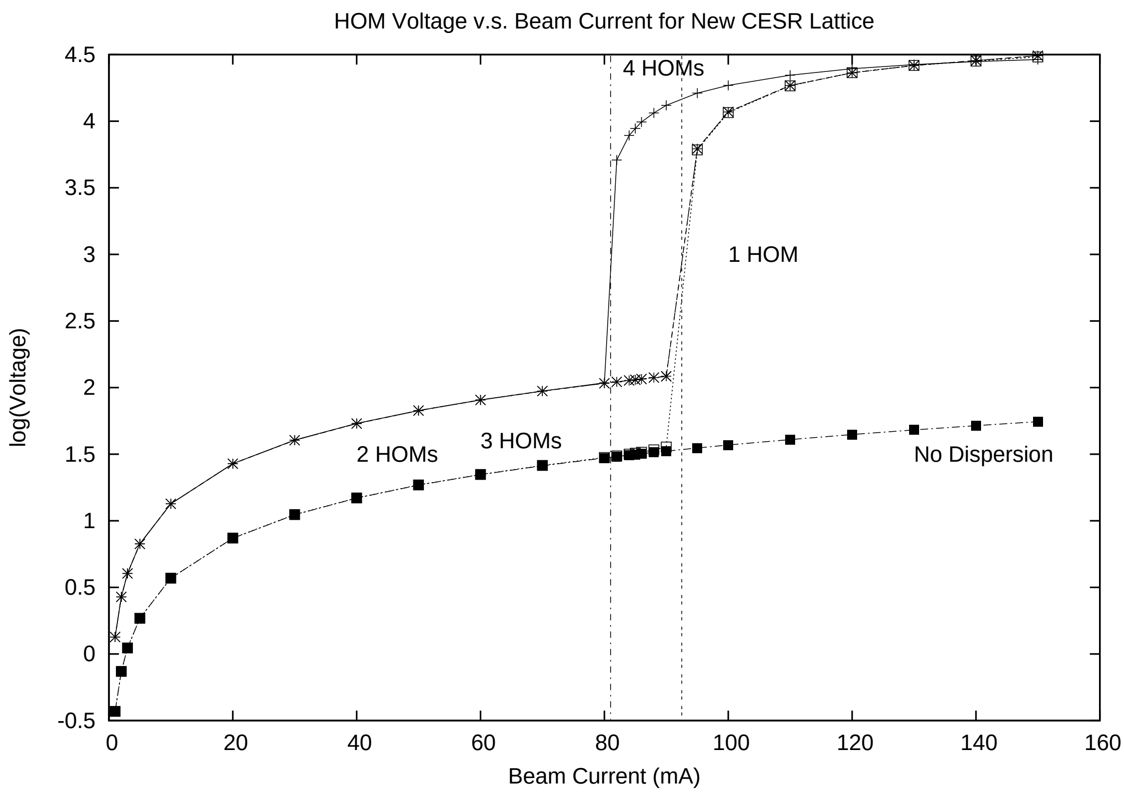

Figure 4: HOM Voltage v.s. Beam Current for the New CESR Lattice

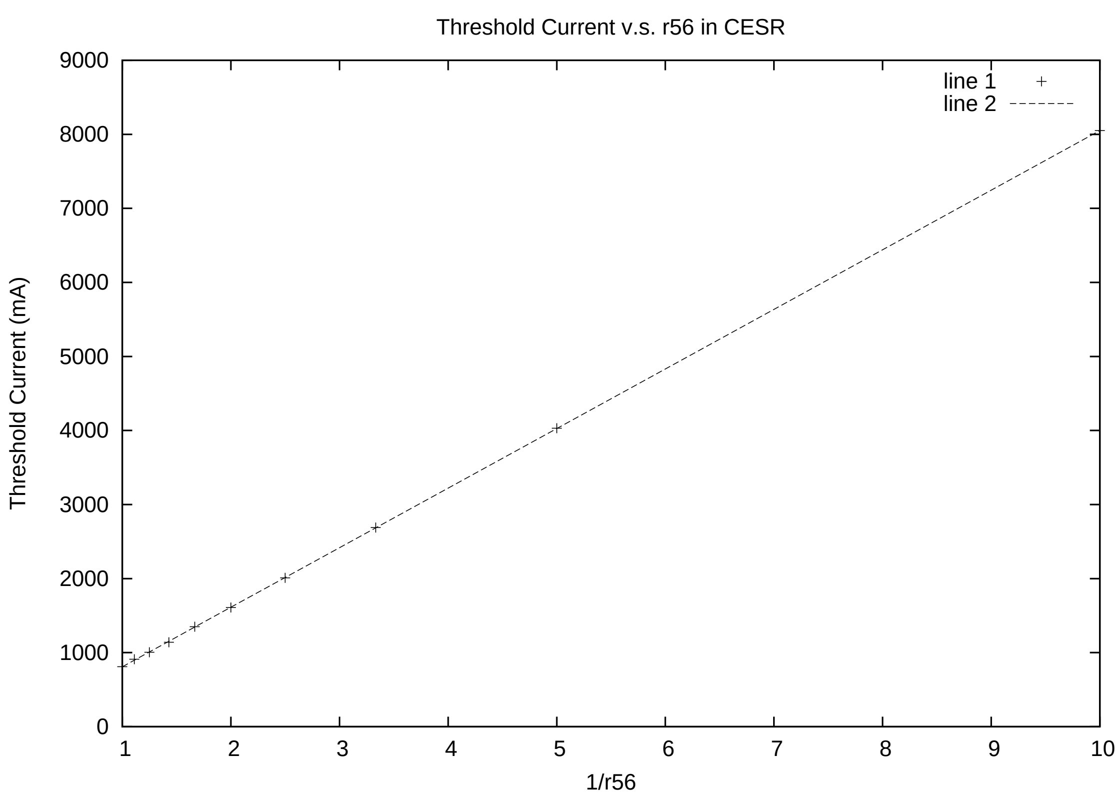

Figure 5: Threshold Current v.s. $1/r_{56}$ in the New CESR Lattice

r_{56}, we carry out simulation by putting different artificial r_{56}s into the lattice and the result is shown in Fig. 5. It is very clear that the threshold current is proportional to the inverse of the r_{56}, which is predicted by the theory in Eq. (38).

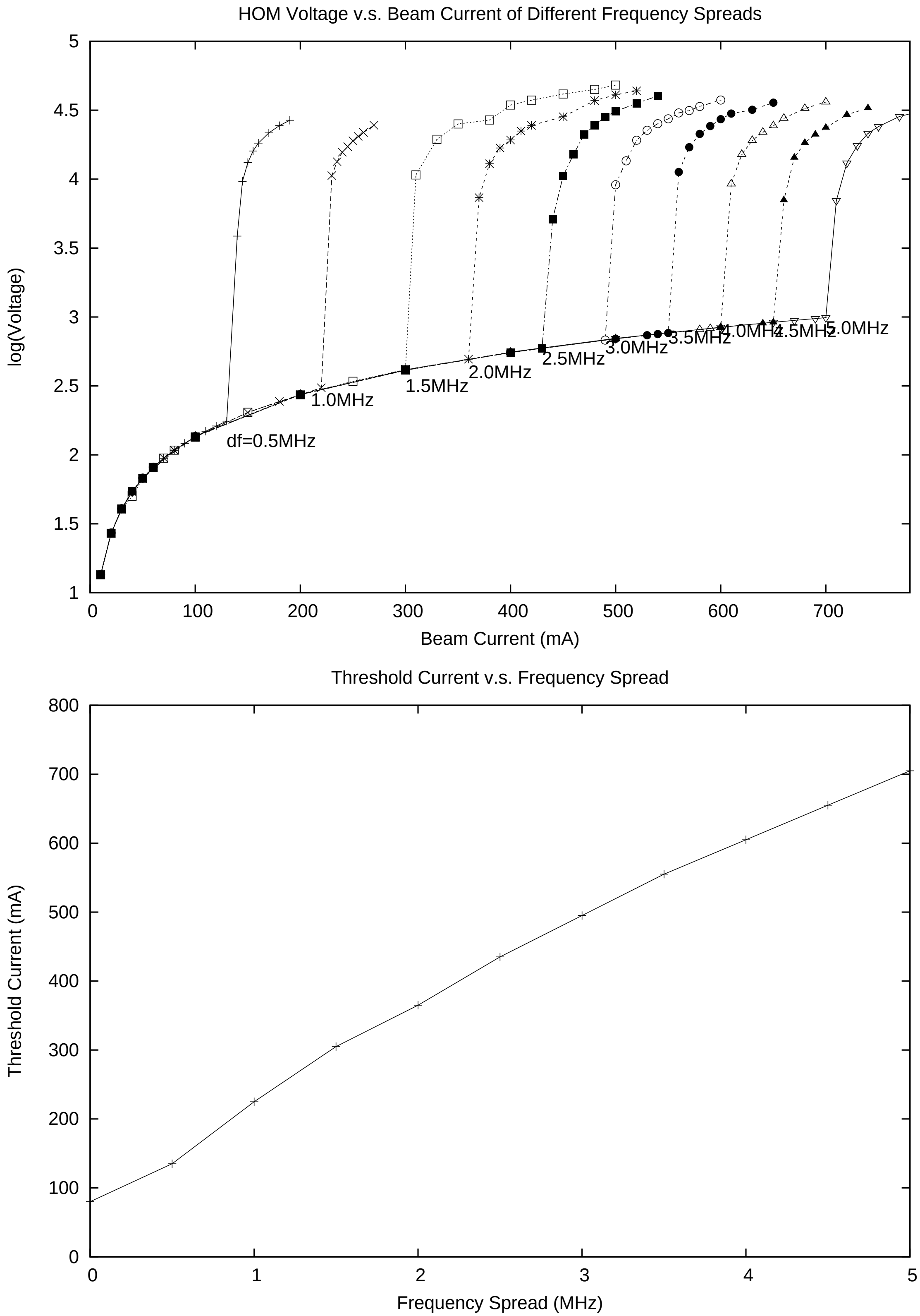

Figure 6: The threshold current v.s. Frequency spread for the New CESR Lattice

Similar to the transverse BBU case, the longitudinal BBU threshold current is also influenced by the frequency spread of HOMs in different RF cavities. Figure 6 shows that the threshold current increases almost linearly with the frequency spread. Thus introducing frequency spread should be a good way to boost the threshold current in both longitudinal and transverse cases.

1.6 The Theory of Longitudinal HOM Power

In a real accelerator the beam current is often limited to below the BBU threshold current because of HOM heating. Thus the HOM power is analyzed for a single-cavity linac with beam currents below the BBU threshold. As we discussed before, below the threshold the longitudinal HOM voltage should be proportional to the beam current. Above the threshold the beam current is modulated and the Fourier component that has the same frequency as the HOM is enhanced. The electron bunches should behave like they are at resonance with the HOM. Thus it is important to study how the HOM voltage depends on the current and HOM frequency.

Here we want to derive an analytical formula of the longitudinal HOM power in a single cavity for a certain beam current that is below the BBU threshold. We assume that the beam current can be written as

$$I = I_0 t_b \sum_{n=-\infty}^{\infty} \delta(t - nt_b) \tag{43}$$

The longitudinal HOM voltage is in the form of

$$\begin{aligned} V(t) &= \int_{-\infty}^{\infty} dt' I(t') W(t - t') \\ W(t - t') &= 0, \quad t - t' < 0 \end{aligned} \tag{44}$$

We know the formula of the voltage excited by a single bunch, which is

$$V(t) = \frac{q\omega_\lambda}{2} \left(\frac{R}{Q}\right)_\lambda \cos\omega_\lambda t e^{-\frac{\omega_\lambda}{2Q_\lambda} t} \tag{45}$$

This formula is derived under the current $I(t) = q\delta(t)$, with which we can derive the formula of the wake function.

$$\begin{aligned} V(t) &= \int_{-\infty}^{\infty} dt' I(t') W(t - t') \\ &= \int_{-\infty}^{\infty} dt' q\delta(t') W(t - t') \\ &= qW(t) \quad t > 0 \end{aligned} \tag{46}$$

Thus we have

$$W(t) = \frac{\omega_\lambda}{2} \left(\frac{R}{Q}\right)_\lambda \cos\omega_\lambda t e^{-\frac{\omega_\lambda}{2Q_\lambda} t}, \quad t > 0 \tag{47}$$

In the case where the beam current is $I = I_0 t_b \sum_{n=-\infty}^{\infty} \delta(t - nt_b)$, we can derive the HOM voltage by the wake function above.

$$\begin{aligned} V(t) &= \int_{-\infty}^{\infty} dt' I(t') W(t - t') \\ &= I_0 t_b \sum_{n=-\infty}^{\infty} \frac{\omega_\lambda}{2} \left(\frac{R}{Q}\right)_\lambda \cos\omega_\lambda (t - nt_b) e^{-\frac{\omega_\lambda}{2Q_\lambda}(t - nt_b)}, \quad nt_b \le t \end{aligned} \tag{48}$$

If we define the average power $\bar{P}$ as the average power over one bunch period, and we start injecting bunch at $t = -\infty$, then the average power will become stable at time $t = 0$, thus the analytical formula of the stablized HOM power can be derived by computing the average power in the period containing $t = 0$.

$$\begin{aligned} \bar{P} &= \frac{1}{t_b} \int_{-\frac{t_b}{2}}^{\frac{t_b}{2}} dt' I(t') V(t') \\ &= I_0 V(0) \\ &= I_0^2 t_b \frac{\omega_\lambda}{2} \left(\frac{R}{Q}\right)_\lambda \sum_{n=0}^{\infty} e^{-\frac{\omega_\lambda}{2Q_\lambda} nt_b} \cos n\omega_\lambda t_b \end{aligned} \tag{49}$$

$$\sum_{n=0}^{\infty} e^{-\frac{\omega_\lambda}{2Q_\lambda}nt_b}\cos n\omega_\lambda t_b = \frac{1-e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda t_b}{1-2e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda t_b + e^{-\frac{\omega_\lambda}{Q_\lambda}t_b}} \tag{50}$$

Thus we have the average power

$$\bar{P} = I_0^2 t_b \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \frac{1-e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda t_b}{1-2e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda t_b + e^{-\frac{\omega_\lambda}{Q_\lambda}t_b}} \tag{51}$$

There are two scenarios that need detailed investigation. One is the bunching frequency being close to the resonant frequency of the HOM, the other one is the bunching frequency being far away from the resonant frequency of the HOM.

In an ERL it is always true that $Q_\lambda \gg \omega_\lambda t_b$, under which different approximations are made in the two scenarios. It is convenient to define

$$\varepsilon = \frac{\omega_\lambda t_b}{2Q_\lambda} \quad . \tag{52}$$

In our discussion, $\varepsilon \ll 1$ always holds.

The first scenario is that the bunching frequency is close to the resonance frequency of the HOM, which means $\omega_\lambda t_b \approx 2n\pi$, $n = 1, 2, 3, \cdots$. Under this condition, $\cos\omega_\lambda t_b$ can be expanded in the vicinity of $2n\pi$ to the second order as

$$\cos\omega_\lambda t_b = 1 - \frac{(\omega_\lambda t_b - 2n\pi)^2}{2} + \cdots \tag{53}$$

Thus the formula of the average power becomes

$$\begin{aligned}\bar{P} &= I_0^2 t_b \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \frac{1-e^{-\varepsilon}(1-\frac{(\omega_\lambda t_b-2n\pi)^2}{2})}{1-2e^{-\varepsilon}(1-\frac{(\omega_\lambda t_b-2n\pi)^2}{2})+e^{-2\varepsilon}} \\ &= I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \left\{1+\frac{1-e^{-2\varepsilon}}{(1-e^{-\varepsilon})^2+e^{-\varepsilon}(\omega_\lambda t_b-2n\pi)^2}\right\}\end{aligned} \tag{54}$$

Since $\varepsilon \ll 1$ we have

$$\begin{aligned}\bar{P} &= I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \left\{1+\frac{e^{\varepsilon}-e^{-\varepsilon}}{(1-e^{-\varepsilon})^2 e^{\varepsilon}+(\omega_\lambda t_b-2n\pi)^2}\right\} \\ &= I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \left\{1+\frac{1+\varepsilon+\frac{\varepsilon^2}{2}+\cdots-1+\varepsilon-\frac{\varepsilon^2}{2}+\cdots}{(\varepsilon-\frac{\varepsilon^2}{2}+\cdots)^2(1+\varepsilon+\frac{\varepsilon^2}{2}+\cdots)+(\omega_\lambda t_b-2n\pi)^2}\right\} \\ &\approx I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \left\{1+\frac{2\varepsilon}{\varepsilon^2+(\omega_\lambda t_b-2n\pi)^2}\right\}\end{aligned} \tag{55}$$

At resonant frequencies, which means $\omega_\lambda t_b = 2n\pi$, we have

$$\bar{P} \approx I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \left\{1+\frac{2}{\varepsilon}\right\} \approx I_0^2 \left(\frac{R}{Q}\right)_\lambda Q_\lambda \tag{56}$$

This result is also confirmed by directly making $\cos\omega_\lambda t_b = 0$ in Eq. (51).

$$\bar{P} = I_0^2 n\pi \left(\frac{R}{Q}\right)_\lambda \frac{1}{1-e^{-\frac{n\pi}{Q_\lambda}}} \tag{57}$$

$$\bar{P} = I_0^2 n\pi \left(\frac{R}{Q}\right)_\lambda \frac{1}{1-1+\frac{n\pi}{Q_\lambda}} = I_0^2 \left(\frac{R}{Q}\right)_\lambda Q_\lambda \tag{58}$$

The second scenario is that the bunching frequency falls in bewteen resonant frequencies, in which we have

$$
\begin{aligned}
\bar{P} &= I_0^2 t_b \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \frac{1-(1-\varepsilon+\frac{\varepsilon^2}{2}-\cdots)\cos\omega_\lambda t_b}{1-2(1-\varepsilon+\frac{\varepsilon^2}{2}-\cdots)\cos\omega_\lambda t_b+1-2\varepsilon+2\varepsilon^2-\cdots} \\
&\approx I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \frac{(1-\cos\omega_\lambda t_b)+\epsilon\cos\omega_\lambda t_b}{(1-\cos\omega_\lambda t_b)(1-\varepsilon)} \\
&\approx I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \left\{(1+\varepsilon+\cdots)+(\varepsilon+\varepsilon^2+\cdots)\frac{\cos\omega_\lambda t_b}{1-\cos\omega_\lambda t_b}\right\}
\end{aligned}
\tag{59}
$$

Since the bunching frequency is not close to any resonant frequency, $\dfrac{\cos\omega_\lambda t_b}{1-\cos\omega_\lambda t_b}$ is not large enough to overcome the effect of a small ε. Thus we only need to keep terms of the lowest order of ε, which are the constant terms.

$$
\bar{P} \approx I_0^2 t_b \frac{\omega_\lambda}{4}\left(\frac{R}{Q}\right)_\lambda \tag{60}
$$

These results indicate that when the system is off resonance, the power load of HOMs is proportional to the HOM resonant frequency. When the bunching frequency approaches to one of the HOM resonant frequencies, the power load becomes a Lorentz curve peaked at the resonant frequency.

An interesting point is at the same average beam current, the ratio between the HOM power on and off resonance is

$$
\frac{\bar{P}_{\text{on}}}{\bar{P}_{\text{off}}} \approx \frac{4Q_\lambda}{\omega_\lambda t_b} \tag{61}
$$

In an ERL there are two beams in the accelerator at the same time. The effective current in such a case can be written as

$$
I = I_0 t_b \sum_{n=-\infty}^{\infty} \delta(t-nt_b)+\delta(t-t_r-nt_b) \tag{62}
$$

where t_r is the return time and can be written as

$$
t_r = n_r t_b + \delta t_b\,, \qquad 0 \le \delta < 1 \tag{63}
$$

Thus the current can also written in terms of δ

$$
I = I_0 t_b \sum_{n=-\infty}^{\infty} \delta(t-nt_b)+\delta(t-nt_b-\delta t_b) \tag{64}
$$

Similar to the single beam case the HOM voltage can be written as

$$
\begin{aligned}
V(t) &= \int_{-\infty}^{\infty} dt' I(t')W(t-t') \\
&= I_0 t_b \sum_{n=-\infty}^{\infty} W(t-nt_b)+W(t-nt_b-\delta t_b)
\end{aligned}
\tag{65}
$$

According to Eq. (44) we have

$$
V(t) = I_0 t_b \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \sum_{n=-\infty}^{(n+\delta)t_b\le t} e^{-\frac{\omega_\lambda}{2Q_\lambda}(t-nt_b)}\cos\omega_\lambda(t-nt_b)+e^{-\frac{\omega_\lambda}{2Q_\lambda}(t-nt_b-\delta t_b)}\cos\omega_\lambda(t-nt_b-\delta t_b) \tag{66}
$$

Again we assume that the beam is turned on at $t=-\infty$ and the HOM voltage becomes stable eventually. Thus we will only calculate the HOM voltage at time $t=\delta t_b$, when the returning bunch arrives at the cavity. (In the simulation code we only record the HOM voltage when a bunch passes through the cavity.)

$$
\begin{aligned}
V(\delta t_b) &= I_0 t_b \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \sum_{n=-\infty}^{0} e^{-\frac{\omega_\lambda}{2Q_\lambda}(\delta t_b-nt_b)}\cos\omega_\lambda(\delta t_b-nt_b)+e^{\frac{\omega_\lambda}{2Q_\lambda}nt_b}\cos\omega_\lambda(nt_b) \\
&= I_0 t_b \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \sum_{n=0}^{\infty} e^{-\frac{\omega_\lambda}{2Q_\lambda}(nt_b+\delta t_b)}\cos\omega_\lambda(nt_b+\delta t_b)+e^{-\frac{\omega_\lambda}{2Q_\lambda}nt_b}\cos\omega_\lambda(nt_b)
\end{aligned}
\tag{67}
$$

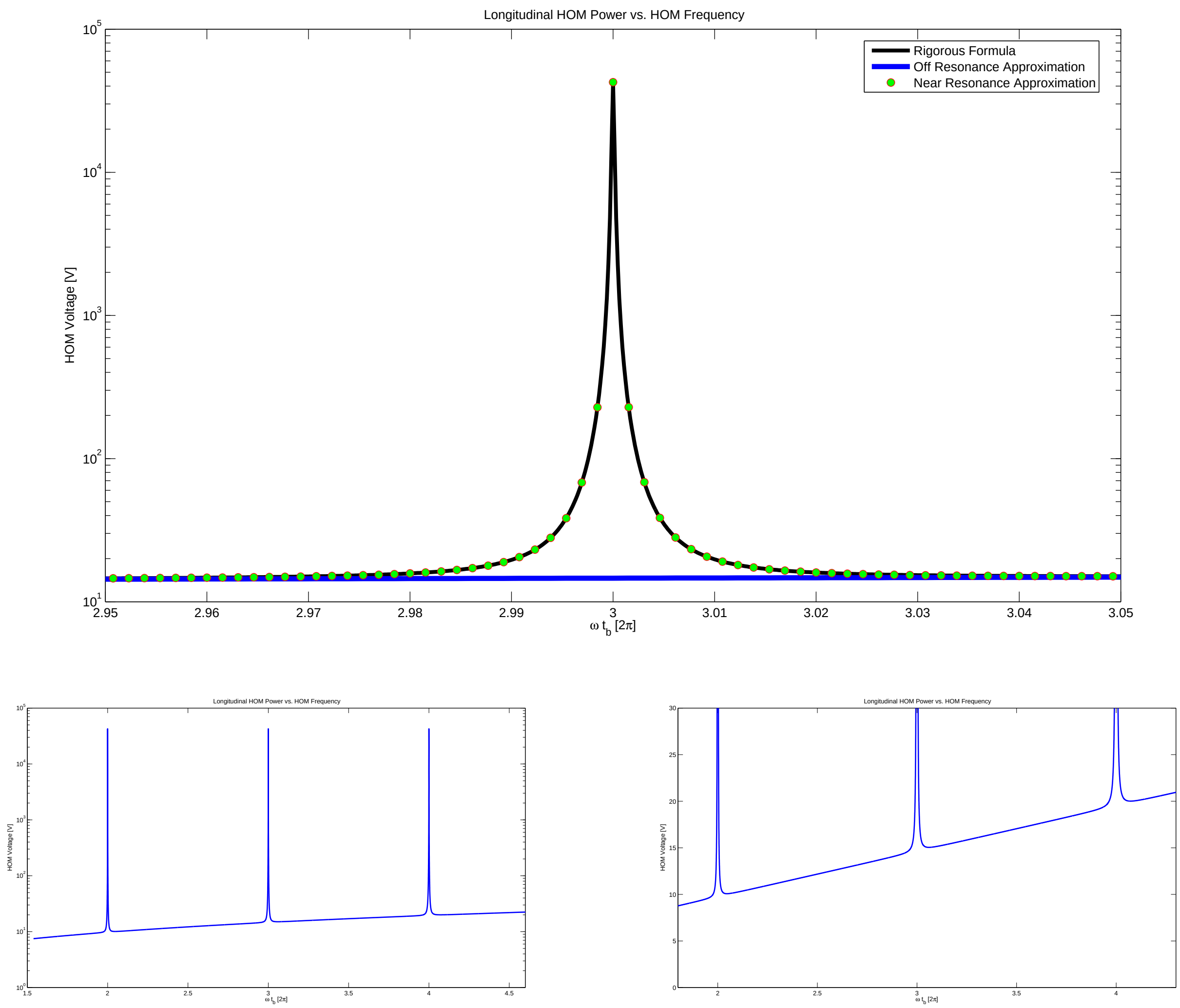

Figure 7: The Average HOM Power v.s. HOM Frequency with Fixed Bunching Frequency $f_b = 1.3$GHz.

Table 2: The HOM voltage for the four dominant longitudinal HOMs in the 7-cell ERL cavity

	f_λ[GHz]	Q_λ	$(R/Q)_\lambda[\Omega]$	HOM Voltage [V]
1	3.85763	13728	31	39.5988
2	2.45658	1778.8	134.5	39.5529
3	5.93396	27887	5.99	4.1437
4	3.85758	40172	2.94	3.7534

According to Eq. (50) and

$$\begin{aligned}
&\sum_{n=0}^{\infty} e^{-\frac{\omega_\lambda}{2Q_\lambda}(nt_b+\delta t_b)} \cos\omega_\lambda(nt_b+\delta t_b) \\
&= e^{-\frac{\omega_\lambda}{2Q_\lambda}\delta t_b} \sum_{n=0}^{\infty} e^{-\frac{\omega_\lambda}{2Q_\lambda}nt_b} \cos\omega_\lambda(nt_b+\delta t_b) \\
&= \frac{e^{-\frac{\omega_\lambda}{2Q_\lambda}\delta t_b}\left[\cos\omega_\lambda\delta t_b - e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda(1-\delta)t_b\right]}{1-2e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda t_b + e^{-\frac{\omega_\lambda}{Q_\lambda}}}
\end{aligned} \tag{68}$$

the HOM voltage at time $t=\delta t_b$ can be written as

$$V(\delta t_b) = I_0 t_b \frac{\omega_\lambda}{2}\left(\frac{R}{Q}\right)_\lambda \frac{1-e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda t_b + e^{-\frac{\omega_\lambda}{2Q_\lambda}\delta t_b}\left[\cos\omega_\lambda\delta t_b - e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda(1-\delta)t_b\right]}{1-2e^{-\frac{\omega_\lambda}{2Q_\lambda}t_b}\cos\omega_\lambda t_b + e^{-\frac{\omega_\lambda}{Q_\lambda}}} \tag{69}$$

In Fig. 9 we can see that if an ERL is operating at $\delta = 0.5$, the effective current becomes

$$I = I_0 t_0 \sum_{n=-\infty}^{\infty} \delta(t - nt_b/2) \tag{70}$$

and the resonance at $\omega_\lambda t_b = 6\pi$ disappears because the effective period becomes $t_b/2$. We also calculate HOM voltages for the four dominant modes in the 7-cell ERL cavity and the results are listed in Tab. 2.

2 Acknowledgments

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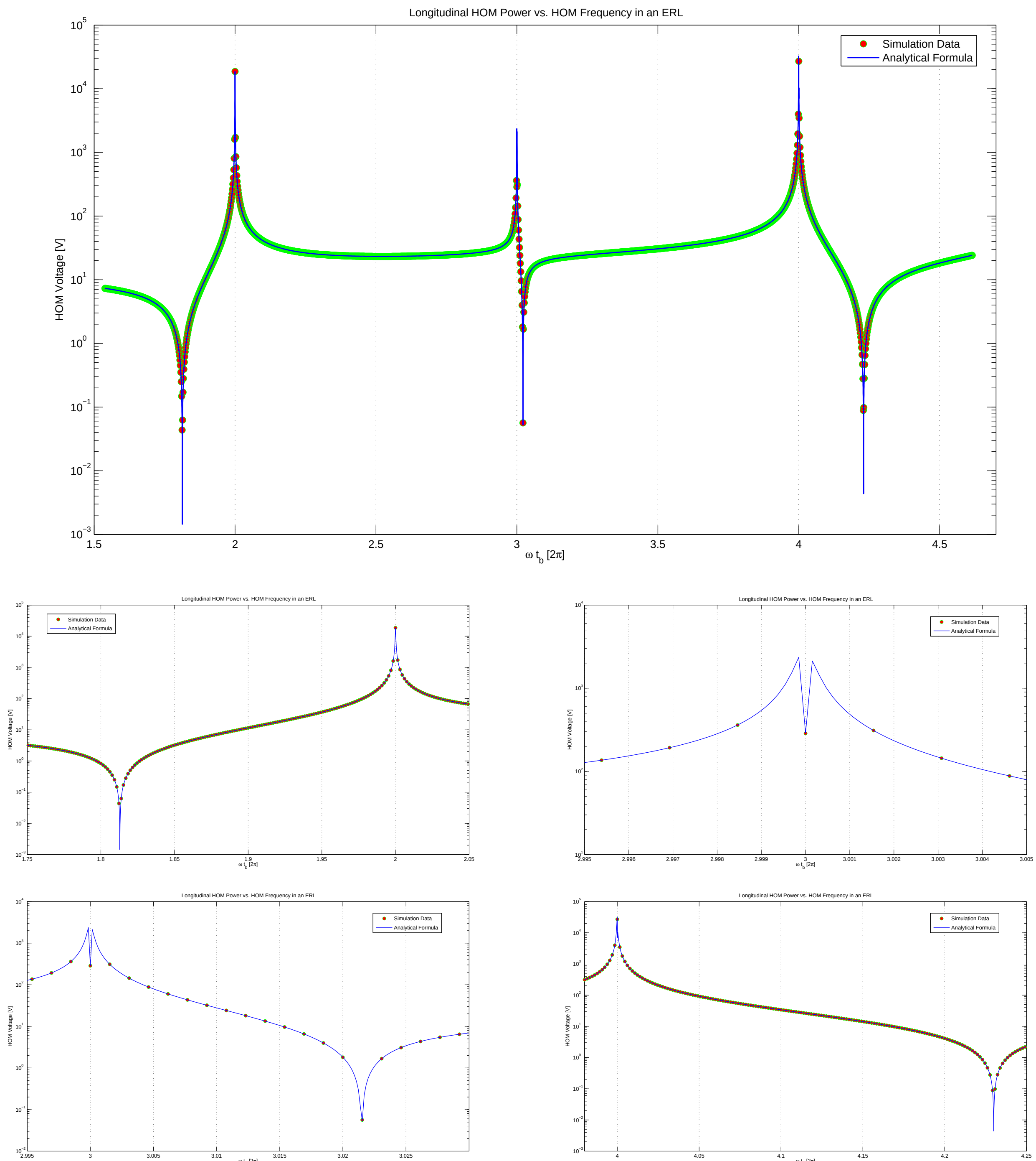

Figure 8: Comparison between Eq. (69) and simulation results with $\delta = 0.8274036$

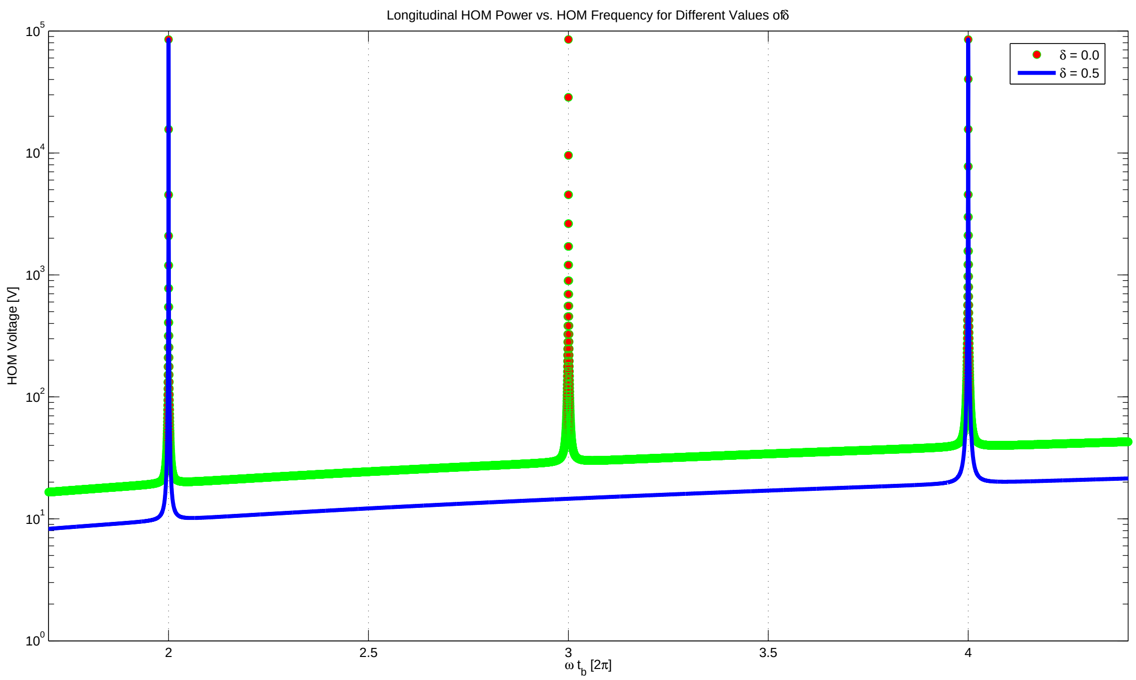

Figure 9: Comparison between a normal linac $\delta = 0$ and an ERL $\delta = 0.5$