Threshold Resummation in Momentum Space from Effective Field Theory

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Methods from soft-collinear effective theory are used to perform the threshold resummation of Sudakov logarithms for the deep-inelastic structure function $F_2(x, Q^2)$ in the endpoint region $x \to 1$ directly in momentum space. An explicit all-order formula is derived, which expresses the short-distance coefficient function C in the convolution $F_2 = C \otimes \phi_q$ in terms of Wilson coefficients and anomalous dimensions defined in the effective theory. Contributions associated with the physical scales Q^2 and $Q^2(1-x)$ are separated from non-perturbative hadronic physics in a transparent way. A crucial ingredient to the momentum-space resummation is the exact solution to the integro-differential evolution equation of the jet function, which is derived. The methods developed in this Letter can be applied to many other hard QCD processes.

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I. INTRODUCTION

A generic problem in applications of perturbative QCD to collider physics or heavy-quark physics is to disentangle contributions associated with different momentum scales, and to resum large logarithms of ratios of such scales to all orders in the perturbative expansion. In processes containing hadronic jets, a scale hierarchy is created by the fact that the invariant mass of a collimated jet is typically much smaller than the hard scale of the process (e.g., the center-of-mass energy). The intricate interplay of soft and collinear emissions then leads to large Sudakov double logarithms. The resummation of these logarithms is conventionally performed in moment space, and predictions for differential cross sections in momentum space are then obtained by an inverse Mellin transformation. This procedure is cumbersome and often leads to unphysical singularities, because the resummation formulae involve integrals over the Landau pole of the running coupling. These singularities are dealt with by means of ad hoc prescriptions, or by introducing artificial infrared cutoffs.

In this Letter we develop an approach based on effective field theory, which allows us to resum Sudakov logarithms for a large class of processes directly in momentum space. The starting point is a factorization theorem for the cross section, in which contributions from different momentum scales are separated in a transparent way. Evolution equations for the various components in the factorization formula are solved exactly in momentum space, in such a way that one never encounters integrals over the Landau pole. We illustrate the procedure with the example of deep-inelastic scattering. However, the same methods can be applied to many other hard QCD processes, such as Drell-Yan lepton-pair production, prompt photon production in hadron-hadron collisions, Higgs-boson production in gluon-gluon fusion, heavy-quark fragmentation, event shapes, and others. Technical details of our derivations will be presented in a forthcoming paper [1].

II. FACTORIZATION FORMULA

We focus on the flavor non-singlet component of the structure function $F_2(x, Q^2)$ in deep-inelastic scattering (DIS) of electrons off a nuclear target, $e^- + N(p) \to e^- + X(P)$, denoting by $q = P - p$ the momentum of the virtual photon. We are interested in the region where the Bjorken scaling variable $x = Q^2/(2p\cdot q)$ is near 1, so that there is a hierarchy of scales $Q^2 \gg Q^2(1-x) \gg \Lambda^2_{\rm QCD}$. The intermediate scale $Q^2(1-x) \approx M_X^2$ is set by the invariant mass M_X of the final-state jet. In this region the structure function can be written in the factorized form [2–4] (with μ_f the factorization scale and e_q the quark electric charge)

$$F_2^{\rm ns}(x, Q^2) = \sum_q e_q^2\, |C_V(Q^2, \mu_f)|^2 \times Q^2 \int_x^1 d\xi\, J\big(Q^2(\xi - x), \mu_f\big)\, \phi_q^{\rm ns}(\xi, \mu_f)\,.$$

This formula is valid to all orders in perturbation theory and at leading power in $(1-x)$ and $\Lambda^2_{\rm QCD}/M_X^2$. Here C_V is a hard matching coefficient, J is a jet function, and $\phi_q^{\rm ns}$ is the non-singlet component of the quark distribution function in the nucleon. As shown in [1], a relatively simple proof of the factorization formula can be given using the technology of soft-collinear effective theory (SCET) [5] (see [6–8] for earlier investigations).

In SCET the hard function C_V is identified with the Wilson coefficient in the matching relation of the QCD vector current onto the unique leading-power current operator in the effective theory. To calculate the Wilson coefficient one must compare perturbative expressions for the photon vertex function in the two theories. The calculation can be simplified by performing these calculations on-shell, in which case all loop graphs in the effective theory are scaleless and vanish in dimensional regularization. The bare on-shell vertex function in QCD (called the on-shell quark form factor) has been studied extensively at two-loop order and beyond [9–12]. The form factor is infrared divergent and must be regularized. When the SCET graphs are subtracted from the QCD result, the infrared poles in $1/\epsilon$ get cancelled and replaced by ultraviolet poles. To obtain the matching coefficient we introduce a renormalization factor Z_V, which absorbs these poles. At one-loop order this gives [6]

$$C_V(Q^2,\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left(-L^2 + 3L - 8 + \frac{\pi^2}{6}\right),$$

where $L = \ln(Q^2/\mu^2)$ and $\alpha_s = \alpha_s(\mu)$. The two-loop expression for C_V can be found in [1]. The scale dependence of the Wilson coefficient is governed by the evolution equation

$$\frac{dC_V(Q^2,\mu)}{d\ln\mu} = \left[\Gamma_{\rm cusp}(\alpha_s)\,\ln\frac{Q^2}{\mu^2} + \gamma^V(\alpha_s)\right] C_V(Q^2,\mu)\,, \tag{1}$$

where $\Gamma_{\rm cusp}$ is the universal cusp anomalous dimension of Wilson loops with light-like segments [13], which is associated with the appearance of Sudakov double logarithms. The quantity γ^V accounts for single-logarithmic evolution effects. The anomalous dimension can be obtained from the coefficient of the $1/\epsilon$ pole term in the renormalization factor Z_V. Using the results of [12] it can be calculated at three-loop order. The result is presented in [1].

The jet function J is defined in terms of the discontinuity of a vacuum correlator of two quark fields, made gauge invariant by the introduction of Wilson lines. It obeys the integro-differential evolution equation [14]

$$\begin{aligned}\frac{dJ(p^2,\mu)}{d\ln\mu} &= -\left[2\Gamma_{\rm cusp}(\alpha_s)\,\ln\frac{p^2}{\mu^2} + 2\gamma^J(\alpha_s)\right] J(p^2,\mu)\\ &\quad - 2\Gamma_{\rm cusp}(\alpha_s)\int_0^{p^2} dp'^2\,\frac{J(p'^2,\mu)-J(p^2,\mu)}{p^2-p'^2}\,.\end{aligned}$$

We encounter again the cusp anomalous dimension, and in addition a new function γ^J, which has been calculated in [14] at two-loop order, and whose three-loop coefficient is determined in [1].

III. SOLUTIONS OF THE RG EQUATIONS

The exact solution to the evolution equation (1) is

$$\begin{aligned}C_V(Q^2,\mu) &= \exp\left[2S(\mu_h,\mu) - a_{\gamma^V}(\mu_h,\mu)\right]\\ &\quad\times\left(\frac{Q^2}{\mu_h^2}\right)^{-a_\Gamma(\mu_h,\mu)} C_V(Q^2,\mu_h)\,,\end{aligned} \tag{2}$$

where $\mu_h \sim Q$ is a hard matching scale, at which the value of the coefficient C_V is calculated using fixed-order perturbation theory. The Sudakov exponent S and the exponents a_n are given by

$$\begin{aligned}S(\nu,\mu) &= -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)}\int\limits_{\alpha_s(\nu)}^{\alpha}\frac{d\alpha'}{\beta(\alpha')}\,,\\ a_\Gamma(\nu,\mu) &= -\int\limits_{\alpha_s(\nu)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)}\,,\end{aligned} \tag{3}$$

and similarly for a_{γ^V}, where $\beta(\alpha_s) = d\alpha_s/d\ln\mu$ is the β-function. The explicit perturbative expansions of these expressions valid at next-to-next-to-leading order (NNLO) in renormalization-group (RG) improved perturbation theory are given in [1].

An important object in the derivation of the solution to the evolution equation for J is the associated jet function $\widetilde{j}$, which has originally been defined in terms of an integral over the jet function followed by a certain replacement rule [15]. More elegantly, it can be obtained by the Laplace transformation

$$\widetilde{j}\Big(\ln\frac{Q^2}{\mu^2},\mu\Big) = \int_0^\infty dp^2\,e^{-sp^2}\,J(p^2,\mu)\,,$$

where $s = 1/(e^{\gamma_E}Q^2)$. The inverse transformation is

$$J(p^2,\mu) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} ds\,e^{sp^2}\,\widetilde{j}\Big(\ln\frac{1}{e^{\gamma_E}s\,\mu^2},\mu\Big)\,. \tag{4}$$

Using the evolution equation for the jet function we find that the associated jet function obeys

$$\begin{aligned}&\frac{d}{d\ln\mu}\,\widetilde{j}\Big(\ln\frac{Q^2}{\mu^2},\mu\Big)\\ &= -\left[2\Gamma_{\rm cusp}(\alpha_s)\,\ln\frac{Q^2}{\mu^2} + 2\gamma^J(\alpha_s)\right]\widetilde{j}\Big(\ln\frac{Q^2}{\mu^2},\mu\Big)\,,\end{aligned}$$

which is analogous to the evolution equation (1) for the hard function. Inserting the solution to this equation into the inverse transformation (4) we obtain

$$\begin{aligned}J(p^2,\mu) &= \exp\left[-4S(\mu_i,\mu) + 2a_{\gamma^J}(\mu_i,\mu)\right]\\ &\quad\times\widetilde{j}(\partial_\eta,\mu_i)\,\frac{e^{-\gamma_E\eta}}{\Gamma(\eta)}\,\frac{1}{p^2}\left(\frac{p^2}{\mu_i^2}\right)^\eta\,,\end{aligned} \tag{5}$$

where $\eta = 2a_\Gamma(\mu_i,\mu)$, and ∂_η denotes a derivative with respect to this quantity. The above form of the result is valid as long as $\eta > 0$ (i.e., $\mu < \mu_i$). For negative η the singularity at $p^2 = 0$ must be regularized using a star distribution [1]. Relation (5) is one of the main results of this Letter. It relates J to the associated jet function $\widetilde{j}$ evaluated at the scale μ_i, where it can be computed using fixed-order perturbation theory. At one-loop order

$$\widetilde{j}(L,\mu) = 1 + \frac{C_F\alpha_s}{4\pi}\left(2L^2 - 3L + 7 - \frac{2\pi^2}{3}\right),$$

where in (5) the argument L is replaced by the derivative operator ∂_η. The two-loop expression for $\widetilde{j}$ can be extracted from [14].

IV. MOMENTUM-SPACE RESUMMATION

We are now ready to write down a resummed expression for the structure function $F_2^{\rm ns}(x,Q^2)$ valid to all orders in perturbation theory and at leading power in $(1-x)$ and $\Lambda_{\rm QCD}^2/M_X^2$. When combining the results (2) and (5) the Sudakov exponents can be simplified. Introducing the short-hand notation $a_{\gamma^\phi}=a_{\gamma^J}-a_{\gamma^V}$, we find after a straightforward calculation

$$F_2^{\rm ns}(x,Q^2)=\sum_q e_q^2\,|C_V(Q^2,\mu_h)|^2\,U(Q,\mu_h,\mu_i,\mu_f) \times\widetilde{j}\Big(\ln\frac{Q^2}{\mu_i^2}+\partial_\eta,\mu_i\Big)\,\frac{e^{-\gamma_E\eta}}{\Gamma(\eta)}\int_x^1 d\xi\,\frac{\phi_q^{\rm ns}(\xi,\mu_f)}{(\xi-x)^{1-\eta}}\,,$$

where

$$U(Q,\mu_h,\mu_i,\mu_f)=\exp\left[4S(\mu_h,\mu_i)-2a_{\gamma_V}(\mu_h,\mu_i)\right] \times\left(\frac{Q^2}{\mu_h^2}\right)^{-2a_\Gamma(\mu_h,\mu_i)}\exp\left[2a_{\gamma^\phi}(\mu_i,\mu_f)\right]\,,$$

and as before $\eta=2a_\Gamma(\mu_i,\mu_f)$. The remaining integral can be performed noting that, on general grounds, the behavior of the parton distribution function near the endpoint can be parameterized as

$$\phi_q^{\rm ns}(\xi,\mu_f)\big|_{\xi\to1}=N(\mu_f)\,(1-\xi)^{b(\mu_f)}\Big[1+\mathcal{O}(1-\xi)\Big]\,,$$

where $b(\mu_f)>0$. This leads to the final expression

$$\frac{F_2^{\rm ns}(x,Q^2)}{\sum_q e_q^2\,x\,\phi_q^{\rm ns}(x,\mu_f)}=|C_V(Q^2,\mu_h)|^2\,U(Q,\mu_h,\mu_i,\mu_f) \times(1-x)^\eta\,\widetilde{j}\Big(\ln\frac{Q^2(1-x)}{\mu_i^2}+\partial_\eta,\mu_i\Big) \times\frac{e^{-\gamma_E\eta}\,\Gamma(1+b(\mu_f))}{\Gamma(1+b(\mu_f)+\eta)}\,. \tag{6}$$

This exact all-order result is independent of the scales μ_h and μ_i, at which the matching coefficient C_V and the associated jet function $\widetilde{j}$ are calculated. The answer simplifies further if we choose the "natural" values $\mu_h=Q$ and $\mu_i=Q\sqrt{1-x}$ (for fixed x). In practical calculations the residual dependence on the matching scales introduced by the truncation of the perturbative expansions of the various objects can be used as an estimator of yet unknown higher-order corrections.

In (6) we have accomplished the resummation of threshold logarithms for F_2 directly in momentum space. The resulting formula is much simpler than corresponding expressions in the literature (see e.g. [16]), which require complicated integrations. The fact that the final answer is a convolution (rather than a product) of a hard-scattering kernel with the parton distribution function is reflected via a non-trivial dependence on the hadronic parameter b describing the large-ξ behavior of $\phi_q^{\rm ns}$. The right-hand side of (6) can be evaluated at any desired order in resummed perturbation theory. Using currently available results, it is possible to include terms at NNLO [1], which is equivalent to the so-called next-to-next-to-next-to-leading double-logarithmic ($\rm N^3LL$) approximation. The resummation is under perturbative control as long as $(1-x)\gg\Lambda_{\rm QCD}^2/Q^2$, since only then the intermediate matching scale $\mu_i\sim Q\sqrt{1-x}$ is a short-distance scale. While the theoretical description thus breaks down very close to the endpoint, we note that weighted integrals of the structure function over an interval $x_0\le x\le1$ can be calculated starting from (6) as long as $Q\sqrt{1-x_0}$ is in the short-distance domain.

It is instructive to compare our result (6) with the conventional approach to threshold resummation in DIS, which proceeds via moment space [2,3]. One defines

$$F_{2,N}^{\rm ns}(Q^2)=\int_0^1 dx\,x^{N-1}F_2^{\rm ns}(x,Q^2) =C_N(Q^2,\mu_f)\,\sum_q e_q^2\,\phi_{q,N}^{\rm ns}(\mu_f)\,,$$

where $\phi_{q,N}^{\rm ns}$ are moments of the parton distribution function. For large values of N the integral is dominated by the endpoint region $(1-x)\sim1/N$. The short-distance coefficient C_N is decomposed as

$$C_N(Q^2,\mu_f)=g_0(Q^2,\mu_f)\,\exp\left[G_N(Q^2,\mu_f)\right]\,,$$

where the prefactor g_0 collects all N-independent terms, and the exponent is written in the form (see [16] for the most up-to-date discussion)

$$G_N(Q^2,\mu_f)=\int_0^1 dz\,\frac{z^{N-1}-1}{1-z} \times\left[\int_{\mu_f^2}^{(1-z)Q^2}\frac{dk^2}{k^2}\,A_q(\alpha_s(k))+B_q\big(\alpha_s(Q\sqrt{1-z})\big)\right]. \tag{7}$$

The resummation for the momentum-space structure function $F_2(x,Q^2)$ itself is obtained from that for the moments $F_{2,N}(Q^2)$ by an inverse Mellin transformation. It is possible to show (see [1] for details) that the outcome of this procedure is equivalent, at any finite order in the perturbative expansion, to the result (6) derived from effective field theory, provided we identify $A_q(\alpha_s)=\Gamma_{\rm cusp}(\alpha_s)$ and

$$\left(1+\frac{\pi^2}{12}\,\nabla^2+\dots\right)B_q(\alpha_s)=\gamma^J(\alpha_s)+\nabla\,\ln\widetilde{j}(0,\mu) -\left(\frac{\pi^2}{12}\,\nabla-\frac{\zeta_3}{3}\,\nabla^2+\dots\right)\Gamma_{\rm cusp}(\alpha_s)\,,$$

where $\nabla=d/d\ln\mu^2$. It follows from this relation that the quantities B_q and γ^J agree at first order in α_s (as observed in [6]), but they differ starting from two-loop order.

There are a few unpleasant features of the conventional approach which are worth pointing out. First, note that

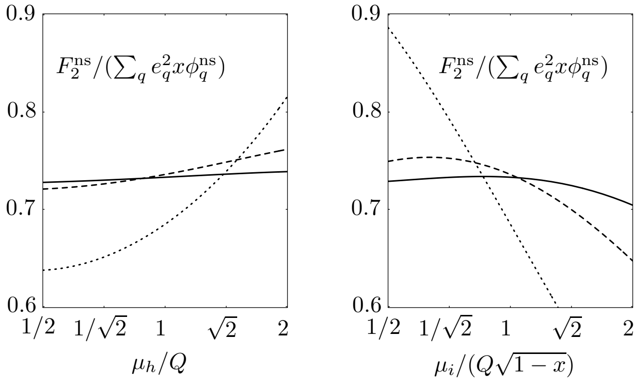

FIG. 1. Dependence of the resummed result for $F_2^{\rm ns}(x,Q^2)$ on the hard (left) and intermediate (right) matching scales, at different orders in RG-improved perturbation theory: LO (dotted), NLO (dashed), and NNLO (solid). We use $Q = 30\,\text{GeV}$, $x = 0.9$, $\mu_f = 5\,\text{GeV}$, and $b(\mu_f) = 4$.

the integrals over the functions A_q and B_q in (7) run over the Landau pole of the running coupling $\alpha_s(\mu)$, introducing an infrared renormalon ambiguity of order $\Lambda^2_{\rm QCD}/M_X^2$. No such problem arises for the integrals (3) in our approach. The particular integral representation of the solution (7) results from the fact that in the conventional approach one solves a set of complicated partial differential equations derived by diagramatic methods instead of the RG evolution equations in SCET [2,3]. Secondly, when performing the inverse Mellin transform one needs to integrate the function G_N over N along a contour parallel to the imaginary axis. This integration involves arbitrarily small physical scales $|k^2| \sim Q^2/|N|$, leading to a second encounter with the Landau pole. Different prescriptions have been proposed in the literature for how to deal with this problem. In our approach integrals over the Landau pole never arise, because factorization and resummation are directly performed in momentum space. The presence of these singularities in the conventional approach is simply an artifact of the way the resummation of large logarithms is implemented. This also implies that there is no physics contained in the Landau-pole singularities of expressions such as (7), and hence it is questionable to what extent it is meaningful to build a "renormalon phenomenology" based on these singularities (see e.g. [17]).

Figure 1 shows the scale dependence of our result for $F_2^{\rm ns}$ obtained with $Q = 30\,\text{GeV}$, $x = 0.9$, $\mu_f = 5\,\text{GeV}$, and $n_f = 5$ light flavors. Varying the matching scales μ_h and μ_i about their default values, we observe that the residual scale dependence is strongly reduced when going to successively higher orders in perturbation theory. In the literature the matching scales are often held fixed, and the perturbative uncertainties are estimated by comparing results at different orders in the expansion. It is evident from the plots that this would underestimate the theoretical uncertainties.

V. CONCLUSIONS

Using methods of effective field theory we have introduced a new approach to resummation of large logarithms for applications of perturbative QCD in which Sudakov logarithms arise. Factorization and resummation are performed directly in momentum space, such that the resulting formulae are free of unphysical infrared sensitivities and provide a transparent separation of the different scales in the problem. As an application we have derived an exact all-order expression for the resummed deep-inelastic structure function F_2, which is much simpler than corresponding results found in the literature.

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