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Subleading Shape Functions in Inclusive B Decays

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Abstract

The contributions of subleading shape functions to inclusive decay distributions of B mesons are derived from a systematic two-step matching of QCD current correlators onto soft-collinear and heavy-quark effective theory. At tree-level, the results can be expressed in terms of forward matrix elements of bi-local light-cone operators. Four-quark operators, which arise at $O(\alpha_s)$, are included. Their effects can be absorbed entirely into a redefinition of other shape functions. Our results are in disagreement with some previous studies of subleading shape-function effects. A numerical analysis of $\bar{B} \to X_u\, l^- \bar{\nu}$ decay distributions suggests that power corrections are small, with the possible exception of the endpoint region of the charged-lepton energy spectrum.

1 Introduction

Inclusive decays of B mesons into final states containing light particles, such as $\bar{B} \to X_u\, l^- \bar{\nu}$ and $\bar{B} \to X_s \gamma$, play a prominent role in the extraction of the Cabbibo-Kobayashi-Maskawa (CKM) matrix element $|V_{ub}|$ and in searches for physics beyond the Standard Model. Necessary experimental cuts in the analysis of these processes restrict the hadronic final state to have large energy, $E_X \sim m_B$, but only moderate invariant mass, $M_X \sim \sqrt{m_B \Lambda_{\rm QCD}}$. In this region of phase space, the inclusive rates can be calculated using a twist expansion, which resums infinite sets of power corrections into non-perturbative shape functions [1, 2, 3].

The impressive performance of the BaBar, Belle, and CLEO experiments demands a continuous effort to reduce the theoretical uncertainties in this approach. Recently, significant progress has been made by systematically incorporating higher-order perturbative corrections [4, 5, 6]. A careful estimate of the theoretical uncertainty on $|V_{ub}|$, extracted using a cut on $P_+ = E_X - |\boldsymbol{P}_X|$, finds an error of 7% due to variations of the leading shape function [7], which can be reduced significantly using information on the $\bar{B} \to X_s \gamma$ photon spectrum. With a 5% relative theoretical error on $|V_{ub}|$, corrections suppressed by a power of $\Lambda_{\rm QCD}/m_b$ are considered the second largest source of uncertainty. The present paper is devoted to a more thorough study of power corrections to inclusive B-meson decay distributions in the shape-function region, using the two-step matching procedure developed in [4, 5, 6]. Subleading shape functions have been investigated first by Bauer, Luke, and Mannel [8], and their effects on various inclusive spectra have been analyzed by several groups [9, 10, 11, 12]. We disagree with the findings of [8, 10] in some important aspects.

The hadronic physics in the inclusive semileptonic decay $\bar{B} \to X_u\, l^- \bar{\nu}$ is encoded in a hadronic tensor $W^{\mu\nu}$ defined via the discontinuity of the forward B-meson matrix element of a correlator of two flavor-changing weak currents $J^\mu = \bar{u}\, \gamma^\mu (1-\gamma_5)\, b$. We define

$$W^{\mu\nu} = \frac{1}{\pi}\, \mathrm{Im}\, \frac{\langle \bar{B}(v)|\, T^{\mu\nu}\, |\bar{B}(v)\rangle}{2m_B}\,, \qquad T^{\mu\nu} = i \int d^4x\, e^{iq\cdot x}\, T\{ J^{\dagger\mu}(0)\, J^\nu(x)\}\,, \tag{1}$$

where v is the B-meson velocity and q the momentum carried by the lepton pair. To be slightly more general, we will consider QCD currents of the type $J_i^\dagger = \bar{b}\, \Gamma_i\, q$ and $J_j = \bar{q}\, \Gamma_j\, b$, where q is a massless quark, and Γ_i, Γ_j are arbitrary Dirac matrices. The corresponding current correlator T_{ij} and hadronic tensor W_{ij} can then also be applied to study the contribution of the dipole operator $Q_{7\gamma}$ to $\bar{B} \to X_s \gamma$ decay.

The current correlator T_{ij} receives contributions from different energy scales, which can be disentangled using effective field theories. Besides the hard scale m_b and the hadronic scale $\Lambda_{\rm QCD}$, an intermediate "hard-collinear" scale $\sqrt{m_b \Lambda_{\rm QCD}}$ set by the typical invariant mass of the hadronic final state is of relevance. Because quantum fluctuations associated with the hard and hard-collinear scales can be treated in perturbation theory, the hadronic tensor trivially factorizes into products of hard functions ($\mu \sim m_b$), jet functions ($\mu \sim \sqrt{m_b \Lambda_{\rm QCD}}$), and shape functions defined in terms of B-meson matrix elements of quark-gluon operators. In a first step, hard fluctuations are integrated out by matching the QCD currents onto soft-collinear effective theory (SCET) [13, 14]. The current correlator is then expanded in terms of light-cone operators in heavy-quark effective theory (HQET) [15], thereby integrating out fluctuations at the hard-collinear scale. The fact that the relevant HQET operators live on the light cone follows from the structure of the multipole expansion of soft fields in SCET. In the present work, we carry out the matching procedure at tree level and to order $\Lambda_{\rm QCD}/m_b$ in the heavy-quark expansion. We indicate how our results would change if loop corrections were included.

2 Short-distance expansion of the hadronic tensor

We begin by recalling some elements of SCET, referring to [14] for a more detailed discussion (see also [13]). We work in a reference frame where the B meson is at rest, $v^\mu = (1,0,0,0)$, and where the lepton 3-momentum $\boldsymbol{q}$ points in the negative z-direction. We introduce two light-like vectors $n^\mu = (1,0,0,1)$ and $\bar n^\mu = (1,0,0,-1)$, with $n\cdot\bar n = 2$, $n\cdot v = \bar n\cdot v = 1$, and $\bar n = 2v - n$. Any 4-vector a^μ can be decomposed as

$$a^\mu = \bar n\cdot a\,\frac{n^\mu}{2} + n\cdot a\,\frac{\bar n^\mu}{2} + a_\perp^\mu \equiv a_-^\mu + a_+^\mu + a_\perp^\mu\,. \tag{2}$$

In this basis, $v_\perp = 0$ and $q_\perp = 0$ by construction. In the kinematic region of interest, the hadronic jet has a hard-collinear momentum, which scales like $(P_-, P_+, P_\perp) \sim m_b(1,\lambda,\sqrt{\lambda})$, where $\lambda \sim \Lambda_{\rm QCD}/m_b$. The momenta of the soft, light constituents of the B meson scale like $p_s^\mu \sim \Lambda_{\rm QCD} \sim m_b\lambda$.

SCET is the appropriate effective field theory for the description of the interactions among soft and hard-collinear degrees of freedom. Its Lagrangian is organized in an expansion in powers of $\sqrt{\lambda}$. The leading-order Lagrangian is

$$\begin{aligned}\mathcal{L}_{\rm SCET}^{(0)} &= \bar\xi\,\frac{\not{\bar n}}{2}\left(in\cdot D_{hc} + gn\cdot A_s(x_-) + i\not{D}_{\perp hc}\,\frac{1}{i\bar n\cdot D_{hc}}\,i\not{D}_{\perp hc}\right)\xi \\ &\quad + \bar q\, i\not{D}_s\, q + \bar h\, iv\cdot D_s\, h + \mathcal{L}_{\rm YM}^{(0)}\,,\end{aligned} \tag{3}$$

where ξ is a hard-collinear quark field, q is a soft, massless quark field, h is a heavy-quark field defined in HQET, A_s is a soft gluon field, and $iD_{hc}^\mu = i\partial^\mu + gA_{hc}^\mu$ is the covariant derivative containing a hard-collinear gluon field. All fields in the above Lagrangian are evaluated at point x, except for the soft gluon field in the first term, which is evaluated at $x_- = \frac12\,(\bar n\cdot x)\,n$. The explicit form of the leading-order Yang-Mills Lagrangian can be found in [14].

The terms up to second order in the expansion in $\sqrt{\lambda}$ are

$$\begin{aligned}\mathcal{L}_{\rm SCET}^{(1)} &= \mathcal{L}_\xi^{(1)} + \mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_{\rm YM}^{(1)}\,, \\ \mathcal{L}_{\rm SCET}^{(2)} &= \mathcal{L}_\xi^{(2)} + \mathcal{L}_{\xi q}^{(2)} + \mathcal{L}_h^{(2)} + \mathcal{L}_{\rm YM}^{(2)}\,,\end{aligned} \tag{4}$$

where

$$\mathcal{L}_h^{(2)} = \frac{1}{2m_b}\left[\bar h\,(iD_s)^2 h + \frac{C_{\rm mag}}{2}\,\bar h\,\sigma_{\mu\nu}\,gG_s^{\mu\nu} h\right] \tag{5}$$

is the next-to-leading term in the expansion of the HQET Lagrangian [15]. Expressions for the remaining Lagrangian corrections have been presented in [14].

While it is consistent to apply a perturbative expansion at the hard and hard-collinear scales, the soft-gluon couplings to hard-collinear fields are non-perturbative and must be treated to all orders in the coupling constant. For instance, the hard-collinear quark propagator derived from (3) should be taken to be the propagator in the background of the soft gluon field, summing up arbitrarily many insertions of the field A_s. The most convenient way of achieving this summation is to decouple the leading-order interactions between the soft gluon field and hard-collinear fields in (3) with the help of a field redefinition, under which [13, 16]

$$\xi(x) = S(x_-)\,\xi^{(0)}(x)\,, \qquad A_{hc}^\mu(x) = S(x_-)\,A_{hc}^{(0)\mu}(x)\,S^\dagger(x_-)\,, \tag{6}$$

where

$$S(x) = P\exp\left(ig\int\limits_{-\infty}^{0} dt\,n\cdot A_s(x+tn)\right) \tag{7}$$

is a soft Wilson line along the n direction. Introducing the new fields into the Lagrangian yields

$$\mathcal{L}_\xi^{(0)} = \bar\xi^{(0)}\,\frac{\not{\bar n}}{2}\left(in\cdot D_{hc}^{(0)} + i\not{D}_{\perp hc}^{(0)}\,\frac{1}{i\bar n\cdot D_{hc}^{(0)}}\,i\not{D}_{\perp hc}^{(0)}\right)\xi^{(0)}\,, \tag{8}$$

and similarly all interactions between soft and hard-collinear gluon fields are removed from the Yang-Mills Lagrangian $\mathcal{L}_{\rm YM}^{(0)}$. The propagator of the new hard-collinear quark field is now given by the simple expression

$$\Delta_\xi(x-y) = \langle 0|\,T\{\xi^{(0)}(x)\,\bar\xi^{(0)}(y)\}\,|0\rangle = \frac{\not{n}}{2}\int\frac{d^4p}{(2\pi)^4}\,\frac{i\bar n\cdot p}{p^2+i\epsilon}\,e^{-ip\cdot(x-y)}\,. \tag{9}$$

The effect of soft-gluon attachments is taken into account by factors of S in the results below.

We now list the expressions for the subleading corrections to the SCET Lagrangian in terms of the redefined fields, using the formalism of gauge-invariant building blocks [17]. We define

$$\mathfrak{X} = W^\dagger\xi^{(0)}\,, \qquad \mathcal{A}_{hc}^\mu = W^\dagger(iD_{hc}^{(0)\mu}W)\,, \tag{10}$$

where

$$W = P\exp\left(ig\int\limits_{-\infty}^{0} dt\,\bar n\cdot A_{hc}^{(0)}(x+t\bar n)\right) \tag{11}$$

is a hard-collinear Wilson line. These "calligraphic" fields are invariant under both hard-collinear and soft gauge transformations. Note that in the light-cone gauge, $\bar n\cdot A_{hc}^{(0)}=0$, we simply have $\mathfrak{X}=\xi$ and $\mathcal{A}_{hc}^\mu = A_{hc}^{(0)\mu}$. In terms of these fields, the results compiled in [14] take the form

$$\mathcal{L}_\xi^{(1)} = \bar{\mathfrak{X}}\,\frac{\not{\bar n}}{2}\,x_\perp^\mu n^\nu\left(S^\dagger gG_{\mu\nu}S\right)_{x_-}\mathfrak{X}\,,$$

$$
\begin{aligned}
\mathcal{L}_{\xi}^{(2)} &= \bar{\mathfrak{X}}\,\frac{\not{\bar{n}}}{2}\left(\frac{n\cdot x}{2}\,\bar{n}^{\mu}n^{\nu}\left(S^{\dagger}gG_{\mu\nu}S\right)_{x_-}+\frac{x_{\perp}^{\mu}x_{\perp}^{\rho}}{2}\,n^{\nu}\left(S^{\dagger}[D_{\rho},gG_{\mu\nu}]S\right)_{x_-}\right)\mathfrak{X}\\
&\quad+\bar{\mathfrak{X}}\,\frac{\not{\bar{n}}}{2}\left(i\not{\mathcal{D}}_{\perp hc}\,\frac{1}{i\bar{n}\cdot\partial}\,\frac{x_{\perp}^{\mu}}{2}\,\gamma_{\perp}^{\nu}\left(S^{\dagger}gG_{\mu\nu}S\right)_{x_-}+\frac{x_{\perp}^{\mu}}{2}\,\gamma_{\perp}^{\nu}\left(S^{\dagger}gG_{\mu\nu}S\right)_{x_-}\,\frac{1}{i\bar{n}\cdot\partial}\,i\not{\mathcal{D}}_{\perp hc}\right)\mathfrak{X}\,,\\
\mathcal{L}_{\xi q}^{(1)} &= \left(\bar{q}S\right)_{x_-}i\not{\mathcal{D}}_{\perp hc}\,\mathfrak{X}+\text{h.c.}\,,
\end{aligned}
\qquad (12)
$$

where $i\mathcal{D}_{hc}^{\mu}=i\partial^{\mu}+\mathcal{A}_{hc}^{\mu}$, and we have dropped the subscript "s" on the soft covariant derivative and field strength. The expression for $\mathcal{L}_{\xi q}^{(2)}$ will not be needed for our analysis. The notation $(\dots)_{x_-}$ indicates that, in interactions with hard-collinear fields, soft fields are multipole expanded and live at position x_-, whereas hard-collinear fields are always evaluated at position x. Because the SCET Lagrangian is not renormalized [14], the above expressions remain valid to all orders in perturbation theory.

Next, we need the expressions for heavy-light current operators in SCET. In general, a QCD current $\bar{q}\,\Gamma\,b$ matches onto

$$
\bar{q}(x)\,\Gamma\,b(x)=e^{-im_b v\cdot x}\left(J_A^{(0)}+J_A^{(1)}+J_A^{(2)}+J_B^{(1)}+J_B^{(2)}+\dots\right), \qquad (13)
$$

where we distinguish between type-A "two-particle" operators and type-B "three-particle" operators [18]. The operators arising at tree level are [14, 18, 19]

$$
\begin{aligned}
J_A^{(0)} &= \bar{\mathfrak{X}}\,\Gamma\left(S^{\dagger}h\right)_{x_-}\,,\\
J_A^{(1)} &= \bar{\mathfrak{X}}\,\Gamma\,x_{\perp}^{\mu}\left(S^{\dagger}D_{\mu}h\right)_{x_-}+\bar{\mathfrak{X}}\,\frac{\not{\bar{n}}}{2}\,i\overleftarrow{\not{\partial}}_{\perp}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma\left(S^{\dagger}h\right)_{x_-}\,,\\
J_A^{(2)} &= \bar{\mathfrak{X}}\,\Gamma\left[\frac{n\cdot x}{2}\left(S^{\dagger}\bar{n}\cdot Dh\right)_{x_-}+\frac{x_{\perp}^{\mu}x_{\perp}^{\nu}}{2}\left(S^{\dagger}D_{\mu}D_{\nu}h\right)_{x_-}+\left(S^{\dagger}\,\frac{i\not{D}}{2m_b}\,h\right)_{x_-}\right]\\
&\quad+\bar{\mathfrak{X}}\,\frac{\not{\bar{n}}}{2}\,i\overleftarrow{\not{\partial}}_{\perp}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma\,x_{\perp}^{\mu}\left(S^{\dagger}D_{\mu}h\right)_{x_-}\,,
\end{aligned}
\qquad (14)
$$

and

$$
\begin{aligned}
J_B^{(1)} &= -\,\bar{\mathfrak{X}}\,\frac{\not{\bar{n}}}{2}\,\not{\mathcal{A}}_{\perp hc}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma\left(S^{\dagger}h\right)_{x_-}-\bar{\mathfrak{X}}\,\Gamma\,\frac{\not{n}}{2m_b}\,\not{\mathcal{A}}_{\perp hc}\left(S^{\dagger}h\right)_{x_-}\,,\\
J_B^{(2)} &= -\,\bar{\mathfrak{X}}\,\Gamma\left(\frac{1}{i\bar{n}\cdot\partial}+\frac{\not{n}}{2m_b}\right)n\cdot\mathcal{A}_{hc}\left(S^{\dagger}h\right)_{x_-}\\
&\quad-\bar{\mathfrak{X}}\left(\frac{\not{\bar{n}}}{2}\,\not{\mathcal{A}}_{\perp hc}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\Gamma+\Gamma\,\frac{\not{n}}{2m_b}\,\not{\mathcal{A}}_{\perp hc}\right)x_{\perp}^{\mu}\left(S^{\dagger}D_{\mu}h\right)_{x_-}\\
&\quad-\bar{\mathfrak{X}}\,\Gamma\,\frac{1}{i\bar{n}\cdot\partial}\,\frac{(i\not{\mathcal{D}}_{\perp hc}\not{\mathcal{A}}_{\perp hc})}{m_b}\left(S^{\dagger}h\right)_{x_-}+\bar{\mathfrak{X}}\,\frac{i\overleftarrow{\not{\mathcal{D}}}_{\perp hc}}{m_b}\,\frac{1}{i\bar{n}\cdot\overleftarrow{\partial}}\,\frac{\not{\bar{n}}}{2}\,\Gamma\,\frac{\not{n}}{2}\,\not{\mathcal{A}}_{\perp hc}\left(S^{\dagger}h\right)_{x_-}\,.
\end{aligned}
\qquad (15)
$$

The expressions for the currents beyond tree level are more complicated, primarily because several new Dirac structures appear. The relevant formulae are known at leading [13] and next-to-leading order [18] in the power expansion. The corresponding results for the currents $J_A^{(2)}$ and $J_B^{(2)}$ have not yet been derived. They would be needed if the analysis in this paper should be extended beyond tree level.

If perturbative corrections at the hard-collinear scale are neglected, the hard-collinear gluon fields can be dropped, and the above expressions for the effective Lagrangians and currents simplify. In this approximation $\mathfrak{X} \to \xi^{(0)}$, $\mathcal{A}_{hc}^\mu \to 0$, $i\mathcal{D}_{hc}^\mu \to i\partial^\mu$, and $J_B^{(n)} \to 0$. While this leads to great simplifications in the calculation, we stress that the structures of soft fields that arise do not simplify. We find operators containing $S^\dagger gG_{\mu\nu}S$, $S^\dagger[D_\rho, gG_{\mu\nu}]S$, $S^\dagger h$, $S^\dagger D_\mu h$, and $S^\dagger D_\mu D_\nu h$, and the same operators would arise if the calculation was extended beyond the tree approximation. The only exception is that we no longer retain Lagrangian corrections containing the soft quark field, because $\mathcal{L}_{\xi q}^{(1)} \to 0$ in the limit where the hard-collinear gluon field is neglected. (The term $\bar{q}S\, i\not{\partial}_\perp \xi^{(0)}$ is forbidden by momentum conservation.) In Section 5, we analyze the subleading shape functions introduced at $O(\alpha_s)$ by two insertions of $\mathcal{L}_{\xi q}^{(1)}$.

With all the definitions in place, we are now ready to evaluate the current correlator T_{ij} including terms of up to second order in $\sqrt{\lambda}$, working at tree level in α_s at the hard and hard-collinear scales. The leading term is readily found to be

$$T_{ij}^{(0)} = -\int d^4x\, e^{i(q-m_b v)\cdot x} \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\, \frac{\bar{n}\cdot p}{p^2 + i\epsilon}\, \left(\bar{h}S\right)_0 \Gamma_i \frac{\not{n}}{2} \Gamma_j \left(S^\dagger h\right)_{x_-} . \tag{16}$$

First-order corrections in $\sqrt{\lambda}$ vanish by rotational invariance in the transverse plane (provided we choose the coordinate system such that $v_\perp = 0$ and $q_\perp = 0$), i.e., $T_{ij}^{(1)} = 0$. At second order in the expansion, the correlator receives several contributions, which can be represented symbolically as

$$J^{\dagger(2)}\, J^{(0)}\,, \quad J^{\dagger(1)}\, J^{(1)}\,, \quad J^{\dagger(0)}\, J^{(2)}\,, \tag{17}$$

$$J^{\dagger(1)}\, J^{(0)} \int d^4z\, \mathcal{L}_\xi^{(1)}\,, \quad J^{\dagger(0)}\, J^{(1)} \int d^4z\, \mathcal{L}_\xi^{(1)}\,, \quad J^{\dagger(0)}\, J^{(0)} \int d^4z \left[\mathcal{L}_\xi^{(2)} + \mathcal{L}_h^{(2)}\right], \tag{18}$$

and

$$J^{\dagger(0)}\, J^{(0)} \int d^4z\, \mathcal{L}_\xi^{(1)} \int d^4w\, \mathcal{L}_\xi^{(1)}\,, \tag{19}$$

where at tree level only the type-A current operators appear. Examples of these time-ordered products are depicted in Figure 1. At $O(\alpha_s)$, one must include a tree-level contribution of the form

$$J^{\dagger(0)}\, J^{(0)} \int d^4z\, \mathcal{L}_{\xi q}^{(1)} \int d^4w\, \mathcal{L}_{\xi q}^{(1)}\,, \tag{20}$$

shown by the last diagram in the figure. Beyond tree level, one would also have to include the contributions from the type-B current operators.

Because the external B-meson states in the definition of the hadronic tensor contain only soft constituents, and because the hard-collinear fields have been decoupled from the soft fields in the leading-order SCET Lagrangian (8), it is possible to contract all hard-collinear fields in the time-ordered products. At tree level, all we need is the hard-collinear quark propagator (9). Derivatives acting on hard-collinear fields give powers of p in momentum space, whereas components of x^μ appearing in the multipole-expanded expressions for the effective

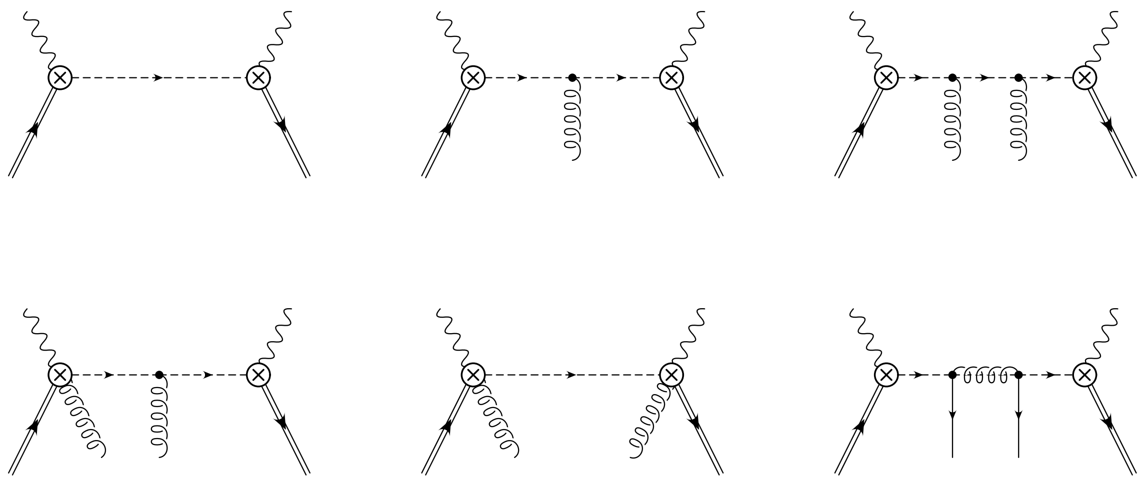

Figure 1: Representative examples of time-ordered products contributing to the current correlator T_{ij} in SCET. Double lines show heavy-quark fields, dashed ones hard-collinear fields, and wavy lines denote the external currents. Lagrangian insertions and higher-order effective current operators are exemplified by soft gluons.

Lagrangians and currents can be turned into derivatives $\partial/\partial p$ acting on the momentum-space amplitudes. Each insertion of a SCET Lagrangian correction introduces an integral over soft fields located along the n light-cone. Consequently, the time-ordered products in (17) lead to expressions involving bi-local operators as in (16), while those in (18), (19), and (20) also lead to tri- and quadri-local operators. At first sight, this would seem to require the introduction of complicated subleading shape functions depending on up to three momentum variables ω_i, defined in terms of the Fourier transforms of the matrix elements of the non-local operators. However, the non-localities can be reduced using partial-fraction identities for the resulting hard-collinear quark propagators. At tree level, it suffices to define shape functions of a single variable ω.

To see how this works, consider the effect of an insertion of the Lagrangian $\mathcal{L}_\xi^{(1)}$ in (12). Since all hard-collinear fields must be contracted, we may consider without loss of generality the expression

$$L(y_1, y_2) = i \int d^4z\, \Delta_\xi(y_1 - z)\, \frac{\not{\bar{n}}}{2}\, z_\perp^\mu\, n^\nu \left(S^\dagger g G_{\mu\nu} S\right)_{z_-} \Delta_\xi(z - y_2)\,. \tag{21}$$

The fact that the soft fields live at position z_- implies that the two hard-collinear quark propagators carry the same momentum components $\bar{n}\cdot p$ and $p_\perp$. In analogy with the definition of the hard-collinear calligraphic gluon field, we now introduce the soft field [17]

$$\mathcal{A}_{s\mu}(x) = S^\dagger(x)\,(iD_\mu S)(x) = -\int\limits_{-\infty}^{0} dt\, n^\nu \left(S^\dagger g G_{\mu\nu} S\right)(x + tn)\,, \tag{22}$$

which allows us to write $n^\nu\,(S^\dagger g G_{\mu\nu} S)_{z_-}$ as a derivative, $-n\cdot\partial_z\,\mathcal{A}_{s\mu}(z_-)$. Integrating by parts in (21), we find

$$L(y_1,y_2) = i\int d^4z\,\Delta_\xi(y_1-z)\,\frac{\not\bar{n}}{2}\,z_\perp^\mu\,\mathcal{A}_{s\mu}(z_-)\,n\cdot\partial_z\,\Delta_\xi(z-y_2)$$
$$+\,i\int d^4z\,[n\cdot\partial_z\,\Delta_\xi(y_1-z)]\,\frac{\not\bar{n}}{2}\,z_\perp^\mu\,\mathcal{A}_{s\mu}(z_-)\,\Delta_\xi(z-y_2). \tag{23}$$

The hard-collinear propagator is a Green's function obeying the differential equation

$$\left(n\cdot\partial+\frac{\partial_\perp^2}{\bar{n}\cdot\partial}\right)\Delta_\xi(x)=\frac{\not\bar{n}}{2}\,\delta^{(4)}(x)\,. \tag{24}$$

This allows us to write

$$L(y_1,y_2) = i\Big[y_{2\perp}\cdot\mathcal{A}_s(y_{2-})-y_{1\perp}\cdot\mathcal{A}_s(y_{1-})\Big]\,\Delta_\xi(y_1-y_2)$$
$$-\,i\int d^4z\,\Delta_\xi(y_1-z)\,\frac{\not\bar{n}}{2}\,z_\perp\cdot\mathcal{A}_s(z_-)\,\frac{\partial_{z\perp}^2}{\bar{n}\cdot\partial_z}\,\Delta_\xi(z-y_2)$$
$$-\,i\int d^4z\left[\frac{\partial_{z\perp}^2}{\bar{n}\cdot\partial_z}\,\Delta_\xi(y_1-z)\right]\frac{\not\bar{n}}{2}\,z_\perp\cdot\mathcal{A}_s(z_-)\,\Delta_\xi(z-y_2). \tag{25}$$

At tree level, we find that the terms involving transverse derivatives vanish, because they provide powers of transverse momenta of external lines, which are zero (recall that $v_\perp=0$ and $q_\perp=0$). This leaves the terms shown in the first line, in which the z integral has been eliminated, and in which the product of two propagators has been reduced to a single propagator. In momentum space, this corresponds to the partial-fraction identity

$$\frac{1}{\bar{n}\cdot p\,n\cdot p+p_\perp^2}\,\frac{1}{\bar{n}\cdot p\,(n\cdot p+\omega)+p_\perp^2}=\frac{1}{\bar{n}\cdot p\,\omega}\left[\frac{1}{\bar{n}\cdot p\,n\cdot p+p_\perp^2}-\frac{1}{\bar{n}\cdot p\,(n\cdot p+\omega)+p_\perp^2}\right], \tag{26}$$

valid for two momenta that differ only in their $n\cdot p$ components.

In the sum of all terms many cancellations and simplifications take place, and we find the rather simple result

$$T_{ij}^{(2)}=-\int d^4x\,e^{i(q-m_bv)\cdot x}\int\frac{d^4p}{(2\pi)^4}\,e^{ip\cdot x}\,\frac{\bar{n}\cdot p}{p^2+i\epsilon}\,\sum_{n=1}^{4}O_n(x_-)\,, \tag{27}$$

where

$$O_1(x_-)=i\int d^4z\,T\{\big(\bar{h}S\big)_0\,\Gamma_i\,\frac{\not n}{2}\,\Gamma_j\,\big(S^\dagger h\big)_{x_-}\,\mathcal{L}_h^{(2)}(z)\}\,,$$
$$O_2(x_-)=\frac{1}{2m_b}\left[\big(\bar{h}S\big)_0\,\Gamma_i\,\frac{\not n}{2}\,\Gamma_j\,\big(S^\dagger i\not D h\big)_{x_-}+\Big(\bar{h}(-i\overleftarrow{\not D})S\Big)_0\,\Gamma_i\,\frac{\not n}{2}\,\Gamma_j\,\big(S^\dagger h\big)_{x_-}\right],$$
$$O_3(x_-)=\frac{1}{\bar{n}\cdot p}\left[\big(\bar{h}S\big)_0\,\Gamma_i\,\frac{\not n\not\bar{n}}{4}\,\gamma_\mu\,\Gamma_j\,\big(S^\dagger iD_\perp^\mu h\big)_{x_-}+\big(\bar{h}iD_\perp^\mu S\big)_0\,\Gamma_i\,\gamma_\mu\,\frac{\not\bar{n}\not n}{4}\,\Gamma_j\,\big(S^\dagger h\big)_{x_-}\right],$$
$$O_4(x_-)=\frac{i}{\bar{n}\cdot p}\int_0^{\bar{n}\cdot x/2}dt\;\big(\bar{h}S\big)_0\,\Gamma_i\,\Big(S^\dagger i\not D_\perp\frac{\not n}{2}\,i\not D_\perp S\Big)_{tn}\,\Gamma_j\,\big(S^\dagger h\big)_{x_-}\,. \tag{28}$$

In deriving these expressions, we have made frequent use of the identity $iD^\mu = S(i\partial^\mu + \mathcal{A}_s^\mu)S^\dagger$ with $\mathcal{A}_s^\mu$ as defined in (22). Note that on the space of forward matrix elements the result for $T_{ij}^{(2)}$ is hermitean, because we can integrate by parts and use translational invariance.

In the last step, we can simplify the operator O_2 by noting that the HQET equation of motion, $iv\cdot Dh = 0$, along with $\not{v}h = h$, implies

$$S^\dagger i\not{D}h = S^\dagger i\not{D}_\perp h + (\not{v} - \not{n})\, in\cdot\partial\left(S^\dagger h\right). \tag{29}$$

It follows that all gauge-covariant derivatives of the heavy-quark fields are perpendicular derivatives. This fact restricts the number of subleading shape functions.

3 Definition of subleading shape functions

The Dirac structure of the operators O_n can be simplified noting that the heavy-quark fields h of HQET are two-component spinor fields, so that between $\bar{h}\dots h$ the Dirac basis collapses to a set of four basis matrices $(\mathbf{1}, \boldsymbol{\sigma})$. In four-component notation, these are the upper-left 2×2 blocks of $(\mathbf{1}, \boldsymbol{\gamma}\gamma_5)$ (in the Dirac representation). Instead of $\boldsymbol{\gamma}\gamma_5$, we are free to take the matrices $\gamma_\perp^\mu\gamma_5$ and $\not{n}\gamma_5$. It follows that, between the $P_v = \frac{1}{2}(1+\not{v})$ projectors supplied by the heavy-quark fields, any Dirac matrix Γ can be decomposed as

$$\Gamma \to \frac{1}{2}\,\mathrm{tr}\left(\Gamma\, P_v\right)\mathbf{1} - \frac{1}{2}\,\mathrm{tr}\left[\Gamma\, P_v\,(\not{v} - \not{n})\,\gamma_5\right]\not{n}\gamma_5 - \frac{1}{2}\,\mathrm{tr}\left(\Gamma\, P_v\,\gamma_{\perp\mu}\gamma_5\right)\gamma_\perp^\mu\gamma_5\,. \tag{30}$$

We denote by

$$\langle \bar{h}\dots h\rangle \equiv \frac{\langle \bar{B}(v)|\,\bar{h}\dots h\,|\bar{B}(v)\rangle}{2m_B} \tag{31}$$

the forward B-meson matrix element of any HQET operator. Rotational invariance in the transverse plane implies that transverse indices can only be contracted using the symmetric and anti-symmetric tensors (we set $\epsilon_{0123} = 1$)

$$g_\perp^{\mu\nu} = g^{\mu\nu} - \frac{n^\mu\bar{n}^\nu + n^\nu\bar{n}^\mu}{2}\,, \qquad \epsilon_\perp^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\,v_\alpha n_\beta\,. \tag{32}$$

It follows that the only non-vanishing matrix elements are

$$\begin{aligned}
\langle \left(\bar{h}S\right)_0 \left(S^\dagger h\right)_{x_-}\rangle &= \int d\omega\, e^{-\frac{i}{2}\omega\bar{n}\cdot x}\, S(\omega)\,,\\
\langle i\int d^4z\, T\{\left(\bar{h}S\right)_0 \left(S^\dagger h\right)_{x_-}\mathcal{L}_h^{(2)}(z)\}\rangle &= \frac{1}{m_b}\int d\omega\, e^{-\frac{i}{2}\omega\bar{n}\cdot x}\, s(\omega)\,,\\
\langle \left(\bar{h}S\right)_0 \gamma_\perp^\rho\gamma_5 \left(S^\dagger iD_\perp^\mu h\right)_{x_-}\rangle &= -\frac{i\epsilon_\perp^{\rho\mu}}{2}\int d\omega\, e^{-\frac{i}{2}\omega\bar{n}\cdot x}\, t(\omega)\,,\\
-i\int_0^{\bar{n}\cdot x/2} dt\,\langle \left(\bar{h}S\right)_0 \left(S^\dagger iD_\perp^\mu iD_\perp^\nu S\right)_{tn} \left(S^\dagger h\right)_{x_-}\rangle &= \frac{g_\perp^{\mu\nu}}{2}\int d\omega\, e^{-\frac{i}{2}\omega\bar{n}\cdot x}\, u(\omega)\,,\\
-i\int_0^{\bar{n}\cdot x/2} dt\,\langle \left(\bar{h}S\right)_0 \not{n}\gamma_5 \left(S^\dagger iD_\perp^\mu iD_\perp^\nu S\right)_{tn} \left(S^\dagger h\right)_{x_-}\rangle &= -\frac{i\epsilon_\perp^{\mu\nu}}{2}\int d\omega\, e^{-\frac{i}{2}\omega\bar{n}\cdot x}\, v(\omega)\,.
\end{aligned} \tag{33}$$

If radiative corrections at the hard scale are included, it would be more appropriate to split up $s(\omega) = s_{\text{kin}}(\omega) + C_{\text{mag}}\, s_{\text{mag}}(\omega)$, where C_{mag} is the Wilson coefficient of the chromo-magnetic operator in the subleading HQET Lagrangian. This ensures that the shape functions remain independent of the heavy-quark mass. The definitions of the functions t, u, v are chosen such that

$$
\begin{aligned}
\langle \bar{h}(0)\, \not{n}\, [0, x_-]\, (i\not{D}_\perp h)(x_-) \rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar{n}\cdot x}\, t(\omega)\,, \\
-i \int\limits_0^{\bar{n}\cdot x/2} dt\, \langle \bar{h}(0)\, [0, tn]\, (iD_\perp)^2(tn)\, [tn, x_-]\, h(x_-) \rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar{n}\cdot x}\, u(\omega)\,, \\
-i \int\limits_0^{\bar{n}\cdot x/2} dt\, \langle \bar{h}(0)\, \frac{\not{n}}{2}\, [0, tn]\, \sigma^\perp_{\mu\nu}\, G^{\mu\nu}_\perp(tn)\, [tn, x_-]\, h(x_-) \rangle &= \int d\omega\, e^{-\frac{i}{2}\omega \bar{n}\cdot x}\, v(\omega)\,,
\end{aligned}
\tag{34}
$$

where $[x, y] \equiv S(x)\, S^\dagger(y)$ is a product of two infinite-length soft Wilson lines, which on the light cone (i.e., for $x, y \parallel n$) collapses to a straight Wilson line of finite length connecting x and y. Note that for operators containing derivatives on Wilson lines it is important to define the Wilson lines off the light cone, in which case our definition of $[x, y]$ is *different* from a straight line connecting the two points.

We also need a variation of the first matrix element in (34), in which the derivative is located at position 0. Using hermitean conjugation and translational invariance, we find that

$$
\langle (\bar{h} i\not{D}_\perp)(0)\, \not{n}\, [0, x_-]\, h(x_-) \rangle = \langle \bar{h}(0)\, \not{n}\, [0, x_-]\, (i\not{D}_\perp h)(x_-) \rangle\,,
\tag{35}
$$

implying that all terms containing a single insertion of $D_\perp$ can be related to the function $t(\omega)$. From this relation, it follows that

$$
\begin{aligned}
\int d\omega\, e^{-\frac{i}{2}\omega \bar{n}\cdot x}\, t(\omega) &= \langle \big(\bar{h} S\big)_0\, \frac{\not{n}}{2} \left[\not{A}_{s\perp}(x_-) - \not{A}_{s\perp}(0) \right] \big(S^\dagger h\big)_{x_-} \rangle \\
&= - \int\limits_0^{\bar{n}\cdot x/2} dt\, \langle \bar{h}(0)\, \frac{\not{n}}{2}\, [0, tn]\, \gamma_\perp^\mu n^\nu g G_{\mu\nu}(tn)\, [tn, x_-]\, h(x_-) \rangle\,,
\end{aligned}
\tag{36}
$$

which defines the function $t(\omega)$ in terms of a matrix element of the field-strength tensor.

It is now straightforward to express the forward matrix element of the current correlator T_{ij} in terms of shape functions. The resulting traces of Dirac matrices can be simplified using identities for the $\epsilon_\perp^{\mu\nu}$ tensor derived in [20]. Taking the imaginary part, we obtain for the hadronic tensor

$$
\begin{aligned}
W_{ij}^{(0)} &= \int d\omega\, \delta(n\cdot p + \omega)\, S(\omega)\, T_1\,, \qquad W_{ij}^{(1)} = 0\,, \\
W_{ij}^{(2)} &= \int d\omega\, \delta(n\cdot p + \omega) \left[\frac{\omega\, S(\omega) + t(\omega)}{m_b}\, T_2 + \frac{s(\omega)}{m_b}\, T_1 + \frac{t(\omega)}{\bar{n}\cdot p}\, T_3 + \frac{u(\omega)}{\bar{n}\cdot p}\, T_1 - \frac{v(\omega)}{\bar{n}\cdot p}\, T_4 \right],
\end{aligned}
\tag{37}
$$

where now $p = m_b v - q$, and

$$T_1 = \frac{1}{4}\,\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,\frac{1+\not{v}}{2}\right], \qquad T_3 = \frac{1}{4}\,\mathrm{tr}\left[\Gamma_i\,\gamma_\rho^\perp\gamma_5\,\Gamma_j\,\frac{1+\not{v}}{2}\,\gamma_\perp^\rho\gamma_5\right],$$
$$T_2 = \frac{1}{8}\,\mathrm{tr}\left[\Gamma_i\,\not{n}\,\Gamma_j\,(\not{v}-\not{n})\right], \qquad T_4 = \frac{1}{4}\,\mathrm{tr}\left[\Gamma_i\,\not{n}\gamma_5\,\Gamma_j\,\frac{1+\not{v}}{2}\,(\not{v}-\not{n})\,\gamma_5\right]. \tag{38}$$

It follows from (37) that the subleading shape functions $s(\omega)$ and $u(\omega)$ always come with the same trace as the leading shape function $S(\omega)$. However, $u(\omega)$ is divided by the kinematic variable $\bar{n}\cdot p$, and so it does *not* enter in a universal (i.e., process-independent) combination with $S(\omega)$.

We can now specialize our result to the case of semileptonic decay, for which $\Gamma_i = \gamma^\mu(1-\gamma_5)$ and $\Gamma_j = \gamma^\nu(1-\gamma_5)$. This yields

$$\begin{aligned} W^{\mu\nu} = \int d\omega\,\delta(n\cdot p+\omega)\,\Bigg\{ &\left(n^\mu v^\nu + n^\nu v^\mu - g^{\mu\nu} - i\epsilon^{\mu\nu\alpha\beta}\,n_\alpha v_\beta\right) \\ &\times\left[\left(1+\frac{\omega}{m_b}\right)S(\omega) + \frac{s(\omega)+t(\omega)}{m_b} + \frac{u(\omega)-v(\omega)}{\bar{n}\cdot p}\right] \\ &- 2(n^\mu v^\nu + n^\nu v^\mu)\,\frac{t(\omega)}{\bar{n}\cdot p} + 2n^\mu n^\nu\left[-\frac{\omega\,S(\omega)}{m_b} - \frac{t(\omega)}{m_b} + \frac{t(\omega)+v(\omega)}{\bar{n}\cdot p}\right]\Bigg\}. \end{aligned} \tag{39}$$

Similarly, for the contribution of the dipole operator $Q_{7\gamma}$ to $\bar{B}\to X_s\gamma$ decay, the Dirac structures are (up to prefactors) $\Gamma_i = \frac{1}{4}[\gamma_\perp^\mu, \not{\bar{n}}](1-\gamma_5)$ and $\Gamma_j = \frac{1}{4}[\not{n}, \gamma_\perp^\nu](1+\gamma_5)$, where the indices μ,ν are contracted with the transverse polarization vector of the photon. In this case we obtain

$$W^{\mu\nu} = (i\epsilon_\perp^{\mu\nu} - g_\perp^{\mu\nu})\int d\omega\,\delta(n\cdot p+\omega)\left[\left(1-\frac{\omega}{m_b}\right)S(\omega) + \frac{s(\omega)-t(\omega)}{m_b} + \frac{u(\omega)-v(\omega)}{\bar{n}\cdot p}\right]. \tag{40}$$

We stress, however, that while at leading power in $\Lambda_{\mathrm{QCD}}/m_b$ the dipole operator gives the only tree-level contribution to the $\bar{B}\to X_s\gamma$ decay rate, this is no longer the case when power corrections are included. For instance, interference terms of the dipole operator with current-current operators can lead to new subleading shape functions even at lowest order in perturbation theory. To derive these structures, it would be necessary to match the entire weak effective Hamiltonian for $\bar{B}\to X_s\gamma$ decay onto SCET operators [6]. This task still has to be completed beyond the leading order in λ. Contrary to claims in [8, 10], a complete description of tree-level subleading shape-function effects in $\bar{B}\to X_s\gamma$ decay is therefore still lacking.

4 Moment relations and comparison with the literature

Moments of the shape functions can be related to forward B-meson matrix elements of local HQET operators [1]. In particular, setting $x = 0$ in the defining relations (33) yields expressions for the normalization integrals of the shape functions. They are

$$\int d\omega\, S(\omega) = 1\,, \qquad \int d\omega\,\omega\,\{s(\omega), t(\omega), u(\omega), v(\omega)\} = 0\,. \tag{41}$$

The vanishing of the norm of all subleading shape functions is a consequence of Luke's theorem [21], and it ensures that there are no first-order $\Lambda_{\rm QCD}/m_b$ corrections to total inclusive decay rates. For the functions t, u, v this is an obvious consequence of the fact that the integration domain in (36) and (34) shrinks to zero in the limit $x \to 0$. The interpretation of (41) is that subleading shape functions lead to local distortions of inclusive spectra, which cancel out when the spectra are integrated over a sufficiently large region in phase space. The first moments characterize the strength of the distortions, while higher moments determine their shape.

Taking a derivative $n\cdot\partial_x$ in the definitions (33) brings down a factor of ω under the integrals on the right-hand side. Setting then $x \to 0$ yields a set of relations for the first moments of the shape functions. The resulting matrix elements can be evaluated by means of the relations [22]

$$\langle \bar h\,\Gamma_{\alpha\beta}\, iD^\alpha iD^\beta h\rangle = \frac{1}{2}\,\mathrm{tr}\left(\Gamma_{\alpha\beta}\,\frac{1+\not v}{2}\left[\left(g^{\alpha\beta} - v^\alpha v^\beta\right)\frac{\lambda_1}{3} + i\sigma^{\alpha\beta}\,\frac{\lambda_2}{2}\right]\frac{1+\not v}{2}\right), \tag{42}$$

and [23]

$$\langle i\int d^4z\, T\{(\bar h iD^\mu h)(0)\,\mathcal{L}_h^{(2)}(z)\}\rangle = -v^\mu\,\langle \mathcal{L}_h^{(2)}(0)\rangle = -v^\mu\,\frac{\lambda_1 + 3C_{\rm mag}\,\lambda_2}{2m_b}\,, \tag{43}$$

where λ_1 and λ_2 are the familiar HQET parameters arising in the parameterization of second-order power corrections to inclusive decay spectra, and $C_{\rm mag} = 1$ at tree level. We obtain

$$\begin{aligned} \int d\omega\,\omega\, s(\omega) &= -\frac{\lambda_1 + 3\lambda_2}{2}\,, \qquad & \int d\omega\,\omega\, u(\omega) &= \frac{2\lambda_1}{3}\,, \\ \int d\omega\,\omega\, t(\omega) &= -\lambda_2\,, & \int d\omega\,\omega\, v(\omega) &= \lambda_2\,. \end{aligned} \tag{44}$$

We also recall that the first two moments of the leading shape function are $\int d\omega\,\omega\, S(\omega) = 0$ and $\int d\omega\,\omega^2\, S(\omega) = -\lambda_1/3$ [1].

It is appropriate at this point to relate our subleading shape functions to those defined by Bauer et al. [8, 10]. Using the notations of their second paper, we find $S(\omega) = f(-\omega)$ at leading order, and

$$s(\omega) = \frac{t(-\omega)}{2}\,, \quad t(\omega) = -h_1(-\omega)\,, \quad u(\omega) = -G_2(-\omega)\,, \quad v(\omega) = -H_2(-\omega) \tag{45}$$

for the subleading functions. Their first paper uses a dimensionless momentum variable, so the relations are $S(\omega) = f(-\omega/m_b)/m_b$ etc. (In comparing, one must take note of the fact that we use a different sign convention for the Levi-Civita tensor. Also, we believe there are typographic errors in the definitions of the functions g_2 and h_2 in [8].)

Comparing our result (37) for the hadronic tensor with the corresponding expressions obtained in [8, 10] we find some sources of disagreement. As mentioned earlier, the function

$u(\omega)$ is divided by $\bar{n}\cdot p$, whereas Bauer et al. state that it enters in the universal combination $S(\omega)+u(\omega)/m_b$ with the leading shape function. Also, it appears to us that the term proportional to the trace T_3, which adds a contribution to the last structure function in (39), is missed by these authors.

5 Contributions from four-quark operators

The last diagram in Figure 1 shows a tree-level contribution to the hadronic tensor involving two insertions of the subleading SCET Lagrangian $\mathcal{L}_{\xi q}^{(1)}$. The exchange of a hard-collinear gluon implies that this graph is of order g^2, and so it vanishes in the limit where α_s is set to zero. The fact that the suppression factor is $\pi\alpha_s$ instead of α_s/π reflects the phase-space enhancement of four-quark tree-level graphs compared with loop diagrams [24]. One might therefore expect that the four-quark contribution is numerically as important as the other tree-level subleading shape-function contributions.

Applying the partial-fraction identity (26) twice, we find that the resulting contribution to the correlator $T_{ij}^{(2)}$ can be written as in (27), adding a fifth operator to the sum. It reads

$$O_5(x_-)=\frac{\pi\alpha_s}{\bar{n}\cdot p}\int\limits_0^{\bar{n}\cdot x/2}dt_1\int\limits_{t_1}^{\bar{n}\cdot x/2}dt_2\,\big(\bar{h}S\big)_0\,\Gamma_i\,\not{\bar{n}}\gamma_\rho^\perp\,t_a\,\big(S^\dagger q\big)_{t_1 n}\,\big(\bar{q}S\big)_{t_2 n}\,\gamma_\perp^\rho\not{n}\,\Gamma_j\,t_a\,\big(S^\dagger h\big)_{x_-}\,, \tag{46}$$

where t_a are the generators of color $\mathrm{SU}(N_c)$. Note that the field insertions are ordered according to "light-cone time" x_-, just as they appear in the Feynman diagram in Figure 1. This is a general result. Because the minus components $\bar{n}\cdot p_{hc}$ of hard-collinear momenta are large, of order m_b, hard-collinear fields always propagate forward in light-cone time. Turning, for instance, a forward-moving hard-collinear quark into a backward-moving hard-collinear antiquark would require a hard quantum fluctuation, which is already integrated out in SCET. As a result, Feynman amplitudes in SCET are ordered with respect to light-cone time, and that ordering is preserved in the matching onto HQET. This discussion explains why all our operators have the property that the coordinates z_- of soft fields range from 0 to x_- in an ordered fashion. The results can therefore always be expressed in terms of bi-local operators depending only on 0 and x_-. However, at present we cannot exclude the possibility of nontrivial weight functions under these integrals, which could arise at higher orders in perturbation theory. If present, they may require a generalization of our definitions of subleading shape functions.

Returning to the case of the four-quark operator in (46), we note that its contribution vanishes in the vacuum-insertion approximation due to the color-octet structure of the heavy-light quark bilinears. While this approximation is admittedly naive, phenomenological evidence based on studies of B-meson lifetimes [24, 25] and theoretical arguments based on lattice calculations [26, 27] and QCD sum rules [28] support the notion that matrix elements which vanish in the vacuum-insertion approximation are numerically suppressed, typically by an order of magnitude. It is thus very unlikely that the four-quark operator in (46) could give a larger contribution than loop-suppressed $O(\alpha_s)$ corrections, which we have neglected. Note also that the contribution of the corresponding local operator to total inclusive rates, which is what remains when the corresponding shape functions are integrated over a sufficiently large domain, is bound to be tiny. The effect is of order $(\Lambda_{\rm QCD}/m_b)^3$, and it is α_s-suppressed with respect to a four-quark contribution discussed by Voloshin [29], whose effect on the total decay rate is believed to be at most 3%.

Nevertheless, it is interesting to study the structure of the operator $O_5(x_-)$ in more detail and define corresponding subleading shape functions. The decomposition (30) implies that the Dirac structure can be rearranged in the form (omitting factors of S, $S^\dagger$ and color indices for simplicity)

$$\Gamma_i \, \not{n} \gamma_\rho^\perp \, q \, \bar{q} \, \gamma_\perp^\rho \not{n} \, \Gamma_j = -\frac{1}{2} \begin{pmatrix} \mathbf{1} \\ \not{n}\gamma_5 \\ -\gamma_\perp^\mu \gamma_5 \end{pmatrix} \bar{q} \, \gamma_\perp^\rho \not{n} \, \Gamma_j \, \frac{1+\not{v}}{2} \begin{pmatrix} \mathbf{1} \\ (\not{n}-\not{v})\gamma_5 \\ \gamma_\mu^\perp \gamma_5 \end{pmatrix} \Gamma_i \, \not{n} \gamma_\rho^\perp \, q \,, \tag{47}$$

where the notation implies a sum over the three rows in the equation. In the next step, we use that between $\not{n} \dots \not{n}$ any Dirac matrix can be decomposed as

$$\Gamma \to \frac{1}{4} \, \mathrm{tr}(\Gamma \, \not{n}) \, \frac{\not{\bar{n}}}{2} - \frac{1}{4} \, \mathrm{tr}(\Gamma \, \not{n}\gamma_5) \, \frac{\not{\bar{n}}}{2} \, \gamma_5 - \frac{1}{4} \, \mathrm{tr}(\Gamma \, \not{n}\gamma_\perp^\alpha) \, \frac{\not{\bar{n}}}{2} \, \gamma_\alpha^\perp \,. \tag{48}$$

Only the first two terms survive after contraction of the index ρ in (47). At this stage we are left with four-quark operators with Dirac structures

$$\bar{h} \, h \, \bar{q} \, \not{n} \, (\gamma_5) \, q \,, \qquad \bar{h} \, \not{n}\gamma_5 \, h \, \bar{q} \, \not{n} \, (\gamma_5) \, q \,, \qquad \bar{h} \, \gamma_\perp^\mu \gamma_5 \, h \, \bar{q} \, \not{n} \, (\gamma_5) \, q \,, \tag{49}$$

where (γ_5) means $\mathbf{1}$ or γ_5. Lorentz invariance implies that only the two Lorentz-scalar structures $\bar{h} \, h \, \bar{q} \, \not{n} \, q$ and $\bar{h} \, \not{n}\gamma_5 \, h \, \bar{q} \, \not{n}\gamma_5 \, q$ have non-zero forward matrix elements between B-meson states. Putting back color factors and soft Wilson lines, we define the corresponding subleading shape functions

$$\begin{aligned}
& 2(-i)^2 \int\limits_0^{\bar{n}\cdot x/2} dt_1 \int\limits_{t_1}^{\bar{n}\cdot x/2} dt_2 \, \langle \big[\big(\bar{h}S \big)_0 \, t_a \big]_k \big[t_a \, \big(S^\dagger h \big)_{x_-} \big]_l \big[(\bar{q}S)_{t_2 n} \big]_l \, \not{n} \big[\big(S^\dagger q \big)_{t_1 n} \big]_k \rangle \\
& = \int d\omega \, e^{-\frac{i}{2}\omega \bar{n}\cdot x} \, f_u(\omega) \,, \\
& 2(-i)^2 \int\limits_0^{\bar{n}\cdot x/2} dt_1 \int\limits_{t_1}^{\bar{n}\cdot x/2} dt_2 \, \langle \big[\big(\bar{h}S \big)_0 \, t_a \big]_k \, \not{n}\gamma_5 \big[t_a \, \big(S^\dagger h \big)_{x_-} \big]_l \big[(\bar{q}S)_{t_2 n} \big]_l \, \not{n}\gamma_5 \big[\big(S^\dagger q \big)_{t_1 n} \big]_k \rangle \\
& = \int d\omega \, e^{-\frac{i}{2}\omega \bar{n}\cdot x} \, f_v(\omega) \,,
\end{aligned} \tag{50}$$

where k, l are color indices. Using the same notation as in (37), the corresponding contributions to the hadronic tensor are given by

$$W_{ij}^{(2)} = -\pi\alpha_s \int d\omega \, \delta(n\cdot p + \omega) \left[\frac{f_u(\omega)}{\bar{n}\cdot p} \, T_1 + \frac{f_v(\omega)}{\bar{n}\cdot p} \, T_4 \right] , \tag{51}$$

with the same traces as defined in (38). The four-quark contributions can thus be absorbed into a redefinition of the subleading shape functions u and v, namely

$$\tilde{u}(\omega) \equiv u(\omega) - \pi\alpha_s\, f_u(\omega)\,, \qquad \tilde{v}(\omega) \equiv v(\omega) + \pi\alpha_s\, f_v(\omega)\,. \tag{52}$$

Note that the definitions (50) imply that the normalization integrals as well as the first moments of the functions $f_u(\omega)$ and $f_v(\omega)$ vanish (because the integration domain is of second order in $\bar{n}\cdot x$), and therefore the new functions $\tilde{u}$ and $\tilde{v}$ have the same first moments as the original ones, see (44).

6 Applications

We are now ready to study the phenomenological implications of our results. We absorb the contributions from four-quark operators into the functions $\tilde{u}(\omega)$ and $\tilde{v}(\omega)$ defined in (52). Similarly, we absorb the contribution from the subleading shape function $s(\omega)$ into the leading shape function, denoting $\tilde{S}(\omega) \equiv S(\omega) + s(\omega)/m_b$. The moment constraints on the shape functions can be summarized by their expansions in distributions [1], which read

$$\begin{aligned}
\tilde{S}(\omega) &= \delta(\omega) - \frac{\lambda_1}{6}\,\delta''(\omega) + \frac{\lambda_1 + 3\lambda_2}{2m_b}\,\delta'(\omega) + \dots\,, \\
t(\omega) &= \lambda_2\,\delta'(\omega) + \dots\,, \quad \tilde{u}(\omega) = -\frac{2\lambda_1}{3}\,\delta'(\omega) + \dots\,, \quad \tilde{v}(\omega) = -\lambda_2\,\delta'(\omega) + \dots\,.
\end{aligned} \tag{53}$$

These expressions will allow us to test our results against existing predictions for inclusive spectra obtained using a conventional heavy-quark expansion. It is convenient to introduce a new variable $P_+ = n\cdot p + \bar{\Lambda} = m_B - n\cdot q = E_X - |\boldsymbol{P}_X|$, which is the plus component of the total hadronic momentum of the final-state jet, and to express the shape functions as functions of $\hat{\omega} = \bar{\Lambda} - \omega$ [4], where $\bar{\Lambda} = m_B - m_b$. The analytic properties of the shape functions are such that they have support for $0 \le \hat{\omega} < \infty$. The δ-function in the tree-level expression for the hadronic tensor sets $\hat{\omega} = P_+$. We denote functions of $\hat{\omega}$ by a hat, e.g., $\hat{S}(\hat{\omega}) \equiv \tilde{S}(\bar{\Lambda} - \hat{\omega})$, and similarly for the other functions.

We start by presenting the triple differential rate for the inclusive semileptonic decay $\bar{B} \to X_u\, l^- \bar{\nu}$ in terms of the kinematic variables

$$P_+\,, \qquad y = \frac{\bar{n}\cdot p}{m_b} = 1 - \frac{\bar{n}\cdot q}{m_b}\,, \qquad \bar{x} = 1 - \frac{2E_l}{m_b}\,. \tag{54}$$

Using the expression for the hadronic tensor in (39) combined with the rate equations derived in [30], we obtain

$$\begin{aligned}
\frac{1}{\Gamma}\,\frac{d^3\Gamma}{d\bar{x}\,dy\,dP_+} &= 12(y-\bar{x}) \Bigg\{ (1-y+\bar{x}) \Bigg[\left(1 + \frac{2(\bar{\Lambda} - P_+)}{m_b} \right) \hat{S}(P_+) + \frac{\hat{t}(P_+)}{m_b} + \frac{\hat{u}(P_+) - \hat{v}(P_+)}{y\,m_b} \Bigg] \\
&\quad + \left(1 - \frac{2\bar{x}}{y} \right) \frac{\bar{\Lambda} - P_+}{m_b}\,\hat{S}(P_+) + \frac{2\bar{x}(1-2y)}{y^2}\,\frac{\hat{t}(P_+)}{m_b} + \frac{2\bar{x}}{y^2}\,\frac{\hat{v}(P_+)}{m_b} \Bigg\}\,.
\end{aligned} \tag{55}$$

This expression is exact at tree level and to order $\Lambda_{\rm QCD}/m_b$ in the heavy-quark expansion. The phase space for these variables is such that

$$0 \le P_+ \le m_B - 2E_l = m_b\,\bar{x} + \bar{\Lambda}\,, \qquad \frac{P_+ - \bar{\Lambda}}{m_b} \le \bar{x} \le y \le 1\,, \tag{56}$$

and the hadronic tensor has support only for $y \ge 0$. Above we assume that $\bar{x}$ and y are of $O(1)$. The collinear expansion breaks down in the region of small y. A restriction to small $\bar{x}$ is allowed and corresponds to the so-called "endpoint region" of the charged-lepton spectrum. In this case, however, it is necessary for consistency to expand the expression above in powers of $\bar{x} = O(\Lambda_{\rm QCD}/m_b)$.

As a first application, we integrate (55) over the lepton energy, finding

$$\begin{aligned}\frac{1}{\Gamma}\,\frac{d^2\Gamma}{dy\,dP_+} &= 2y^2(3-2y)\left[\left(1 + \frac{2(\bar{\Lambda}-P_+)}{m_b}\right)\hat{S}(P_+) + \frac{\hat{t}(P_+)}{m_b} + \frac{\hat{u}(P_+)}{y\,m_b}\right]\\ &\quad + 2y(6-5y)\,\frac{\bar{\Lambda}-P_+}{m_b}\,\hat{S}(P_+) + 2y(1-2y)\,\frac{2\hat{t}(P_+) - \hat{v}(P_+)}{m_b}\,.\end{aligned} \tag{57}$$

The variables y and P_+ allow us to reconstruct any kinematic variable characterizing the final-state hadronic system, such as the total hadronic energy, $2E_X = P_+ + ym_b + \bar{\Lambda}$, or the invariant hadronic mass, $M_X^2 = P_+(ym_b + \bar{\Lambda})$. Of particular interest for a measurement of $|V_{ub}|$ are the spectra in the variables P_+ and M_X. For the distribution in P_+, we obtain

$$\frac{1}{\Gamma}\,\frac{d\Gamma}{dP_+} = \left[1 + \frac{14(\bar{\Lambda}-P_+)}{3m_b}\right]\hat{S}(P_+) + \frac{\hat{t}(P_+) + 5\hat{u}(P_+) + \hat{v}(P_+)}{3m_b}\,. \tag{58}$$

Next, denoting $s_H = M_X^2$ and $\Delta_s = s_H/m_B$, we find for the hadronic invariant mass spectrum

$$\frac{1}{\Gamma}\,\frac{d\Gamma}{ds_H} = \frac{1}{m_B}\int_{\Delta_s}^{\infty}\frac{dP_+}{P_+}\,f(P_+,\Delta_s/P_+)\,, \tag{59}$$

where

$$\begin{aligned}f(P_+,r) &= 2r^2(3-2r)\left[\left(1 + \frac{2(3\bar{\Lambda}-P_+)}{m_b}\right)\hat{S}(P_+) + \frac{\hat{t}(P_+)}{m_b} + \frac{\hat{u}(P_+)}{r\,m_b}\right]\\ &\quad - 2r\left[(6-5r)\,\frac{P_+}{m_b} + 2r\,\frac{\bar{\Lambda}}{m_b}\right]\hat{S}(P_+) + 2r(1-2r)\,\frac{2\hat{t}(P_+) - \hat{v}(P_+)}{m_b}\,.\end{aligned} \tag{60}$$

A comment is in order concerning the fact that the integration over P_+ is extended to infinity, while kinematically $P_+ \le \sqrt{s_0}$. The point is that near its kinematic limit the momentum $P_+ \sim \sqrt{m_B\Lambda_{\rm QCD}}$ is no longer of order $\Lambda_{\rm QCD}$, and hence the collinear expansion breaks down. However, the contribution to the rate resulting from the vicinity of this region is suppressed by more than a single power of $\Lambda_{\rm QCD}/m_b$ [4]. It is therefore required to set the upper limit to infinity so as to avoid spurious higher-order power corrections. Once this is done, it is theoretically consistent to treat P_+ as a quantity of order $\Lambda_{\rm QCD}$.

As a final application, we integrate (55) over the hadronic variables y and P_+ to obtain the charged-lepton energy distribution

$$\frac{1}{\Gamma}\frac{d\Gamma}{dE_l} = 4\int_0^{m_B-2E_l}\frac{d\hat\omega}{m_B-\hat\omega}\left[\hat S(\hat\omega) + \frac{6(\bar\Lambda-\hat\omega)}{m_b}\,\hat S(\hat\omega) + \frac{\hat t(\hat\omega)+3\hat u(\hat\omega)-3\hat v(\hat\omega)}{m_b}\right]. \tag{61}$$

For purposes of comparison with previous authors, we also give the result for the dipole-operator contribution to the $\bar B\to X_s\gamma$ photon spectrum, stressing once again that this does *not* provide a complete description of all tree-level subleading shape-function effects. Taking into account that the photon spectrum is proportional to the expression in (40) times a factor E_γ^3, we find

$$\frac{1}{\Gamma}\frac{d\Gamma}{dE_\gamma} = \left[1+\frac{2(\bar\Lambda-P_+)}{m_b}\right]\hat S(P_+) + \frac{-\hat t(P_+)+\hat u(P_+)-\hat v(P_+)}{m_b} + \dots\,, \tag{62}$$

where $P_+ = m_B - 2E_\gamma$ in this case.

Despite the discrepancies mentioned earlier, we agree with [8] on the result for the $\bar B\to X_s\gamma$ photon spectrum (having, however, added the new contribution from four-quark operators), and we also find complete agreement with a recent calculation of the hadronic invariant mass spectrum for semileptonic $\bar B\to X_u\,l^-\bar\nu$ decay by Luke et al. [12]. However, we do not confirm the result for the charged-lepton energy spectrum obtained in [10], when converted to our notation. These authors have (apart from an overall missing factor of 2) a coefficient of 2 instead of our 6 in front of the term proportional to $(\bar\Lambda-\hat\omega)$, and a coefficient of 1 instead of our 3 in front of $\hat u(\hat\omega)$. Our results can be tested against established formulae by using the moment expansions in (53). We find that (58), (61), and (62) are consistent with expressions for the corresponding spectra derived in [7], [23], and [1], respectively. Amusingly, the result for the lepton spectrum found in [10] is also consistent with the moment expansion. The reason is that this spectrum differs from our result by a term involving the combination $4(\omega/m_b)\,\tilde S(\omega)+2\tilde u(\omega)$, in which the $\delta'(\omega)$ terms from (53) cancel each other.

Of particular phenomenological importance are various fractions of all $\bar B\to X_u\,l^-\bar\nu$ decays that pass certain experimental cuts, which are chosen so as to eliminate (or reduce) the background from semileptonic decays with charm hadrons in the final state. The most common cuts are $E_l\ge E_0$ with $\Delta_E = m_B-2E_0\le m_D^2/m_B$ ("lepton endpoint"), $s_H\le s_0$ with $\Delta_M = s_0/m_B\le m_D^2/m_B$ ("low hadronic mass"), and $P_+\le\Delta_P$ with $\Delta_P\le m_D^2/m_B$ ("low hadronic plus momentum"). We define corresponding fractions $F_E(\Delta_E)$, $F_M(\Delta_M)$, and $F_P(\Delta_P)$. While the result for $F_P(\Delta_P)$ is simply an integral of the spectrum in (58) over the range $0\le P_+\le\Delta_P$, the resulting expressions for the other two event fractions are more complicated, and we quote them here for completeness. We obtain

$$F_E(\Delta_E) = \int_0^{\Delta_E} d\hat\omega\,\frac{2(\Delta_E-\hat\omega)}{m_B-\hat\omega}\left[\hat S(\hat\omega)+\frac{6(\bar\Lambda-\hat\omega)}{m_b}\,\hat S(\hat\omega)+\frac{\hat t(\hat\omega)+3\hat u(\hat\omega)-3\hat v(\hat\omega)}{m_b}\right], \tag{63}$$

and

$$F_M(\Delta_M) = \int\limits_0^{\Delta_M} dP_+ \left[\left(1 + \frac{14(\bar\Lambda - P_+)}{3m_b} \right) \hat S(P_+) + \frac{\hat t(P_+) + 5\hat u(P_+) + \hat v(P_+)}{3m_b} \right] + \int\limits_{\Delta_M}^{\infty} dP_+\, g(P_+, \Delta_M/P_+)\,, \tag{64}$$

where $\Delta_M = s_0/m_B$, and

$$g(P_+,r) = r^3(2-r) \left[\left(1 + \frac{2(3\bar\Lambda - P_+)}{m_b} \right) \hat S(P_+) + \frac{\hat t(P_+)}{m_b} \right] - \frac{2r^2}{3} \left[(9-5r)\frac{P_+}{m_b} + 2r\,\frac{\bar\Lambda}{m_b} \right] \hat S(P_+) + \frac{r^2}{3}\,(9-4r)\frac{\hat u(P_+)}{m_b} + \frac{r^2}{3}\,(3-4r)\,\frac{2\hat t(P_+) - \hat v(P_+)}{m_b}\,. \tag{65}$$

In order to illustrate the numerical impact of subleading shape functions, we focus on the single differential spectra in (58), (59), and (61) in semileptonic B decay. For the leading shape function we use the parametrization

$$S_{\rm model}(\hat\omega) = \frac{b^b}{\Gamma(b)}\,\frac{\hat\omega^{b-1}}{\bar\Lambda^b}\,e^{-b\,\hat\omega/\bar\Lambda} \tag{66}$$

with $\bar\Lambda = 0.63\,\text{GeV}$ and $b = 2.93$, corresponding to the default choice in [4]. We employ two models for the subleading shape functions based on the moment expansion in (53), one where $\delta'(\omega)$ is replaced by a derivative $-S'_{\rm model}(\hat\omega)$ of the leading shape function, and one where it is replaced by $[3(\bar\Lambda-\hat\omega)/\lambda_1]\,S_{\rm model}(\hat\omega)$. Both models satisfy the moment constraints identically, while giving rather different shapes for the subleading structure functions. We take $\lambda_2 = 0.12\,\text{GeV}^2$, $m_b = 4.65\,\text{GeV}$, and determine $\lambda_1 = -0.41\,\text{GeV}^2$ from the second moment of the model function (66).

Results for the three spectra and the corresponding integrated event fractions are presented in Figure 2. In each plot, the dashed-dotted line shows the leading term in the heavy-quark expansion, the dashed gray curve includes the effect of the "kinematic" power corrections proportional to $\hat S(\hat\omega)$, which are predicted model independently, and the two solid lines include the model predictions for the subleading shape-function effects (the lighter of the two lines corresponds to the first model). While the effects are clearly visible on the spectra themselves, the integrated rate fractions with a cut on P_+ or s_H receive only moderate power corrections, which alter the leading-power results by typically 10% or less in the region near the charm threshold (located at $m_D^2/m_B \approx 0.66\,\text{GeV}$). Note that the kinematic corrections alone overestimate the effect, especially for the case of the P_+ spectrum.

The situation is rather different for the charged-lepton energy spectrum, where power corrections can have a drastic effect and tend to reduce the event fraction by a large amount. This effect can be traced to the particular weight function under the integral in (63) [11]. Even for large values of $\hat\omega$, the effect of subleading shape functions does not diminish, because

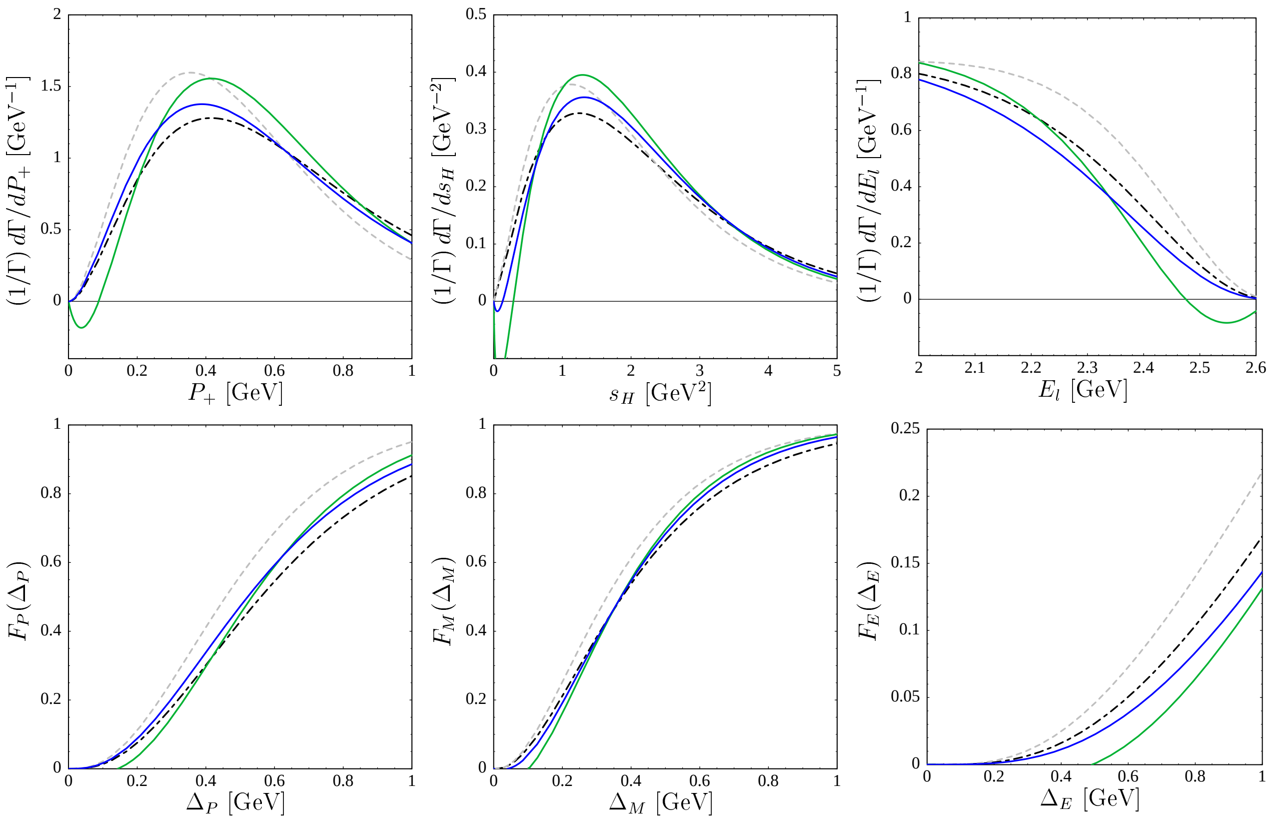

Figure 2: Predictions for the P_+, s_H, and E_l spectra (upper plots) and event fractions (lower plots). The dash-dotted curves refer to the heavy-quark limit, while the solid lines include subleading shape-function effects. The gray dashed curves include only those power corrections proportional to the leading shape function.

the linear term in the weight function picks out the first moments of the subleading shape functions. As a result, there is a significant theoretical uncertainty in the prediction for the event fraction $F_E(\Delta_E)$ near the charm threshold, which imposes limitations on the precision with which $|V_{ub}|$ can be determined from the charged-lepton endpoint region.

No such large effects are seen for the event fractions with a cut on P_+ or hadronic invariant mass. With $\Delta_{P,M} \approx m_D^2/m_B$, about 65–85% of all $\bar{B} \to X_u\, l^- \bar{\nu}$ events are retained with these cuts, and the corresponding fractions are enhanced further when radiative corrections are included [4]. Once the efficiency is so high, possible uncertainties due to four-quark operator contributions ("weak annihilation") become negligible. Our analysis supports the study of theoretical uncertainties presented in [7], where the uncertainty due to power corrections in the value of $F_P(\Delta_P)$ near the charm threshold was estimated as 10%. It provides further credibility to the strategy of obtaining highly efficient, precise measurements of $|V_{ub}|$ using experimental cuts on P_+ or s_H. We recall at this point that the P_+ spectrum offers the additional advantage that it can be directly related to the $\bar{B} \to X_s\gamma$ photon spectrum, cf. (58) and (62), whereas a prediction of the hadronic invariant mass spectrum requires knowledge of the shape function over a range not accessible in radiative B decay. Note also that the hadronic invariant mass distribution receives fractional power corrections of order $(\Lambda_{\rm QCD}/m_b)^\gamma$ with $\gamma \approx 1.35$ from a region of phase space where the collinear expansion breaks down [4]. While these effects are formally beyond the accuracy of the present work, unless they can be shown to cancel they introduce an additional theoretical uncertainty that is difficult to quantify.

The smallness of subleading shape-function effects for some observables can be understood based on the moment expansions in (53). For instance, in the P_+ spectrum (58) the first moments of the functions $\hat{t}$ and $\hat{v}$ cancel in the combination $\hat{t}+\hat{v}$, and the first moment of $\frac{5}{3}\,\hat{u}$ cancels against that of $\frac{10}{3}\,(\bar{\Lambda}-P_+)\,\hat{S}$, leaving a small residual term $-\frac{4}{9}\,\lambda_1\,\delta'(P_+-\bar{\Lambda})$. In weighted integrals over the P_+ spectrum, this can give rise to small second-order power corrections. Higher moments may spoil these cancellations, but their effects are only of order $(\Lambda_{\rm QCD}/m_b)^3$ or higher when the spectrum is integrated over a sufficiently large domain. Similar cancellations occur for the hadronic invariant mass spectrum (59), and for the photon spectrum (62), for which the first moments cancel entirely. In all three cases, we have checked that taking a mix of the two models for the subleading shape functions, in which different higher moments prevent perfect cancellations, does not change the results by a significant amount. The case of the charged-lepton spectrum (61) is again different. The first moment of $3\hat{u}$ cancels against that of $6(\bar{\Lambda}-\hat{\omega})\,\hat{S}$. However, the sum $\hat{t}-3\hat{v}$ has a rather large first moment given by $-4\lambda_2\,\delta'(\hat{\omega}-\bar{\Lambda})$, which in conjunction with the linear weight factor in (63) gives rise to a significant power correction to the event fraction F_E [9, 10, 11].

7 Conclusions

With the development of soft-collinear effective theory (SCET), theoretical predictions for inclusive B-meson decay distributions have recently received renewed attention. While these processes had been studied extensively during the past decade, the ever increasing precision of the measurements at the B factories requires a new level of accuracy. This is an area where SCET is not only a useful conceptual tool, but where it provides concrete means of pushing the limits of theoretical calculations.

Earlier this year, two groups have performed systematic analyses of short-distance corrections in the semileptonic decay $\bar{B}\to X_u\,l^-\,\bar{\nu}$ [4, 5] and radiative decay $\bar{B}\to X_s\gamma$ [6], including a complete scale separation and resummation of Sudakov logarithms. As a result, the behavior of the leading shape function under renormalization is now well understood. This leaves power corrections to the heavy-quark limit as the principal source of theoretical uncertainties.

In the present work, we have used the formalism of SCET to perform a systematic study of such power-suppressed effects. At tree level, the results can be expressed in terms of a set of subleading shape functions defined via the Fourier transforms of forward matrix elements of bi-local light-cone operators in heavy-quark effective theory. We have identified a new contribution arising from four-quark operators, which was not considered previously. We have shown that, when shape functions appearing in process-independent combinations are combined into single functions, then a total of three subleading shape functions are required to describe arbitrary current-induced decay distributions of B mesons into light final-state particles. While subleading shape-function effects had been studied in the past, our results do not agree with those found in the original papers [8, 10].

In the last part of this work, we have presented analytical expressions for a variety of distributions in $\bar{B} \to X_u \, l^- \bar{\nu}$ decay, which can be used directly for the analysis of experimental data. We have also given a formula for the triple differential rate, which allows for arbitrary cuts on kinematic variables. While this concludes the problem of tree-level power corrections in semileptonic decay, we have stressed that no complete (tree-level) analysis of power-suppressed corrections to the $\bar{B} \to X_s \gamma$ decay exists to date. The formalism developed in this work can, however, readily be extended to this case.

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