CLNS 03/1822
hep-ph/0303082

Renormalization-Group Evolution of the B-Meson Light-Cone Distribution Amplitude

Björn O. Lange and Matthias Neubert
Newman Laboratory for Elementary-Particle Physics, Cornell University, Ithaca, NY 14853, USA

An integro-differential equation governing the evolution of the leading-order B-meson light-cone distribution amplitude is derived. The anomalous dimension in this equation contains a logarithm of the renormalization scale, whose coefficient is identified with the cusp anomalous dimension of Wilson loops. The exact analytic solution of the evolution equation is obtained, from which the asymptotic behavior of the distribution amplitude is derived. These results can be used to resum Sudakov logarithms entering the hard-scattering kernels in QCD factorization theorems for exclusive B decays.

12.38.Cy,12.39.Hg,12.39.St,13.25.Hw

1. Introduction. It has recently become apparent that the light-cone structure of the B meson is of great phenomenological interest. Using standard perturbative QCD methods for hard exclusive processes, B-meson light-cone distribution amplitudes (LCDAs) were first introduced in a study of the asymptotic behavior of heavy-meson form factors at large momentum transfer [1]. Two such amplitudes, called $\phi_\pm^B(\omega,\mu)$, arise in the parameterization of B-meson matrix elements of bilocal heavy-light current operators at leading-order in heavy-quark effective theory (HQET). The interest in these quantities was revived when it was found that the amplitudes for many exclusive B-meson decays can be simplified significantly using QCD factorization theorems [2]. At leading power in $\Lambda_{\rm QCD}/m_b$ these amplitudes can be expressed in terms of convolution integrals of perturbative hard-scattering kernels with the leading-order LCDAs.

The B-meson LCDAs appear in processes where light energetic particles are emitted into the final state, such as $B\to\pi\pi$ and $B\to K^*\gamma$, and at leading power arise from hard interactions of collinear partons with the soft spectator quark inside the B meson. In almost all applications of QCD factorization theorems only the function $\phi_+^B(\omega,\mu)$ contributes. Generically, the hard-spectator term in a QCD factorization formula is of the form

$$\int_0^\infty \frac{d\omega}{\omega}\,T(\omega,\mu)\,\phi_+^B(\omega,\mu)\,, \tag{1}$$

where the kernel T is calculable in perturbation theory. The factor $1/\omega$ ensures convergence of the integral for large ω. The characteristic scale for soft-collinear interactions is $2E\omega\sim m_b\Lambda_{\rm QCD}$, where $E\sim m_b$ is the energy of a collinear particle in the B-meson rest frame. At fixed order in perturbation theory the kernel depends logarithmically on this scale, and it is independent of ω to lowest order in perturbation theory. In addition, the kernel can depend on hard scales of order m_b. We will assume below that T is defined such that the convolution integral (1) is renormalization-group (RG) invariant.

A thorough understanding of factorization requires controlling the scale dependence of the LCDA and the kernel under the convolution integral (1) using evolution equations. This is crucial for a clean separation of physics associated with different mass scales and for the systematic resummation of large (Sudakov) logarithms, which enter the kernel at every order in perturbation theory. In this Letter we derive the RG equations for the LCDA $\phi_+^B(\omega,\mu)$ and for the kernel $T(\omega,\mu)$, present their exact analytic solutions, and extract model-independent results for the asymptotic behavior for $\omega\to 0$ and $\omega\to\infty$.

2. Evolution equations. The LCDA is given by the Fourier transform

$$\phi_+^B(\omega,\mu)=\frac{1}{2\pi}\int d\tau\,e^{i\omega\tau}\,\widetilde{\phi}_+^B(\tau,\mu) \tag{2}$$

of a function $\widetilde{\phi}_+^B(\tau,\mu)$ defined in terms of a B-meson matrix element in HQET. Denoting by h the effective heavy-quark field and by q_s the soft spectator quark, and using a mass independent normalization of meson states, we write [1]

$$\begin{aligned}&\langle\,0\,|\,\bar q_s(z)\,S_n(z,0)\,\not{n}\,\Gamma\,h(0)\,|\bar B(v)\rangle\\&\qquad=-\frac{iF(\mu)}{2}\,\widetilde{\phi}_+^B(\tau,\mu)\,{\rm tr}\left(\not{n}\,\Gamma\,\frac{1+\not{v}}{2}\,\gamma_5\right).\end{aligned} \tag{3}$$

Here $z\parallel n$ with $n^2=0$ is a null vector, v is the B-meson velocity, Γ represents an arbitrary Dirac matrix, and $\tau=v\cdot z-i0$. The gauge string $S_n(z,0)$ represents a soft Wilson line connecting the points 0 and z on a straight light-like segment. The quantity $F(\mu)$ is the HQET matrix element corresponding to the asymptotic value of the product $f_B\sqrt{m_B}$ in the heavy-quark limit. The distribution amplitude $\widetilde{\phi}_+^B(\tau,\mu)$ is normalized to 1 at $\tau=0$. The analytic properties of this function in the complex τ plane imply that $\phi_+^B(\omega,\mu)=0$ if $\omega<0$.

We denote by $O_+(\omega)$ the Fourier transform of the bilocal HQET operator in (3) and write the relation between bare and renormalized operators in the form

$$O_+^{\rm ren}(\omega,\mu) = \int d\omega'\, Z_+(\omega,\omega',\mu)\, O_+^{\rm bare}(\omega')\,, \qquad (4)$$

where $Z_+(\omega,\omega',\mu) = \delta(\omega-\omega')$ at lowest order. Operators with different momentum ω' can mix under renormalization since they have the same quantum numbers. In the $\overline{\rm MS}$ scheme the function $Z_+(\omega,\omega',\mu)$ is defined so as to subtract the ultraviolet (UV) poles in the matrix elements of the bare operators.

The B-meson matrix element of the renormalized operator $O_+^{\rm ren}(\omega,\mu)$ is, up to a Dirac trace, given by the product $F(\mu)\,\phi_+^B(\omega,\mu)$. It follows that the LCDA obeys the evolution equation

$$\frac{d}{d\ln\mu}\,\phi_+^B(\omega,\mu) = -\int_0^\infty d\omega'\,\gamma_+(\omega,\omega',\mu)\,\phi_+^B(\omega',\mu) \qquad (5)$$

with the anomalous dimension (unless otherwise indicated, $\alpha_s \equiv \alpha_s(\mu)$)

$$\gamma_+(\omega,\omega',\mu) = -\int d\tilde\omega\,\frac{dZ_+(\omega,\tilde\omega,\mu)}{d\ln\mu}\,Z_+^{-1}(\tilde\omega,\omega',\mu) - \gamma_F(\alpha_s)\,\delta(\omega-\omega')\,. \qquad (6)$$

Here γ_F is the universal anomalous dimension of local heavy-light currents in HQET, which determines the scale dependence of $F(\mu)$.

At one-loop order the result for $Z_+(\omega,\omega',\mu)$ is obtained by evaluating the UV poles of the one-loop HQET diagrams shown in Figure 1. The loop integrals contain δ-functions whose arguments depend on the value of the "plus component" $k_+ = n\cdot k$ of the loop momentum. The integration over the orthogonal "minus component" is most easily performed using the theorem of residues, and the remaining integration over transverse components can be evaluated using standard techniques of dimensional regularization (setting $d_\perp = 2-2\epsilon$ for the dimension of the transverse space). In the first two diagrams the transverse loop integrations give rise to UV poles, whereas the third diagram is UV finite. In units of $C_F\alpha_s/4\pi$, the corresponding contributions to $Z_+(\omega,\omega',\mu)$ are given by ($k \equiv -k_+$)

$$D_1 = -\frac{2}{\epsilon}\int_0^\infty \frac{dk}{k}\left(\frac{k}{\mu}\right)^{-2\epsilon}\Big[\delta(k+\omega'-\omega)-\delta(\omega-\omega')\Big]\,,$$
$$D_2 = -\frac{2}{\epsilon}\int_0^{\omega'}\frac{dk}{k}\,\frac{\omega'-k}{\omega'}\Big[\delta(k+\omega-\omega')-\delta(\omega-\omega')\Big]\,. \qquad (7)$$

These expressions must be understood in the sense of distributions, and only pole terms in $1/\epsilon$ are kept after integration over ω' in (4). Note that the k-integral in the first diagram in Figure 1 gives rise to a $1/\epsilon^2$ pole times $\delta(\omega-\omega')$. Accounting for wave-function renormalization of the external quark fields we obtain

$$Z_+^{(1)}(\omega,\omega',\mu) = \left(\frac{1}{\epsilon^2}+\frac{2}{\epsilon}\,\ln\frac{\mu}{\omega}-\frac{5}{2\epsilon}\right)\delta(\omega-\omega') - \frac{2}{\epsilon}\left[\frac{\omega}{\omega'}\,\frac{\theta(\omega'-\omega)}{\omega'-\omega}+\frac{\theta(\omega-\omega')}{\omega-\omega'}\right]_+ . \qquad (8)$$

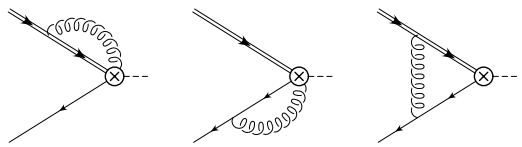

FIG. 1. One-loop diagrams contributing to the calculation of the anomalous dimension. The crossed circle denotes an insertion of the operator $O_+^{\rm bare}(\omega')$. The double lines represent heavy-quark fields in HQET.

Here and below a superscript "(1)" is used to indicate one-loop coefficients in units of $C_F\alpha_s/4\pi$. The plus distribution is defined such that, when Z_+ is integrated with a function $f(\omega')$, one must replace $f(\omega') \to f(\omega') - f(\omega)$ under the integral. The non-diagonal terms in (8) agree with those found in [1]; however, the double pole and logarithm in the first term were missed in that paper. For the anomalous dimension in (6) we now obtain

$$\gamma_+(\omega,\omega',\mu) = \left[\Gamma_{\rm cusp}(\alpha_s)\,\ln\frac{\mu}{\omega}+\gamma(\alpha_s)\right]\delta(\omega-\omega') + \omega\,\Gamma(\omega,\omega',\alpha_s)\,, \qquad (9)$$

where $\int d\omega'\,\Gamma(\omega,\omega',\alpha_s) = 0$ by definition. Taking into account that $\gamma_F^{(1)} = -3$ is the one-loop coefficient of the anomalous dimension of heavy-light currents, it follows from (8) that $\Gamma_{\rm cusp}^{(1)} = 4$, $\gamma^{(1)} = -2$, and

$$\Gamma^{(1)}(\omega,\omega') = -\Gamma_{\rm cusp}^{(1)}\left[\frac{\theta(\omega'-\omega)}{\omega'(\omega'-\omega)}+\frac{\theta(\omega-\omega')}{\omega(\omega-\omega')}\right]_+ . \qquad (10)$$

The anomalous dimension in (9) is of the Sudakov type. It contains a logarithmic dependence on the renormalization scale in addition to the dependence through the running coupling $\alpha_s(\mu)$. This feature distinguishes the kernel for the B-meson LCDA from the familiar Brodsky–Lepage kernel for the LCDA of a light pseudoscalar meson [3]. The extra logarithm has its origin in the renormalization properties of Wilson lines with light-like segments [4]. Using the fact that a heavy quark in HQET can be described by a Wilson line $h(0) = S_v(0,-\infty)\,h_0$, where h_0 is a sterile field without QCD interactions, it follows that the interacting fields in the operator in (3) can be written as $\bar q_s(z)\,S_n(z,0)\,S_v(0,-\infty)$. This corresponds to a gauge string extending from minus infinity to 0 along the v direction, another string extending from 0 to z along the light-like n direction, and a light quark field located at the end of the string at position z. This Wilson line has a cusp singularity at the origin, which gives rise to a local, single-logarithmic term in the anomalous dimension. From (7) it is seen that the logarithm arises indeed from the gluon exchange between the two strings S_n and S_v (first diagram in Figure 1). Such cusp singularities are universal and of geometric origin. The corresponding anomalous dimension $\Gamma_{\rm cusp}$ is process independent and has been computed at two-loop order in [4]. The important observation following from

this discussion is that also in higher orders there is only a single logarithm present in the anomalous dimension $\gamma_+(\omega,\omega',\mu)$. With this knowledge, the evolution equation can be solved in RG-improved perturbation theory.

From the scale independence of the convolution integral (1) it follows that the hard-scattering kernel obeys the evolution equation

$$\frac{d}{d\ln\mu}\,T(\omega,\mu) = \int_0^\infty d\omega'\,\frac{\omega}{\omega'}\,\gamma_+(\omega',\omega,\mu)\,T(\omega',\mu)\,. \quad (11)$$

The explicit result for the anomalous dimension given above shows that $\frac{\omega}{\omega'}\,\gamma_+(\omega',\omega,\mu) = \gamma_+(\omega,\omega',\mu)$ (at least to one-loop order). Hence, apart from an overall sign the LCDA and the kernel obey the same RG equation.

3. Analytic solutions. The general solution of the evolution equations (5) and (11) can be obtained using the fact that on dimensional grounds

$$\int d\omega'\,\omega\,\Gamma(\omega,\omega',\alpha_s)\,(\omega')^a = \omega^a\,\mathcal{F}(a,\alpha_s)\,, \quad (12)$$

where the function $\mathcal{F}$ only depends on the exponent a and the coupling constant, and $\mathcal{F}(0,\alpha_s) = 0$ by definition. The integral on the left-hand side is convergent as long as $-1 < \mathrm{Re}\,a < 1$. The corresponding integral with $\Gamma(\omega,\omega',\alpha_s)$ replaced by $\Gamma(\omega',\omega,\alpha_s)$, which is relevant to the evolution equation (11) for the kernel, is given by $\mathcal{F}(-a,\alpha_s)$. This follows from the fact that $\Gamma(\omega,\omega',\alpha_s) = \omega^{-2}\,f(\omega'/\omega)$ on dimensional grounds. At one-loop order we find from (10)

$$\mathcal{F}^{(1)}(a) = \Gamma_{\rm cusp}^{(1)}\Big[\psi(1+a) + \psi(1-a) + 2\gamma_E\Big]\,, \quad (13)$$

where $\psi(z)$ is the logarithmic derivative of the Euler Γ-function.

Relation (12) implies that the ansatz

$$f(\omega,\mu,\mu_0,\eta) = \left(\frac{\omega}{\mu_0}\right)^{\eta+g(\mu,\mu_0)} \exp U(\mu,\mu_0,\eta) \quad (14)$$

with

$$\begin{aligned} g(\mu,\mu_0) &= \int\limits_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} d\alpha\,\frac{\Gamma_{\rm cusp}(\alpha)}{\beta(\alpha)}\,, \\ U(\mu,\mu_0,\eta) &= -\int\limits_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha}{\beta(\alpha)}\,\Big[g(\mu,\mu_\alpha) + \gamma(\alpha) \\ &\qquad + \mathcal{F}\big(\eta + g(\mu_\alpha,\mu_0),\alpha\big)\Big]\,, \end{aligned} \quad (15)$$

provides a solution to the evolution equation (5) with initial condition $f(\omega,\mu_0,\mu_0,\eta) = (\omega/\mu_0)^\eta$ at some scale μ_0. Here μ_α is defined such that $\alpha_s(\mu_\alpha) = \alpha$, the β-function is $\beta(\alpha_s) = d\alpha_s/d\ln\mu$, and η can be an arbitrary complex parameter. Note that $g(\mu,\mu_0) > 0$ if $\mu > \mu_0$. Given the exact expressions in (15) one can derive approximate results for the functions g and U at a given order in RG-improved perturbation theory. The explicit forms arising at next-to-leading order can be found in [5].

We now assume that the function $\phi_+^B(\omega,\mu_0)$ is given at some low scale $\mu_0 \sim$ few $\times\,\Lambda_{\rm QCD}$ and define its Fourier transform with respect to $\ln(\omega/\mu_0)$ through

$$\phi_+^B(\omega,\mu_0) = \frac{1}{2\pi}\int_{-\infty}^\infty dt\,\varphi_0(t)\left(\frac{\omega}{\mu_0}\right)^{it}\,, \quad (16)$$

where $\varphi_0(0) = 1/\lambda_B$ is determined in terms of the first inverse moment of the LCDA at the scale μ_0 [2]. It follows that the result for the LCDA at a different scale μ is given by

$$\phi_+^B(\omega,\mu) = \frac{1}{2\pi}\int_{-\infty}^\infty dt\,\varphi_0(t)\,f(\omega,\mu,\mu_0,it)\,. \quad (17)$$

It is instructive to work out the t and ω-dependence of the integrand at leading order in RG-improved perturbation theory. Using the one-loop result (13) we obtain

$$f(\omega,\mu,\mu_0,it) \propto \left(\frac{\omega}{\mu_0}\right)^{it+g} \frac{\Gamma(1-it-g)\,\Gamma(1+it)}{\Gamma(1+it+g)\,\Gamma(1-it)}\,, \quad (18)$$

where $g = (2C_F/\beta_0)\,\ln[\alpha_s(\mu_0)/\alpha_s(\mu)]$ is the leading-order contribution to the function $g(\mu,\mu_0)$, and the formula can only be trusted if $g < 1$ (which is justified for all reasonable parameter values). The expression above is an analytic function in the complex t-plane with singularities along the imaginary axis. For $\mu > \mu_0$ the nearest singularities are located at $t = -i(1-g)$ and $t = i$. The locations of the nearest singularities in the product $\varphi_0(t)\,f(\omega,\mu,\mu_0,it)$ determine the asymptotic behavior of the LCDA for $\omega \to 0$ (lower half-plane) and $\omega \to \infty$ (upper half-plane). If we assume that the function $\phi_+^B(\omega,\mu_0)$ at the low scale vanishes like ω^δ for small ω and falls off like $\omega^{-\xi}$ for large ω (exponential fall-off would correspond to $\xi \to \infty$), then the nearest poles in $\varphi_0(t)$ are located at $t = -i\delta$ and $t = i\xi$, respectively. It follows that after evolution effects

$$\phi_+^B(\omega,\mu) \sim \begin{cases} \omega^{\min(1,\delta+g)}\,; & \text{for } \omega \to 0, \\ \omega^{-\min(1,\xi)+g}\,; & \text{for } \omega \to \infty. \end{cases} \quad (19)$$

Irrespective of the initial behavior of the LCDA, evolution effects drive it toward a linear growth at the origin and generate a radiative tail that falls off slower than $1/\omega$ even if the initial function has an arbitrarily rapid fall-off. This implies, in particular, that the normalization integral of $\phi_+^B(\omega,\mu)$ is UV divergent for large values of ω. This fact, which was already noted in [1], is not an obstacle to our analysis. Convolution integrals appearing in QCD factorization theorems are always of the form (1), in which the extra $1/\omega$ factor suppresses the contributions from large ω and renders the integral finite.

The solution of the evolution equation (11) for the hard-scattering kernel proceeds in a similar way, except that in the expression for the function f in (14)

one must replace $\mu_0 \to \mu_i$, $g(\mu,\mu_0) \to -g(\mu,\mu_i)$, and $U(\mu,\mu_0,\eta) \to -U(\mu,\mu_i,-\eta)$. As mentioned in the Introduction, at fixed order in perturbation theory and at an intermediate scale $\mu_i \sim \sqrt{m_b\Lambda}$, the kernel depends on ω only through logarithms of the type $\ln(2E\omega/\mu_i^2)$, which are of order 1 and so do not need to be resummed. We can therefore write $T(\omega,\mu_i) \equiv \mathcal{T}[\ln(2E\omega/\mu_i^2),\ldots]$, where the dots represent other arguments independent of ω. (Some of these arguments may contain large logarithms, which must be resummed separately.) It follows that by considering derivatives of the solution with respect to η (evaluated at $\eta=0$) we can satisfy arbitrary initial conditions at the scale μ_i. We can then solve (11) and compute the hard-scattering kernel at a scale $\mu < \mu_i$. The exact solution is given by

$$T(\omega,\mu) = \mathcal{T}[\nabla_\eta,\ldots] \left(\frac{2E\omega}{\mu_i^2}\right)^\eta \left(\frac{\omega}{\mu_i}\right)^{g(\mu_i,\mu)} \times \exp[-U(\mu,\mu_i,-\eta)]\Big|_{\eta=0}, \qquad (20)$$

where the notation $\mathcal{T}[\nabla_\eta,\ldots]$ means that one must replace each logarithm of the ratio $2E\omega/\mu_i^2$ in the initial condition by a derivative with respect to the auxiliary parameter η. It follows that the kernel scales like $T \sim \omega^{g(\mu_i,\mu)}$ modulo logarithms.

From the explicit solutions given above it can be seen that the convolution integral in (1) is indeed independent of the renormalization scale. For this, it is necessary to move the integration contour in (17) into the upper complex t plane by an amount $\eta + g(\mu_i,\mu_0)$. Note, however, that the product $T(\omega,\mu)\,\phi_+^B(\omega,\mu)$ for fixed ω is not scale independent. We also stress that evolution effects mix different (fractional) moments of the LCDA. For instance, the first inverse moment of the LCDA at a scale μ, which is sometimes called $1/\lambda_B(\mu)$, is related to a fractional inverse moment of order $1-g(\mu,\mu')$ at a different scale μ'. As a result, the scale dependence of the parameter $\lambda_B(\mu)$ is not calculable in perturbation theory. Controlling it would require knowledge of the functional form of the LCDA.

To illustrate our results we consider a scenario where the hard-scattering kernel at a high scale μ_i with $\alpha_s(\mu_i) = 0.3$ takes the simple form $T(\omega,\mu_i) = 1$, and where the LCDA at a low hadronic scale μ_0 with $\alpha_s(\mu_0) = 1$ assumes the form $\phi_+^B(\omega,\mu_0) = (\omega/\lambda_B^2)\,e^{-\omega/\lambda_B}$ suggested in [1], for which

$$\varphi_0(t) = \frac{1}{\lambda_B}\left(\frac{\mu_0}{\lambda_B}\right)^{it} \Gamma(1-it)\,. \qquad (21)$$

Figure 2 shows the results for the LCDA and the kernel at three different values of the renormalization scale. For simplicity, the anomalous dimensions and β-function are evaluated at one-loop order. Whereas the kernel exhibits a smooth power-like behavior, the most characteristic feature of the evolution of the LCDA is the development of a radiative tail for large values of ω. In this example the value of the convolution integral (1) is about 25% smaller than the value $1/\lambda_B$ which one would obtain without evolution effects.

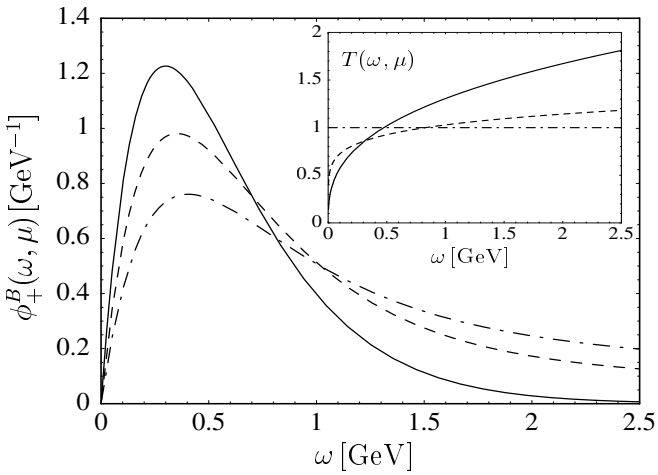

FIG. 2. Results for the LCDA and the kernel (inset) for different values of the renormalization scale such that $\alpha_s = 1$ (solid), 0.5 (dashed), 0.3 (dashed-dotted). We use $\lambda_B = 0.3\,\mathrm{GeV}$ at $\mu = \mu_0$.

4. Conclusions. We have derived evolution equations for the leading-order B-meson LCDA and for the hard-scattering kernels entering the spectator term in QCD factorization theorems for exclusive B decays into light particles. Simple scaling relations are obtained for the asymptotic behavior of these quantities. Exact analytic solutions to the evolution equations are given in terms of integrals over anomalous dimension functions. This accomplishes the resummation of large Sudakov double logarithms to all orders in perturbation theory.

Acknowledgment: We are grateful to Richard Hill for useful discussions. This research was supported by the National Science Foundation under Grant PHY-0098631.

[1] A. G. Grozin and M. Neubert, Phys. Rev. D **55**, 272 (1997) [hep-ph/9607366].

[2] M. Beneke, G. Buchalla, M. Neubert and C. T. Sachrajda, Phys. Rev. Lett. **83**, 1914 (1999) [hep-ph/9905312]; Nucl. Phys. B **591**, 313 (2000) [hep-ph/0006124]; Nucl. Phys. B **606**, 245 (2001) [hep-ph/0104110].

[3] G. P. Lepage and S. J. Brodsky, Phys. Rev. D **22**, 2157 (1980).

[4] G. P. Korchemsky and A. V. Radyushkin, Nucl. Phys. B **283**, 342 (1987); I. A. Korchemskaya and G. P. Korchemsky, Phys. Lett. B **287**, 169 (1992).

[5] S. W. Bosch, R. J. Hill, B. O. Lange and M. Neubert, preprint hep-ph/0301123.