Revised α^4 term of lepton $g-2$ from the Feynman diagrams containing an internal light-by-light scattering subdiagram

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(Dated: October 24, 2002)

The α^4 contribution to the lepton $g-2$ from a gauge-invariant set of 18 Feynman diagrams containing a light-by-light scattering subdiagram internally has been reevaluated by a method independent of the previous approach. Comparison of two methods revealed a program error in the first version. Correcting this error, the contributions of these 18 diagrams become -0.990 72 (10)$(\alpha/\pi)^4$ and -4.432 43 (58)$(\alpha/\pi)^4$ for the electron and muon $g-2$, respectively.

PACS numbers: 14.60.Cd, 14.60.Ef, 12.20.Ds, 13.40.Em

Precise theoretical and experimental values of lepton anomalous magnetic moments ($a_l = (g_l-2)/2$) provide one of the most stringent tests of QED [1]. Furthermore, since the muon anomalous magnetic moment a_μ is sensitive to short-distance physics, high precision measurement of a_μ at Brookhaven National Laboratory may be able to open the first window to "new physics" [2]. Before taking the discovery of "new physics" in the muon $g-2$ seriously, however, we must make sure that the old physics, namely the Standard Model, is known with sufficiently high precision.

The largest source of theoretical uncertainty for a_μ is the hadronic contribution [3, 4]. Unfortunately we are currently unable to deal with the hadronic correction from first principles because of nonperturbative nature of QCD. On the other hand, the QED correction can be treated precisely by perturbation theory. In order to achieve the precision comparable to that of measurements, the QED calculation for lepton $g-2$ must include terms of up to eighth-order of perturbation theory. Leading contributions of tenth-order are also relevant for a_μ [5, 6].

The purpose of this paper is to correct a program error in the previous calculation of gauge-invariant set of 18 Feynman diagrams contributing to the α^4 QED term [5]. This was accomplished by constructing alternative forms of integrals for these diagrams. As a consequence all 891 Feynman diagrams contributing to the eighth-order term of a_e, and additional diagrams contributing to $a_\mu - a_e$, have now been verified by independent calculations and/or checked by analytic comparison with lower-order integrals. This enables us to pursue with confidence an order of magnitude improvement in numerical precision of all α^4 terms of a_μ and a_e. The results will be reported shortly elsewhere [7, 8].

Let us now describe how the error was discovered and corrected. The contribution of the QED diagrams to a_μ can be written in the general form

$$a_\mu(\text{QED}) = A_1 + A_2(m_\mu/m_e) + A_2(m_\mu/m_\tau) + A_3(m_\mu/m_e, m_\mu/m_\tau), \quad (1)$$

where m_e, m_μ, and m_τ are the masses of the electron, muon, and tau, respectively. A similar equation holds for a_e. Throughout this article we shall use the values m_e = 0.510 998 902 (21) MeV/c^2, m_μ =105.658 357 (5) MeV$/c^2$, and m_τ = 1 776.99 (+29,-26) MeV$/c^2$ [9].

The renormalizability of QED guarantees that the functions A_1, A_2,

and A_3 can be expanded in power series in α/π with finite calculable coefficients:

$$A_i = A_i^{(2)}\left(\frac{\alpha}{\pi}\right) + A_i^{(4)}\left(\frac{\alpha}{\pi}\right)^2 + A_i^{(6)}\left(\frac{\alpha}{\pi}\right)^3 + \ldots, \quad i = 1,2,3. \quad (2)$$

$A_1^{(2)}$, $A_1^{(4)}$ and $A_1^{(6)}$ are known analytically [10]. Most terms contributing to $A_1^{(8)}$ have not yet been obtained by analytic means. The available value of $A_1^{(8)}$ is a consequence of numerical integration. Its precision is being improved by an extensive computer calculation right now [7]. For the purpose of evaluating a_μ(QED), however, it is sufficient to use $A_1^{(8)}$ derived from the measured value of the electron anomaly a_e [11] corrected for small contributions of muon, hadron, and weak interactions.

The terms $A_2^{(4)}(m_\mu/m_e)$ and $A_2^{(6)}(m_\mu/m_e)$ are known exactly [12]. The situation is quite different for $A_2^{(8)}(m_\mu/m_e)$ since most terms contributing to it are known only by numerical means.

There are altogether 469 Feynman diagrams contributing to $A_2^{(8)}(m_\mu/m_e)$. Of these diagrams 343 have been checked by more than one independent methods, some

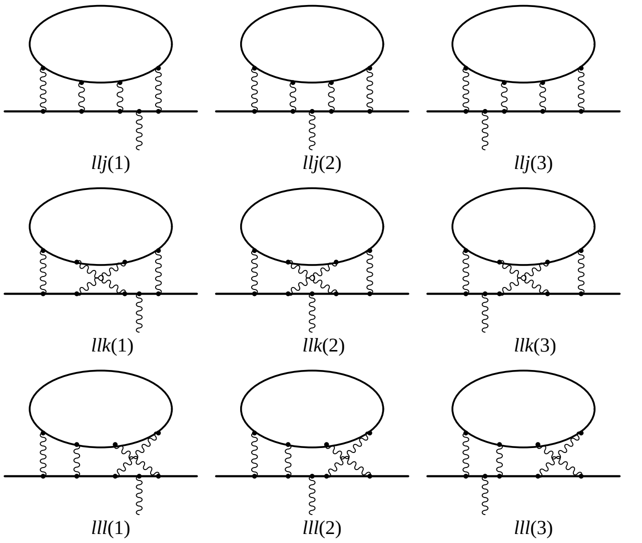

FIG. 1: Vertex diagrams containing a light-by-light scattering subdiagram internally. There are altogether 18 such diagrams.

of which being even analytic. Other 108 diagrams, all of which contain an *external* light-by-light scattering subdiagram, have been checked analytically by comparison with the exactly known sixth-order vertices [8]. Unfortunately, this was not the case for the remaining 18, all generated by inserting a light-by-light scattering subdiagram *internally* in a fourth-order vertex diagram (see Fig. 1). Here *external* and *internal* means whether one of the attached photon lines represents an external magnetic field or not.

These 18 diagrams form a gauge-invariant set and share the same basic algebraic structure. Unfortunately, it was not possible to examine their analytic structure by comparison with lower-order diagrams since they are not reducible to such diagrams in the UV and/or IR limits. Besides having been checked by two people working independently [13], the only check made was mutual consistency among these 18 diagrams. This is not sufficient to eliminate the possibility that they share the same program error. Clearly, in order to enhance the credibility of the QED calculation of a_l, it is highly desirable to reevaluate these 18 diagrams by more than one method.

This paper reports the consequence of our effort to reexamine the previous result by construction of new and independent integrals. Before describing the new approach, let us quickly go over the initial approach. In that approach (which will be called *Version A*)[13], we put together vertex diagrams, for instance $llj(1), llj(2), llj(3)$, all obtained from the self-energy-like diagram llj of Fig. 2 as terms linear in the magnetic field in the weak field expansion. This is to take advantage of their shared structure and the tendency of partial cancellation among them. With the help of Ward-Takahashi identity the

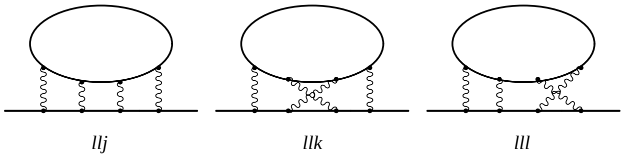

FIG. 2: Self-energy-like diagrams in which lepton lines propagate in the magnetic field. First-order terms in the weak magnetic field expansion correspond to the diagrams of Fig. 1.

sum $\Lambda^\nu(p,q)$ of all vertex diagrams thus related to a self-energy diagram $\Sigma(p)$ can be expressed as

$$\Lambda^\nu(p,q) \approx -q^\mu \left[\frac{\partial \Lambda_\mu(p,q)}{\partial q_\nu}\right]_{q=0} - \frac{\partial \Sigma(p)}{\partial p_\nu}, \qquad (3)$$

where $(p+q/2)^2 = (p-q/2)^2 = m_l^2$. The $g-2$ term is projected out from the right-hand side of (3). In terms of Feynman parameters $z_1, z_2, \dots z_N$, the n-th order magnetic moment derived from (3) has a form

$$\begin{aligned} M^{(2n)} &= \Big(\frac{-1}{4}\Big)^n (n-1)! \int (dz) \Big(\frac{\boldsymbol{E}+\boldsymbol{C}}{n-1} \frac{1}{U^2 V^{n-1}} \\ &+ (\boldsymbol{N}+\boldsymbol{Z}) \frac{1}{U^2 V^n} \Big) , \end{aligned} \qquad (4)$$

where bold letters $\boldsymbol{E}, \boldsymbol{C}, \boldsymbol{N}$, and $\boldsymbol{Z}$ stand for the projection operators adapted for Fig. 2. See [14] for definitions of projection operators, U, V and (dz).

In order to check the validity of *Version A* based on Fig. 2 we evaluated these diagrams by a second method, called *Version B*, which is actually a straightforward parametrization of individual vertex diagrams of Fig. 1, without relying on the Ward-Takahashi identity. In this approach it is convenient to evaluate the sum $jkl(n) \equiv llj(n) + llk(n) + lll(n)$, where $n = 1, 2, 3$, since partial cancellation of singular terms occurs resulting in less singular behavior.

Diagrams of Fig. 1 (or Fig. 2) form a (formal) gauge-invariant set. But individual diagrams are UV-divergent, and must be regularized in advance, for instance by dimensional regularization, to enforce gauge invariance. For numerical evaluation, however, it turns out to be more convenient to combine it with a subtractive regularization. Let $F_m(d)$ be one of the integrals defined in d-dimension, where m is any one of $llj(1), \dots, lll(3)$. Let $G_m(d)$ be the subtraction term containing the light-by-light scattering tensor with zero external momenta $\Pi_{\mu\nu\sigma\rho}(0,0,0,0)$ as well as terms containing a sixth-order charge renormalization diagram. Let us rewrite $F_m(d)$ symbolically as

$$(F_m(d) - G_m(d)) + G_m(d), \qquad (5)$$

where "symbolically" means that subtraction is performed on the integrand before the integration is carried out. Now, we can safely take the limit $d \to 4$ for the first term since its integrand does not cause UV divergence. Of course, the second term $G_m(d)$ is singular for $d \to 4$. However, gauge invariance guarantees that the sum over all diagrams of Fig. 1 vanishes for any dimension d:

$$\sum_m G_m(d) = 0. \tag{6}$$

Thus we have only to compute $(F_m(4) - G_m(4))$ in the end.

The integrands of both *Version A* and *Version B* were generated by an algebraic program FORM [15]. Numerical integration is carried out by an adaptive-iterative Monte-Carlo routine VEGAS [16].

When we compared the numerical results of *Version A* and *Version B*, we were surprised to find that their values were significantly different. After an extensive detective work, we located a programming error in *Version A*, which resulted from an incomplete implementation of the $\boldsymbol{E}$ operation of (4) in the algebraic manipulation program: It left out some terms referring to the light-by-light loop subdiagram. Such a program error can be readily detected if the integral exhibits UV or IR divergence after renormalization is carried out. Unfortunately the particular error in the 18 diagrams caused no divergence and escaped scrutiny of two people. Once this error was corrected, both approaches gave identical numerical results. Thus we now have two sets of independent codes for the 18 diagrams that have been fully verified.

Numerical evaluation of these diagrams requires an enormous amount of computational effort. A systematic algorithm of computation to minimize human error is indispensable. Such a scheme was developed originally for the calculation of the sixth-order lepton $g-2$ [14, 17], and were thoroughly tested over the years [18]. It was later extended to the eighth-order [19]. Early results for the eighth-order term remained rather crude for many years. This is mainly due to the enormous size of the integrands which could not be handled adequately by the computers then available. More precise values have become available only in this decade thanks to the development of massively parallel computer, which enabled us to vastly increase the sampling statistics of VEGAS.

Enlarging sampling statistics, however, amplified the difficulty caused by a previously poorly understood problem in estimating errors in a computer calculation. This arises from the fact that computer calculation always deals with a finite number of digits. This means that the error estimate based on the assumption of normal distribution of errors must be modified to take the effect of rounding-off of digits into account. In our formulation in which subtractive renormalization of UV divergence as well as separation of IR divergences are carried out on the computer, this *digit deficiency problem* can distort error estimates seriously, or even prevent further iteration. Besides increasing the number of effective digits from real*8 to real*16 arithmetic, which was the most obvious and effective cure, various methods had to be devised to deal with the *digit deficiency error* [20].

New results of numerical integration of a_μ by *Versions A* and *B* are listed in Table I. The values of llj, llk, and lll listed were obtained using 5×10^9 sampling points per iteration and iterated 110, 219, 220 times, respectively. They were evaluated on $v1$ cluster at Cornell Theory Center. The calculation of *Version B* was carried out on Fujitsu VPP700E at the Computer and Information Division of RIKEN. For $jkl(1,3)$ 4.6×10^9 sampling points per iteration were used for 131 iterations. The program $jkl(2)$ shows less singular behavior. It was evaluated using 4.6×10^9 sampling points per iteration and iterated 60 times.

Results for the electron are listed in Table II. The values of llj, llk, and lll listed were obtained using 2×10^9 sampling points per iteration and iterated 160, 220, 180 times, respectively, on $v1$ cluster at Cornell Theory Center. For $jkl(1,3)$ 4.6×10^9 sampling points per iteration were used for 60 iterations on VPP700E. For $jkl(2)$ 4.6×10^9 sampling points per iteration were used and iterated 60 times.

Combining the results from Tables I and II, which we treat as statistically independent, we obtain the best estimate of the contribution to a_μ from the 18 Feynman diagrams of Fig. 1:

$$a^{(8)\mu}_{IV(d)} = -4.432\ 43\ (58) \left(\frac{\alpha}{\pi}\right)^4, \tag{7}$$

and a corresponding result for a_e

$$a^{(8)e}_{IV(d)} = -0.990\ 72\ (10) \left(\frac{\alpha}{\pi}\right)^4. \tag{8}$$

Here, superscripts $(8)\mu$ and $(8)e$ refer to the eight-order muon and electron $g-2$, the subscript IV to the group of all diagrams containing light-by-light scattering subdiagrams, and (d) to its subgroup shown in Fig. 1, which consists of all diagrams containing an internal light-by-light scattering diagram.

The new results (7) and (8) supersede the earlier values $-3.4387\ (533)\ (\alpha/\pi)^4$ [5] and $-0.7503\ (60)\ (\alpha/\pi)^4$ [21], respectively. The effect of this modification on a_μ(QED) is less than 1% of the overall eighth-order term (which is of the order of 130 $(\alpha/\pi)^4$), and thus does not affect comparison of experiment and theory significantly. On the other hand, the effect on a_e is $\sim -7.0 \times 10^{-12}$ which is about 16 % of the entire eighth-order term and is larger than the measurement uncertainty 4.3×10^{-12}. As a consequence, it reduces the inverse fine structure constant α^{-1} obtained from theory and measurement of a_e by $\sim 0.82 \times 10^{-6}$ or ~ 6 ppb. Currently all α^4 terms are being upgraded by an extensive numerical integration. Precise values of a_μ(QED) and a_e(QED) including these terms will be reported in [7, 8].

TABLE I: Muon $g-2$ contributions from the diagrams of Fig. 1. In *Version A* the Ward-Takahashi-summed $llj \equiv llj(1) + llj(2) + llj(3)$, $llk \equiv llk(1) + llk(2) + llk(3)$, and $lll \equiv lll(1) + lll(2) + lll(3)$ are calculated, while in *Version B* $jkl(1,3) \equiv llj(1) + llj(3) + llk(1) + llk(3) + lll(1) + lll(3)$, and $jkl(2) \equiv llj(2) + llk(2) + lll(2)$ are calculated.

Version A		*Version B*	
llj	6.389 802 (460)	jkl(1,3)	-3.509 978 (802)
llk	-7.763 474 (537)	$jkl(2)$	-0.921 589 (89)
lll	-3.059 704 (452)		
sum	—4.433 376 (840)	sum	-4.431 567 (806)

TABLE II: Electron $g-2$ contributions from the diagrams of Fig. 1. In *Version A* the Ward-Takahashi-summed $llj \equiv llj(1) + llj(2) + llj(3)$, $llk \equiv llk(1) + llk(2) + llk(3)$, and $lll \equiv lll(1) + lll(2) + lll(3)$ are calculated, while in *Version B* $jkl(1,3) \equiv llj(1) + llj(3) + llk(1) + llk(3) + lll(1) + lll(3)$, and $jkl(2) \equiv llj(2) + llk(2) + lll(2)$ are calculated.

Version A		*Version B*	
llj	2.551 223 (78)	jkl(1,3)	-0.872 717 (138)
llk	-1.873 801 (72)	$jkl(2)$	-0.117 959 (28)
lll	-1.668 182 (80)		
sum	-0.990 760 (133)	sum	-0.990 675 (141)

T. K. should like to thank RIKEN for the hospitality extended to him while a part of this work was carried out. The part of the material presented by T. K. is based upon work supported by the National Science Foundation under Grant No. PHY-0098631. The numerical work for *Version A* has been carried out on the "velocity cluster" computer at Cornell Theory Center, which receives funding from Cornell University, New York State, the National Center for Research Resources at the National Institute of Health, the National Science Foundation, the Defense Department Modernization Program, the United States Department of Agriculture, and the corporate partners. The numerical work of *Version B* has been carried out on the Fujitsu VPP700E at Computer and Information Division of RIKEN.

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