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Spectator Interactions in Soft-Collinear Effective Theory

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Abstract

Soft-collinear effective theory is generalized to include soft massless quarks in addition to collinear fields. This extension is necessary for the treatment of interactions with the soft spectator quark in a heavy meson. The power counting of the relevant fields and the construction of the effective Lagrangian are discussed at leading order in Λ/m_b. Several novel effects occur in the matching of full-theory amplitudes onto effective-theory operators containing soft light quarks, such as the appearance of an intermediate mass scale and large non-localities of operators on scales of order $1/\Lambda$. Important examples of effective-theory operators with soft light quarks are studied and their renormalization properties explored. The formalism presented here forms the basis for a systematic analysis of factorization and power corrections for any exclusive B-meson decay into light particles.

1 Introduction

Processes involving energetic light particles play an important role in particle physics. Examples are jet production in e^+e^- annihilation, B-meson decays into light particles, and many other hard QCD processes. The theoretical description of such processes is often complicated by the presence of soft and collinear singularities, which invalidate the application of the (local) operator product expansion. In some cases factorization theorems have been established, which provide a simplified description of the relevant observables at leading order in the limit $E \gg \Lambda$, where E is the characteristic energy of the process, and $\Lambda \sim \Lambda_{\rm QCD}$ is the scale of non-perturbative hadronic physics. Formal proofs of these factorization theorems are difficult and typically rely on a diagrammatic analysis of different momentum regions giving rise to leading-order contributions to the amplitude. It would be desirable to facilitate these proofs and make them more transparent. Even more challenging is to develop a systematic framework for the parameterization and classification of power corrections for observables that do not admit an expansion in local operators.

The proposal of an effective field theory for collinear and soft particles by Bauer et al. [1, 2, 3, 4] is an important step toward achieving this goal. This "soft-collinear effective theory" (SCET) lets us discuss factorization theorems and power corrections in terms of fields and operators rather than momentum regions of Feynman diagrams. While power counting in the effective theory is non-trivial due to the fact that the relevant operators are non-local, there is nevertheless hope for a controlled expansion of amplitudes in terms of hadronic matrix elements of SCET operators (multiplied by perturbative Wilson coefficient functions), whose structure is constrained by gauge invariance and Lorentz symmetry.

SCET has been applied to prove factorization and resum Sudakov logarithms for the endpoint region of the photon energy spectrum in $B \to X_s\gamma$ decays [1, 2], and to prove QCD factorization as established in [5] for the weak decays $B \to D\pi$ [6]. Applications to hard processes outside B physics, such as deep-inelastic scattering, Drell–Yan production, and deeply virtual Compton scattering, have been considered in [7]. Recently, the formulation of SCET has been extended beyond leading power [8, 9]. These analyses constitute a significant first step toward a theory of power corrections that is more general than a local operator product expansion.

In this work we present an extension of the previous formulation of SCET, which is necessary for the discussion of exclusive B decays into light particles, such as $B \to \pi\pi$, $B \to K^*\gamma$, and heavy-to-light form factors at large recoil. (The "soft contribution" to heavy-to-light form factors has been discussed previously in the context of SCET, both at leading order [2] and beyond [8, 9]. However, no systematic treatment of all leading-power contributions has been presented so far.) For all these cases QCD factorization theorems have been proposed [10, 11, 12, 13, 14], but have not yet been proved beyond next-to-leading order in perturbation theory. In fact, factorization theorems for exclusive B-meson decays into light particles are more complicated than those for the decays into a heavy-light final state such as $B \to D\pi$. In addition to a form-factor term, a hard-scattering contribution appears at leading power, which results from hard gluon exchange with the spectator quark in the B meson [10, 11]. This contribution involves a convolution of a hard-scattering kernel with light-cone distribution amplitudes for the final-state hadrons and the initial B meson. A proof of factorization for such spectator contributions has not been attempted so far. The formulation of SCET developed here provides for the first time the framework for a systematic discussion of factorization in all of these cases and others [15, 16, 17].

Whereas for highly energetic light mesons the relevance of light-cone distribution amplitudes to the description of exclusive processes is familiar from many applications of perturbative QCD, relatively little is known about the light-cone structure of heavy hadrons. At first sight, even the appearance of light-cone distributions for the B meson seems surprising, because (in the B-meson rest frame) all characteristic momentum scales are soft, of order Λ. Unlike for a fast light meson, there is thus no hierarchy between the different components of the momenta of the B-meson constituents. However, the kinematics of heavy-to-light decay processes ensures that (in some cases) only the projection of the soft spectator momentum along some light-like direction enters the decay amplitudes at leading power in Λ/m_b. Because of the softness of the relevant momentum scales the notion of twist is not appropriate for the characterization of B-meson light-cone distribution amplitudes, which instead should be categorized according to their canonical dimension. Some distinctive features between B-meson and light-meson distribution amplitudes have already been noted in the literature. Whereas there exists a single leading-twist distribution amplitude for light pseudoscalar and vector mesons, two independent B-meson distribution amplitudes appear at leading order in the heavy-quark expansion [18]. In the case of light mesons, the equations of motion imply relations between higher-twist distribution amplitudes and connect them with amplitudes of lower twist (see, e.g., [19, 20]). For heavy mesons instead, the equations of motion relate the leading-order two-particle amplitudes to certain three-particle amplitudes (corresponding to quark–antiquark–gluon Fock components) of higher dimension [14, 21].

The intrinsic softness of the B-meson internal dynamics complicates the understanding of factorization properties of decay amplitudes. A new element is the appearance of an intermediate scale of order $m_b\Lambda$, which arises from the scalar product of the soft spectator momentum l with a collinear momentum p_c. While this scale is perturbative (since formally $p_c \cdot l \gg \Lambda^2$) and thus should be integrated out from the low-energy effective theory, it nevertheless depends on the B-meson dynamics and is not simply fixed by kinematics. This is different from previous applications of heavy-quark effective theory (HQET) and SCET, where the large scales were fixed in terms of the b-quark mass and the large energy $E \sim m_b$ carried by collinear fields. The appearance of an intermediate scale naturally leads to non-local operators integrated along light-like directions, whose matrix elements define the B-meson distribution amplitudes. The presence of three widely separated scales ($m_b^2 \gg m_b\Lambda \gg \Lambda^2$) also complicates the perturbative structure of decay amplitudes. Sudakov double logarithms appear at every order in perturbation theory and must be resummed.

The remainder of this paper is organized as follows: In Section 2 we present the construction of the SCET relevant to exclusive B decays, introduce the relevant fields, discuss their power counting and gauge transformations, and derive the effective Lagrangian. Interactions between soft and collinear fields are studied in Section 3, and are shown to be absent at leading order. In Section 4 we discuss in detail the matching of current operators containing a soft light quark and a collinear quark onto operators in the effective theory. Several new features appear in this calculation, such as the emergence of the intermediate scale, large non-localities of operators on a scale $1/\Lambda$, and non-trivial couplings between soft quarks and collinear gluons. In Section 5 we show how reparameterization invariance can be used to constrain the functional dependence of short-distance coefficient functions on the separation between the component fields of non-local operators. The renormalization of such operators and the related resummation of Sudakov logarithms are briefly discussed. The matching of local four-quark operators onto operators in the SCET is studied in Section 6. This application is of relevance to QCD factorization theorems for many exclusive B decays. A surprising result is that, generically, higher-twist three-particle distribution amplitudes of a light final-state meson can contribute to the decay amplitude at leading power. In Section 7 we illustrate the implications of these findings for the factorization properties of B-meson decay amplitudes in some toy models. We present two examples, one where a standard QCD factorization formula can be established at leading power, and one where the factorization formula must be generalized due to non-trivial interactions of the soft spectator quark with collinear gluons. The results derived in this work suggest a new formulation of the SCET, in which operators are composed out of gauge-invariant building blocks, replacing the original quark and gluon fields. This formalism is developed in Section 8. In the new formulation operators are automatically gauge invariant and their structure is constrained only by Lorentz invariance. Finally, in Section 9 we explain how our power-counting scheme can be applied to describe the soft overlap contribution to heavy-to-light form factors, which in previous work was analyzed using a different formulation of the SCET [8, 9]. A summary of our findings is given in Section 10.

2 Ingredients of the effective theory

We start by discussing in detail the properties of the fields present in the low-energy theory, using the coordinate-space formulation developed by Beneke et al. [9] (and avoiding the hybrid momentum–position space representation and label-operator formalism employed in earlier papers on SCET). Although part of this discussion is a repetition of similar results presented in earlier work (see, in particular, the review [4]), our power counting is different from the one employed in [9], and consequently our results for the effective Lagrangian and external operators in SCET will be different from those already discussed in the literature.

Our focus in this paper is on exclusive B-meson decays into final states containing light, energetic particles. The invariant masses of the final-state hadrons are of order Λ, and the momenta of their constituents are predominantly collinear. It is often convenient to decompose momenta and gauge fields in the light-cone basis constructed with the help of two light-like vectors

$$n^\mu = (1,0,0,1)\,, \qquad \bar{n}^\mu = (1,0,0,-1)\,, \tag{1}$$

which obey $n^2 = \bar{n}^2 = 0$, and $n\cdot\bar{n} = 2$. An arbitrary 4-vector can be expanded as $p^\mu = \frac{1}{2}(p_+\bar{n}^\mu + p_- n^\mu) + p_\perp^\mu$ with $p_+ = n\cdot p$ and $p_- = \bar{n}\cdot p$. The components $(p_+, p_-, p_\perp)$ of a collinear momentum scale like $p_c \sim E(\lambda^2, 1, \lambda)$, where $E \sim m_b$ is the large energy release in the process, and $\lambda \sim \Lambda/E$ is the expansion parameter of the SCET. The initial B meson, on the other hand, consists of soft partons with momenta scaling like $p_s \sim E(\lambda,\lambda,\lambda)$. (This assumes that we work in the B-meson rest frame and subtract the static piece $m_b\, v^\mu$ from the b-quark momentum.) Note that in kinematical situations different from the ones considered here the scaling relations of soft and collinear momenta (and of the corresponding SCET fields) can be different. For instance, when the photon energy in inclusive $B\to X_s\gamma$ decays is near the kinematic endpoint, the hadronic final state X_s has an invariant mass of order $\sqrt{E\Lambda}$, which is much larger than Λ. It is then appropriate to introduce an expansion parameter $\lambda \sim \sqrt{\Lambda/E}$ [2].

In the kinematical situation of relevance to our discussion, the fields appearing in the low-energy effective theory are either soft or collinear. As in HQET, we introduce a soft heavy-quark field h_v defined in terms of the QCD field b by

$$h_v(x) = e^{im_b v\cdot x}\,\frac{1+\not{v}}{2}\,b(x)\,, \qquad \text{with} \quad \not{v}\,h_v = h_v\,, \tag{2}$$

where v is the B-meson velocity. The Fourier modes of the field h_v carry the soft residual momentum $k^\mu = p_b^\mu - m_b\, v^\mu$. The two components of the spinor b projected out in the definition of h_v are integrated out in the construction of the HQET. The soft light-quark field q_s and soft gluon fields A_s^μ are simply given by the corresponding QCD fields, restricted to the subspace of soft Fourier modes. Using the identity $n\cdot\bar{n}=2$, the Dirac field ψ_c of a collinear quark can be decomposed into two 2-component spinors

$$\xi_n = \frac{\not{n}\not{\bar{n}}}{4}\,\psi_c\,, \qquad \eta_n = \frac{\not{\bar{n}}\not{n}}{4}\,\psi_c\,, \qquad \text{with} \quad \not{n}\,\xi_n = \not{\bar{n}}\,\eta_n = 0\,. \tag{3}$$

The components of η_n are suppressed with respect to those of ξ_n by a factor $\lambda \sim \Lambda/E$ and are integrated out in the construction of the SCET. The collinear gluon fields $A_{c,n}^\mu$ are defined as in QCD, restricted however to the subspace of Fourier modes with collinear momenta. For the sake of simplicity, we will from now on drop the label v on heavy-quark fields, and the label n on collinear quark and gluon fields. It is understood that collinear fields always have their large momentum component in the n-direction, with $\bar{n}\cdot p_c > 0$.

From the scaling behavior of the various two-point functions of two soft or two collinear fields one can derive the scaling properties of these fields with the expansion parameter λ. The strategy here is to define the kinetic terms in the action to have scaling

λ^0, so that factors of λ only appear in vertices of the effective theory. It follows that the soft fields scale like h, $q_s \sim \lambda^{3/2}$ and $A_s^\mu \sim (\lambda, \lambda, \lambda)$, whereas the collinear fields scale like $\xi \sim \lambda$, $\eta \sim \lambda^2$, and $A_c^\mu \sim (\lambda^2, 1, \lambda)$. Note that the covariant derivatives $iD_s^\mu \equiv i\partial^\mu + gA_s^\mu$ and $iD_c^\mu \equiv i\partial^\mu + gA_c^\mu$ have homogeneous scaling laws when acting on soft or collinear fields, respectively.

Previous discussions of SCET have often introduced ultrasoft modes with momentum scaling $p_{us} \sim E(\lambda^2, \lambda^2, \lambda^2)$, because these modes can be coupled to collinear particles without taking them far off their mass shell. In cases where the expansion parameter scales like $\lambda \sim \sqrt{\Lambda/E}$ ultrasoft fields simply correspond to what we call soft modes in the present paper. In our case, where $\lambda \sim \Lambda/E$, there is no need to introduce ultrasoft modes as degrees of freedom in the effective theory, because there are no external ultrasoft particles present. Such modes would correspond to color fields extending over large distance scales of order m_b/Λ^2, which do not appear in QCD because of confinement.

Operators in the effective theory must be invariant under residual gauge transformations in the collinear and soft sectors (i.e., transformations that leave the scaling properties of the fields unaltered). Under a collinear gauge transformation $U_c(x)$ the collinear fields transform according to $\xi \to U_c\,\xi$ and $A_c^\mu \to U_c\,A_c^\mu\,U_c^\dagger + (i/g)\,U_c\,(\partial^\mu U_c^\dagger)$, whereas soft fields remain invariant. Likewise, under a soft gauge transformation $U_s(x)$ we have $h \to U_s\,h$, $q_s \to U_s\,q_s$, and $A_s^\mu \to U_s\,A_s^\mu\,U_s^\dagger + (i/g)\,U_s\,(\partial^\mu U_s^\dagger)$, while collinear fields remain invariant. Gauge invariance of operators built out of these fields can be restored by the introduction of collinear and soft Wilson lines defined as

$$\begin{aligned} W(x) &= P\exp\left(ig\int_{-\infty}^0 ds\,\bar n\cdot A_c(x+s\bar n)\right), \\ S(x) &= P\exp\left(ig\int_{-\infty}^0 dt\,n\cdot A_s(x+tn)\right), \end{aligned} \tag{4}$$

which can be visualized as color strings attaching to a quark field at point x and extending to infinity. (It does not matter whether the integrals in these expressions run from $-\infty$ to 0 or from 0 to $+\infty$.) The path-ordering symbol "P" is defined such that the gluon fields are ordered from left to right in order of decreasing s or t values. We also need the conjugate operators $W^\dagger(x)$ and $S^\dagger(x)$, which are given by analogous expressions with ig replaced by $-ig$, and with the opposite ordering of the fields. Under a collinear gauge transformation $W(x) \to U_c(x)\,W(x)\,U_c^\dagger(-\infty)$, whereas $S(x)$ remains invariant. Similarly, under a soft gauge transformation $S(x) \to U_s(x)\,S(x)\,U_s^\dagger(-\infty)$, whereas $W(x)$ remains invariant. If, without loss of generality, we agree that the fields do not transform at infinity, then it follows that $W^\dagger(x)\,\xi(x)$ is gauge invariant, as are $S^\dagger(x)\,h(x)$ and $S^\dagger(x)\,q_s(x)$. The Wilson lines satisfy several important properties, the most useful ones being

$$W^\dagger\,i\bar n\cdot D_c\,W = i\bar n\cdot\partial\,, \qquad \frac{1}{i\bar n\cdot D_c + i\epsilon} = W\,\frac{1}{i\bar n\cdot\partial + i\epsilon}\,W^\dagger\,, \tag{5}$$

and corresponding relations for S. Also note that $(i\bar n\cdot D_c\,W) = 0$ and $(in\cdot D_s\,S) = 0$ by definition.

A particularly important object in our discussion below is the combination $\mathcal{A}_c^\mu = W^\dagger(iD_c^\mu W)$, which is invariant under both collinear and soft gauge transformations. By definition $\bar{n}\cdot\mathcal{A}_c = 0$, but the other components of $\mathcal{A}_c^\mu$ are non-zero. In the light-cone gauge $\bar{n}\cdot A_c = 0$, we have $W = 1$ and hence $\mathcal{A}_c^\mu = gA_c^\mu$. In an arbitrary gauge, we find the useful representation

$$\mathcal{A}_c^\mu(x) = \left[W^\dagger(iD_c^\mu W)\right](x) = \int_{-\infty}^0 dw\, \bar{n}_\alpha \left[W^\dagger g G_c^{\alpha\mu} W\right](x+w\bar{n})\,, \tag{6}$$

which makes explicit that $\mathcal{A}_c^\mu$ is a pure color octet, $\mathcal{A}_c^\mu = \mathcal{A}_c^{\mu,a}\, t_a$. (Here and below color indices on gluon fields will appear as superscripts, while subscripts "c" and "s" always refer to "collinear" or "soft", respectively.) To derive this formula one notes that both sides are gauge invariant, and that the result is obviously correct in the light-cone gauge. The reader should think of the object $\mathcal{A}_c$ as an insertion of a collinear gluon field, remembering however that this quantity is gauge invariant.

The effective Lagrangian for soft and collinear fields can be derived by systematically integrating out the hard modes (including the small-component fields for heavy and collinear quarks) from the QCD Lagrangian, and expanding the result in powers of λ. It is convenient to split up the answer into several terms,

$$\mathcal{L}_{\rm SCET} = \mathcal{L}_h + \mathcal{L}_s + \mathcal{L}_c + \mathcal{L}_g + \mathcal{L}_{sc}\,. \tag{7}$$

The effective Lagrangian for heavy quarks is the familiar HQET Lagrangian [22]

$$\mathcal{L}_h = \bar{h}\, iv\cdot D_s\, h + \frac{1}{2m_b}\left[\bar{h}\,(iD_{s\perp})^2\, h + C_{\rm mag}(\mu)\,\bar{h}\,\frac{g}{2}\,\sigma_{\mu\nu}\, G_s^{\mu\nu}\, h\right] + O(1/m_b^2)\,. \tag{8}$$

Let us note parenthetically that interactions between heavy quarks and soft gluons are, strictly speaking, not allowed in the low-energy theory, since they put the heavy quark off-shell by an amount $(m_b v + k)^2 - m_b^2 \sim E\Lambda$. These interactions can be integrated out, leading to the replacement $h \to S_v\, h_0$, where S_v is a Wilson line defined in analogy with (4) but with n replaced by the B-meson velocity v. The field h_0 is sterile in the sense that it does not couple to soft or collinear gluons. The heavy-quark Lagrangian is then simply $\mathcal{L}_h = \bar{h}_0\, iv\cdot\partial\, h_0 + O(1/m_b)$. In all expressions for operators containing heavy-quark fields the replacement $h \to S_v\, h_0$ must be made as well. Since this has no effect on the Feynman rule for the soft-gluon couplings, rather than introducing the new string operator S_v we will follow the usual convention of using the HQET field h with its soft interactions as given by (8).

The Lagrangian for a soft massless quark is the usual Dirac Lagrangian (it would be straightforward to include a small mass term)

$$\mathcal{L}_s = \bar{q}_s\, i\rlap{\,/}D_s\, q_s\,. \tag{9}$$

The effective Lagrangian for collinear quarks, which is obtained by integrating out the small-component field η, has a more interesting structure [2]. With our notations it reads

$$\begin{aligned}\mathcal{L}_c(x) &= \bar\xi\,\frac{\not{\bar n}}{2}\,in\cdot D_c\,\xi + \bar\xi\,\frac{\not{\bar n}}{2}\,i\not{D}_{c\perp}\,\frac{1}{i\bar n\cdot D_c + i\epsilon}\,i\not{D}_{c\perp}\,\xi \\ &= \bar\xi(x)\,\frac{\not{\bar n}}{2}\,in\cdot D_c\,\xi(x) - i\int_{-\infty}^0 ds\,\big[\bar\xi\,i\overleftarrow{\not{D}}_{c\perp}W\big](x)\,\frac{\not{\bar n}}{2}\,\big[W^\dagger i\not{D}_{c\perp}\,\xi\big](x+s\bar n)\,. \qquad (10)\end{aligned}$$

In [9] it has been argued that the choice of the $+i\epsilon$ prescription is arbitrary and not dictated by the QCD Lagrangian. A regularization of the inverse differential operator is necessary but bears no physical implications. The Lagrangian $\mathcal{L}_c$ sums up an infinite number of leading-order couplings between collinear quarks and (scalar or longitudinal) gluons. In the absence of sources the collinear Lagrangian is related to the QCD Lagrangian by a Lorentz boost, and so the two must be equivalent. As a result, the collinear Lagrangian is exact to all orders in λ, and it is not renormalized [9]. Finally, the pure-glue Lagrangian $\mathcal{L}_g$ in (7) takes the same form as in QCD, including gauge-fixing and ghost terms. However, it is understood that no term in this Lagrangian couples soft to collinear gluon fields. Those couplings, if present, would be part of $\mathcal{L}_{sc}$.

3 Soft-collinear interactions

As explained in [9], there are no vertices in the effective theory that connect the two heavy-quark fields to any number of collinear fields, because such interactions are kinematically forbidden at tree level. By the Coleman–Norton theorem [23] they can therefore not lead to on-shell singularities in any QCD diagram, and hence there is no need to include such interactions as part of the effective Lagrangian. However, the same argument does not apply to the interactions between collinear particles and soft light fields. We will now investigate these interactions in more detail.

It is kinematically allowed to couple soft fields to collinear fields only if the total soft momentum $p_s^{\rm tot}$ satisfies the condition $n\cdot p_s^{\rm tot} = O(E\lambda^2)$, and the difference between the ingoing and outgoing collinear momenta is such that $\bar n\cdot(p_c^{\rm out} - p_c^{\rm in}) = O(E\lambda)$. These constraints imply a power counting for the measure d^4x in the soft-collinear action $\int d^4x\,\mathcal{L}_{sc}$ that is different from that for the interactions of soft or collinear fields among themselves. Usually the measure d^4x is counted as λ^{-4}, since it eliminates either a collinear or a soft momentum, with $d^4p_c \sim d^4p_s \sim \lambda^4$. As a result, the operators appearing at leading order in the Lagrangians $\mathcal{L}_h$, $\mathcal{L}_s$, $\mathcal{L}_c$ and $\mathcal{L}_g$ scale like λ^4. On the other hand, when a term in the soft-collinear Lagrangian $\mathcal{L}_{sc}$ is integrated over d^4x, this produces δ-functions

$$\delta(\bar n\cdot p_c^{\rm out} - \bar n\cdot p_c^{\rm in} - \bar n\cdot p_s^{\rm tot})\,\delta(n\cdot p_c^{\rm out} - n\cdot p_c^{\rm in} - n\cdot p_s^{\rm tot})\,\delta^{(2)}(p_{c\perp}^{\rm out} - p_{c\perp}^{\rm in} - p_{s\perp}^{\rm tot})\,. \qquad (11)$$

In each term we must eliminate one of the largest momentum components, so that the remaining momenta are unconstrained. It follows that the first δ-function eliminates an integral over a minus component of a collinear momentum, the second δ-function

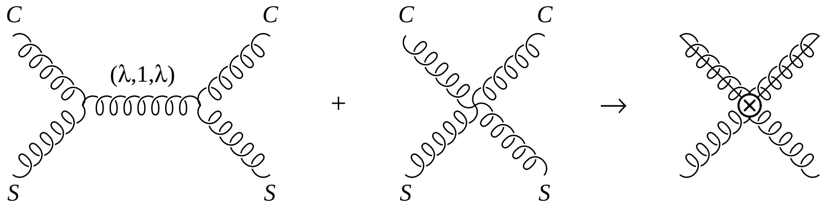

Figure 1: Effective four-gluon coupling obtained after integrating out the off-shell gluon in the first diagram and adding the corresponding local QCD vertex. The resulting SCET operator is denoted by a crossed circle. Collinear gluons in SCET are drawn as springs with a line. Soft and collinear gluons in full QCD are labeled by S and C.

eliminates an integral over a plus component of a soft momentum, and the last δ-function eliminates two components of a transverse momentum. It is important that the second δ-function must not be used to eliminate a plus component of a collinear momentum such as $n\cdot p_c^{\rm out}$, since for generic soft momenta the combination $n\cdot p_c^{\rm in}+n\cdot p_s^{\rm tot}$ would not be of order $E\lambda^2$. The eliminated momenta combined scale like $E^4\lambda^3$, and hence we must count the measure in the soft-collinear action as λ^{-3}. It follows that, at leading order in λ, only operators scaling like λ^3 can appear in the soft-collinear interaction Lagrangian.

We start by discussing soft-collinear interactions induced by the exchange of off-shell modes with momentum scaling like $E(\lambda,1,\lambda)$, which are generically produced when soft fields are coupled to collinear ones. In order to find the exact form of the corresponding terms in the Lagrangian one would have to integrate out the off-shell modes in the path integral. This is a difficult problem, whose solution we leave for future work. In the following we choose a "pedestrian" approach and match the resulting interaction terms involving two soft and two collinear fields perturbatively. We find that at leading order in λ there are interactions between two soft quarks and two collinear gluons, two collinear quarks and two soft gluons, and two soft and two collinear gluons. In the first two cases an off-shell quark propagator is integrated out, while in the latter case an off-shell gluon propagator occurs. As shown in Figure 1, in this case one must also include the local four-gluon vertex present in full QCD. We find that the resulting soft-collinear interactions connecting two collinear and two soft partons are obtained from the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{sc}^{(\rm induced)} &= -g^2\,\bar q_s\,A_{c-}\,\frac{\not{n}}{2}\,\frac{1}{i\partial_-}\,A_{c-}\,q_s - g^2\,\bar\xi\,A_{s+}\,\frac{\not{\bar n}}{2}\,\frac{1}{i\partial_+}\,A_{s+}\,\xi \\
&\quad - \frac{g^2}{4}\,f_{abe}f_{mne}\,\Bigg\{ A_{s+}^m\,\frac{1}{i\partial_+}\,A_{s+}^n\Big[A_{c-}^a\,i\partial_+\,A_{c-}^b - A_{c\perp\mu}^a\,i\partial_-\,A_{c\perp}^{\mu,b} + 2A_{c\perp\mu}^a\,i\partial_\perp^\mu\,A_{c-}^b\Big] \\
&\qquad + A_{c-}^a\,\frac{1}{i\partial_-}\,A_{c-}^b\Big[A_{s+}^m\,i\partial_-\,A_{s+}^n - A_{s\perp\mu}^m\,i\partial_+\,A_{s\perp}^{\mu,n} + 2A_{s\perp\mu}^m\,i\partial_\perp^\mu\,A_{s+}^n\Big] \\
&\qquad - \frac{1}{2}\,i\partial_{\perp\mu}\Big(A_{s+}^m\,\frac{1}{i\partial_+}\,A_{s+}^n\Big)\,i\partial_\perp^\mu\Big(A_{c-}^a\,\frac{1}{i\partial_-}\,A_{c-}^b\Big)\Bigg\}\,,
\end{aligned}
\tag{12}
$$

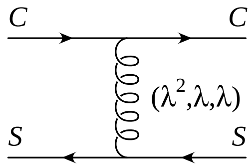

Figure 2: Example of a long-distance soft-collinear interaction induced by the exchange of a soft messenger gluon.

where we use the short-hand notation $A_+ = n\cdot A$ and $A_- = \bar{n}\cdot A$ etc. for brevity. Note that this Lagrangian is symmetric under the exchange of soft and collinear fields combined with $n \leftrightarrow \bar{n}$. That this is a symmetry of the soft-collinear interaction Lagrangian follows from the fact that a longitudinal Lorentz boost can be used to turn collinear fields into soft ones and vice versa.

It is also possible to couple a single soft field to two or more collinear fields. In that case the δ-functions in (11) enforce that the momentum of the soft field must scale like $E(\lambda^2,\lambda,\lambda)$. The smallness of the plus component of this momentum implies a phase-space suppression, which however is already taken into account by assigning scaling λ^{-3} rather than λ^{-4} to the measure d^4x in the action. Note that the soft parton is still off-shell by an amount of order Λ^2, as is any other soft mode. As illustrated in Figure 2, the soft field produced in such an interaction can interact with other soft particles, thereby acting as a messenger between the soft and collinear sectors of SCET. Let us add that, alternatively, we could study couplings of a single collinear field to two or more soft fields. In that case the δ-functions in (11) enforce that the momentum of the collinear field must scale like $E(\lambda^2,\lambda,\lambda)$, and the phase-space suppression is now reflected in the smallness of the minus component of this momentum. In order to avoid double counting, let us agree that collinear fields always have a large momentum component, so a field with momentum scaling like $E(\lambda^2,\lambda,\lambda)$ is considered part of a soft mode, not a collinear one. This convention breaks the "soft-collinear symmetry" mentioned at the end of the last paragraph. It is nevertheless reasonable, since the B meson defines a particular Lorentz frame, in which it is natural to consider a $(\lambda^2,\lambda,\lambda)$ mode as part of a soft mode. The asymmetry introduced by this choice could be avoided by introducing separate "soft-collinear" fields for the $(\lambda^2,\lambda,\lambda)$ modes and studying their interactions with soft and collinear particles. But since we will show that these modes are irrelevant at leading order, this would only lead to an unnecessary proliferation of notation.

There are three elementary vertices in QCD which are of order λ^3 and couple a soft field to collinear fields: the coupling of a soft gluon to two collinear quarks, the coupling of a soft gluon to two collinear gluons, and the coupling of a soft gluon to three collinear gluons. These interactions follow from the Lagrangian

$$\begin{aligned}\mathcal{L}_{sc}^{(\text{direct})} &= g\,\bar{\xi}\,\frac{\not{\bar{n}}}{2}\,A_{s+}\,\xi + \frac{g}{2}\,f_{abm}\,A_{s+}^m\,A_{c\mu}^a\big(2\partial^\mu A_{c-}^b - \partial_- A_c^{\mu,b}\big)\\ &\quad - \frac{g^2}{4}\,f_{abe}f_{mne}\,A_{c\mu}^a\,A_c^{\mu,m}\,A_{c-}^b\,A_{s+}^n\,. \end{aligned} \tag{13}$$

An important observation following from the calculations presented above is that at leading order in λ the soft-collinear interactions involve at least one A_{s+} or A_{c-} gluon field. We expect this feature to pertain also to interactions involving more than four partons. These interaction terms are "unphysical" in the sense that they involve scalar or longitudinal gluon polarizations and can be made to vanish by choosing the light-cone gauge conditions $n\cdot A_s = 0$ and $\bar{n}\cdot A_c = 0$. We will argue in Section 8 that any gauge-invariant operator in SCET can be constructed out of gauge-invariant building blocks, which are non-zero in light-cone gauge. This suggests that the interactions in (12) and (13) can be removed by field redefinitions (i.e., they vanish by the equations of motion). Let us demonstrate this explicitly for the terms involving quark fields. Consider the first term in (12), which couples two collinear gluons to two soft quarks. Since we have derived this term by matching an amplitude with two external gluons, we are free to replace $i\partial_-$ in the denominator by a covariant derivative. Next we use the identity

$$g\bar{n}\cdot A_c\, q_s = -i\bar{n}\cdot D_c\,(W-1)\, q_s + (W-1)\, i\bar{n}\cdot\partial\, q_s\,, \tag{14}$$

which follows from (5). The second term on the right-hand side is suppressed with respect to the first one by a power of λ and can be neglected. Repeated application of this identity (and its Dirac conjugate) yields, to leading order in λ,

$$\begin{aligned}
-g^2\,\bar{q}_s\,\frac{\not{n}}{2}\,A_{c-}\,\frac{1}{i\partial_-}\,A_{c-}\,q_s \to -\bar{q}_s\,\frac{\not{n}}{2}\,(W^\dagger-1)\,i\bar{n}\cdot D_c\,(W-1)\,q_s \\
&= -\bar{q}_s\,\frac{\not{n}}{2}\,(W^\dagger\, i\bar{n}\cdot D_c\, W - i\bar{n}\cdot D_c)\,q_s - 2\,\bar{q}_s\,\frac{\not{n}}{2}\,g\bar{n}\cdot A_c\,q_s \\
&= -\bar{q}_s\,\frac{\not{n}}{2}\,g\bar{n}\cdot A_c\,q_s\,.
\end{aligned} \tag{15}$$

This vanishes by momentum conservation (enforced by the integration over d^4x in the action), since with our convention for collinear fields it is impossible to couple a single collinear particle to soft fields. A derivation analogous to (15), but with all soft and collinear fields interchanged, can be used to show that the terms in (12) and (13) coupling two collinear quarks to one or two soft gluons vanishes at leading order:

$$-g^2\,\bar{\xi}\,A_{s+}\,\frac{\not{\bar{n}}}{2}\,\frac{1}{i\partial_+}\,A_{s+}\,\xi + g\,\bar{\xi}\,\frac{\not{\bar{n}}}{2}\,A_{s+}\,\xi \to 0\,. \tag{16}$$

Similar arguments should apply for the pure-glue interactions. We conclude that the soft-collinear Lagrangian in (7) vanishes to leading order, $\mathcal{L}_{sc}^{(\mathrm{LO})} = 0$, and that the $(\lambda^2,\lambda,\lambda)$ messenger modes of the type shown in Figure 2 are irrelevant to leading power in λ. These observations will be of crucial importance to the discussion of factorization in Sections 6 and 7.

We stress at this point that non-trivial soft-collinear interactions do occur at next-to-leading order in λ. For instance, a gauge-invariant coupling of two soft quarks to two collinear gluons can be written in the form

$$-\bar{q}_s\,S\,\not{A}_{c\perp}\,\frac{\not{n}}{2}\,\frac{1}{in\cdot\partial}\,\not{A}_{c\perp}\,S^\dagger q_s\,, \tag{17}$$

which cannot be made to vanish in any gauge. A similar interaction (with soft and collinear fields interchanged) can also be written for the coupling of two collinear quarks to two soft gluons. These terms are of order λ^4 and contribute at subleading power to the Lagrangian $\mathcal{L}_{sc}$. Because the soft and collinear Lagrangians (9) and (10) are exact, and the heavy-quark Lagrangian (8) is known beyond the leading order, a complete derivation of the soft-collinear interaction terms of order λ^4 is the last missing step in the construction of the SCET Lagrangian at next-to-leading order. We leave this derivation for future work.

4 Soft-collinear currents

An important application of the SCET formalism concerns the representation of external current operators (such as the flavor-changing operators arising in weak interactions) in terms of effective-theory fields. For the case of heavy-light currents the corresponding matching relation is [4]

$$\left[\bar{\psi}(x)\,\Gamma\,b(x)\right]_{\rm QCD} \to e^{-im_b v\cdot x}\sum_i C_i(m_b,E,\mu)\,\left[\bar{\xi}\,W\right](x)\,\Gamma_i\,\left[S^\dagger h\right](x) + \dots\,, \qquad (18)$$

where $E = \bar{n}\cdot p_c^{\rm tot}$ is the large component of the total collinear momentum, which is fixed by kinematics, and the dots represent higher-order terms in λ. The Dirac matrices Γ_i have the same transformation property as the original matrix Γ. For instance, in the case of a vector current we have $\Gamma = \gamma^\mu$ and $\Gamma_i = \{\gamma^\mu, v^\mu, n^\mu\}$. The SCET current operators are manifestly invariant under soft and collinear gauge transformations. Note that the string operators W and S sum up an infinite set of couplings (involving collinear and soft gluons) allowed at leading power. That this is done correctly can be checked explicitly by perturbative matching [2]. When a soft gluon couples to a collinear quark it produces a fluctuation with momentum scaling like $E(\lambda,1,\lambda)$, which is off-shell by an amount of order $E\Lambda$. Similarly, when a collinear gluon couples to a heavy quark it produces a fluctuation that is off-shell by an amount of order E^2. These modes remain off their mass shell when further soft or collinear gluons are coupled to them. When the off-shell modes are integrated out in the path integral, one reproduces the product of the two Wilson lines in (18) [4]. It is sufficient to keep the leading terms in the λ expansion at any stage in this derivation. The propagator for an off-shell collinear quark scales like $1/\Lambda$, and it combines with the soft gluon field $n\cdot A_s \sim \Lambda$ to give a leading-order contribution (other components of the soft gluon field do not contribute at leading power, since $\bar{\xi}\,\not{A}_s\,\not{n} = 2\,\bar{\xi}\,n\cdot A_s$). Similarly, the propagator for an off-shell heavy quark scales like $1/E$, and so only the leading component $\bar{n}\cdot A_c \sim E$ of the collinear gluon field must be kept.

Let us now study the analogous situation in which the soft heavy quark is replaced by a soft light quark. (We have been unable to find a phenomenological application of the resulting light-light soft-collinear current, so that this example is of academic value. Nevertheless, it will elucidate the novel features encountered in the presence of soft light quarks. The results derived in this section can readily be generalized to more realistic situations.) The naive guess $\bar\xi\, W\, \Gamma_i\, S^\dagger q_s$ for the resulting current operators in SCET is wrong for two reasons: first, the presence of an intermediate mass scale will lead to a non-locality of the resulting operators at large scales of order $1/\Lambda$; secondly, a more complicated structure of collinear fields will be induced in the matching process.

The appearance of an intermediate scale can already be seen at one-loop order in perturbation theory. When the matching is performed using on-shell external quark states (which is legitimate, since the Wilson coefficients are insensitive to infrared physics) with momenta l (soft) and $p_c = En$ (collinear), the only non-zero invariant is $l\cdot p_c = E\, n\cdot l$, which is a perturbative scale of order $E\Lambda$. However, the value of this scale depends on non-perturbative hadronic physics through its dependence on a component of a soft momentum. A one-loop matching calculation yields an expression of the form (the Dirac structure is preserved, so there is no need for a summation over matrices Γ_i)

$$C_\Gamma\left(\frac{El_+}{\mu^2}\right) \bar u_\xi(p_c)\, \Gamma\, u_{q_s}(l)\,, \tag{19}$$

where $l_+ = n\cdot l - i0$. In order to write this as the matrix element of an operator we replace l_+ by a derivative on the light-quark field and obtain

$$\int dl_+\, C_\Gamma\left(\frac{El_+}{\mu^2}\right) \bar\xi\, \Gamma\, \delta(l_+ - in\cdot\partial)\, q_s\,. \tag{20}$$

The δ-function indicates that the resulting operator is non-local on a scale of order $1/\Lambda$. Introducing the Fourier transform of the Wilson coefficient, the above result can be rewritten as

$$\int dt\, \widetilde C_\Gamma(t,E,\mu)\, \big[\bar\xi\, W\big](x)\, \Gamma\, \big[S^\dagger q_s\big](x+tn)\,, \tag{21}$$

where

$$\widetilde C_\Gamma(t,E,\mu) = \frac{1}{2\pi}\int dl_+\, e^{il_+ t}\, C_\Gamma\left(\frac{El_+}{\mu^2}\right). \tag{22}$$

The string operators W and S have been inserted here so that the resulting expression is gauge invariant. Note that by definition l_+ is now the plus component of the total momentum carried by the soft fields $S^\dagger q_s$. Likewise, $2E$ is the minus component of the total momentum carried by the collinear fields $\bar\xi\, W$. The non-locality of the soft-collinear current operator in SCET is a novel feature, which makes (21) more complicated than the corresponding expression (18) for heavy-light currents. (At tree level, however, $C_\Gamma = 1$ and $\widetilde C_\Gamma = \delta(t)$, so the non-locality is absent.) In the discussion of four-quark operators in Section 6, this will naturally introduce gauge-invariant soft quark bilinears of the form $[\bar h\, S](0)\dots[S^\dagger q_s](nt)$. The B-meson matrix elements of such operators define the leading-order light-cone distribution amplitudes [18].

Surprisingly, eq. (21) is still not the final answer for the representation of the current in SCET. In order to understand this, let us study in more detail what happens when a collinear gluon hits a massless soft quark. As indicated in Figure 3, the resulting off-shell

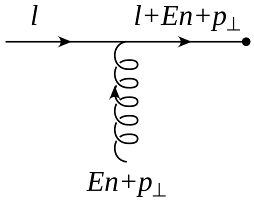

Figure 3: Attachment of a collinear gluon to a soft light quark.

mode has momentum $l+En+p_\perp$ scaling like $E(\lambda,1,\lambda)$. Keeping terms up to subleading order in λ, the corresponding propagator and vertex yield

$$\frac{i(E\not{n}+\not{p}_\perp+\not{l})}{2En\cdot l}\,ig\left(\frac{\not{n}}{2}\,\bar{n}\cdot A_c+\not{A}_{c\perp}\right)q_s=-g\left(\frac{\bar{n}\cdot A_c}{2E}+\frac{\not{n}}{2l_+}\left[\not{A}_{c\perp}-\not{p}_\perp\,\frac{\bar{n}\cdot A_c}{2E}\right]\right)q_s\,, \tag{23}$$

where we have used the equation of motion $\not{l}\,q_s=0$ for the light quark. The key point to note about this result is that the off-shell propagator scales like $1/\Lambda$, while the largest component of the collinear gluon field scales like E. The superficially largest term of order E/Λ (which would upset power counting) cancels. However, leading contributions arise from the subleading terms in both the propagator and the gluon field. While the first term in the above result corresponds to the expansion of the Wilson line W, the remaining terms correspond to the object $(i\not{D}_{c\perp}W)$. The factor $1/l_+$ associated with these terms gives rise to a non-locality even at tree level. A careful analysis (using perturbative matching) reveals that there is a second type of current operator in the effective theory, given by

$$-\int dt\,ds\,\widetilde{D}_\Gamma(t,s,E,\mu)\,\big[\bar{\xi}\,W\big](x)\,\Gamma\,\frac{\not{n}}{2}\,\not{\mathcal{A}}_{c\perp}(x+s\bar{n})\,\big[S^\dagger q_s\big](x+tn)\,, \tag{24}$$

where $\mathcal{A}_c^\mu$ is the gauge-invariant object defined in (6). The Wilson coefficient $\widetilde{D}_\Gamma$ is related by a double Fourier transformation to a momentum-space coefficient function,

$$\widetilde{D}_\Gamma(t,s,E,\mu)=\frac{1}{(2\pi)^2}\int dl_+\,dp_-\,e^{il_+t}\,e^{-ip_-s}\,D_\Gamma(l_+,p_-,E,\mu)\,, \tag{25}$$

where $p_-=\bar{n}\cdot p_g$ is the minus component of the collinear momentum carried by the field $\mathcal{A}_c$, and $E=\frac{1}{2}\,\bar{n}\cdot p_c^{\rm tot}$ is the large energy carried by all collinear fields. At tree level, we find $D_\Gamma=1/l_+$ and hence $\widetilde{D}_\Gamma=i\theta(t)\,\delta(s)$. We have checked by explicit calculation that the sum of the two expressions in (21) and (24) reproduces (at tree level) the current matrix elements with an arbitrary number of collinear gluons and no soft gluon, an arbitrary number of soft gluons and no collinear gluon, and one soft and one collinear gluons. The relevant diagrams for the latter case are shown in Figure 4. One must add to these graphs a contribution from the equation of motion $i\not{\partial}\,q_s=-g\not{A}_s\,q_s$ for the soft quark applied to the graph with only one external collinear gluon. Note that the transverse collinear gluon field in (24) appears together with a factor of $\not{n}$. We will show

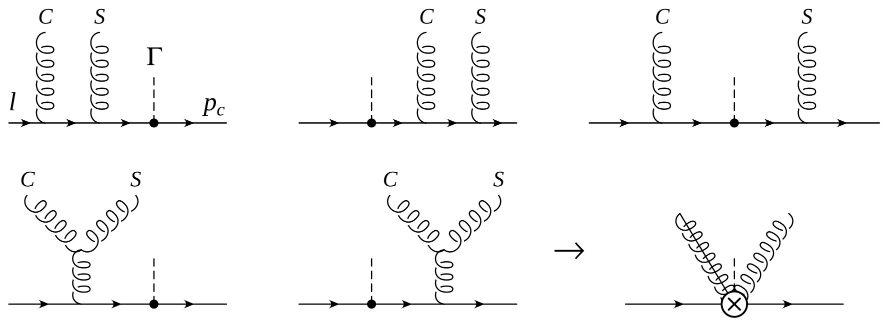

Figure 4: Diagrams contributing to the matching calculation for the soft-collinear current for the case of one soft and one collinear external gluon, and the resulting non-local operator in the SCET. The dashed line represents the current insertion.

in the next section that this is, in fact, required by reparameterization invariance. It is therefore not possible to obtain more than one insertion of the product $\not{n}\,\not{A}_{c\perp}$. Eq. (24) then gives the most general operator with a transverse collinear gluon insertion that is allowed by gauge and reparameterization invariance.

A formal argument justifying the result (24) for an arbitrary number of soft and collinear gluons would have to be based on integrating out the off-shell $(\lambda, 1, \lambda)$ modes of quarks and gluons in the path integral, generalizing the discussion presented in Appendix A of [4]. This is more complicated in the present case, however, because the transverse components and transverse derivatives of the gauge fields cannot be ignored. For the sake of simplicity we consider here only the case of an arbitrary number of collinear gluons attached to a soft quark, ignoring soft gluons. We then need to integrate out off-shell $(\lambda, 1, \lambda)$ modes ψ_X of the light quark, but there is no need to consider off-shell gluon fields. The QCD Lagrangian gives rise to the following interactions between the relevant on-shell and off-shell fields:

$$\mathcal{L}_X = \bar{\psi}_X\, i\not{D}_c\, \psi_X + \bar{\psi}_X\, g\not{A}_c\, q_s + \bar{q}_s\, g\not{A}_c\, \psi_X\,. \tag{26}$$

In the kinetic term for the off-shell field the full covariant derivative should appear, but within the approximation just described we need to keep only the collinear gauge field. Since the off-shell field carries a large momentum in the n direction it is useful to split it up into two 2-component fields ψ_n and $\psi_{\bar{n}}$ defined such that $\not{n}\,\psi_n = 0$ and $\not{\bar{n}}\,\psi_{\bar{n}} = 0$. The equations of motion satisfied by these fields are

$$\begin{aligned} in\cdot D_c\,\psi_n + \frac{\not{n}}{2}\, i\not{D}_{c\perp}\,\psi_{\bar{n}} + \frac{\not{n}}{2}\left(g\not{A}_{c\perp} + \frac{\not{\bar{n}}}{2}\, gn\cdot A_c\right) q_s = 0\,, \\ i\bar{n}\cdot D_c\,\psi_{\bar{n}} + \frac{\not{\bar{n}}}{2}\, i\not{D}_{c\perp}\,\psi_n + \frac{\not{\bar{n}}}{2}\left(g\not{A}_{c\perp} + \frac{\not{n}}{2}\, g\bar{n}\cdot A_c\right) q_s = 0\,. \end{aligned} \tag{27}$$

Solving the equation for $\psi_{\bar{n}}$ to leading order in λ we obtain

$$\psi_{\bar{n}} = -\frac{\not{\bar{n}}\not{n}}{4}\,\frac{1}{i\bar{n}\cdot D_c}\, g\bar{n}\cdot A_c\, q_s + \dots = \frac{\not{\bar{n}}\not{n}}{4}\,(W-1)\, q_s + \dots\,, \tag{28}$$

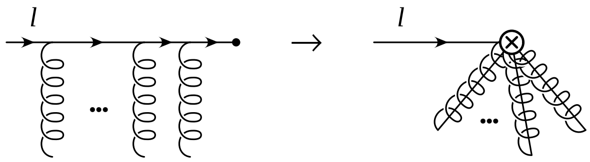

Figure 5: Multiple attachments of collinear gluons to a soft light quark, and the resulting non-local interaction in SCET.

where the dots denote higher-order terms, and to arrive at the second equality we have used the identity (14). Inserting the solution for $\psi_{\bar{n}}$ into the first equation in (27) yields

$$\begin{aligned} in\cdot D_c\,\psi_n &= -\frac{\not{n}}{2}\,(i\not{D}_{c\perp}W)\,q_s - \frac{\not{n}}{2}\,(W-1)\,i\not{\partial}_\perp\,q_s + \dots \\ &= -\frac{\not{n}}{2}\,(i\not{D}_{c\perp}W)\,q_s + \frac{\not{n}\,\not{\bar{n}}}{4}\,(W-1)\,in\cdot\partial\,q_s + \dots\,, \end{aligned} \tag{29}$$

where in the last step the equation of motion $i\not{\partial}\,q_s = 0$ for the light quark (in the absence of soft gluons) has been used. Because the off-shell field has momentum $n\cdot p_X \sim \lambda$, which is larger than $n\cdot A_c \sim \lambda^2$, we can replace the covariant derivative on the left-hand side of this equation by an ordinary derivative. Combining then the two results in (28) and (29), and using that the derivative $in\cdot\partial$ commutes with collinear fields to leading order in λ, we obtain

$$\psi_X = (W-1)\,q_s - \frac{\not{n}}{2}\,(i\not{D}_{c\perp}W)\,\frac{1}{in\cdot\partial}\,q_s + \dots\,. \tag{30}$$

Figure 5 illustrates that this result can be understood as a tree-level matching relation for a light quark (in the absence of soft gluons),

$$\begin{aligned} q(x)\big|_{\rm QCD} \to \big[q_s + \psi_X\big](x) &= W\left(1 - \frac{\not{n}}{2}\left[W^\dagger(i\not{D}_{c\perp}W)\right]\frac{1}{in\cdot\partial - i\epsilon}\right) q_s \\ &= W(x)\left[q_s(x) - i\,\frac{\not{n}}{2}\,\not{A}_{c\perp}(x)\int_0^\infty dt\,q_s(x+tn)\right], \end{aligned} \tag{31}$$

where we have chosen a $-i\epsilon$ prescription to regularize the inverse derivative on the light-quark field. This choice is consistent with the Feynman prescription for the propagator of an off-shell quark obtained by coupling a final-state collinear gluon to an initial state soft quark. Performing this replacement, along with $\psi \to \xi$, in the QCD current operator $\bar{\psi}\,\Gamma\,q$ precisely reproduces the tree-level structure of collinear fields in (21) and (24). The extension of this argument to include soft fields is left for future work. Note that the integro-differential operator on the right-hand side of the first equation in (31) is nilpotent in the sense that $\exp(1-\dots) = (1-\dots)$. Therefore, insertions of transverse collinear gluons on a soft light-quark line do not exponentiate.

It is, at first sight, surprising that the operator in (24) is of leading order in SCET power counting, because the extra transverse gluon field $\not{A}_{c\perp}$, which is absent in (21), scales like a power of λ. However, this suppression is compensated by the different behavior of the Wilson coefficient functions, $\widetilde{C}_\Gamma \sim \lambda$ and $\widetilde{D}_\Gamma \sim 1$. (In momentum space, the transverse derivative is compensated by the factor $1/l_+$ associated with the transverse terms in (23).) It is evident from this example that the presence of large non-localities on the scale $1/\Lambda$ can upset naive SCET power counting. From our discussion so far it follows that one needs to know the t-dependence of the short-distance coefficients (or the l_+ dependence of the corresponding coefficients in momentum space) before one can decide whether a non-local operator such as (21) or (24) contributes at leading order in power counting. Fortunately, it is possible to deduce the t-dependence of the Wilson coefficients to all orders in perturbation theory without an explicit calculation. This is discussed in the next section.

5 Reparameterization invariance

Operators in SCET must be invariant under redefinitions of the light-cone basis vectors n and $\bar{n}$ that leave the scaling properties of fields and momenta unchanged [8, 24]. This property is referred to as reparameterization invariance, and it can be used to derive constraints on the Wilson coefficients of SCET operators, often relating the coefficients of some operators to those of others. Reparameterization invariance is a consequence of the invariance of QCD under Lorentz transformations, which is not explicit (but still present) in SCET because of the introduction of the light-cone vectors.

It is useful to distinguish between three classes of infinitesimal transformations, corresponding to two different transverse boosts and a longitudinal boost:

$$\begin{array}{llll} \text{Type I:} & n^\mu \to n^\mu + \epsilon_\perp^\mu \,, & \bar{n}^\mu \text{ invariant} & (\text{with } \epsilon_\perp^\mu \sim \lambda) \\ \text{Type II:} & \bar{n}^\mu \to \bar{n}^\mu + e_\perp^\mu \,, & n^\mu \text{ invariant} & (\text{with } e_\perp^\mu \sim 1) \\ \text{Type III:} & n^\mu \to n^\mu/\alpha \,, & \bar{n}^\mu \to \alpha \bar{n}^\mu & (\text{with } \alpha \sim 1) \end{array} \tag{32}$$

In parenthesis we give the scaling properties for the parameters of the corresponding finite transformations, which are relevant for power counting. Using the properties of the fields and Wilson lines under these transformation, as compiled in Table I of [24], it is straightforward to show that the current operators in (21) and (24) are separately invariant under type I and type II transformations to leading order in λ. In other words, reparameterization invariance links these operators with operators that appear at subleading order in λ. The only non-trivial point in this analysis concerns the transverse collinear derivative $D_{c\perp}^\mu$, which has non-vanishing variations at leading order in λ under both type I and type II transformations,

$$D_{c\perp}^\mu \overset{\text{type I}}{\to} D_{c\perp}^\mu - \frac{\epsilon_\perp^\mu}{2}\, \bar{n}\cdot D_c + O(\lambda^2)\,, \qquad D_{c\perp}^\mu \overset{\text{type II}}{\to} D_{c\perp}^\mu - \frac{n^\mu}{2}\, e_\perp \cdot D_{c\perp} + O(\lambda^2)\,. \tag{33}$$

However, in both cases the object $\not{n}\, W^\dagger (i\not{D}_{c\perp} W) = \not{n}\, \not{A}_{c\perp}$ is left invariant. For type I transformations this follows from $(\bar{n}\cdot D_c W) = 0$, whereas for type II transformations it

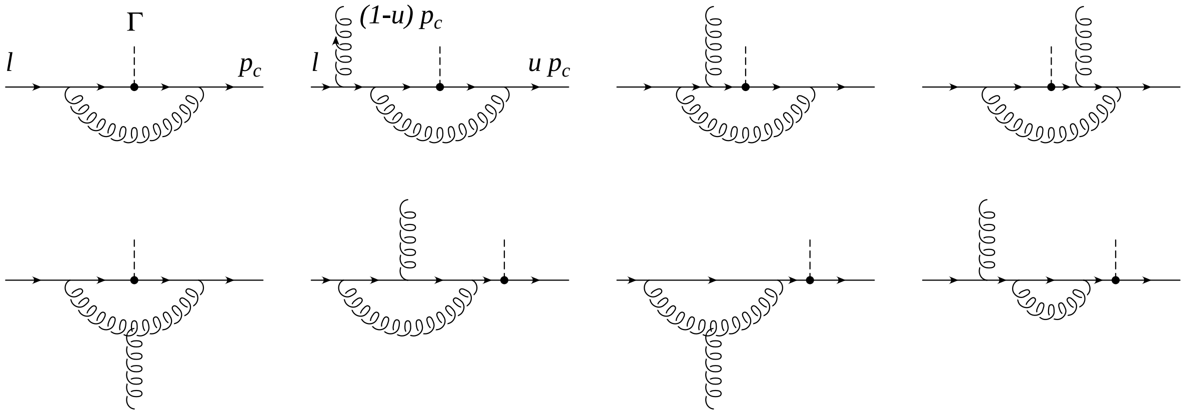

Figure 6: One-loop diagrams required to determine the matching coefficients C_Γ and D_Γ. The first diagram suffices to find the coefficient C_Γ, while the evaluation of the remaining graphs with an external collinear gluon is required to obtain D_Γ.

follows since $\not{n}^2 = 0$. We stress that, without the extra factor of $\not{n}$, the operator in (24) would not be invariant under a type II reparameterization.

The type III transformations have non-trivial consequences. We find that the current operators are invariant under these transformations only if their Wilson coefficients obey the homogeneity relations

$$\widetilde{C}_\Gamma(t,E,\mu) = \alpha\,\widetilde{C}_\Gamma(\alpha t,\alpha E,\mu)\,, \qquad \widetilde{D}_\Gamma(t,s,E,\mu) = \frac{1}{\alpha}\,\widetilde{D}_\Gamma(\alpha t, s/\alpha, \alpha E,\mu)\,. \tag{34}$$

Taking into account the canonical dimensions of these coefficients, it follows that

$$\begin{aligned} \widetilde{C}_\Gamma(t,E,\mu) &= \delta(t)\,c_\Gamma^{(1)}[\alpha_s(\mu)] + \frac{1}{t}\,c_\Gamma^{(2)}\big[\mu^2\,t/E,\alpha_s(\mu)\big]\,, \\ \widetilde{D}_\Gamma(t,s,E,\mu) &= \delta(s)\,d_\Gamma^{(1)}\big[\mu^2\,t/E,\alpha_s(\mu)\big] + \frac{1}{s}\,d_\Gamma^{(2)}\big[\mu^2\,t/E, sE,\alpha_s(\mu)\big]\,, \end{aligned} \tag{35}$$

where the coefficient functions $c_\Gamma^{(i)}$ and $d_\Gamma^{(i)}$ are dimensionless. Since the dependence of the Wilson coefficients on the renormalization scale is logarithmic, we conclude that to all orders of perturbation theory $\widetilde{C}_\Gamma(t,E,\mu) \sim 1/t \sim \Lambda$ and $\widetilde{D}_\Gamma(t,s,E,\mu)\,ds \sim 1$ modulo logarithms. With this information, it is now evident that the two types of current operators contribute at the same order in power counting.

The above argument based on longitudinal boost invariance determines the behavior of the momentum-space coefficients on the soft momentum l_+ to all orders in perturbation theory. This provides valuable information about the convergence of convolution integrals of hard-scattering kernels (Wilson coefficients) with the B-meson light-cone distribution amplitudes, which will be an important ingredient to factorization proofs.

It may be instructive to illustrate our results for the soft-collinear current with a concrete example. For the case of the vector current, the explicit expressions for the matching coefficients C_V and D_V obtained at one-loop order (in the $\overline{\rm MS}$ scheme, but before subtraction of the pole terms in $\epsilon = 2 - d/2$) read

$$\begin{aligned} C_V &= 1 + \frac{C_F\,\alpha_s(\mu)}{4\pi}\left(\frac{2El_+}{\mu^2}\right)^{-\epsilon}\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{\pi^2}{6}\right), \\ D_V &= \frac{1}{l_+}\left\{1 + \frac{\alpha_s(\mu)}{4\pi}\left(\frac{2El_+}{\mu^2}\right)^{-\epsilon}\left[C_F\,k_F(u,\epsilon) - \frac{C_A}{2}\,k_A(u,\epsilon)\right]\right\}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} k_F(u,\epsilon) &= -\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(1 - \frac{2\ln u}{1-u}\right) - \frac{\ln^2 u}{1-u} + \frac{4\ln u}{1-u} - 3 + \frac{\pi^2}{6}, \\ k_A(u,\epsilon) &= \frac{1}{\epsilon}\,\frac{2\ln u}{1-u} - \frac{\ln^2 u}{1-u} + \frac{4\ln u}{1-u} + \frac{2\ln(1-u)}{u}. \end{aligned} \tag{37}$$

The coefficient D_V depends in a non-trivial way on the fraction u of the total collinear momentum carried by the collinear quark (we define $\bar n\cdot p_q = 2uE$ and $\bar n\cdot p_g = 2(1-u)E$). To obtain these results we work with on-shell external quark and gluon states and use dimensional regularization to regulate both ultraviolet and infrared divergences. This ensures that all SCET loop diagrams vanish (since there is no large scale left in the low-energy theory), and so the matching calculation is reduced to the calculation of vertex graphs in the full theory [25, 26]. While the computation of C_V is a simple exercise, to obtain D_V requires evaluating the vertex and box diagrams with an external collinear gluon shown in Figure 6. A subtle point in this calculation is that there is a contribution to D_V resulting from the application of the equation of motion for the collinear quark in the first diagram. It is a highly non-trivial check of our result that the sum of the seven diagrams with an external gluon can be represented as the sum of two contributions corresponding to the two operators in (21) and (24), with the coefficient C_V of the first operator fixed by the u-independent expression in (36) obtained from the first diagram shown in the figure.

Taking the Fourier transforms of these results, we find the position-space coefficient functions

$$\begin{aligned} \widetilde C_V &= \delta(t) + \frac{C_F\,\alpha_s(\mu)}{4\pi}\,\frac{\theta(t)}{t}\left(\frac{\mu^2\,it\,e^{\gamma_E}}{2E}\right)^{\epsilon}\epsilon\left(-\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 + \frac{\pi^2}{6}\right), \\ \widetilde D_V &= i\theta(t)\left\{\delta(s) + \frac{\alpha_s(\mu)}{4\pi}\left(\frac{\mu^2\,it\,e^{\gamma_E}}{2E}\right)^{\epsilon}\right. \\ &\qquad \left.\times\,\frac{E}{\pi}\int du\,e^{-2iEs(1-u)}\left[C_F\,k_F(u,\epsilon) - \frac{C_A}{2}\,k_A(u,\epsilon)\right]\right\}, \end{aligned} \tag{38}$$

in accordance with the general forms in (35) predicted by reparameterization invariance. When taking the limit $\epsilon\to 0$ in the first result the pole at $t=0$ must be regularized, e.g. by using a plus distribution.

The scale dependence of the Wilson coefficient functions (in momentum or position space) is governed by renormalization-group equations, for instance

$$\mu^2 \frac{\partial}{\partial\mu^2} C_V(\mu) = \left[\frac{C_F\,\alpha_s(\mu)}{4\pi} \left(-2\ln\frac{\mu^2}{2El_+} - 3 \right) + O(\alpha_s^2) \right] C_V(\mu)\,. \tag{39}$$

The "anomalous dimension" in this equation contains logarithms involving the ratio of μ^2 to the intermediate scale El_+. This is a characteristic feature of amplitudes involving Sudakov logarithms. The integration of the renormalization-group equation in such cases leads to the exponentiation of the leading Sudakov double logarithms [1, 2].

6 Matching of four-quark operators

The discussion of the soft-collinear current presented in the previous two sections has elucidated many new features encountered in the interactions of a soft light quark with collinear gluons. We will now consider a slightly more complicated example of a matching calculation, which however is of greater phenomenological importance. Our goal is to match a local color-singlet four-quark operator of the type $O_{4q} = \bar\psi\,\Gamma_1\,T_1\,\psi\,\bar b\,\Gamma_2\,T_2\,q$ onto operators in SCET, in a kinematical situation where the quarks q and $\bar b$ are soft, while ψ and $\bar\psi$ are collinear. Here $\Gamma_{1,2}$ are arbitrary Dirac structures, and $T_1 \otimes T_2 = \mathbf{1}\otimes\mathbf{1}$ or $t_a \otimes t_a$ are color structures. The resulting operators in SCET have the spinor content $\bar\xi\dots\xi\,\bar h\dots q_s$. Such operators would arise, e.g., in the discussion of factorization for the hard-scattering term in the exclusive decay $B \to K^*\gamma$. (However, in general they are obtained by matching a non-local full-theory amplitude onto a four-quark operator in the SCET.)

The matching calculation for four-quark operators proceeds in analogy to the matching for the soft-collinear current discussed in Section 4. We find that again two types of operators appear at leading order in λ. At tree level, the result is (setting $x = 0$ for simplicity)

$$\begin{aligned} O_{4q}(0) \to\ & \big[\bar\xi\,W\,\Gamma_1\,T_1\,W^\dagger\xi\big](0)\,\big[\bar h\,S\,\Gamma_2\,T_2\,S^\dagger q_s\big](0) \\ & - \frac{i}{2}\int_0^\infty dt\,\big[\bar\xi\,W\,\Gamma_1\,T_1\,W^\dagger\xi\big](0)\,\big[\bar h\,S\,\Gamma_2\,T_2\,\not{n}\,\not{\mathcal{A}}_{c\perp}\big](0)\,\big[S^\dagger q_s\big](tn)\,. \end{aligned} \tag{40}$$

For final states containing up to two external gluons, we have checked the correctness of this expression explicitly using perturbative matching. Note, in particular, that the structure of collinear gluon fields follows from the matching relation (31).

When radiative corrections are included, the above result gets generalized in several ways. First, Dirac structures different from those of the original operator can be induced. Secondly, both color structures arise, since they mix under renormalization. Finally, the various components of the SCET operators become non-local. The most general gauge-invariant matching relation at leading order in λ is of the form

$$O_{4q}(0) \to \sum_{i,j} \sum_{C=S,O} \left\{ \int dr\, dt\, \widetilde{C}_{ij}^{(C)}(r,t,E,m_b,\mu)\, Q_{ij}^{(C)}(r,t) \right.$$

$$\left. - \frac{1}{2} \int dr\, ds\, dt\, \widetilde{D}_{ij}^{(C)}(r,s,t,E,m_b,\mu)\, R_{ij}^{(C)}(r,s,t) \right\}, \qquad (41)$$

where

$$\begin{aligned} Q_{ij}^{(C)}(r,t) &= \big[\bar\xi\, W\big](-r\bar n)\,\Gamma_i\, T_1 \big[W^\dagger \xi\big](r\bar n)\, \big[\bar h\, S\big](0)\,\Gamma_j\, T_2 \big[S^\dagger q_s\big](tn)\,, \\ R_{ij}^{(C)}(r,s,t) &= \big[\bar\xi\, W\big](-r\bar n)\,\Gamma_i\, T_1 \big[W^\dagger \xi\big](r\bar n)\, \big[\bar h\, S\big](0)\,\Gamma_j\, T_2\, \not{n}\, \not{\mathcal{A}}_{c\perp}(s\bar n)\, \big[S^\dagger q_s\big](tn) \end{aligned} \qquad (42)$$

are non-local operators, and the color label $C = S$ or O refers to the color singlet-singlet and color octet-octet structures, respectively. As in the case of the soft-collinear current both operators contribute at the same order in λ, since $\widetilde{C}_{ij}^{(C)} \sim \lambda$ while $\widetilde{D}_{ij}^{(C)} \sim 1$.

The form of the operators in (41) is determined by gauge invariance. To see this, note that instead of working in the singlet–octet basis of operators with flavor structure $\bar\xi \dots \xi\, \bar h \dots q_s$ we could have chosen instead to work with operators containing only products of color-singlet currents. Gauge invariance would then require that these operators be of the type

$$\begin{aligned} &\big[\bar\xi\, W\big]\,\Gamma_i \big[W^\dagger \xi\big]\, \big[\bar h\, S\big]\,\Gamma_j\, (\not{n}\, \not{\mathcal{A}}_{c\perp})\, \big[S^\dagger q_s\big] \\ \text{or}\quad &\big[\bar\xi\, W\big]\,\Gamma_i'\, (\not{n}\, \not{\mathcal{A}}_{c\perp})\, \big[S^\dagger q_s\big]\, \big[\bar h\, S\big]\,\Gamma_j' \big[W^\dagger \xi\big]\,, \end{aligned} \qquad (43)$$

where each bracket $[\dots]$ can be located at a different point, and the factor $(\not{n}\, \not{\mathcal{A}}_{c\perp})$ may or may not be present. Note that the second operator consists of a product of a soft-collinear current considered in Section 4 with a heavy-collinear current derived in [2]. With these results at hand, one can now perform a Fierz transformation to the flavor basis $\bar\xi \dots \xi\, \bar h \dots q_s$ and express the result in terms of color singlet-singlet and color octet-octet operators, as shown in (41).

Consider now the hadronic matrix elements of the resulting SCET operators between an initial-state B meson and a light, highly energetic final-state meson M. Since at leading order in λ there are no QCD interactions between soft and collinear fields, it follows that these matrix elements factorize into two parts, one containing only soft fields and one containing only collinear fields. Using the fact that matrix elements of color-octet currents between physical states vanish, and recalling from (6) that the object $\mathcal{A}_c^\mu$ is a pure color octet, we obtain

$$\langle M|\, Q_{ij}^{(S)}(r,t)\, |B\rangle = \langle M|\, \big[\bar\xi\, W\big](-r\bar n)\,\Gamma_i \big[W^\dagger \xi\big](r\bar n)\, |\, 0\, \rangle\, \langle\, 0\, |\, \big[\bar h\, S\big](0)\,\Gamma_j \big[S^\dagger q_s\big](tn)\, |B\rangle$$

$$\begin{aligned} \langle M|\, R_{ij}^{(O)}(r,s,t)\, |B\rangle = \frac{1}{2N_c}\, &\langle M|\, \big[\bar\xi\, W\big](-r\bar n)\,\Gamma_i\, \mathcal{A}_c^\mu(s\bar n)\, \big[W^\dagger \xi\big](r\bar n)\, |\, 0\, \rangle \\ &\times \langle\, 0\, |\, \big[\bar h\, S\big](0)\,\Gamma_j\, \not{n}\, \gamma_{\perp\mu} \big[S^\dagger q_s\big](tn)\, |B\rangle\,, \end{aligned} \qquad (44)$$

while

$$\langle M|\, Q_{ij}^{(O)}(r,t)\,|B\rangle = 0\,, \qquad \langle M|\, R_{ij}^{(S)}(r,s,t)\,|B\rangle = 0\,. \tag{45}$$

The matrix elements of the singlet operators $Q_{ij}^{(S)}$ have precisely the form expected from familiar applications of QCD factorization. The two matrix elements of bilocal currents define the leading-twist light-cone distribution amplitude of the light meson M [19, 20] and the leading-order (in the heavy-quark expansion) distribution amplitudes of the B meson [18], respectively. The fact that there may appear a non-vanishing, leading-power contribution from matrix elements of the octet operators $R_{ij}^{(O)}$ is a surprising result of our analysis, which has not been anticipated in the literature. These matrix elements correspond to higher-twist projections onto the light meson M, which contribute at leading power because (in momentum space) they are enhanced with respect to the singlet-operator matrix elements by a factor of $1/l_+$, where l is the soft spectator momentum. Introducing the definition $W(x,y) \equiv W(x)\,W^\dagger(y)$ for a collinear Wilson line connecting two points x and y, and using relation (6), the corresponding matrix element can be recast into the form

$$\begin{aligned}
&\langle M|\,\big[\bar\xi\,W\big](-r\bar n)\,\Gamma_i\,\mathcal{A}_c^\mu(s\bar n)\,\big[W^\dagger\xi\big](r\bar n)\,|\,0\rangle \\
&= \int_{-\infty}^{s} dw\,\langle M|\,\bar\xi(-r\bar n)\,W(-r\bar n,w\bar n)\,\Gamma_i\,\bar n_\alpha\,gG_c^{\alpha\mu}(w\bar n)\,W(w\bar n,r\bar n)\,\xi(r\bar n)\,|\,0\,\rangle\,,
\end{aligned} \tag{46}$$

which involves the conventional definition of a higher-twist, three-particle light-cone distribution amplitude [19, 20]. It remains to be seen whether such higher Fock-state contributions will also arise in cases where a non-local amplitude is matched onto four-quark operators in the SCET.

The momentum-space Wilson coefficients corresponding to the coefficients $\widetilde{C}_{ij}^{(C)}$ and $\widetilde{D}_{ij}^{(C)}$ in (41) are, in general, complicated functions of the scales m_b^2, m_bE, El_+, and μ^2 (where $E \sim m_b$), and of dimensionless variables u and v measuring the longitudinal momentum fractions of the collinear quark and gluon (in the case of the operators $R_{ij}^{(C)}$) inside the light final-state meson. It is impossible to eliminate all large ratios of these scales by a choice of the renormalization point μ. In such a case the resummation of large logarithms of the type $\ln(E/\Lambda)$ can be achieved by performing the matching onto the low-energy effective theory in two steps. First one integrates out hard modes as well as the couplings of collinear gluons to the heavy quark. This yields (non-local) operators in an intermediate effective theory which still contains $(\lambda,1,\lambda)$ modes as dynamical degrees of freedom. The Wilson coefficients arising in this step are functions of m_b^2, m_bE and μ^2 (as well as u). They can be calculated perturbatively at a scale $\mu^2 \sim m_b^2$ and evolved down to a scale $\mu^2 \sim E\Lambda$ using the renormalization group. In the second step the off-shell $(\lambda,1,\lambda)$ modes are integrated out, yielding the SCET as constructed in this work. This gives rise to Wilson coefficients that depend on the scales El_+ and μ^2 (as well as u and v). Solving the renormalization-group equations in SCET these coefficients can then be evolved down to scales $\mu^2 \ll E\Lambda$, at which the operators in SCET are renormalized. Concrete examples of such a two-step matching procedure will be discussed elsewhere.

We stress that, while the resummation of large logarithms arising from the evolution between the two hard scales m_b^2 and $m_b\Lambda$ is necessary to obtain reliable perturbative predictions for the Wilson coefficient functions, it is not required for the discussion of the factorization properties of matrix elements in the low-energy effective theory.

7 Implications for factorization theorems

The presence of non-trivial interactions between the soft spectator quark and collinear gluons complicates the understanding of the factorization properties of B-meson decay amplitudes. We will now illustrate this fact with the help of some toy examples. Realistic examples such as $B \to K^*\gamma$ or $B \to \pi\pi$ are more complicated and will be discussed elsewhere.

Consider the effective weak Hamiltonian

$$\mathcal{H}_{\rm eff} = C^{(S)}\,\Phi\,\bar u\,\Gamma_1\,s\,\bar b\,\Gamma_2\,u + C^{(O)}\,\Phi\,\bar u\,\Gamma_1\,t_a\,s\,\bar b\,\Gamma_2\,t_a\,u \tag{47}$$

mediating the exclusive decay $B^+ \to K^{(*)+}\Phi$, where the color-singlet field Φ can be a scalar, vector, or any other field, depending on the quantum numbers of the Dirac matrices $\Gamma_{1,2}$. This interaction is simpler than in the Standard Model since it is local, but otherwise it shares many similarities with terms that occur in the effective weak Hamiltonian of the Standard Model.

Based on the fact that the outgoing kaon contains a highly energetic, collinear $(\bar s u)$ pair and so decouples from soft gluons ("color transparency"), one would expect, following [10, 11], that at leading power in Λ/m_b the decay amplitude should obey the QCD factorization formula

$$\mathcal{A} = \Phi \int du \int dl_+\,\phi_{K^{(*)}}(u)\,\phi_B(l_+)\,T(u,l_+)\,, \tag{48}$$

where $\phi_{K^{(*)}}$ and ϕ_B are leading-order light-cone distribution amplitudes defined, e.g., in [5], and T is the hard-scattering kernel, which in the context of the SCET would be identified with a Wilson coefficient function. It is evident from our discussion in the previous section that for the simple result (48) to be correct the matrix elements of the operators $R_{ij}^{(C)}$ would have to vanish. Otherwise, the factorization formula would have to be generalized to include a term involving higher-twist distribution amplitudes of the kaon.

Matrix elements of the operators $R_{ij}^{(C)}$ in (42) are zero if the product $\Gamma_j\,\not{\bar n}$ of Dirac matrices vanishes, or if the B-meson matrix element of the structure $\bar h\,\Gamma_j\,\not{\bar n}\,\gamma_{\perp\mu}\,q_s$ vanishes by rotational invariance in the transverse plane. We believe that in many cases of phenomenological interest this is indeed what happens. Consider, as an example, the decay $B^+ \to K^+\Phi^0$, where Φ^0 is a fictitious light scalar. In this case the Lorentz indices of Γ_1 must always be contracted with those of Γ_2. Let us, for simplicity, work at tree level, so that the Dirac structures $\Gamma_i \otimes \Gamma_j$ appearing in the SCET operators are the same as those in the original operators. Between the collinear quark spinors $\bar\xi$ and ξ the Dirac basis matrices $\{1, \gamma_5, \gamma^\mu, \gamma^\mu\gamma_5, [\gamma^\mu, \gamma^\nu]\}$ are projected to $\{0, 0, \frac{1}{2}n^\mu \not{\bar{n}}, \frac{1}{2}n^\mu \not{\bar{n}}\gamma_5, \not{\bar{n}}(n^\mu\gamma_\perp^\nu - n^\nu\gamma_\perp^\mu)\}$. It follows that after contraction of Lorentz indices the product $\Gamma_i \otimes \Gamma_j$ can only take the forms $\not{\bar{n}}(\gamma_5) \otimes \not{n}(\gamma_5)$ or $[\not{\bar{n}}, \gamma_\perp^\nu] \otimes [\not{n}, \gamma_{\perp\nu}]$. This guarantees that $\Gamma_j \not{n} = 0$ for all possible Dirac structures, and hence the operators $R_{ij}^{(C)}$ vanish. We conclude that for this particular process the factorization formula (48) would hold, at least at tree level.

The mechanism just described does not appear to be universal, however. Consider, as a counter example, the case where $\Phi = F_{\alpha\beta}$ is the electromagnetic field, and where the Dirac matrices in (47) are chosen to be $\Gamma_1 \otimes \Gamma_2 = \sigma^{\alpha\beta} \otimes \gamma_5$ for the color singlet-singlet term and $\gamma^\alpha \otimes \gamma^\beta\gamma_5$ for the octet-octet term. This effective Hamiltonian mediates the radiative decay $B^+ \to K^{*+}\gamma$ (but not in the Standard Model). At tree level, the corresponding operators $Q_{ij}^{(C)}$ and $R_{ij}^{(C)}$ in SCET have Dirac structures $\Gamma_i \otimes \Gamma_j = \not{\bar{n}}\,\gamma_\perp^\beta \otimes \gamma_5$ times $n^\alpha F_{\alpha\beta}$ and $\Gamma_i \otimes \Gamma_j \not{n}\,\gamma_{\perp\mu} = \not{\bar{n}} \otimes \gamma^\beta\gamma_5 \not{n}\,\gamma_{\perp\mu}$ times $n^\alpha F_{\alpha\beta}$, respectively. After projection onto the B meson, the resulting Lorentz structures in the two cases are

$$\int dr\, dt\, \widetilde{C}_{ij}^{(S)} \langle K^* | \, Q_{ij}^{(S)} \, | B \rangle \sim E^{\frac{1}{2}} \Lambda^{\frac{3}{2}} \, n^\alpha F_{\alpha\beta} \, \langle K^* | \, \bar{\xi} \not{\bar{n}} \gamma_\perp^\beta \, \xi \, | \, 0 \, \rangle \sim E^{\frac{3}{2}} \Lambda^{\frac{5}{2}} \, n^\alpha \, \varepsilon_\perp^{*\beta} F_{\alpha\beta} \, ,$$

$$\int dr\, ds\, dt\, \widetilde{D}_{ij}^{(O)} \langle K^* | \, R_{ij}^{(O)} \, | B \rangle \sim E^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \, n^\alpha F_{\alpha\beta} \, g_{\perp\mu}^\beta \, \langle K^* | \, \bar{\xi} \not{\bar{n}} \mathcal{A}_{c\perp}^\mu \, \xi \, | \, 0 \, \rangle \sim E^{\frac{3}{2}} \Lambda^{\frac{5}{2}} \, n^\alpha \, \varepsilon_\perp^{*\beta} F_{\alpha\beta} \, , \tag{49}$$

where $\varepsilon_\perp$ is the transverse polarization vector of the K^* meson, and we have used the well-known scaling relations for current matrix elements of heavy and light mesons [22]. It is evident that for this example both operators contribute at the same order in power counting, and hence the simple QCD factorization formula in (48) must be generalized to include a term involving twist-3 light-cone distribution amplitudes of the kaon.

It follows that the question of factorization for the hard-spectator term in a QCD factorization formula is far from trivial. Whether a simple QCD factorization formula holds, or whether it must be generalized to include higher-twist distribution amplitudes, requires a case by case study. It is conceivable that in many cases of phenomenological importance it will be possible to exclude the presence of the operators $R_{ij}^{(C)}$ based on some symmetry, such as rotational invariance in the transverse plane or reparameterization invariance. It remains to find a general argument supporting this assertion.

8 Gauge-invariant building blocks for operators in SCET

The careful reader will have noticed that, as a result of gauge invariance, the external current and four-quark operators in SCET discussed in the previous sections always contain products of the Wilson lines W and S with soft or collinear quark fields, respectively. This suggests introducing new fields

$$\mathcal{H} = S^\dagger h \, , \qquad \mathcal{Q}_s = S^\dagger q_s \, , \qquad \mathcal{X} = W^\dagger \xi \, , \tag{50}$$

which are manifestly gauge invariant and have the same scaling relations in λ as the original fields. In the case where one chooses the light-cone gauge conditions $n \cdot A_s = 0$ and $\bar{n} \cdot A_c = 0$ the new fields agree with the original ones. In terms of the new fields, the four-quark operators in (42) take the simple form

$$\begin{aligned} Q_{ij}^{(C)}(r,t) &= \bar{\mathcal{X}}(-r\bar{n})\,\Gamma_i\,T_1\,\mathcal{X}(r\bar{n})\,\bar{\mathcal{H}}(0)\,\Gamma_j\,T_2\,\mathcal{Q}_s(tn)\,, \\ R_{ij}^{(C)}(r,s,t) &= \bar{\mathcal{X}}(-r\bar{n})\,\Gamma_i\,T_1\,\mathcal{X}(r\bar{n})\,\bar{\mathcal{H}}(0)\,\Gamma_j\,T_2\,\not{n}\,\not{\mathcal{A}}_{c\perp}(s\bar{n})\,\mathcal{Q}_s(tn)\,, \end{aligned} \tag{51}$$

and similar simplified expressions hold for the current operators in (18), (21), and (24).

While the definition of these new fields seems merely a matter of convenience of notation, we will now argue that, in fact, any operator in SCET can be built from a small set of gauge-invariant fields. Once these fields are known, gauge invariance of operators containing them is guaranteed, and the only remaining rules for the construction of SCET operators are Lorentz and reparameterization invariance. Therefore, the formalism we will introduce in this section will simplify the construction of SCET operators, which is important, in particular, for going beyond the leading order in λ.

The QCD Lagrangian contains, besides quark (and ghost) fields, the gauge-covariant derivative. In SCET we have distinguished between the collinear and soft covariant derivatives D_c^μ and D_s^μ. We can construct gauge-invariant objects from these quantities by using the field $\mathcal{A}_c^\mu$ introduced in (6) and a corresponding object defined in the soft sector:

$$\mathcal{A}_s^\mu(x) = \left[S^\dagger(iD_s^\mu S)\right](x) = \int_{-\infty}^0 dw\, n_\alpha \left[S^\dagger g G_s^{\alpha\mu} S\right](x+wn)\,. \tag{52}$$

The gauge-invariant fields $\mathcal{A}_c^\mu$ and $\mathcal{A}_s^\mu$ obey the constraints $\bar{n}\cdot\mathcal{A}_c = 0$ and $n\cdot\mathcal{A}_s = 0$ in any gauge. (Recall that in the light-cone gauge these fields coincide with the corresponding gluon fields gA_c^μ and gA_s^μ.) It follows that the scaling relations of the new fields are $\mathcal{A}_c^\mu \sim (\lambda^2, 0, \lambda)$ and $\mathcal{A}_s^\mu \sim (0, \lambda, \lambda)$. We finally define new, gauge-invariant objects

$$i\mathcal{D}_c^\mu = W^\dagger\, iD_c^\mu\, W = i\partial^\mu + \mathcal{A}_c^\mu\,, \qquad i\mathcal{D}_s^\mu = S^\dagger\, iD_s^\mu\, S = i\partial^\mu + \mathcal{A}_s^\mu\,. \tag{53}$$

With the help of these definitions, the different parts of the leading-order SCET Lagrangian can be rewritten in the form

$$\begin{aligned} \mathcal{L}_h &= \bar{\mathcal{H}}\, iv\cdot\mathcal{D}_s\,\mathcal{H}\,, \qquad \mathcal{L}_s = \bar{\mathcal{Q}}_s\, i\not{\mathcal{D}}_s\,\mathcal{Q}_s\,, \\ \mathcal{L}_c &= \bar{\mathcal{X}}\,\frac{\not{\bar{n}}}{2}\, in\cdot\mathcal{D}_c\,\mathcal{X} + \bar{\mathcal{X}}\,\frac{\not{\bar{n}}}{2}\, i\not{\mathcal{D}}_{c\perp}\,\frac{1}{i\bar{n}\cdot\partial}\, i\not{\mathcal{D}}_{c\perp}\,\mathcal{X}\,. \end{aligned} \tag{54}$$

The appearance of an ordinary derivative in the expression for $\mathcal{L}_c$ is not in conflict with gauge invariance, since all the component fields are gauge invariant by themselves. Finally, also the pure-glue Lagrangians for soft and collinear fields take a simple form. Since we can replace $G_c^{\mu\nu}$ by $W^\dagger G_c^{\mu\nu} W = (i/g)\,[\mathcal{D}_c^\mu, \mathcal{D}_c^\nu]$ and $G_s^{\mu\nu}$ by $S^\dagger G_s^{\mu\nu} S = (i/g)\,[\mathcal{D}_s^\mu, \mathcal{D}_s^\nu]$ in the Lagrangian (using the cyclicity of the color trace), the resulting pure-glue Lagrangians are simply obtained by replacing all gluon fields in the usual QCD Lagrangian with the new fields $\mathcal{A}_c$ and $\mathcal{A}_s$.

Let us briefly summarize the properties of the gauge-invariant building blocks under the three types of reparameterizations discussed in Section 5. We find that the redefined quark fields as well as the soft gluon field $\mathcal{A}_s^\mu$ are invariant up to higher-order terms in λ. However, the collinear gluon field has non-trivial transformations at leading order. Under a type I transformation $n\cdot\mathcal{A}_c \to n\cdot\mathcal{A}_c + \epsilon_\perp\cdot\mathcal{A}_c$ while $\mathcal{A}_{c\perp}$ remains invariant. Under a type II transformation $\mathcal{A}_{c\perp}^\mu \to \mathcal{A}_{c\perp}^\mu - \frac{1}{2}\,n^\mu\,e_\perp\cdot\mathcal{A}_{c\perp}$ while $n\cdot\mathcal{A}_c$ remains invariant. The corresponding transformation properties of the "collinear derivative" are $n\cdot\mathcal{D}_c \to n\cdot\mathcal{D}_c + \epsilon_\perp\cdot\mathcal{D}_c$ and $\mathcal{D}_{c\perp}^\mu \to \mathcal{D}_{c\perp}^\mu - \frac{1}{2}\,\epsilon_\perp^\mu\,\bar n\cdot\partial$ (type I), and $\mathcal{D}_{c\perp}^\mu \to \mathcal{D}_{c\perp}^\mu - \frac{1}{2}\,n^\mu\,e_\perp\cdot\mathcal{D}_{c\perp}$ while $n\cdot\mathcal{D}_c$ remains invariant (type II). Invariance under type III reparameterizations requires that every occurrence of a vector n in the numerator must be accompanied by that of a vector $\bar n$, or by a factor of $n\cdot\partial$ in the denominator (and vice versa). From these rules it follows, e.g., that the sum of the two terms in the collinear Lagrangian $\mathcal{L}_c$ in (54) is reparameterization invariant, but not the two operators separately.

After the introduction of the new fields the expression for any operator in SCET looks like an expression in the light-cone gauge; however, we have not imposed light-cone gauge but rather redefined the fields by a unitary transformation. In the new formulation the Wilson lines have disappeared, and all fields in the theory scale like at least one power of λ (the large collinear field $\bar n\cdot A_c \sim 1$ has been removed). Finally, gauge invariance is no longer a constraint on the construction of operators in the effective theory.

9 Kinematics of heavy-to-light form factors

Our goal in this paper was to find a formulation of SCET suitable for the systematic study of the factorization properties and power corrections for any exclusive B-meson decay into light particles. It is important, then, to have a theory that describes the form-factor term and the hard-scattering term in a factorization formula in terms of the same fields and scaling rules. However, previous work on the soft component of exclusive heavy-to-light form factors has been based on the scaling assumption $\lambda \sim \sqrt{\Lambda/E}$ [8, 9], which is different from our hypothesis. In this case soft fields carrying momenta of order Λ scale like $E(\lambda^2,\lambda^2,\lambda^2)$ and are thus called "ultrasoft". The main difference with our approach is that the "collinear modes" in this formulation have momenta scaling like $(\Lambda, E, \sqrt{E\Lambda})$. The resulting effective theory thus appears to be different from the one constructed here.

The challenge is to understand the soft contribution to heavy-to-light form factors, in which the B-meson spectator quark enters the light final-state meson as a soft (or ultrasoft) quark, so that this meson is produced in a highly asymmetric state. Consider a $B\to\pi$ transition for concreteness. The scaling $\lambda \sim \sqrt{\Lambda/E}$ was assumed based on the following kinematical consideration. The pion emitted in a heavy-to-light decay at large recoil carries momentum scaling like $p_\pi \sim (\Lambda^2/E, E, \Lambda)$, and making this up by combining a soft quark with momentum $p_s \sim (\Lambda,\Lambda,\Lambda)$ and a collinear jet requires that this jet have invariant mass squared $p_c^2 \sim E\Lambda$. Given that for a collinear particle $p_c^2 \sim \lambda^2 E^2$, it then follows that $\lambda \sim \sqrt{\Lambda/E}$. Although this argument might seem compelling, it has the

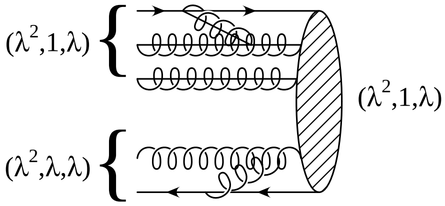

Figure 7: How to make a pion in a highly asymmetric configuration? The drawing illustrates the scaling of the collinear and soft components proposed in the present work, where $\lambda \sim \Lambda/E$ is assumed. In earlier papers a different scaling of collinear momenta was used.

unattractive feature that the external pion momentum now scales like $p_\pi \sim E(\lambda^4, 1, \lambda^2)$, which cannot be built up from the combination of a generic ultrasoft momentum $p_s \sim E(\lambda^2, \lambda^2, \lambda^2)$ with a generic collinear momentum $p_c \sim E(\lambda^2, 1, \lambda)$. In other words, the plus component of the collinear jet must cancel the plus component of the soft spectator quark with a relative precision of order $\lambda^2 \sim \Lambda/E$, and the transverse component of the jet has to be smaller than its generic size by a factor $\lambda \sim \sqrt{\Lambda/E}$. These two constraints combined imply that the soft overlap mechanism is strongly suppressed in this picture (which a priori is no problem, since heavy-to-light form factors are suppressed in the heavy-quark limit). This discussion can also be summarized in more physical terms by saying that it is unlikely that a collinear jet of particles with invariant mass of order $\sqrt{E\Lambda}$ will absorb a soft quark and become a light meson with mass of order Λ.

Here we wish to suggest a different possibility for interpreting the soft overlap contribution to heavy-to-light form factors. As illustrated in Figure 7, we assume the scaling $\lambda \sim \Lambda/E$ adopted throughout this work, so that the pion momentum scales like any other collinear momentum. In order to make a light meson out of collinear particles and soft particles, the only thing that is required is that the plus component of the total soft momentum, which generically would scale like $E\lambda$, is accidentally small, of order $E\lambda^2$. This implies a phase-space suppression of order Λ/E. It appears more intuitive to us to assume that only a jet of particles with invariant mass of order Λ can ultimately hadronize into a single light meson.

It is not evident whether the choice of λ is simply a matter of convenience, or whether it corresponds to a different kinematical situation. Naively, we would expect that our predictions for heavy-to-light form factors would differ from the ones obtained in [8, 9]. For instance, we expect that violations of heavy-quark symmetry relations between form factors start at order Λ/m_b, while the power counting adopted in these papers would allow for the presence of $\sqrt{\Lambda/m_b}$ corrections.

10 Summary and conclusions

The development of an effective field theory for the strong interactions of soft and collinear partons is a significant step toward the systematic study of factorization and a field-theoretical description of power corrections for observables that do not admit an operator product expansion. Power counting in this soft-collinear effective theory is non-trivial due to the presence of non-local operators integrated along light-like directions, but it appears feasible to construct a controlled heavy-quark expansion of amplitudes in terms of hadronic matrix elements of effective theory operators.

In this paper we have constructed the extension of the previous formulation of soft-collinear effective theory necessary for the description of exclusive B-meson decays into light particles. QCD factorization theorems for such processes are complicated, because in addition to a form-factor term a hard-scattering contribution appears at leading power. It results from hard gluon exchange with the soft spectator quark in the B meson. The intrinsic softness of the B-meson dynamics complicates the understanding of factorization properties of decay amplitudes. A new intermediate mass scale of order $m_b\Lambda$ arises and leads to large non-localities of effective-theory operators on a scale $1/\Lambda$, thus upsetting naive power counting. Power counting can be restored, however, using reparameterization invariance, which gives control over the dependence of Wilson coefficient functions on the light-like separation between the component fields of non-local operators.

The version of soft-collinear effective theory relevant to the discussion of exclusive B decays into light particles contains soft and collinear fields. We have constructed the effective Lagrangian at leading order in Λ/m_b, finding that it does not contain soft-collinear interaction terms. Such interactions would, however, appear at subleading order. We have then discussed in detail the matching of current and four-quark operators from the full theory onto their effective-theory counterparts. This matching is complicated by the presence of non-trivial interactions between the soft spectator quark and collinear gluons. The most surprising finding of our analysis is that, generically, there are two types of four-quark operators present in the effective theory, one of which contains an insertion of a transverse collinear gluon field. Upon evaluating hadronic matrix elements of such operators between a B meson and a light, energetic meson, one finds that in addition to the leading-twist distribution amplitude of the light meson also three-particle distribution amplitudes of subleading twist contribute at leading power. This suggests that, in some cases, QCD factorization formulae may have to be generalized. We have presented a toy example where this extension is indeed necessary.

Our results for the matching of currents and four-quark operators suggest a reformulation of soft-collinear effective theory in terms of operators composed out of gauge-invariant building blocks replacing the original quark and gluon fields. In the new formulation gauge invariance is automatic, and the form of operators is only constrained by Lorentz invariance. We anticipate that this observation will facilitate the extension of our results beyond the leading order.

The formalism developed in this work provides for the first time the basis for a systematic discussion of factorization and power corrections for any B-meson decay into light particles. Phenomenological applications of this framework will be discussed elsewhere.

Note added: While this paper was in writing the work [27] appeared, in which factorization for the leptonic radiative decay $B \to \gamma e \nu$ is discussed in the context of soft-collinear effective theory. The formalism used in that work differs from the one developed here and would not be applicable (at least not in an obvious way) to exclusive decays with light hadrons in the final state. Nevertheless, part of the discussion presented by these authors realizes the idea of two-step matching mentioned at the end of Section 6.

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