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Noncommutative Integrable Field Theories in 2d

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Abstract

We study the noncommutative generalization of (euclidean) integrable models in two-dimensions, specifically the sine- and sinh-Gordon and the $U(N)$ principal chiral models. By looking at tree-level amplitudes for the sinh-Gordon model we show that its naïve noncommutative generalization is *not* integrable. On the other hand, the addition of extra constraints, obtained through the generalization of the zero-curvature method, renders the model integrable. We construct explicit non-local non-trivial conserved charges for the $U(N)$ principal chiral model using the Brezin-Itzykson-Zinn-Justin-Zuber method.

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1 Introduction

Noncommutative field theories (ncft's)have attracted a great deal of attention recently, due to their relation to string theory, where they arise as a limit of type IIB theories with a B-field turned on [1]. Besides this important connection, ncft's are interesting on their own setting, with a very rich and unexpected structure, such as the UV/IR mixing for example [2], and applications to the quantum Hall effect [3].

It has been shown in general that the introduction of space-time noncommutativity leads to non-unitary theories [4], but it is conceivable that some specific models could evade some of these arguments [5, 6]. Since in a noncommutative theory in two-dimensions we necessarily have space-time noncommutativity, we have to be careful in defining the theory properly. One way to avoid these complications is to consider two-dimensional *euclidean* models.

We argue that after introducing noncommutativity, obtained by considering the replacement of the product of the fields in the action by their $\star$-products, some of these models are still integrable classically, whereas others are not. We show that models obtained in this way that are not integrable, can be redefined by a suitable generalization of the zero-curvature method [7] and then shown to be integrable.

This paper is organized as follows. In the next section we briefly review perturbative non-commutative field theory. In section 3 we discuss some of the generalities of integrable field theories, introduce the models we are going to study, show the non-integrability of the noncommutative sG and shG models, discuss the noncommutative generalization of the zero-curvature formalism, and show how the integrability of the sG and shG models may be restored and present soliton (localized) solutions. We also discuss the $U(N)$ pcm and show that its noncommutative generalization is integrable. In this case we construct non-local charges following the method of [10]. In section 4 we present our conclusions and comment on future directions to pursue. Some of the technical aspects are presented in the appendices.

2 Non-Commutative Field Theories

Let us consider scalar field theories for simplicity. We construct a ncft [8] from a given quantum field theory (qft) by replacing the product of fields by the $\star$-product

$$\phi_1(x)\phi_2(x) \rightarrow \phi_1(x) \star \phi_2(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu^{x_1}\partial_\nu^{x_2}} \phi_1(x_1)\phi_2(x_2)|_{x_1=x_2=x} \tag{2.1}$$

This deformation of the usual product implies in a change in the Feynman rules. We refer the reader to Filk's paper [9] for a more complete discussion of Feynman rules in ncft (see also [2], [8]). Here we review the essential aspects to our discussion.

A simplifying aspect in the analysis of ncft's is that the propagator of a ncft is the same as the one of its commuting version. This is due to the fact that, for a manifold without boundaries,

$$\int dx f(x) \star g(x) = \int dx f(x) g(x) \tag{2.2}$$

Therefore, the quadratic part of the action is the same for the noncommutative version of the model, providing the same propagator.

In the following we will refer to functions of operators in the noncommutative deformation by a $\star$ sub-index, for example $\phi^n_\star = \phi \star \phi \star \ldots \star \phi$.

If on one hand propagators are the same as in the commutative versions, vertices will pick up phases. For example, if we consider a $\phi^n_\star$ term in a two-dimensional scalar field theory, we obtain in momentum space

$$\int dx \phi(x) \star \ldots \star \phi(x) = \int \prod_{i=1}^{n} dp_i e^{-\frac{i}{2}\sum_{k<m}(p_k)_\mu \theta^{\mu\nu}(p_m)_\nu} \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_n) \delta(p_1 + \ldots + p_n) \tag{2.3}$$

Notice that already at tree-level, there will be differences in the scattering amplitudes of a commutative theory and its noncommutative counterpart, since the vertices are modified.

Let us see what are the changes in the cases that will be of interest to us, namely the 4- and 6-point vertices in a scalar theory. The 4-point vertex changes according to

$$\int dx \phi^4_\star = \int dx (\exp(-i \sum_{i<j} k_i \wedge k_j)) \phi(k_1)\phi(k_2)\phi(k_3)\phi(k_4)\delta(k_1 + k_2 + k_3 + k_4) \ , \tag{2.4}$$

where we introduced the notation $k_i \wedge k_j = \frac{1}{2}((k_i)_\mu \theta^{\mu\nu}(k_j)_\nu)$. We should remark that in two dimensions $k_i \wedge k_j = \frac{\theta}{2}((k_i)_1(k_j)_2 - (k_i)_2(k_j)_1)$, since $\theta^{\mu\nu} = \theta\epsilon^{\mu\nu}$. In general the Moyal deformation of vertices does not preserve the permutation symmetry, but in the case of a single scalar boson, we can actually symmetrize the integrand, and replace the phases in 2.4 by

$$G_4(k_1, k_2, k_3, k_4) = \frac{1}{4!} \sum_{perm.} \exp(-i \sum_{i<j} k_i \wedge k_j) = \frac{1}{3}(\cos(k_1 \wedge k_2)\cos(k_3 \wedge k_4) + \tag{2.5}$$

$$\cos(k_1 \wedge k_3)\cos(k_2 \wedge k_4) + \cos(k_1 \wedge k_4)\cos(k_2 \wedge k_3)) \ , \tag{2.6}$$

The analysis of the 6-point vertex is very similar, and gives

$$G_6(k_1, k_2, k_3, k_4, k_5, k_6) = \frac{1}{6!} \sum_{perm.} \exp(-i \sum_{i<j} k_i \wedge k_j) \tag{2.7}$$

We will leave the 6-point vertex in this form, since there is no simpler way to write it, as in the case of the 4-point vertex. All one has to do in order to compute amplitudes in a noncommutative scalar field theory is to write down exactly the same Feynman graphs as in the commutative theory and replace the vertices by expressions like 2.4 and 2.7 (and their analogous for higher order vertices).

3 Noncommutative Integrable Field Theories

The existence of non-trivial higher-spin conserved charges has dramatic consequences in the dynamics of two-dimensional qft's: there is no particle production in a scattering process, the set of in and out momenta is the same, multi-particle amplitudes are factorized into products of two-body processes, and the two-body S-matrix satisfy the Yang-Baxter equation, besides the usual analiticity and crossing-symmetry properties ([11], [12]).

We will consider the noncommutative extensions of the sine- and sinh-Gordon (sG and shG) models and of the $U(N)$ principal chiral model (pcm). In the sG and shG models the naïve Moyal deformation leads to non-integrable field theories, whereas in the case of the $U(N)$ pcm, integrability is preserved. On the other hand it is possible to introduce new constraints in the sG and shG models in such a way to restore integrability.

3.1 Noncommutative sine- and sinh-Gordon

The lagrangian of the shG model is

$$\mathcal{L}_{shG} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{\beta^2}(1 - \cosh(\beta\phi)) \ . \tag{3.8}$$

As it is well known, the shG model is integrable, and is related to the sG model, by replacing $\beta \to i\beta$. The equation of motion of the shG model can be easily derived from 3.8 to be

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^2}{\beta}\sinh(\beta\phi) = 0 \tag{3.9}$$

One may be tempted at guessing that the noncommutative version of the shG model, obtained by replacing the products of local fields in the action by $\star$-products, would lead to an integrable model. This turns out not to be the case. Notice that the classical non-integrability of the shG model implies the same for the sG model.

Consider the Moyal deformation of the shG lagrangian

$$\mathcal{L}^{\star}_{shG} = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{\beta^2}(1 - \cosh(\beta\phi)_\star) \ . \tag{3.10}$$

where

$$\cosh(\beta\phi)_\star = \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{\beta\phi}{2}\right)^{2n}_\star \tag{3.11}$$

The corresponding equation of motion is

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \frac{m^2}{\beta}\sinh(\beta\phi)_\star = 0 \tag{3.12}$$

The fact that the amplitude for particle production processes vanishes exactly in an integrable model means that they vanish to each order in a loop expansion, that is, in powers of $\hbar$. In particular it should vanish at tree-level, which corresponds to the classical limit of the theory, and is the hallmark of classical integrability. Therefore, if any tree-level amplitude for a particle production process is non-zero, we may be sure that this model is not classically integrable. This strategy of showing non-integrability for a given model was used in [13].

In the following we compute the tree-level amplitude for $2 \to 4$ particles and show that it vanishes in the shG model (see [12]), but that it does *not* vanish in its naïve noncommutative deformation [1]. For this specific computation we need only to consider the truncated lagrangian

$$\widetilde{\mathcal{L}}_{shG} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{m^2\beta^2}{4!}\phi^4 - \frac{m^2\beta^4}{6!}\phi^6 \tag{3.13}$$

Let us denote the in-momenta p_1 and p_2, and the out-momenta p_3, p_4, p_5, and p_6. The amplitude for $p_1 + p_2 \to p_3 + p_4 + p_5 + p_6$ will be denoted by $\mathcal{M}_{2\to4} = (2\pi)^4\delta(p_1 + p_2 - p_3 - p_4 - p_5 - p_6)T(p_1, p_2; p_3, p_4, p_5, p_6)$. Using the rapidity variable we can write the in- and out-momenta as $p_i = m(\cosh(\theta_i), \sinh(\theta_i))$. In light-cone coordinates it becomes $p_i^\pm = p_i^0 \pm p_i^1 = m\exp(\pm\theta_i)$. We will denote the numbers $\exp(\theta_i) = a_i$. In the following we will consider T alone.

The amplitude $T(p_1, p_2; p_3, p_4, p_5, p_6)$ gets contributions from three types of diagrams, as shown in figure 1, where we still have to sum over possible permutations of the in- and out-lines.

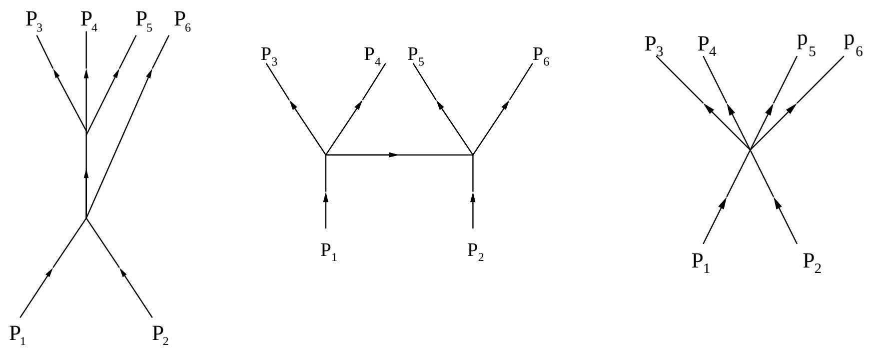

Fig. 1 The three types of diagrams contributing to T

We will call the amplitude for the first type of diagrams $A(p_1, p_2; p_3, p_4, p_5, p_6)$ and for the second $B(p_1, p_2; p_3, p_4, p_5, p_6)$. The third type is simply the ϕ^6 vertex. It is easy to see that

$$A(p_1, p_2; p_3, p_4, p_5, p_6) = \frac{1}{(p_1 + p_2 - p_6)^2 - m^2} = -\frac{1}{m^2}\frac{a_1 a_2 a_6}{(a_1 + a_2)(a_1 - a_6)(a_2 - a_6)}$$

[1] In this subsection we compute the amplitudes in Minkowski space, since there is no problem with unitarity at tree-level.

$$B(p_1,p_2;p_3,p_4,p_5,p_6) = \frac{1}{(p_1-p_3-p_4)^2-m^2} = -\frac{1}{m^2}\frac{a_1a_3a_4}{(a_1-a_3)(a_1-a_4)(a_3+a_4)} \quad (3.14)$$

The final scattering amplitude has factors that depend on the external legs, which are the same for all diagrams, and therefore unimportant in our computation.

The scattering amplitude for the $2 \to 4$ process is, therefore, proportional to

$$\begin{aligned} T(p_1,p_2;p_3,p_4,p_5,p_6) &= A(p_1,p_2;p_3,p_4,p_5,p_6) + A(p_1,p_2;p_4,p_5,p_6,p_3) + \\ &A(p_1,p_2;p_5,p_6,p_3,p_4) + A(p_1,p_2;p_6,p_3,p_4,p_5) + B(p_1,p_2;p_3,p_4,p_5,p_6) + \\ &B(p_1,p_2;p_3,p_5,p_4,p_6) + B(p_1,p_2;p_3,p_6,p_5,p_4) + B(p_1,p_2;p_5,p_6,p_3,p_4) + \\ &B(p_1,p_2;p_4,p_6,p_3,p_5) + B(p_1,p_2;p_5,p_4,p_3,p_6) + 1 \end{aligned} \quad (3.15)$$

where the 1 is the contribution from the 6-point vertex. By using energy-momentum conservation, which corresponds to $a_1 + a_2 = a_3 + a_4 + a_5 + a_6$ and $1/a_1 + 1/a_2 = 1/a_3 + 1/a_4 + 1/a_5 + 1/a_6$, it can be shown that the above expression vanishes! In particular this means that the contribution coming from the 4-point vertices (amplitudes A and B) add up to a constant (-1), and the constant contribution from the 6-point vertex (+1) precisely cancels it.

Let us consider the noncommutative amplitude now. Using formula 2.4 the noncommutative amplitudes $\tilde{A}$ and $\tilde{B}$ become

$$\begin{aligned} \widetilde{A}(p_1,p_2;p_3,p_4,p_5,p_6) &= A(p_1,p_2;p_3,p_4,p_5,p_6)G_4(p_1,p_2,p_1+p_2-p_6,p_6) \\ &\quad G_4(p_1+p_2-p_6,p_3,p_4,p_5) \\ \widetilde{B}(p_1,p_2;p_3,p_4,p_5,p_6) &= B(p_1,p_2;p_3,p_4,p_5,p_6)G_4(p_1,p_3,p_4,p_1-p_3-p_4) \\ &\quad G_4(p_1-p_3-p_4,p_2,p_5,p_6) \end{aligned} \quad (3.16)$$

the amplitude for the $2 \to 4$ process, is now

$$\begin{aligned} \widetilde{T}(p_1,p_2;p_3,p_4,p_5,p_6) &= \widetilde{A}(p_1,p_2;p_3,p_4,p_5,p_6) + \widetilde{A}(p_1,p_2;p_4,p_5,p_6,p_3) + \\ &\widetilde{A}(p_1,p_2;p_5,p_6,p_3,p_4) + \widetilde{A}(p_1,p_2;p_6,p_3,p_4,p_5) + \widetilde{B}(p_1,p_2;p_3,p_4,p_5,p_6) + \\ &\widetilde{B}(p_1,p_2;p_3,p_5,p_4,p_6) + \widetilde{B}(p_1,p_2;p_3,p_6,p_5,p_4) + \widetilde{B}(p_1,p_2;p_5,p_6,p_3,p_4) + \\ &\widetilde{B}(p_1,p_2;p_4,p_6,p_3,p_5) + \widetilde{B}(p_1,p_2;p_5,p_4,p_3,p_6) + G_6(p_1,p_2;p_3,p_4,p_5,p_6) \end{aligned} \quad (3.17)$$

where G_6 is given by 2.7. Once again, we have to take into account the energy-momentum conservation constraint in evaluating this expression. As expected, the zeroth order in θ is the amplitude 3.15, and therefore it vanishes. On the other hand, the expression 3.17 does not vanish to the next order in θ (which is actually θ^2), and so this model is *not* integrable: the Moyal deformation of the shG model is not integrable. This does not mean that there is no noncommutative version of the shG and sG models, but only that our first attempt does not work. We will see now how to define the noncommutative shG and sG models in such a way to obtain integrable theories that reduce to the appropriate limits as $\theta \to 0$.

3.2 Zero-Curvature Condition

The definition of the noncommutative shG and sG models as the Moyal deformation of the action of their actions does not give an integrable field theory. On the other hand we can define the noncommutative sG and shG models through the noncommutative generalization of the zero-curvature condition, which will provide, by construction, a theory with an infinite number of conserved charges, and gives the usual sG and shG models in the limit $\theta \to 0$. We start by reviewing the zero-curvature method.

The equations of motion of an integrable field theory in two dimensions can be written in the form

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0 \ , \tag{3.18}$$

where U and V are two given potentials, which depend on space and time, and a spectral parameter λ, and $[U, V] = UV - VU$. This is the so-called zero-curvature condition, which encodes the integrable structure of the theory. It corresponds to the compatibility of the following pair of differential equations

$$\frac{\partial F}{\partial x} = UF \qquad \text{and} \qquad \frac{\partial F}{\partial t} = VF \tag{3.19}$$

where F is an auxiliary vector. We introduce now the $\star$-zero-curvature condition. Similarly to the usual zero-curvature condition, the $\star$-zero curvature condition arises from the compatibility of the following pair of differential equations

$$\frac{\partial F}{\partial x} = U \star F \qquad \text{and} \qquad \frac{\partial F}{\partial t} = V \star F \tag{3.20}$$

We will show now, that the 3.20 implies the existence of an infinite number of conserved charges.

We will consider our theory to be defined on the interval $[-L, L]$, and that the operators U and V satisfy periodic boundary conditions. The equation satisfied by the monodromy operator $T_\lambda(x)$ is

$$\frac{\partial T_\lambda}{\partial x} = U \star T_\lambda \tag{3.21}$$

with the boundary condition $T_\lambda(-L) = 1$. The solution of 3.21 is easily seen to be

$$T_\lambda(x) = P_\star \exp\left(-\int_{-L}^{x} dz \ U(z; \lambda)\right) \tag{3.22}$$

where $P_\star$ is the $\star$-path-ordered operator. By taking the time derivative of 3.21 and using the $\star$-zero-curvature condition, we obtain

$$\begin{aligned}
\frac{\partial^2 T_\lambda}{\partial_x \partial_t} &= \frac{\partial U}{\partial t} \star T_\lambda + U \star \frac{\partial T_\lambda}{\partial t} \\
&= \frac{\partial V}{\partial x} \star T_\lambda - [U, V]_\star \star T_\lambda + U \star \frac{\partial T_\lambda}{\partial t} \\
&= \frac{\partial V}{\partial x} \star T_\lambda - U \star V \star T_\lambda + V \star \frac{\partial T_\lambda}{\partial x} + U \star \frac{\partial T_\lambda}{\partial t} \\
&= \frac{\partial}{\partial x}(V \star T_\lambda) + U \star \left(\frac{\partial T_\lambda}{\partial t} - V \star T_\lambda \right)
\end{aligned} \tag{3.23}$$

which implies

$$\frac{\partial}{\partial x} \left(\frac{\partial T_\lambda}{\partial t} - V \star T_\lambda \right) = U \star \left(\frac{\partial T_\lambda}{\partial t} - V \star T_\lambda \right) \tag{3.24}$$

This means that

$$\frac{\partial T_\lambda(x)}{\partial t} = V(x) \star T_\lambda(x) + T_l(x) \star K \tag{3.25}$$

where K is an x-independent operator. By using the boundary condition for T_λ we obtain

$$\frac{\partial T_\lambda(x)}{\partial t} = V(x) \star T_\lambda(x) - T_\lambda(x) \star V(-L) \tag{3.26}$$

Evaluating 3.26 at $x = L$ and using the boundary condition for the operator V, we obtain

$$\frac{\partial T_\lambda(x)}{\partial t} = [V(L), T_\lambda(L)]_\star \tag{3.27}$$

Once we have managed to write the time derivative of $T_\lambda(L)$ as a commutator, it follows straightforwardly that $\mathrm{tr}(T_\lambda(L))$ is time independent, and we can read-off conserved charges from the expansion of $T_\lambda(L)$ in powers of λ. In this derivation we had to use the fact that the $\star$-product is associative and that the $\star$-inverse of certain operators exist.

Before we proceed in constructing the $\star$-zero-curvature condition for the sinh-Gordon model we should mention one important aspect: the noncommutative generalization of a given term is not necessarily unique. For example, in going from $\theta = 0$ to $\theta \neq 0$ the derivative of ϕ can be written as $\partial\phi$ or as $\frac{1}{2}(e_\star^{-\phi} \star \partial e_\star^{\phi} - e_\star^{\phi} \star \partial e_\star^{-\phi})$, and as we will see later, this ambiguity leads to different equations of motion.

We will work in light-cone coordinates now, where $x_\pm = (x_0 \pm x_1)/2$. We can write the equation of motion for the shG model as a zero-condition equation, by introducing a two component vector potential A and $\bar{A}$,

$$A = -\frac{m\lambda}{2}(e^{\beta\phi}\,\sigma_- + e^{-\beta\phi}\,\sigma_+) \qquad \text{and} \qquad \bar{A} = \frac{m}{2\lambda}(\sigma_- + \sigma_+) - \frac{\beta}{2}\bar{\partial}\phi\,\sigma_3 \tag{3.28}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, σ_i are the usual Pauli matrices, and λ is the spectral parameter. A and $\bar{A}$ satisfy

$$\bar{\partial} A - \partial \bar{A} + [A, \bar{A}] = 0 \ , \tag{3.29}$$

It is a simple computation to verify that the zero-curvature condition 3.29 with the functions 3.28 are equivalent to the equation of motion for the shG model 3.8. Notice that in showing this, the diagonal elements of the matrix equation 3.29 are the equation of motion and the off-diagonal elements vanish.

We *define* now the noncommutative sinh-Gordon model by the following $\star$-zero-curvature equation

$$\bar{\partial} A - \partial \bar{A} + [A, \bar{A}]_\star = 0 \ , \tag{3.30}$$

where $[A, \bar{A}]_\star = A \star \bar{A} - \bar{A} \star A$. The equation of motion derived from 3.30 is

$$\partial\bar{\partial}\phi + \frac{m^2}{\beta}\sinh_\star(\beta\phi) = 0 \tag{3.31}$$

which is exactly the same equation one would obtain from the Moyal deformation of the shG action. There are, though, two more constraints, coming from the off-diagonal elements, and which read

$$\bar{\partial}(e_\star^{-\beta\phi}) + \frac{\beta}{2}(e_\star^{-\beta\phi} \star \bar{\partial}\phi + \bar{\partial}\phi \star e_\star^{-\beta\phi}) = 0 \tag{3.32}$$

$$\bar{\partial}(e_\star^{\beta\phi}) - \frac{\beta}{2}(e_\star^{\beta\phi} \star \bar{\partial}\phi + \bar{\partial}\phi \star e_\star^{\beta\phi}) = 0 \tag{3.33}$$

It is easy to show that these constraints can be written as total derivatives, and that they vanish in the limit $\theta \to 0$.

3.3 The Grisaru and Penati Proposal

In [14] Grisaru and Penati have proposed a system of equations for the noncommutative sine-Gordon model, using the method of bicomplexes. We will show that it is possible to obtain their equation of motion from the $\star$-zero-curvature equation. On the other hand, the constraints we find are, apparently, different from theirs.

If we take the gauge potential A and $\bar{A}$ to be

$$\begin{aligned} A &= -\frac{m\lambda}{2}(e_\star^{\beta\phi}\ \sigma_- + e_\star^{-\beta\phi}\ \sigma_+) \\ \bar{A} &= \frac{m}{2\lambda}(\sigma_- + \sigma_+) + \frac{1}{4}(e_\star^{\beta\phi} \star \bar{\partial}(e_\star^{-\beta\phi}) - e_\star^{-\beta\phi} \star \bar{\partial}(e_\star^{\beta\phi}))\ \sigma_3 \end{aligned} \tag{3.34}$$

we obtain the same equation of motion [2] as in [14]

$$\partial(e_\star^{\beta\phi} \star \bar{\partial}(e_\star^{-\beta\phi}) - e_\star^{-\beta\phi} \star \bar{\partial}(e_\star^{\beta\phi})) = 2m^2 \sinh_\star(\beta\phi) \tag{3.35}$$

[2]After replacing β by $i\beta$.

but different additional constraints

$$\bar{\partial}(e_\star^{-\beta\phi}) - \frac{1}{4}\{e_\star^{-\beta\phi}, e_\star^{\beta\phi} \star \bar{\partial}(e_\star^{-\beta\phi}) - e_\star^{-\beta\phi} \star \bar{\partial}(e_\star^{\beta\phi})\}_\star = 0 \qquad (3.36)$$

$$\bar{\partial}(e_\star^{\beta\phi}) - \frac{1}{4}\{e_\star^{\beta\phi}, e_\star^{-\beta\phi} \star \bar{\partial}(e_\star^{\beta\phi}) - e_\star^{\beta\phi} \star \bar{\partial}(e_\star^{-\beta\phi})\}_\star = 0 \ . \qquad (3.37)$$

It is straightforward to show that these constraints can be written as total derivatives, and that they vanish in the limit $\theta \to 0$.

3.4 Euclidean Solitons

In this section we will study soliton solutions for the ncsG model defined in the previous sections, equations 3.31 and 3.33. We should remark here that by "euclidean solitons" we refer to solutions to the equations of motion for the ncsG model that correspond to the usual soliton solution for the sG model, when setting $\theta = 0$.

Since the solution for the field ϕ itself is a function of the noncommutative parameter θ, we have to make a double expansion. Initially we expand the field ϕ in a power series in θ

$$\phi = \sum_{n=0}^{\infty} \phi_n \theta^n \qquad (3.38)$$

We can use this expansion to find the equations of motion and constraints to first order in θ. Notice that there will be a dependence in θ arising from the expansion 3.38 and also from the definition of the $\star$-product. We have

$$\begin{aligned}\phi_\star^n = \phi_0^n + n\phi_0^{n-1}\phi_1\theta + (n\phi_0^{n-1}\phi_2 + \frac{n(n-1)}{2}\phi^{n-2}\phi_1^2 + \\ + \frac{n(n-1)(n-2)}{24}\phi_0^{n-3}\mathcal{B}_1 + \frac{n(n-1)}{8}\phi_0^{n-2}\mathcal{B}_2)\theta^2 + o(\theta^3)\end{aligned} \qquad (3.39)$$

where $\mathcal{B}_1$ and $\mathcal{B}_2$ are given by

$$\begin{aligned}\mathcal{B}_1 &= (\partial\phi_0)^2(\bar{\partial}^2\phi_0) + (\bar{\partial}^2\phi_0)(\partial\phi_0)^2 - 2\partial\bar{\partial}\phi_0\partial\phi_0\bar{\partial}\phi_0 \\ \mathcal{B}_2 &= \partial^2\phi_0\bar{\partial}^2\phi_0 - (\partial\bar{\partial}\phi_0)^2\end{aligned} \qquad (3.40)$$

See appendix 1 for a derivation of 3.39 and 3.40. The equations of motion are, to order θ^2

$$\partial\bar{\partial}\phi_0 + \frac{m^2}{\beta}\sin(\beta\phi_0) = 0 \qquad (3.41)$$

$$\partial\bar{\partial}\phi_1 + m^2\phi_1\cos(\beta\phi_0) = 0 \qquad (3.42)$$

$$\partial\bar{\partial}\phi_2 + m^2\phi_2\cos(\beta\phi_0) - \frac{m^2\beta}{2}\phi_1^2\sin(\beta\phi_0) = 0 \qquad (3.43)$$

The first two are the same as found by Grisaru and Penati in [14].

The solutions for ϕ_0, ϕ_1, and ϕ_2 are readily found, and we refer to appendix 2 for the details. The solutions are

$$\phi_0 = \frac{4}{\beta}\tan^{-1}(\exp(\frac{m}{\sqrt{1-v^2}}(x-x_0))) \tag{3.44}$$
$$\phi_1 = K\phi_0' \tag{3.45}$$
$$\phi_2 = \frac{K^2}{2}\phi_0'' \tag{3.46}$$

Where K is a constant of integration. Using these expressions we see that the series for $\phi(x)$ can be partially resummed to give $\phi(x + A\theta)$. Actually, it is easy to show that this is indeed the case to all orders, which establishes the fact that *the one soliton solution for the commutative theory solves the noncommutative equations of motion.* To show that, we start by establishing that if $f(x_0, x_1)$ and $g(x_0, x_1)$ depend on their arguments as a linear function of x_1 and x_2, say, $x_1 - v\,x_0$, then their $\star$-product coincides with their classical product ($\theta = 0$). This can be easily seen by using the Fourier decomposition of f and g,

$$\begin{aligned} f\star g &= \exp(\frac{i}{2}\theta_{\mu\nu}\partial_x^\mu\partial_y^\nu)\int dpdq\tilde{f}(p)\tilde{g}(q)\exp(ip(x_1-vx_0)+iq(y_1-vy_0))\Big|_{x=y} = \\ &= \int dpdq\tilde{f}(p)\tilde{g}(q)\exp(i(p+q)(x_1-vx_0)) = fg \end{aligned} \tag{3.47}$$

Therefore the $\sin(\beta\phi)_\star = \sin(\beta\phi)$ and the equation of motion turns out to be the same as in the usual sine-Gordon model. The next step to be verified is to check if the constraints are satisfied. This again is easily shown to be the case, since from 3.47 we can take the constraints to be evaluated at $\theta = 0$, when they become trivial.

The study of multi-soliton solutions is not as simple as the one-soliton case, and we shall not pursue it here.

3.5 Noncommutative Principal Chiral Model

In the previous subsections we studied models where the naïve noncommutative version fails to be integrable. In this subsection we will study a model where the naïve construction works. This is the $U(N)$ principal chiral model (pcm).

The action of the $U(N)$ pcm is

$$S_{pcm} = \frac{1}{2g_0^2}\int d^2x \mathrm{Tr}(\partial_\mu g^{-1}\partial^\mu g) \tag{3.48}$$

where g takes values in $U(N)$. The equation of motion of the pcm is easily seen to be $\partial_\mu(g^{-1}\partial^\mu g) = 0$. The Moyal deformation of the pcm is given by the action

$$S_{pcm}^{*} = \frac{1}{2g_0^2}\int d^2x \mathrm{Tr}(\partial_\mu g^{-1}\star\partial^\mu g) \tag{3.49}$$

and the field g is required to satisfy $g \star g^{\dagger} = g^{\dagger} \star g = 1$. The reason why we should be specific about the group to which g belongs is that not all groups allow noncommutative extensions, for example, there is no noncommutative $SU(N)$. Therefore we will restrict our analysis to the $U(N)$ pcm. This model was recently studied in [15].

Notice that, as we mentioned earlier, the quadratic part of a noncommutative action in a manifold without boundaries is equivalent to the commutative action. Therefore the actions 3.48 and 3.49 are equivalent [3]. The main difference between the commutative and noncommutative models relies therefore not in their actions, but in the constraints that the field g satisfies.

We would like to construct nontrivial conserved charges for this model, in order to show that it is integrable. We can do so by following the Brezin-Itzykson-Zinn-Justin-Zuber (BIZZ) method [10], which is extremely simple and yet powerful. For the sake of completeness we summarize the BIZZ construction here.

In [10] BIZZ start by assuming that there exists a set of matrices $A_{\mu}^{\alpha\beta}$ satisfying the following conditions:

- The field A_{μ} is a pure gauge, that is, one can find a nonsingular matrix h such that $A_{\mu} = h^{-1}\partial_{\mu}h$
- As a consequence of the equations of motion we should have $\partial_{\mu}A_{\mu} = 0$

Based on these two requirements, it can be shown that the recursively defined currents $J_{\mu}^{(n+1)} = D_{\mu}\chi^{(n)}$, $n \geq 0$, are conserved, where $D_{\mu}^{\alpha\beta} = \delta^{\alpha\beta}\partial_{\mu} + A_{\mu}^{\alpha\beta}$ is a covariant derivative satisfying the zero-curvature condition $[D_{\mu}, D_{\nu}] = 0$, and the fields $\chi^{(n)}$ are defined by $J_{\mu}^{(n)} = \epsilon_{\mu\nu}\partial_{\nu}\chi^{(n)}$, $n \geq 1$, and we start with $\chi^{(0)} = 1$. It is a simple exercise to show that $\partial_{\mu}J_{\mu} = 0$. By construction, these are nonlocal charges.

This construction was used by BIZZ to establish the integrability of the pcm, since we can take $A_{\mu} = g^{-1}\partial\mu g$, as we see from the equation of motion of the pcm, which automatically satisfies both requirements stated above. In order to carry out the derivation, though, we have to establish one crucial point: the Moyal deformed commutator $[D_{\mu}, D_{\nu}]_{\star} = D_{\mu} \star D_{\nu} - D_{\nu} \star D_{\mu}$, should vanish. This is easily seem to be the case, bearing in mind that the identities $\partial_{\mu}g^{-1} \star g = -g^{-1} \star \partial_{\mu}g$ and so on, are still valid. One such conserved charge that we can write is

$$\mathcal{Q}^{(2)} = -\int_{-\infty}^{+\infty} dx\ g^{-1} \star \partial_1 g + \int_{-\infty}^{+\infty} dx\ g^{-1} \star \partial_0 g \star \int_{-\infty}^{x} dx'\ g^{-1} \star \partial_0 g \tag{3.50}$$

[3]Provided the field g falls off fast enough at infinity.

3.6 Zero-Curvature Equation for the Noncommutative PCM

We can also write the noncommutative equation of motion of the $U(N)_\star$ pcm as a zero-curvature condition. Consider the potentials

$$U(\lambda) = \frac{1}{2}\frac{l_0 + l_1}{1-\lambda} - \frac{1}{2}\frac{l_0 - l_1}{1+\lambda} \tag{3.51}$$

$$V(\lambda) = \frac{1}{2}\frac{l_0 + l_1}{1-\lambda} + \frac{1}{2}\frac{l_0 - l_1}{1+\lambda} \tag{3.52}$$

where

$$l_0(x,t) = \frac{\partial g}{\partial t} * g^{-1} \qquad \text{and} \qquad l_1(x,t) = \frac{\partial g}{\partial x} * g^{-1} \tag{3.53}$$

Introducing this on the zero-curvature condition 3.18 we obtain

$$\frac{\partial^2 g}{\partial t^2} - \frac{\partial^2 g}{\partial x^2} = \frac{\partial g}{\partial t} * g^{-1} * \frac{\partial g}{\partial t} - \frac{\partial g}{\partial x} * g^{-1} * \frac{\partial g}{\partial x} \tag{3.54}$$

which is the equation of motion of the pcm, and it can be rewritten in the more compact form

$$\partial_\mu (g^{-1} * \partial^\mu g) = 0 \ . \tag{3.55}$$

Contrary to the noncommutative sG model, there are no further constraints in the noncommutative $U(N)$ pcm.

4 Conclusions

We have seen that the Moyal deformation of a given 2d integrable model does not necessarily provide a integrable field theory. In the case of the sinh-Gordon model (and by replacing $\beta \to i\beta$, the sine-Gordon model) we were able to establish their *non*-integrability by computing the amplitude for $2 \to 4$ particles at the tree-level, and verifying it is non-zero. On the other hand, the noncommutative $U(N)$ principal chiral model defined through the Moyal deformation of the action and constraints of the $U(N)$ principal chiral model, does provide an integrable field theory, where the elegant method of Brezin et al [10] works as well as in the commutative case.

The equations of motion we found initially 3.31 and 3.33 are different from the ones proposed by Grisaru and Penati in [14]. Upon a change in the definition of the noncommutative version of $\partial\phi$ we were able to find the same equation of motion as [14], but it is not trivial to establish the equality of the constraints.

By looking at the equations of motion in a perturbative form, we found the "euclidean solitons" for the noncommutative sine-Gordon model, and showed that the 1-soliton solution of the sine-Gordon model solves the equations of motion and constraints of the noncommutative version.

There are several interesting directions to pursue. Initially, it would be nice to have a more thorough understanding of the conservation laws, verifying for example, that these charges are in involution. Next one could consider the noncommutative versions of different models, such as the affine Toda theories. And last but not least, the investigation of the quantization of these models is a fascinating, if somewhat difficult problem.

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A Expansion of $\phi_\star^n$ to order θ^2

We want to find $\phi_\star^n$ to order θ^2, where $\phi = \sum_{n=0}^{\infty} \phi_n \theta^n$. We write $\phi_\star^n$ as

$$\phi_\star^n = \phi_0^n + \theta A_n + \theta^2 B_n + o(\theta^3) \ . \tag{A.56}$$

Using the associativity of the $\star$-product, $\phi_\star^{n+1} = \phi_\star^n \star \phi$, we find the following recurrence equation for A_n

$$A_{n+1} = A_n \phi_0 + \phi_0^n \phi_1 \tag{A.57}$$

which is readily solved by $A_n = n\phi_0^{n-1}\phi_1$. For B_n it is convenient to introduce

$$B_n = \alpha_n \phi_0^{n-2}\phi_1^2 + \beta_n \phi_0^{n-1}\phi_2 + \gamma_n \phi_0^{n-3}\mathcal{B}_1 + \delta_n \phi_0^{n-2}\mathcal{B}_2 \tag{A.58}$$

where $\mathcal{B}_1 = (\partial\phi_0)^2\bar\partial^2\phi_0 + (\bar\partial\phi_0)^2\partial^2\phi_0 - 2\partial\phi_0\bar\partial\phi_0\partial\bar\partial\phi_0$ and $\mathcal{B}_2 = \partial^2\phi_0\bar\partial^2\phi_0 - (\partial\phi_0\bar\partial\phi_0)^2$. We find the following recurrence relations

$$\begin{aligned} \alpha_{n+1} &= \alpha_n + n \\ \beta_{n+1} &= \beta_n + 1 \\ \gamma_{n+1} &= \gamma_n + \frac{n(n-1)}{8} \\ \delta_{n+1} &= \delta_n + \frac{n}{4} \end{aligned} \tag{A.59}$$

which are solved by $\alpha_n = n(n-1)/2$, $\beta_n = n$, $\gamma_n = n(n-1)(n-2)/24$, and $\delta_n = n(n-1)/8$. The final form, is then

$$\begin{aligned} \phi_\star^n = \phi_0^n + n\phi_0^{n-1}\phi_1\theta + \theta^2(n\phi_0^{n-1}\phi_2 + \frac{n(n-1)}{2}\phi_0^{n-2}\phi_1^2) \\ + \frac{n(n-1)}{8}\phi_0^{n-2}\mathcal{B}_2\theta^2 + \frac{n(n-1)(n-2)}{24}\phi_0^{n-3}\mathcal{B}_1\theta^2 \end{aligned} \tag{A.60}$$

Notice that $\mathcal{B}_1$ and $\mathcal{B}_2$ vanish for soliton solutions.

B Solution of the Equation of Motion to Order θ^2

In section 3.4 we found the classical equations of motion for the noncommutative sine-Gordon model. Taking $\beta = 1$, their static form to order θ^0, θ^1, and θ^2, are

$$\phi_0'' = C \sin\phi_0 \tag{B.61}$$

$$\phi_1'' = C\phi_1 \cos\phi_0 \tag{B.62}$$

$$\phi_2'' = C\phi_2\cos\phi_0 - \frac{C}{2}\phi_1^2\sin\phi_0 \tag{B.63}$$

where the primes indicate space derivatives, and C is a constant ($C = \frac{m^2}{1-v^2}$, and v is the soliton velocity). To solve the first equation B.61 we multiply it by $\phi_0^{'}$ and integrate it, using the boundary conditions $\phi_0 = 0$ at $x = 0$ and $\phi_0 = 2\pi$ for $x \to \infty$

$$\phi_0^{'} = \pm 2\sqrt{C}\sin\frac{\phi_0}{2} \tag{B.64}$$

The plus (minus) sign corresponds to the soliton (anti-soliton) solution. The solution of B.64 is easily found to be

$$\phi_0 = 4\tan^{-1}(\exp(\sqrt{C}(x - x_0))) \tag{B.65}$$

In order to solve B.62 we multiply it by $\phi_0^{'}$ and use B.61 to obtain

$$\phi_0^{'}\phi_1^{''} = C\phi_1\phi_0^{'}\cos\phi_0 = \phi_1\frac{d}{dx}(C\sin\phi_0) = \phi_1\phi_0^{''} \tag{B.66}$$

and the last equation together with the fact that the derivatives of ϕ vanish for $x \to \infty$ implies that

$$\phi_1\phi_0^{''} - \phi_1{'}\phi_0^{'} = 0 \tag{B.67}$$

This equation is easily solved by

$$\phi_1 = \frac{K}{\cosh(\sqrt{C}(x - x_0))} \tag{B.68}$$

where K is an integration constant. The solution for ϕ_2 can be found in the following way: take two derivatives of B.61 and multiply it by $\frac{K^2}{2}$, to get

$$(\frac{K^2}{2}\phi_0^{''})^{''} = C(\frac{K^2}{2}\phi_0^{''})\cos\phi_0 - \frac{C}{2}(K\phi_0^{'})^2\sin\phi_0 \tag{B.69}$$

which is the same as B.63. Since this is a first order equation for ϕ_2 and the boundary conditions at infinity are satisfied automatically, we can write the solution for ϕ_2 as

$$\phi_2 = \frac{K^2}{2}\phi_0^{''} \tag{B.70}$$

This solves the equations of motion for the ncsG model o second order in θ.

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