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The M theory lift of two O6$^-$ planes and four D6 branes

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Abstract: We solve for the effective actions on the Coulomb branches of a class of $N = 2$ supersymmetric theories by finding the complex structure of an M5 brane in an appropriate background hyperkahler geometry corresponding to the lift of two O6$^-$ orientifolds and four D6 branes to M theory. The resulting Seiberg-Witten curves are of finite order, unlike other solutions proposed in the literature. The simplest theories in this class are the scale invariant Sp(k) theory with one antisymmetric and four fundamental hypermultiplets and the SU(k) theory with two antisymmetric and four fundamental hypermultiplets. Infinite classes of related theories are obtained by adding extra SU(k) factors with bifundamental matter and by turning on masses to flow down to various asymptotically free theories. The $N = 4$ supersymmetric SU(k) theory can be embedded in these asymptotically free theories, allowing a derivation of a subgroup of its S duality group as an exact equivalence of quantum field theories.

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Contents

1. Introduction and summary

A large class of supersymmetric gauge field theories can be realized as open string excitations confined to branes in string theory. Following [1] the low energy effective actions on the Coulomb branches of $N = 2$ supersymmetric four dimensional theories with product SU gauge groups and fundamental matter were constructed by using IIA/M theory duality to realize the Seiberg-Witten curve [2, 3] as the complex structure of an M5 brane. This construction was generalized in [4] to include a matter hypermultiplet in a symmetric or an antisymmetric representation of the gauge group by adding an orientifold O6 plane to the configuration of D4, NS5, and D6 branes [1, 4, 5] (see also [6]). Theories with two such matter representations can also be realized as a IIA brane configuration by adding a second O6 plane [5]; however the resulting lift to an M5 brane curve was thought to be more difficult in this case. Furthermore, an alternative approach based on "reverse engineering" the curve from computations of one-instanton corrections to the Coulomb branch effective coupling led to an infinite order Seiberg-Witten curve [7, 8, 9].

In this paper we carry through the lift of a IIA string configuration of two $O6^-$ planes and four D6 branes to an M theory gravitational background. We then show how to write curves describing appropriate holomorphic embeddings of M5 branes in this background geometry so as to preserve 8 global supersymmetries, corresponding to $N = 2$ supersymmetric theories in four dimensions.

The IIA configurations we are interested in all involve various D4 and NS5 branes (whose geometry is detailed in section 2) in the presence of a parallel set of two $O6^-$ orientifold planes and four D6 branes. The heavy branes—the two $O6^-$ planes and the four D6 branes—form a background in which many NS5 and D4 brane configurations can be placed. We identify the various $N = 2$ supersymmetric four dimensional weakly coupled gauge theory limits that are obtained in this way. They include the $\mathrm{Sp}(k)$ theory with four fundamental hypermultiplets and one traceless-antisymmetric matter hypermultiplet with arbitrary masses, the $\mathrm{SU}(k)$ theory with four fundamental and two antisymmetric hypermultiplets, the $\mathrm{Sp}(k) \times \mathrm{Sp}(k)$ theory with two fundamentals in each factor and one bifundamental hypermultiplet, as well as infinite classes of related theories that are obtained by adding extra $\mathrm{SU}(k)$ factors with bifundamental matter and by turning on masses to flow down to various asymptotically free theories.

The transverse space to the heavy branes—the two $O6^-$ planes and the four D6 branes—lifts in M theory to a smooth hyperkahler four-manifold Q_0, which we will refer to as the background surface. In section 3 we show that one of its complex structures can be described as a surface in the 3 complex dimensional space $\mathbb{CP}^2_{(1,1,2)} \times \mathbb{C}$ coordinatized by $(w, x, y) \in \mathbb{CP}^2_{(1,1,2)}$ with

$$(\lambda w, \lambda x, \lambda^2 y) \simeq (w, x, y), \qquad \lambda \in \mathbb{C}^*, \tag{1.1}$$

and $z \in \mathbb{C}$. The equation of the surface Q_0 is then given by

$$y^2 = z \prod_{i=1}^{4} (x - e_i w) + \sum_{j=1}^{4} \mu_j^2 w \prod_{k \neq j} [(x - e_k w)(e_j - e_k)]. \tag{1.2}$$

Only one combination of the e_i is an invariant of the complex structure of Q_0, and has the interpretation as a complex gauge coupling of the associated scale invariant $N = 2$ theories. All four of the μ_j are complex structure invariants, and have the interpretation of linear combinations of the bare masses of certain hypermultiplet matter fields in the $N = 2$ theories. Indeed the μ_j are the residues of the poles of a meromorphic Seiberg-Witten one form on Q_0, given by

$$\lambda = \frac{y(w dx - x dw)}{\prod_i (x - e_i w)}. \tag{1.3}$$

Thus the μ_j and the e_i coordinatize a total of five complex deformations of Q_0.

There is another deformation of this background geometry, associated with a complex parameter M which is important in the $N=2$ gauge theories. It is closely related to the "shift" of the elliptic models introduced in [1]. In Section 3.4 we describe how to implement this shift of Q_0, which we denote Q_M, in an indirect way by specifying a submanifold of Q_0 to be excised, and the modified boundary conditions that holomorphic functions on Q_M must satisfy at this submanifold. In terms of the four-dimensional physics, the parameter M of this shift corresponds to the mass of an antisymmetric hypermultiplet.

The particular arrangements of NS5 and D4 branes in the IIA string theory described in section 2 are lifted to a single M5 brane in M theory. The construction of the complex structure of the curves describing the embedding of the M5 brane in Q_M is given in Section 4. In the case of no shift, these complex curves in Q_0 can be written in the general form of sums of polynomials in z times meromorphic functions in $\mathbb{CP}^2_{(1,1,2)}$. The order of the polynomials in z corresponds to the number of D4 branes (the rank of the gauge group factors), while the positions of the poles in $\mathbb{CP}^2_{(1,1,2)}$ encode the positions of the NS5 branes (corresponding to the relative strengths of the gauge couplings of the gauge group factors). The complex deformation parameters appearing in these curves have the interpretation of Coulomb branch vevs and bifundamental masses in the field theory. In the case where the shift parameter is turned on, the curves lie in Q_M and are more complicated, though explicit prescriptions for their construction are given in Section 4.3. In particular, all the relevant formulas are collected in equation (4.29)—(4.34). The resulting curves are all finite genus Riemann surfaces.

As a check on our construction, and for later use, in section 5 we match our curves at weak coupling to perturbative results and to known curves of other theories in certain decoupling limits. This determines explicit mappings between the various complex parameters appearing in the background and M5 brane curves and the physical couplings, masses, and vevs. In this section we also derive some curves for asymptotically free theories which can be found upon decoupling fundamental hypermultiplets by sending their masses to infinity. In the IIA picture this corresponds to sending the D6 branes to infinity, leaving only the two $\mathrm{O6}^-$ planes as the background geometry.

In section 6 we show that by going to an appropriate submanifold of the Coulomb branch of certain of these asymptotically free theories (and by also turning on appropriate hypermultiplet vevs) we can embed the $N=4$ supersymmetric $\mathrm{SU}(n)$ superYang-Mills theory in $N=2$ supersymmetric theories. Although the enhanced $N=4$ supersymmetry is an accidental symmetry at long wavelengths, this embedding can be used to derive a subgroup of the S duality group of the $N=4$ theory, following the general arguments of [10]. Assuming the validity of the curves derived in section 6 for the low energy effective action of the $N=2$ theories on the Coulomb branch, we derive the complex structure of the image of the coupling space of the embedded $N=4$ theory. This, together with a global discrete symmetry of the asymptotically free theory are enough to imply that the $N=4$ theory has a subgroup of the $N=4$ S duality group. Furthermore, by taking a scaling limit towards the singular submanifold of the Coulomb branch of the asymptotically free theory, we can show, to any given precision, that all the correlators of the $N=4$ (and not just the low energy effective action or supersymmetric states) are duality invariant under this subgroup.

Finally, it would be interesting to compare the predictions of the curves found in this paper with those of the alternative infinite order ones proposed in [7, 8, 9]. It is possible that the two sets of curves are equivalent in the sense that there is an infinite group of identifications on the infinite-dimensional Jacobian torus of the infinite order curves under which they becomes equivalent to the Jacobians of our curves; however, it seems difficult to find such a "folding" in practice. One test of the equivalence of the two sets of curves is to compare their predictions for the multi-instanton contributions to the prepotential. Such a check would have to take into account the full set of allowed non-perturbative redefinitions of the parameters of the two curves as in [11].

As this paper was being written up, the paper [12] appeared which also discusses curves describing branes in the presence of two $O6^-$ planes and four D6 branes in the special case with the deformation parameters (fundamental masses) $\mu_i = 0$.

2. IIA string construction

We begin by describing a set of IIA string constructions whose weak string coupling and low energy limits describe the set of four-dimensional scale invariant $N = 2$ supersymmetric gauge theories we are interested in.

Let x_0—x_9 be the coordinates on the ten-dimensional spacetime of type IIA string theory with x_6 compactified on a circle. The number and type of IIA string theory objects appearing in the configuration, as well as the dimensions along which they extend, are as follows:

- 2 $O6^-$ orientifolds along x_0—x_3, x_7—x_9,
- 4 D6 branes along x_0—x_3, x_7—x_9,
- q NS5 branes along x_0—x_5 and intersecting an $O6^-$,
- r NS5 branes along x_0—x_5, not intersecting any $O6^-$,
- k D4 branes along x_0—x_3, x_6.

As an example, figure 1 shows an x_4—x_6 cross-section of a fundamental domain of a configuration with $q = 2$, $r = 1$, and $k = 2$ after modding out by the orbifold identifications associated with the orientifold planes. The two ends of the domain in x_6 are identified, and correspond to the location of one of the $q = 2$ NS5 branes, with the other midway between the two ends along x_6. The intersections of the $q = 2$ NS5 branes with the diagonal boundary of the fundamental domain are two $\mathbb{Z}_2$ orbifold points corresponding to the positions of the O6 planes. D4 branes suspended between each pair of NS5 branes are shown. As they cross the diagonal boundary of the domain, they emerge on the other side of boundary (where the middle NS5 naturally partitions the domain into two sides).

More explicitly, the $O6^-$ planes have the same x_4-x_5 coordinates, but they are separated along a diameter of x_6. The D6 branes are parallel to the $O6^-$ planes. Define $v \equiv x_4 + ix_5$, and let $2L$ be the circumference of the x_6 circle. If we put one $O6^-$ plane at $(v, x_6) = (0, 0)$ and the other at $(L, M/2)$, then the x_6-v space is orbifolded by the identifications:

$$x_6 \simeq -x_6 \quad \text{and} \quad v \simeq -v, \tag{2.1}$$

and

$$x_6 \simeq x_6 + 2L \quad \text{and} \quad v \simeq v + M. \tag{2.2}$$

We will refer to M as the "shift" in v in the remainder of the paper.

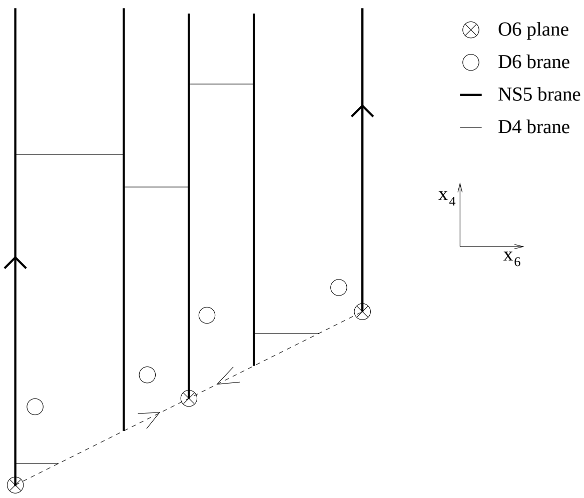


Figure 1: A fundamental domain in the x_4—x_6 plane of an arrangement of two $O6^-$, three NS5 (two of which intersect the $O6^-$'s), two D4 (each split into two segments), and four D6 branes.

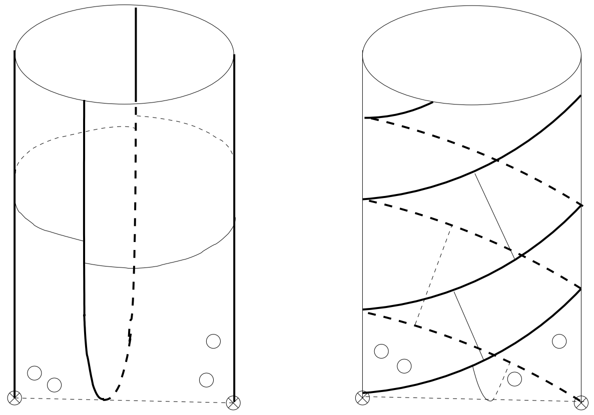
The NS5 branes have common x_7—x_9 coordinates but different x_6 coordinates. We count the $r+q$ NS5 branes as follows. If it does not intersect an $O6^-$ plane, then we count as contributing one to r a NS5 brane together with its image under (2.1); thus in the orbifolded background r counts the asymptotic NS5 brane charge. If it intersects an $O6^-$ plane, then it has no image 5 brane, and we count it as contributing one to q, even though it contributes only 1/2 to the asymptotic 5 brane charge. We consider only $q = 0$, 1 or 2 where at most one of the NS5 branes coincide with a given $O6^-$ plane. The interesting cases with two or more NS5 branes coinciding with a given $O6^-$ plane will be discussed elsewhere.

For the scale invariant models, the D4 branes wrap the x_6 circle. Since they have the same x_7—x_9 coordinates as the NS5 branes, they can split into a set of segments suspended between pairs of NS5 branes. It will prove convenient to let k count the number of D4 branes wrapping the x_6 circle *before* the orbifolding (2.1), *i.e.* a D4 brane and its image under (2.1) contributes *two* units to k. The reason for this is that two different segments of one D4 brane can end up, upon orbifolding, being suspended between the same pair of NS5 branes.

This is illustrated in figure 2(a), where the snocone geometry of figure 1 is made explicit in the simple case where $M = 0$ (the non-shifted case). A snocone is a semi-infinite cylinder-like object with a closed end, flat except at the two $\mathbb{Z}_2$ orbifold points. Note, for example, that there are two D4 branes suspended between the middle U-shaped 5 brane and the one on the right side (which ends on an O6 plane). Figure 2(b) attempts to show the shifted case which is equivalent to introducing a relative tilt between the directions in which the NS5 and the snocone surfaces extend.

The low energy excitations of open strings confined to the D4 branes of the above configurations give rise to the Coulomb branch of low energy theories in four dimensions, since the brane configurations are translationally invariant along x_0—x_3. Each stack of k D4 branes suspended between two NS5 branes results in an SU(k) gauge factor [13], unless there is also an O6$^-$ plane between the two NS5 branes, in which case the gauge factor is Sp(k) [4]. (We use the notation where Sp(k) has rank $k/2$.) Thus with r NS5 branes (not intersecting an O6 plane) there will be $r+1$ gauge group factors; if there are no 5 branes intersecting an O6 plane ($q=0$) then the "end" group factors are Sp(k), but if a 5 brane does intersect an O6 plane then that end factor is changed to SU(k).

Matter hypermultiplets enter in three ways. Firstly, fundamental hypermultiplets arise from D6 branes passing between NS5 branes, and their excitations correspond to strings extending from the D4 brane stack to the D6 brane. The mass of the fundamental is therefore proportional to the distance in the x_4-x_5 plane of the D6 brane from the D4 stack. The D6 branes are associated in pairs to the two O6 planes in our construction, and should be thought of as contributing a pair of fundamental hypermultiplets to each of the gauge group factors (D4 brane stacks) nearest the O6 planes. Secondly, antisymmetric matter hypermultiplets arise when an O6$^-$ brane intersects an NS5 brane, and correspond to strings extending from the D4 brane stack to the O6$^-$ plane. The mass of the antisymmetric is therefore proportional to the distance in the x_4-x_5 plane of the O6$^-$ plane from the D4 stack. Finally, neighboring gauge factors have a bifundamental hypermultiplet from strings stretching between neighboring stacks of D4 branes, and their masses are proportional to the distance in the x_4-x_5 plane between the two neighboring D4 stacks.



(a) (b)

Figure 2: The topology of the region shown in figure 1 in the cases of (a) $q=2$, $r=1$, $k=2$, and $M=0$, and (b) $q=2$, $r=0$, $k=4$, and $M\neq 0$.

Since the 6 brane charges cancel and each 5 brane has equal numbers of 4 branes extending to the left and right, the asymptotic 5 branes are flat, implying that the corresponding gauge groups should all be scale invariant. In particular, the asymptotic separations along x_6 between the r non-stuck 5 branes as well as the O6 planes correspond to the dimensionless couplings of the various gauge group factors. Upon lifting to M theory, the 5 brane separations along the extra x_{10} circle correspond to the theta angles of the gauge factors.

To summarize, the parameters of the IIA configurations are identified with parameters and vevs of the four dimensional gauge theories as follows [1, 4, 5]. The coupling of each gauge group factor is identified with the separation of the pair of neighboring NS5 branes associated with it. The scalar components of the adjoint multiplet correspond to the relative motions of pairs of D4 branes. Meanwhile, the masses of the fundamental, antisymmetric and bifundamental multiplets correspond respectively to the relative motions of D4 branes and

D6 branes, D4 branes and $O6^-$ planes and mean positions of the D4 branes on either side of an NS5 brane. Thus, in particular, the parameter M can be interpreted as the difference of the masses of the two antisymmetric hypermultiplets.

The resulting $N = 2$ gauge theories can be classified by the number (q) of NS5 branes which intersect the two $O6^-$ planes. We list here the gauge group and hypermultiplet content for the $q = 0, 1, 2$ theories. We also count all the complex parameters describing the Coulomb branch geometry of each theory for later use.

- $q = 0$: $\mathrm{Sp}(k) \times \mathrm{SU}(k)^{(r-1)} \times \mathrm{Sp}(k)$ with bifundamentals between each pair of neighboring gauge group factors and two fundamentals in each $\mathrm{Sp}(k)$ factor. There are $r + 1$ couplings, 4 fundamental masses, r bifundamental masses, and $2(k/2) + (r - 1)(k - 1)$ Coulomb branch vevs, for a total of $6 + r + kr$ parameters.

- $q = 1$: $\mathrm{Sp}(k) \times \mathrm{SU}(k)^{(r-1)} \times \mathrm{SU}(k)$ with bifundamentals between neighboring pairs, two fundamentals in $\mathrm{Sp}(k)$ and two fundamentals and an antisymmetric in the last $\mathrm{SU}(k)$ factor. There are $r + 1$ couplings, 4 fundamental masses, r bifundamental masses, 1 antisymmetric mass, and $(k/2) + r(k - 1)$ Coulomb branch vevs, for a total of $6 + r + kr + (k/2)$ parameters.

- $q = 2$: $\mathrm{SU}(k) \times \mathrm{SU}(k)^{(r-1)} \times \mathrm{SU}(k)$ with bifundamentals between neighboring pairs, and two fundamentals and an antisymmetric in the first and last $\mathrm{SU}(k)$ factors. There are $r + 1$ couplings, 4 fundamental masses, r bifundamental masses, 2 antisymmetric masses, and $(r+1)(k-1)$ Coulomb branch vevs, for a total of $6+r+kr+k$ parameters.

In the $q = 0$ and $q = 1$ cases above, k is even, since there is no $\mathrm{Sp}(k)$ theory with k odd. In the $q = 2$ case, on the other hand, odd k is allowed. In terms of the brane configuration, because of the 5 branes intersecting each O6 plane, having an odd number of D4 branes (recall that k counts both D4 branes and their orbifold images) does not mean that one is stuck on the O6 plane, since it can move by splitting along the stuck NS5 brane. The curves we will derive in section 4 incorporate these rules in a natural way.

The above list degenerates for $r = 0$ as follows [5]:

- $q = 2$: $\mathrm{SU}(k)$ with 2 antisymmetric hypermultiplets and 4 fundamentals, which has 1 coupling, 4 fundamental masses, 2 antisymmetric masses, and $k - 1$ vevs for a total of $k + 6$ parameters.

- $q = 1$: $\mathrm{Sp}(k)$ with a traceless-antisymmetric and 4 fundamentals, which has 1 coupling, 4 fundamental masses, 1 antisymmetric mass, and $(k/2)$ vevs for a total of $(k/2) + 6$ parameters.

- $q = 0$: also yields $\mathrm{Sp}(k)$ with a traceless-antisymmetric and 4 fundamentals, but only for $M = 0$ (no shift) which corresponds to the antisymmetric hypermultiplet being massless. Thus this model has a total of $(k/2) + 5$ parameters.

The $q = r = 0$ configuration only exists for zero shift because otherwise the 4 branes would not close upon traversing the x_6 circle. The equivalence of the $q = r = 0$ and $q = 1$, $r = 0$ configurations at zero shift reflects the fact that the stuck 5 branes can move off along the O6 plane when the D4 branes close upon traversing the x_6 circle. In terms of the four dimensional gauge theory this reflects the existence of a mixed Coulomb-Higgs branch at $M = 0$ which includes the whole Coulomb branch as a subvariety. Even with $M \neq 0$, if the D4 branes line up on either side of one of the stuck 5 branes in the $q = 2$ $r = 0$ theory, that 5 brane can also be moved off, giving the $q = 1$ $r = 0$ theory. In field theory terms, giving a vev to one of the antisymmetrics in the SU(k) theory Higgses it to Sp(k) with one antisymmetric. These and many other relations between the theories we are considering found by tuning masses and vevs and taking decoupling limits are reflected in the structure of the curves we derive below.

From the counting of couplings, masses, and vevs given above, we see that the total number of parameters on the Coulomb branch is $6+r+kr+(qk)/2$ (except for the degenerate $r = q = 0$ model). Also, all these theories have $r + 1$ couplings and 4 fundamental masses. This counting will provide a simple check of the curves found in section 4 below.

Finally, it will be important for a later argument to consider the effect of moving a D6 brane along a cycle enclosing one of the $\mathbb{Z}_2$ orbifold points, which corresponds to taking $m_i \to e^{2\pi i} m_i$ where m_i is the fundamental mass parameter associated with the chosen D6 brane. As the D6 brane crosses an NS5 brane, it pulls a D4 brane with a definite orientation behind it (the type IIA equivalent of the Hanany-Witten effect [14]). Moving the D6 brane across the diagonal boundary in figure 1, we obtain a D6 brane on the other side with the D4 brane orientation reversed. Finally, as the D6 traverses the NS5 brane a second time, a D4 brane with opposite orientation to the first D4 brane is created, and annihilates the first D4, leaving only a D6 brane on the other side of the NS5. The D6 can then recross the diagonal boundary and return to its original position. Thus, only when the D6 circles the O6$^-$ twice is the original configuration recovered. We will return to this point in section 4 to resolve an apparent sign ambiguity in our curves.

3. M theory lift of two O6$^-$ planes

The next step is to lift the IIA string constructions described above to M theory [1], thereby obtaining the low energy effective action on the Coulomb branch of the associated $N = 2$ gauge theories. All the IIA brane setups have the same arrangement of two O6$^-$ planes and four D6 branes which are transverse to the x_4, x_5, and x_6 directions. The lift of this transverse geometry to M theory will give a hyperkahler four-manifold, Q_M, which we call the background geometry or surface (since it is two complex dimensional). In lifting to M theory, the D6 branes become multi-center Taub-NUT manifolds, while the O6$^-$ planes become Atiyah-Hitchin manifolds [15, 16, 17, 18]. We focus only on the complex structure of this space, and do not determine its metric.

Meanwhile, all the NS5 and D4 branes lift to a single M5 brane, two directions of which are embedded on a complex curve in the background geometry Q_M. Thus we adopt a convenient splitting of the M theory lift into two steps: in this section we derive the hyperkhaler geometry Q_M associated with the O6$^-$ and the D6 branes, and then in the next section we embed the M5 brane corresponding to our D4 and NS5 brane configuration in Q_M. We subdivide both steps of the lift into three parts as follows. First, we consider the case where a pair of D6 branes is coincident with each O6$^-$ and M is set to zero (no shift), so that the M theory lift is simply an orbifold of the x_6-x_{10} torus cross the v plane (section 3.1). Next, we allow the D6 branes to move off the O6$^-$, introducing deformations of the orbifolded torus corresponding to four fundamental hypermultiplet masses (section 3.2). Finally, we discuss how to account for non-zero shift M in the background (section 3.4). Along the way, we will derive the Seiberg-Witten one-form needed to complete our description of the low energy action, and relate it to parameters in the background geometry (section 3.3).

3.1 $\mathbb{C} \times T^2/\mathbb{Z}_2$: Orbifolding the elliptic model

While the M theory lift of individual D6 branes and O6$^-$ planes involves non-trivial geometries, the lift of an O6$^-$ plane coincident with a pair of D6 branes is simply a $\mathbb{Z}_2$ orbifold known as a D_2 singular space [17]. It is therefore convenient to begin by considering the $\mathbb{Z}_2$ orbifold of the v-plane cross the torus, and then use physical requirements to constrain the deformations of the orbifolded torus corresponding to moving the D6 branes off the O6$^-$ planes and to separating the O6$^-$ planes along v.

In lifting to M theory, we grow a circular dimension x_{10} of radius R. Define the flat variable

$$s \equiv (x_{10} + ix_6)/(2\pi R). \tag{3.1}$$

Thus the orbifold identifications (2.1) and (2.2) are lifted to

$$s \simeq -s \quad \text{and} \quad v \simeq -v, \tag{3.2}$$

$$s \simeq s+1 \quad \text{and} \quad v \simeq v, \tag{3.3}$$

and

$$s \simeq s+\tau \quad \text{and} \quad v \simeq v+M, \tag{3.4}$$

where τ encodes the complex structure of the torus. In this and the next two subsections we will set the shift parameter $M = 0$.

Before orbifolding, the M theory background space $\widetilde{Q}$ is just the flat space $\widetilde{Q} = \mathbb{C} \times T^2$ with coordinates $v \in \mathbb{C}$ and $s \in T^2$. We wish to orbifold this space under the $\mathbb{Z}_2$ identication $(v, s) \simeq (-v, -s)$, which can be done by rewriting $\widetilde{Q}$ in terms of orbifold-invariant coordinates, thus providing good coordinates on $\widetilde{Q}/\mathbb{Z}_2$.

We first introduce a single-valued coordinate description of T^2 by writing the compact torus as a complex curve in the weighted projective space $\mathbb{CP}^2_{(1,1,2)}$ defined as the space of all complex (w, x, η) minus the point $(0, 0, 0)$ and modulo the identification

$$(\lambda w, \lambda x, \lambda^2\eta) \simeq (w, x, \eta), \qquad \lambda \in \mathbb{C}^*. \tag{3.5}$$

Then we can write the torus as

$$\eta^2 = \prod_{i=1}^{4}(x - e_i w) \tag{3.6}$$

where the numbers e_i encode the complex structure τ of the torus in the usual way [19].

If we use $\mathbb{CP}^2_{(1,1,2)}$ scaling to set $\lambda = \frac{1}{w}$ (or $\lambda = \frac{1}{x}$ if $w = 0$), we can relate the single-valued variables x and η to the multivalued variable s via the Weierstrass $\mathcal{P}$-function with an appropriate choice of the e_i's, but the explicit form of the relation will not be needed here. It is nevertheless useful to know how to translate qualitatively between the s-plane and the (w, x, η) space. First, the unique $\mathbb{Z}_2$ automorphism fixing a point ($s = 0$) of the torus is $s \to -s$, while the obvious $\mathbb{Z}_2$ automorphism of (3.6) is $\eta \to -\eta$ with w and x fixed. The $\mathbb{Z}_2$ identification $(v, s) \simeq (-v, -s)$ therefore reads $(v, w, x, \eta) \simeq (-v, w, x, -\eta)$ in (v, w, x, η) language. The fixed points of this map are the half-periods $s = 0, \tau/2, 1/2, (1+\tau)/2$, or the branch points $x = e_i w$. Thus, we can choose to match these fixed points up in some way, for example,

$$\begin{aligned}
(w, x, \eta) &= (1, e_1, 0) \leftrightarrow s = 0,\\
(w, x, \eta) &= (1, e_2, 0) \leftrightarrow s = \frac{1}{2},\\
(w, x, \eta) &= (1, e_3, 0) \leftrightarrow s = \frac{\tau}{2},\\
(w, x, \eta) &= (1, e_4, 0) \leftrightarrow s = \frac{1}{2}(\tau + 1),
\end{aligned} \tag{3.7}$$

although any other ordering would do as well.

Next, we orbifold the space $\widetilde{Q}$ under the identification $(v, w, x, \eta) \simeq (-v, w, x, -\eta)$ by using the single-valued variables:

$$y \equiv v\eta, \qquad\qquad z \equiv v^2, \tag{3.8}$$

(and w and x unchanged) so that the orbifolded background space $Q_0 = \widetilde{Q}/\mathbb{Z}_2$ is given by the surface

$$y^2 = z\prod_{i=1}^{4}(x - e_i w), \tag{3.9}$$

in $\mathbb{C} \times \mathbb{CP}^2_{(1,1,2)}$, which follows simply by multiplying (3.6) by v^2 and changing variables as in (3.8). At any finite value of z this curve describes a torus, but at $z = 0$ this becomes $y = 0$ which is a sphere in the weighted projective space. This can be understood by noting that for $v \neq 0$, the orbifold $(v, w, x, \eta) \to (-v, w, x, -\eta)$ identifies two different tori, while at $v = 0$ it identifies the two η-sheets of one torus, which effectively removes any non-trivial cycles and reduces the torus to a sphere. Thus the M theory lift of this IIA brane configuration can be pictured as being similar to the snocone geometry pictured in figure 2, with, however, the constant $v \neq 0$ ($z \neq 0$) cross sections being similar tori, instead of circles, and the $v = 0$ ($z = 0$) cross section being a sphere instead of a line segment. This corresponds to the naive construction of the M theory lift of the IIA configuration by adding circles.

As a further check on this background surface, note that it is singular (has $\mathbb{Z}_2$ orbifold points) at the four points $(w,x,y,z)=(1,e_i,0,0)$. It should reduce to the D_2 singular space in the vicinity of a pair of these singularities. To see that this is indeed the case, take, say, $e_1=1$, $e_2=-1$ and e_3 and e_4 very large (which we can do for any pair of nearby points by an appropriate coordinate change). Then, in the vicinity of the e_1 and e_2 orbifold points we can set $w=1$ (this is just choosing an appropriate coordinate patch in the $\mathbb{CP}^2$), rescale z to absorb the e_3e_4 factor, and get approximately

$$y^2 = z(x^2-1). \tag{3.10}$$

This is precisely the form of the D_2 singularity found in the literature, see *e.g.* [17].

3.2 Q_0: Deforming the orbifold

Now we deform this background space by (in the IIA picture) moving the D6 branes off the orbifold points. This corresponds to complex deformations of the background surface (3.9) which do not change its asymptotic (large z) behavior. Physically it corresponds to turning on masses for the four fundamental flavors.

The most general holomorphic terms that can be added to the right hand side of (3.9) consistent with the surface having weight four in $\mathbb{CP}^2_{(1,1,2)}$ are of the form

$$z^{n_2}y^2 + z^{n_1}yP_2(x,w) + z^{n_0}P_4(x,w) \tag{3.11}$$

where $P_m(x,w)$ is an arbitrary homogeneous polynomial with constant coefficients of weight m in x and w, and the n_i are integers whose possible values we will now determine. The basic requirement on the n_i is that they do not lead to terms which are either singular at finite z or change the topology of the surface at large z. The first requirement implies that all the n_i must be non-negative. The second is easily seen to imply that $n_0=0$ or 1 while $n_1=n_2=0$. Not all of these terms are deformations of (3.9), though, since some can be reabsorbed in holomorphic redefinitions of the coordinates. Such redefinitions which are regular as $|z|\to\infty$ and respect the $\mathbb{CP}^2_{(1,1,2)}$ structure of the background are of the form

$$\begin{aligned} y &\to Ay + P_2(x,w),\\ x &\to P_1(x,w),\\ w &\to P_1(x,w),\\ z &\to Bz + C, \end{aligned} \tag{3.12}$$

where A, B and C are constants, and the $P_i(x,w)$ are again homogeneous constant coefficient polynomials (and the two P_1's need not be the same). These changes of variables can be used to absorb the $n_2=0$ term in (3.11) by a rescaling of y, the $n_1=0$ term by a P_2 shift in y (completing the square), the $n_0=1$ term by adjusting the e_i and rescaling z, and the coefficient of the x^4 term in P_4 in the $n_0=0$ term by a shift in z. Thus we are left with precisely four complex deformations:

$$y^2 = z \prod_{i=1}^{4} (x - e_i w) - \tilde{\mu}_1 x^3 w - \tilde{\mu}_2 x^2 w^2 - \tilde{\mu}_3 x w^3 - \tilde{\mu}_4 w^4, \tag{3.13}$$

where the $\tilde{\mu}_i$ are the four deformation parameters (which we will relate to the masses below).

Note that the only part of the (3.12) coordinate redefinitions left which preserve the form of the surface (3.13) are the three complex parameter $\mathrm{SL}(2, \mathbb{C})$ transformations $x \to Ax + Bw$, $w \to Cx + Dw$, with $AD - BC = 1$ (accompanied by appropriate rescalings or shifts of y and z), and the scale transformation $z \to \Delta^2 z$, $y \to \Delta y$, leaving x and w fixed. The $\mathrm{SL}(2, \mathbb{C})$ transformations leave appropriate combinations of the $\tilde{\mu}_i$ deformation parameters invariant, but transform the e_i by fractional linear transformations:

$$e_i \to (Ae_i + B)/(Ce_i + D) \tag{3.14}$$

(see section 5.3 below for the details). Thus of the four e_i parameters, only one combination is coordinate invariant. Under the scale transformation the e_i are invariant, but the deformation parameters scale as $\tilde{\mu}_i \to \Delta^2 \tilde{\mu}_i$. But the normalization of an additional structure on the surface, the Seiberg-Witten differential λ to be constructed in the next subsection, fixes the the scale transformation. Thus the complex structure of the Q_0 surface (3.13) is described by five complex parameters, the four deformation parameters $\tilde{\mu}_i$ (which we will see have the interpretation as fundamental hypermultiplet masses) and the invariant combination of the e_i (which corresponds to a dimensionless coupling constant in the gauge theory).

It is easy to check that turning on the $\tilde{\mu}_i$ indeed resolves the four singularities at $x = e_i w$. Rewriting (3.13) as

$$y^2 = (z+1) \prod_{i=1}^{4} (x - e_i w) - \prod_{i=1}^{4} (x - [\delta_i + e_i] w) \tag{3.15}$$

where the δ_i are functions of the $\tilde{\mu}_i$ deformation parameters, it is apparent that turning on δ_i resolves the singularity at $x = e_i w$.

Another check is to compare our surface to the resolution of the D_2 singular space appearing in the literature. Recall that taking $e_1 = 1$, $e_2 = -1$, e_3 and e_4 large, and rescaling by $e_3 e_4$ reduced the undeformed surface (3.9) to the D_2 singularity. Doing the same to the deformed surface (3.15) gives (in the $w = 1$ coordinate patch)

$$y^2 = z(x^2 - 1) + (\delta_1 + \delta_2) x + (\delta_1 - \delta_2 - \delta_1 \delta_2). \tag{3.16}$$

This is the same as the deformation of the D_2 singularity given in [20],[1] $y^2 = z(x^2 - 1) - 2\mu_1 \mu_2 x - (\mu_1^2 + \mu_2^2)$ with an appropriate map between their μ_i's and our δ_i's.

[1]This is equation (40) of [20] with the change of variables $x \to y$, $y \to x$, and $z \to -z$.

3.3 Hyperkahler structure and one-form

It follows from [1] that the Seiberg-Witten differential λ depends solely on the background M theory hyperkahler geometry Q_0 in which the M5 branes are embedded. Here we will use this fact to construct λ for the unshifted ($M=0$) background. In the next subsection when we turn on the shift, we will show how it is modified.

λ can be computed [21] as the solution of

$$\omega = d\lambda \tag{3.17}$$

where ω is a suitably chosen holomorphic (2,0)-form on the background space Q_0 given by (3.13). ω is determined by the hyperkahler structure of Q_0 which implies that Q_0 has three independent complex structures I, J and K, satisfying $I^2 = J^2 = K^2 = -1$ and $IJ = K$. If we choose the coordinates (w, x, y, z) to be holomorphic with respect to the I complex structure, then ω is defined by

$$\omega = \omega_J + i\omega_K, \tag{3.18}$$

where $\omega_{J,K}$ are the Kahler forms associated with the J, K complex structures of Q_0, respectively.

At constant z, Q_0 is a torus, and therefore admits the holomorphic one-form $(wdx - xdw)/y$. Hence the general holomorphic (2,0)-form on Q_0 can be written as

$$\omega = f(z)\frac{(wdx - xdw)\wedge dz}{y} \tag{3.19}$$

with $f(z)$ a polynomial in z to ensure holomorphicity at $z=0$. $f(z)$ can be determined by consistency with (3.18) and the metric on Q_0. Specifically, as $|z| \to \infty$ we know from the M theory construction that the metric on Q_0 approximates the $\mathbb{Z}_2$ orbifold (3.2) of the flat $\mathbb{C}\times T^2$ space coordinatized by (v, s). In the (x, η) coordinates on T^2 (going to a $w=1$ patch of $\mathbb{CP}^2_{(1,1,2)}$) the flat line element is

$$ds^2 = dv\, d\bar{v} + |\alpha|^2 \frac{dx\, d\bar{x}}{|\eta|^2}, \tag{3.20}$$

where α is a complicated constant which depends on the conventional way the flat T^2 coordinate s is expressed in terms of x. Since α will only enter in the overall normalization of λ we need not specify its precise value.

Going to the (x, y, z) orbifold coordinates using (3.8) gives the metric at large z as

$$ds^2 = \frac{dz\, d\bar{z}}{4|z|} + |\alpha|^2 \frac{|z| dx\, d\bar{x}}{|y|^2}. \tag{3.21}$$

Using (3.18) to write $\omega_J = \frac{1}{2}(\omega + \bar{\omega})$, and the relation $J = g^{-1}\omega_J$ between the complex structure, metric, and Kahler form, we compute the nonzero components of J at large z to be

$$J_z^{\bar{x}} = \frac{\bar{y} f}{2|\alpha|^2|z|}, \qquad J_{\bar{x}}^{z} = -\frac{2|z|\bar{f}}{\bar{y}},$$
$$J_{\bar{z}}^{x} = \frac{y \bar{f}}{2|\alpha|^2|z|}, \qquad J_{x}^{\bar{z}} = -\frac{2|z| f}{y}. \tag{3.22}$$

Consistency with the hyperkahler structure of Q_0 implies that $J^2 = -1$ which implies $|f|^2 = |\alpha|^2$ at large z. Since f is polynomial in z this fixes

$$f(z) = \alpha. \tag{3.23}$$

It is convenient to choose $\alpha = 1/2$ so that the final form for the (2,0)-form on Q_0 is

$$\omega = \frac{(w dx - x dw) \wedge dz}{2y}. \tag{3.24}$$

We now solve (3.17) for λ. Since $d\omega = 0$ a local solution always exists. It is not too hard to see that the global solution is given by

$$\lambda = \frac{y(x dw - w dx)}{\prod_{i=1}^{4}(x - e_i w)} \tag{3.25}$$

which is unique up to a total derivative.

The residues of λ at its eight poles at $x = e_i w$ yield linear combinations of the hypermultiplet bare masses appearing in the classical Lagrangian. The residues μ_i are readily obtained by performing contour integrals of (3.25) about the poles e_i yielding:

$$\mu_i = \frac{1}{2\pi i} \oint \lambda = \pm \frac{\sqrt{y(x = e_i, w = 1)}}{\prod_{j \neq i}(e_i - e_j)}. \tag{3.26}$$

This allows us to rewrite the deformation parameters $\tilde{\mu}_j$ of (3.13) in terms of the residues μ_j so that the background surface Q_0 becomes

$$y^2 = z \prod_i (x - e_i w) + \sum_j \mu_j^2 w \prod_{k \neq j} [(x - e_k w)(e_j - e_k)]. \tag{3.27}$$

Note that in this way of writing the surface, w and x are dimensionless, y has dimension of mass, and z has dimension of mass-squared. Note also that the overall multiplicative factor in relating μ_j to the mass parameters appearing in the Lagrangian has been left undetermined because we did not determine the normalization α of λ, but simply chose it to a convenient value. This factor will be determined in section 5 by matching to weak coupling limits of the theories.

Equations (3.27) and (3.25) are our main results so far, describing the complex structure and one-form of the background surface Q_0.

3.4 Q_M: Implementing the shift

Now we turn on the shift in v (in the IIA picture), which, as discussed in section 2, translates into turning on a (relative) antisymmetric mass in the field theories. We follow the discussion of the shift in the elliptic models of [1], rephrasing the essentials of incorporating the shift in (s, v) coordinates. We then formulate specific conditions on the background and curve in terms of the orbifold-invariant (w, x, y, z) coordinates. This gives the shifted background in the undeformed ($\mu_i = 0$) case. As a final step, we show how to deform this procedure to implement the shift in the $\mu_i \neq 0$ background, which corresponds to turning on the four fundamental masses.

Denote by X_M the affine bundle over T^2 formed from $\widetilde{Q} = T^2 \times \mathbb{C}$ by making the two identifications (3.3) and (3.4), and denote by $X_M \backslash P$ the manifold X_M minus the fiber (copy of the v-plane) over a conveniently chosen point $P : \{s = s_0\} \in T^2$ in the base torus. Then, as discussed in [1], the complex structure of $X_M \backslash P$ is the same as that of $X_0 \backslash P$, *i.e.* of the unshifted bundle over T^2. So we can specify the complex structure of X_M by specifying an affine analytic change of variables on the fibers

$$\widetilde{v} = v + g(s), \tag{3.28}$$

between coordinates (s, v) on $X_0 \backslash P$ and the coordinates $(s, \widetilde{v})$ on $X_M \backslash P$. In particular, a holomorphic function on X_M is an arbitrary holomorphic function on $X_0 \backslash P$ (*i.e.* a meromorphic function on X_0 with singularities only at the P fiber) which is regular in the $(s, \widetilde{v})$ variables at $s = s_0$. As discussed in [1], $g(s)$ has a simple pole at P with residue proportional to M. Since the regularity conditions on functions on X_M need only be determined in a vicinity of P, we need only specify g in a neighborhood of P. Thus any function with a simple pole at P will do. It is easy to translate this prescription to the $\mathbb{CP}^2_{(1,1,2)}$ coordinates (w, x, η) on T^2. The change of variables (3.28) becomes

$$\widetilde{v} = v + g(w, x, \eta) \tag{3.29}$$

where g has a simple pole at P with residue proportional to M. Without loss of generality, we can choose to work in a $\mathbb{CP}^2_{(1,1,2)}$ coordinate patch with $w = 1$, which we will do from now on. If the point P has coordinates $(w, x, \eta) = (1, x_0, \eta_0)$, then one convenient choice of the function g is

$$g = M \frac{\eta + \eta_0}{x - x_0}. \tag{3.30}$$

It is easy to check that it has a simple pole at $x = x_0$ on the sheet of the torus where $\eta = \eta_0$ and is regular on the other sheet where $\eta = -\eta_0$.

We now want to orbifold this construction to obtain a description of the shifted M theory background, which we will denote Q_M. Recall that the orbifolding makes the $\mathbb{Z}_2$ identification $(v, w, x, \eta) \simeq (-v, w, x, -\eta)$, which fixes the four branch points of the torus. We can orbifold the above prescription directly if we take the point P to be one of the branch points. However, it turns out to be more convenient to choose a general point on the torus to remove. Call this point P_+ with coordinates which we take to be $(w,x,\eta)=(1,x_0,\eta_0)$. Now the image of this point under the $\mathbb{Z}_2$ identification on the torus is another point P_- with coordinates $(w,x,\eta)=(1,x_0,-\eta_0)$. Thus to orbifold X_M we need to generalize the construction of X_M of [1] to the case where not one but two fibers of X_0 are excised.

But this is easy to do: we define the complex structure of X_M by identifying its holomorphic functions as those on $X_0\backslash\{P_+\cup P_-\}$ which are regular in a neighborhood of the $P_\pm$ fibers when reexpressed in terms of the new fiber coordinates

$$\widetilde{v}_\pm = v + g_\pm(x,\eta) \tag{3.31}$$

where $g_\pm$ have simple poles at $P_\pm$ respectively, and the sum of their residues is proportional to M. An obvious way to respect the $\mathbb{Z}_2$ orbifold symmetry is to choose

$$g_\pm = M\frac{\eta\pm\eta_0}{x-x_0}. \tag{3.32}$$

We perform the orbifolding by rewriting these changes of variables in terms of $\mathbb{Z}_2$-invariant coordinates. Away from $x=x_0$ ($P_\pm$) recall that these are $y=v\eta$ and $z=v^2$ (as well as w and x which are already $\mathbb{Z}_2$-invariant). It follows from (3.31) that the $\mathbb{Z}_2$ action identifies the new coordinates in the neighborhoods of the two fibers by

$$\widetilde{v}_+ \leftrightarrow -\widetilde{v}_-, \qquad \eta\leftrightarrow-\eta, \tag{3.33}$$

(while x and w are left invariant). Thus a basis of algebraically independent $\mathbb{Z}_2$-invariant coordinates can be taken to be

$$\widetilde{z}=\frac{1}{4}(\widetilde{v}_+ + \widetilde{v}_-)^2, \qquad \widetilde{y}=\frac{1}{2}(\widetilde{v}_+ + \widetilde{v}_-)\eta, \tag{3.34}$$

since the other possible invariant, $\widetilde{v}_+ - \widetilde{v}_-$ is just a function of x by (3.32), and so is not independent. Plugging in from (3.31) and (3.32) then gives the desired singular change of variables in a neighborhood of the $x=x_0$ "fiber" of Q_0:

$$\begin{aligned}\widetilde{y} &= y\left(1+M\frac{y}{z(x-x_0)}\right),\\ \widetilde{z} &= z\left(1+M\frac{y}{z(x-x_0)}\right)^2.\end{aligned} \tag{3.35}$$

Note that what before the $\mathbb{Z}_2$ orbifold identification was two fibers over $(x,\eta)=(x_0,\pm\eta_0)$, has become a single "fiber" in Q_0 at $x=x_0$. This is apparent from the equation of the Q_0 surface (3.9), which we recall is

$$y^2 = zP(x) \qquad \text{where} \qquad P(x)=\prod_{i=1}^{4}(x-e_i), \tag{3.36}$$

since the two roots of y at $x=x_0$ are joined at $z=0$.

The specific change of variables (3.35) is not unique; many other particular forms are possible since it is only the local behavior near the $x = x_0$ "fiber" that is important. This local behavior can be made more apparent by expanding out (3.35) using the equation of the surface Q_0 (3.36) giving

$$\begin{aligned}
\widetilde{y} &= y + M\frac{P(x)}{(x-x_0)}, \\
\widetilde{z} &= z + 2M\frac{y}{(x-x_0)} + M^2\frac{P(x)}{(x-x_0)^2}.
\end{aligned} \tag{3.37}$$

Thus y is shifted by a term with a simple pole at x_0 plus regular terms, while z is shifted by a double pole there plus single pole and regular terms.

Nevertheless, the specific change of variables (3.35) has some handy properties. The inverse is given by simply changing M to $-M$:

$$y = \widetilde{y}\left(1 - M\frac{\widetilde{y}}{\widetilde{z}(x-x_0)}\right), \qquad z = \widetilde{z}\left(1 - M\frac{\widetilde{y}}{\widetilde{z}(x-x_0)}\right)^2. \tag{3.38}$$

Also, in their patch the $\widetilde{y}$ and $\widetilde{z}$ coordinates satisfy the same relation defining the Q_0 surface as y and z do (3.36), namely

$$\widetilde{y}^2 = \widetilde{z}P(x). \tag{3.39}$$

These properties will prove useful in section 4 for describing an explicit recipe for writing down M5 brane curves in Q_M, so we will strive to preserve them in generalizing the above change of variables to the case where the D6 brane (fundamental mass) deformations μ_i are turned on.

The generalization of the shift to the case with the μ_i deformations turned on is straightforward. Recall that the equation (3.27) for the mass-deformed Q_0 surface is

$$y^2 = zP(x) + Q(x) \tag{3.40}$$

where

$$Q(x) = \sum_j \mu_j^2 \prod_{k\neq j}[(x-e_k)(e_j-e_k)], \tag{3.41}$$

$P(x)$ is as in (3.36), and $j, k = 1, \ldots, 4$. For general μ_i the deformation only significantly changes the complex structure of Q_0 in a (z-dependent) neighborhood of the $x = e_i$ branch points or for small z (compared to the μ_i^2). Therefore the only region of Q_0 where the change of variables (3.35) near $x = x_0$ might break down is for small z. But, since in a small enough neighborhood of the $x = x_0$ "fiber" the surface Q_0 is smooth whether or not the μ_i vanish, in fact the change of variables (3.35) applies to the deformed case without change.

However, it will be useful to let the change of variables depend on the deformation parameters in such a way as to preserve the handy features mentioned above. When the $\mu_i = 0$, the two branches of the $x = x_0$ "fiber" join at $z = 0$ as noted above; for non-zero μ_i

it follows from (3.40) that they now join at $z = -Q(x_0)/P(x_0)$. This suggests that a natural way to generalize (3.35) is to shift z to $z + Q(x)/P(x)$, giving

$$\widetilde{y} = y\left(1 + M\frac{y}{\left(z + \frac{Q(x)}{P(x)}\right)(x - x_0)}\right),$$
$$\widetilde{z} + \frac{Q(x)}{P(x)} = \left(z + \frac{Q(x)}{P(x)}\right)\left(1 + M\frac{y}{\left(z + \frac{Q(x)}{P(x)}\right)(x - x_0)}\right)^2. \tag{3.42}$$

It is easy to check that this form of the change of variables has the properties that its inverse is given just by interchanging $y \leftrightarrow \widetilde{y}$ and $z \leftrightarrow \widetilde{z}$ and changing $M \to -M$; and that the new variables $(x, \widetilde{y}, \widetilde{z})$ in the $x \sim x_0$ patch satisfy the same equation (3.40) for the unshifted Q_0 surface. Furthermore, since near $x = x_0$ (3.42) differs from (3.35) just by a constant shift in z, it has the same analyticity properties. In fact, expanding (3.42) out using (3.40) gives back precisely (3.37).

Finally, a short computation shows that the one-form λ has the same form (3.25) in both the y, z and the $\widetilde{y}$, $\widetilde{z}$ coordinates, and so is not modified in the presence of the shift M. Though M is interpreted as a mass, it does not appear as the residue of a pole in λ on Q_M. Instead, the curve describing how the NS5 brane lies in Q_M will have poles (*i.e.*, will go off to infinity in Q_M) inducing poles in the restriction of λ to this curve. The mass M will thus appear as the residue of the induced pole at $x = x_0$. The same is also true of other masses in the theory, such as the bifundamental hypermultiplet masses, which appear as residues of induced poles in λ corresponding to the asymptotic infinities of the NS5 branes in the IIA picture.

In summary, we have found that we can implement the shift in the background surface (whether deformed or not) from Q_0 given by (3.40) to Q_M through the change of variables (3.37) in the neighborhood of an arbitrary point x_0. In particular, in section 4.4 we will write curves for M5 branes on Q_M by writing them as curves on Q_0 with singularities at $x = x_0$ such that they are regular at $x = x_0$ when written in terms of the $(w, x, \widetilde{y}, \widetilde{z})$ variables defined in (3.37).

4. M5 brane curve

By lifting the D6 branes and $O6^-$ planes of the IIA string configuration described in section 2 to M theory, we have (in section 3) derived the complex structure of the M theory background geometry. This geometry is a deformed $\mathbb{Z}_2$ orbifold of the product of the v-plane and the x-y torus $(\mathbb{C} \times T^2)$, with a shift in v along one of the torus cycles. We now want to embed an M5 brane curve corresponding to the D4 and NS5 branes in this background. First, in section 4.1, we explain the form of the simple curves describing the M theory lift of isolated D4 branes, NS5 branes, NS5 branes stuck at an orbifold point, and intersecting D4 and NS5 branes. With this formulaic dictionary under our belts, we then proceed to writing down the general curve describing the intersection of many 4 and 5 branes. We do this following the three steps introduced in section 3: in section 4.2 we write the curve in the undeformed background and impose orbifold invariance on it; in section 4.3 we determine how the curve must be modified in the mass-deformed background; in section 4.4 we implement the shift.

4.1 M5 brane lifts of NS5 and D4 branes

NS5 brane lifts

In the geometry of section 2 an NS5 brane lies at constant x_6 and is extended along the v directions. Its M theory lift (in the non-orbifolded case—the elliptic model of [1]) thus is just the fiber above a point $(w, x, \eta) = (1, x_0, \eta_0)$ on the torus:

$$x = x_0 \qquad \text{and} \qquad \eta = \eta_0. \tag{4.1}$$

(Here and henceforth we are working in a $w = 1$ coordinate patch in $\mathbb{CP}^2_{1,1,2}$; thus η_0 is one of the roots of $\eta_0^2 = P(x_0)$ with $P(x_0)$ given in (3.36).)

Now consider the corresponding curve in the orbifolded model of the undeformed Q_0 background (*i.e.* two $O6^-$ planes each with a pair of coincident D6 branes). The orbifold-invariant variables are x, $y = v\eta$, and $z = v^2$, so the curve for an NS5 brane at $x = x_0$ becomes simply

$$x = x_0. \tag{4.2}$$

Note that since on the orbifolded background

$$y^2 = zP(x) \qquad \text{where} \quad P(x) = \prod_{i=1}^{4}(x - e_i), \tag{4.3}$$

there is a qualitative difference between an NS5 brane at a generic point $x = x_0$, and a "stuck" 5 brane at one of the four branch points $x = e_i$. This is apparent from the $z \neq 0$ behavior of these curves: the generic (unstuck) NS5 brane is at two values of y for a given z, while the stuck brane is at only one (namely $y = 0$). Thus an unstuck NS5 brane can be thought of as forming a double cover of the z-plane, while a stuck NS5 brane ony covers it once. This corresponds to the fact that the unstuck brane carries twice the charge of a stuck brane. Thus, despite appearances, making an arbitrarily small change in the 5 brane curve (4.2) from $x = e_i$ to $x = e_i + \delta x$ is in fact a singular transformation.

In this way of describing the M theory lift of NS5 branes the existence of stuck NS5 branes looks dependent on the existence of orbifold points. This is not actually the case, as we can see by deforming the orbifold by moving the D6 branes off the $O6^-$ planes (or, equivalently, by turning on the μ_i masses). Recall the surface (3.27) for the mass-deformed Q_0 background:

$$y^2 = zP(x) + Q(x) \tag{4.4}$$

where

$$Q(x) = \sum_j \mu_j^2 \prod_{k\neq j}[(x - e_k)(e_j - e_k)]. \tag{4.5}$$

Clearly the curve $x = x_0$ for a generic (unstuck) NS5 brane lift in the undeformed case retains the same topology and asymptotics when set in the deformed background: it still forms a double cover of the z-plane, and at large z its y coordinate values asymptote to the same values as in the undeformed case.

The lift of a stuck brane, $x = e_i$, however, has a qualitatively different behavior in the deformed background. It describes two *disconnected* single covers of the z plane, each at constant $y = \pm\sqrt{Q(e_i)} = \pm\mu_i \prod_{k\neq i}(e_i - e_k)$. In order to keep the asymptotic topology unchanged (*i.e.* that it be a single cover of the z plane) we must choose one or the other of these two solutions as the deformation of the stuck NS5 brane. (We do not really have to think of these as two distinct deformations of the stuck NS5 brane since once we have chosen one, the other can be realized by changing the phase $\mu_i \to e^{i\pi}\mu_i$ of the relevant deformation parameter.) In summary, a stuck NS5 brane in the mass-deformed Q_0 background is described by a pair of equations

$$x = e_i \quad \text{and} \quad y = y_i \equiv \sqrt{Q(e_i)}, \tag{4.6}$$

and covers the z plane once. It is stuck since it admits no continuous deformation, since for any $x \neq e_i$ there are only the topologically distinct unstuck NS5 branes which are double covers of the z plane.

D4 brane lifts

A D4 brane lies at constant v and is extended along x_6. Its M theory lift in the (non-orbifolded) elliptic model thus wraps the x_6-x_{10} torus, and so is described in the (v, x, η) coordinates of the $\mathbb{C} \times T^2$ transverse space simply by

$$v = v_0. \tag{4.7}$$

Orbifolding under $v \to -v$ by placing an image brane at $v = -v_0$, implies that the M5 brane lift of a D4 brane in the orbifold-invariant (x, y, z) coordinates of (4.3) becomes simply

$$z = z_0. \tag{4.8}$$

This has no asymptotic region, but wraps the x-y torus once.

There is also a "stuck" version of the lift of a D4 brane, namely the one at $z = 0$, which has the topology of a sphere—the $y = 0$ subspace of $\mathbb{CP}^2_{(1,1,2)}$—which passes through all four orbifold points of the Q_0 background. Upon mass-deforming the Q_0 background, this curve is deformed to $z = -Q(x)/P(x)$ in order to retain its spherical topology on the x-y torus, since it still has $y = 0$ as its solution.[2] But this curve has poles (infinities) in z at the e_i, and so cannot be considered a small deformation of the stuck D4 brane no matter how small the μ_i. So the stuck D4 brane only exists in the $\mathbb{C} \times T^2/\mathbb{Z}_2$ orbifold geometry, and should be considered as a part of a special degeneration of a generic D4 brane. This agrees with

[2]More general deformations are possible: $z = (R^2 - Q)/P$ where R is any degree two polynomial in x which vanishes as the deformation is turned off also has the spherical topology. The above argument still applies, however.

the conclusion reached in section 2 based on the IIA brane geometry, and is also implicitly supported by [4], where stuck D4 branes were not found in the M theory lift of a single $O6^-$, the Atiyah-Hitchin space.

D4-NS5 intersections

Now we would like to describe the intersection of a D4 brane with an unstuck NS5 brane in a neighborhood of the 5 brane. In the IIA limit they are described by $(z-z_0)(x-x_0)=0$. This is the equation for two curves intersecting transversely at two points since for $x=x_0$ and $z=z_0$ there are two values of y satisfying (4.3). But in the M theory lift these intersections will typically be smoothed out. So the question is what are the complex deformations of these intersections which preserve the topology in the vicinity of (but not at) the intersection points, and also have the same asymptotics along the NS5 brane far from the intersection points? Since an unstuck NS5 brane $x=x_0$ is described by two values of y for each $z \neq 0$, preserving the topology means that for each $x \neq x_0$ (but close enough to x_0) there should be precisely two solutions for (y,z) also satisfying (4.3). Having the right asymptotics means that for large z the curve should approach the unstuck NS5 brane curve $x=x_0$. There are only two possible complex deformations of the intersection satisfying these conditions:

$$(z-z_0)(x-x_0)=\epsilon+\delta y \tag{4.9}$$

for arbitrary complex ϵ and δ. The ϵ term smooths out intersection and clearly does not affect the asymptotics. That the δ term does not either is less obvious; but since for fixed x, $y \propto \sqrt{z}$ by (4.3), the δ term is subleading for large z. Finally, it is not too hard to show that for any ϵ and δ, an x close enough to x_0 can be found for which there are always two solutions to (4.9) and (4.3).

The intersection of a D4 ($z=z_0$) with a stuck NS5 ($x=e_i$ and $y=y_i \equiv \sqrt{Q(e_i)}$), however, has only a single deformation:

$$(z-z_0)(x-e_i)=\delta(y+y_i), \tag{4.10}$$

for arbitrary complex δ. This is because it is only for this form that the asymptotics of the stuck NS5 brane are preserved. In particular, for fixed large z there is only one solution to (4.10) and (4.4), and so (4.10) describes asymptotically in a neighborhood of $x=e_i$ only a single-sheeted cover of the z plane, which is moreover asymptotically close to $y=y_i$. This is easiest to see by rewriting (4.10) as

$$z=\delta\frac{y+y_i}{x-e_i}+z_0. \tag{4.11}$$

Then large z occurs only at poles of the right side which are at $x=e_i$ and $x=\infty$. (Recall that $x=\infty$ is a regular point, corresponding to finite x with $w=0$ in a different coordinate patch of $\mathbb{CP}^2_{(1,1,2)}$.) But the pole at $x=\infty$ is not in a neighborhood of the asymptotic NS5 brane, so should be discarded in this local argument. There are generically two poles at $x=e_i$ since there are two points on the (fixed z) torus with this coordinate: $(x,y)=(e_i,\pm y_i)$. However, precisely because of the form of the numerator in (4.11) the pole at $(e_i, -y_i)$ is cancelled, leaving only a simple pole at $(e_i, +y_i)$, as desired.

The form of the curve for the smoothed intersection of a NS5 brane and a D4 brane stuck at $z = 0$ also bears discussion. In the $\mathbb{C}\times T^2/\mathbb{Z}_2$ background, following the prescription developed above, the curve is just (4.9) with $z_0 = 0$, which, as before, should be thought of as being valid in the patch of the x-y torus away from $x = \infty$. For the special case of $\epsilon = 0$, even though this curve does not factorize, it actually describes two intersecting curves. This follows since not only is there the usual solution in which z blows up as x approaches x_i, but there is also the solution $z = 0$ for all x since by (4.3) $y = 0$ if $z = 0$ independent of x. These two branches of solutions intersect at the orbifold points $(x, y, z) = (e_j, 0, 0)$, and it is not hard to check that near the intersections the two branches are described by $z \propto x - e_j$ and $z = 0$, which, in terms of the single-valued local complex coordinate $\xi = \sqrt{x - e_j}$, describes a tangent intersection of two surfaces. In constrast, the curve describing a generic D4-NS5 intersection, *i.e.* (4.9) with either of ϵ or z_0 non-zero, or with the mass deformation $Q(x)$ in (4.4) turned on, describes only a single curve. This is another way of seeing that the stuck D4 brane only exists in the $\mathbb{C} \times T^2/\mathbb{Z}_2$ orbifold geometry.

4.2 M5 brane curve in Q_0

The previous subsection described the M5 brane lifts of intersecting NS5 and D4 branes in the Q_0 background geometry. However this description was only local in the compact x-y ("torus") directions. The global problem of writing a curve with a given number q of stuck NS5 branes at asymptotic (large z) positions $x = e_i$ and r of unstuck NS5 branes at asymptotic postions $x = x_i$ in Q_M is the subject of the rest of this section. We start in this subsection by considering such curves in the $\mathbb{C} \times T^2/\mathbb{Z}_2$ orbifolded background and then include the mass deformations of the Q_0 space, and in the next subsection we include the shift of the general Q_M space.

A convenient starting point for the construction of the curve for q stuck and r unstuck NS5 branes and k D4 branes in the orbifolded background is to consider the curve for Witten's elliptic model [1] with $q+2r$ NS5 branes and k D4 branes. To respect the orbifold invariance[3] $(v, x, \eta) \to (-v, x, -\eta)$ we must distribute q of the 5 branes among the various branch points $(x, \eta) = (e_p, 0)$, and put the rest at the r pairs of points $(x_s, \pm\eta_s)$. We will denote the coordinates of all these points by (x_I, η_I) for $I = 1, \ldots, q + 2r$, and will assume without loss of generality that none of the $x_I = \infty$.

Having located the NS5 branes in M theory, we can write down the elliptic model curve in terms of the unorbifolded coordinates (v, x, η) following [1]:

$$0 = F(v, x, \eta) = v^k + \sum_{\ell=1}^{k} v^{k-\ell} f_\ell(x, \eta). \tag{4.12}$$

[3] We are working for the rest of this section in the $w = 1$ $\mathbb{CP}^2_{(1,1,2)}$ coordinate patch.

The f_ℓ can have at most simple poles at the locations of the NS5 branes on the torus. The most general such functions can be written as

$$f_\ell(x,\eta) = \widetilde{A}_\ell + \sum_{I=1}^{q+2r} \widetilde{B}_{I\ell} \frac{\eta+\eta_I}{x-x_I} \tag{4.13}$$

where $\widetilde{A}_\ell$ and $\widetilde{B}_{I\ell}$ are complex constants constrained by

$$\sum_I \widetilde{B}_{I\ell} = 0 \qquad \text{for all} \quad \ell. \tag{4.14}$$

This last constraint is needed to prevent a pole at $x=\infty$.

Next, we arrange the curve symmetrically with respect to the O6$^-$ plane reflection $(v,x,\eta) \to (-v,x,-\eta)$:

$$F(v,x,\eta) = F(-v,x,-\eta). \tag{4.15}$$

Since the coefficient of the v^k term is one, this can only be satisfied for k even. This condition implies

$$f_\ell(x,-\eta) = (-)^\ell f_\ell(x,\eta), \tag{4.16}$$

and it follows from (4.13) that

$$f_\ell = \begin{cases} A_\ell + \sum_{s=1}^{r} \frac{B_{s\ell}}{x-x_s} & \ell \text{ even} \\ \eta\left(\sum_{s=1}^{r} \frac{C_{s\ell}}{x-x_s} + \sum_{p=1}^{q} \frac{D_{p\ell}}{x-e_p}\right) & \ell \text{ odd} \end{cases} \tag{4.17}$$

where A_ℓ, $B_{s\ell}$, $C_{s\ell}$, and $D_{p\ell}$ are arbitrary complex coefficients (formed from combinations of the (4.13) coefficients $\widetilde{A}_\ell$, $\widetilde{B}_{I\ell}$ and η_I) and subject to the constraint that

$$\sum_{s=1}^{r} C_{s\ell} + \sum_{p=1}^{q} D_{p\ell} = 0, \qquad \text{for all odd } \ell. \tag{4.18}$$

We can now write the curve in terms of the orbifold-invariant variables x, $y=v\eta$ and $z=v^2$. The results are

$$0 = z^n + A(z) + \sum_{s=1}^{r} \frac{B_s(z)+yC_s(z)}{x-x_s} + \sum_{p=1}^{q} \frac{yD_p(z)}{x-e_p} \qquad \text{for} \quad k=2n, \tag{4.19}$$

where A, B_s, C_s, and D_p are arbitrary polynomials in z of order $n-1$, subject to the constraints that

$$\sum_{s=1}^{r} C_s(z) + \sum_{p=1}^{q} D_p(z) = 0 \qquad \text{for all} \quad z. \tag{4.20}$$

It might be thought that k odd solutions in which $F(-v,x,-\eta) = -F(v,x,\eta)$ should also be allowed. For example, a single D4 brane stuck at the orbifold invariant point $v=0$ clearly describes an orbifold invariant submanifold even though its equation changes sign. Allowing this behavior, and then expressing $vF(v,x,\eta)$ in terms of the orbifold-invariant variables gives

$$0 = z^n + z\widetilde{A}(z) + \sum_{s=1}^{r} \frac{z\widetilde{B}_s(z) + yC_s(z)}{x - x_s} + \sum_{p=1}^{q} \frac{yD_p(z)}{x - e_p} \qquad \text{for} \quad k = 2n-1, \tag{4.21}$$

where now $\widetilde{A}(z)$ and $\widetilde{B}(z)$ are arbitrary polynomials of order $n-2$ in z. This is just a specialization of the k-even curve, realized by setting the constant terms of $A(z)$ and $B_s(z)$ to zero. This is precisely the specialization of the generic D4-NS5 intersection discussed at the end of the last subsection: the constant and $z^0(x-x_s)^{-1}$ terms are set to zero. But as discussed there, this specialization does not survive any deformation of the background, and so should not be counted as a separate case. So we will consider only k even from now on.

In the case with two stuck NS5 branes (the $q=2$ cases of section 2), the brane picture allows an odd number of D4 branes (corresponding to having SU gauge factors of odd rank). Since the naive k odd solutions were ruled out above, the question arises as to how to construct them when $q=2$. As we will show in section 5.4, in precisely the $q=2$ case of two stuck 5 branes, a decoupling limit which embeds the scale-invariant SU$(2n-1)$ theory in the scale-invariant SU$(2n)$ theory is possible, and gives the k odd curve. It differs from the k even curve essentially by just deleting the leading z^n term in (4.19).

Next, we determine how the curve must be deformed once we deform the orbifold space to Q_0 by turning on masses as in (4.4). First, note that we cannot change the order of the curve in z, since this corresponds to changing the number of D4 branes in the type IIA picture. Similarly, we must keep the same number of simple poles in the coefficients of each power of z corresponding to the locations of the NS5 branes. Finally, from the discussion in the last subsection, it follows that the positions of the q stuck NS5 branes must be kept the same, *i.e.* at $x=e_p$, but that their numerators should be shifted from y to $y-y_p$ where

$$y_p \equiv \sqrt{Q(e_p)}. \tag{4.22}$$

It follows that the curve in the Q_0 background is

$$0 = F(x,y,z) \equiv z^n + A(z) + \sum_{s=1}^{r} \frac{B_s(z) + yC_s(z)}{x - x_s} + \sum_{p=1}^{q} \frac{(y-y_p)D_p(z)}{x - e_p} \qquad \text{for} \quad k = 2n. \tag{4.23}$$

It might seem that the choice of sign of the square root in (4.22) for each p give us many distinct curves. However it is not too hard to see that the different choices of sign correspond to the same theory albeit at different values of its parameters. For we can always write the deformed background in the form

$$y^2 = (z-1)\prod_{i=1}^{4}(x-e_i) - \prod_{i=1}^{4}(x-e_i-\delta_i), \tag{4.24}$$

where the δ_i are functions of the μ_i. When the δ_i vanish the branch points of the torus (at a given z) are at the points $x_i = b_i$ with $b_i = e_i$. When the δ_i are all small but non-zero, a power series expansion for the branch points b_i gives

$$b_i = e_i + \frac{1}{z}\delta_i + \mathcal{O}(\delta^2). \tag{4.25}$$

Now in the curve (4.23) the choice was whether to put the pole at a given $x = e_i$ on one sheet or the other. But by (4.25) as $\delta_i \to e^{2\pi i}\delta_i$, the position of the pole traverses a complete circle around the branch point in the x-plane, thus shifting from one sheet to the other. Thus we learn that we can go from one choice of any one sign of the square root in (4.22) to the other by a continuous change in the deformation parameters. This was anticipated in the IIA brane picture by a deformation argument at the end of section 2.1.

4.3 M5 brane curve in Q_M

Now we generalize the results of the previous subsection from the non-shifted ($M = 0$) case to the case of an arbitrary shift parameter M. As discussed in section 3.4, the curve for the shifted model is like the curve for the non-shifted model except that in addition to the poles at $x = x_s$ and $x = e_p$ in (4.23), we should also allow singularites at $x = x_0$ (the excised "fiber" of Q_0), subject to the constraint that the curve be regular in a coordinate patch covering $x = x_0$ with coordinates $(x, \widetilde{y}, \widetilde{z})$ related to (x, y, z) as in (3.37).

Precisely which order singularities should be allowed in (4.23) can be determined by the form of the change of variables (3.37) to the $(x, \widetilde{y}, \widetilde{z})$ coordinate patch which we repeat here:

$$\begin{aligned}\widetilde{y} &= y + M\frac{P(x)}{(x - x_0)},\\ \widetilde{z} &= z + 2M\frac{y}{(x - x_0)} + M^2\frac{P(x)}{(x - x_0)^2}.\end{aligned} \tag{4.26}$$

Recalling that the inverse change of variables is of the same form with $y \leftrightarrow \widetilde{y}$, $z \leftrightarrow \widetilde{z}$ and $M \to -M$, it follows that the inverse change of variables can only introduce or remove poles at $x = x_0$ or $x = \infty$, and since $F(x, y, z)$ in (4.23) should be regular at $x = x_0$ when written in the $(x, \widetilde{y}, \widetilde{z})$ coordinates, the general form of the transformed F, $\widetilde{F}(x, \widetilde{y}, \widetilde{z})$, must be that of (4.23) but with possible extra poles at infinity:

$$0 = \widetilde{F}(x, \widetilde{y}, \widetilde{z}) \equiv \widetilde{z}^n + \sum_{\alpha=0}^{\infty}\left[\widetilde{A}_\alpha(\widetilde{z}) + \widetilde{y}\widetilde{E}_\alpha(\widetilde{z})\right]x^\alpha + \sum_{s=1}^{r}\frac{\widetilde{B}_s(\widetilde{z}) + \widetilde{y}\widetilde{C}_s(\widetilde{z})}{x - x_s} + \sum_{p=1}^{q}\frac{(\widetilde{y} - y_p)\widetilde{D}_p(\widetilde{z})}{x - e_p} \tag{4.27}$$

where the $\widetilde{A}_\alpha$, $\widetilde{B}_s$, $\widetilde{C}_s$, $\widetilde{D}_p$, and $\widetilde{E}_\alpha$ are arbitrary polynomials of order $n - 1$ in $\widetilde{z}$. (The additional terms compared to (4.23) are the $\widetilde{A}_{\alpha\neq 0}$ and $\widetilde{E}_\alpha$ polynomials which parametrize the possible singularities at $x = \infty$.) Now substituting in the change of variables (4.26) into (4.27) and using the Q_0 equation $y^2 = zP(x) + Q(x)$ to reexpress all higher powers of y in terms of polynomials in x, y, and z that are at most linear in y, the result is a curve of the form

(4.23) but with additional poles at x_0. Computing the order of these poles is then a simple matter of counting. Since the change of variables (4.26) for $\widetilde{z}$ is of the form $\widetilde{z} = z + P_1 y + P_2$ and $\widetilde{y} = y + P_1$, where P_i refer to rational functions of x with poles of up to order i at x_0, then just counting powers of y, z, and the order of poles at x_0 gives

$$\begin{aligned} \widetilde{z}^\ell &\sim z^\ell + \sum_{a=1}^{\ell} z^{\ell-a}(P_{2a} + yP_{2a-1}), \\ \widetilde{y}\widetilde{z}^\ell &\sim \sum_{a=0}^{\ell} z^{\ell-a}(P_{2a+1} + yP_{2a}). \end{aligned} \tag{4.28}$$

Since the highest power of $\widetilde{z}$ alone is $\ell = n$, and of $\widetilde{y}\widetilde{z}^\ell$ is $\ell = n-1$ in (4.27), it follows that the highest order pole at $x = x_0$ in the coefficient of a term with $z^{n-\ell}$ is 2ℓ, while for that of a term with $yz^{n-\ell}$ it is $2\ell - 1$.

Thus, the general form of the curve with k D4 branes, q stuck NS5 branes, and r unstuck NS5 branes in Q_M is

$$0 = z^n + A(z,x) + yE(z,x) + \sum_{s=1}^{r} \frac{B_s(z) + yC_s(z)}{x - x_s} + \sum_{p=1}^{q} \frac{(y - y_p)D_p(z)}{x - e_p} \qquad \text{for} \quad k = 2n, \tag{4.29}$$

where

$$y_p \equiv \sqrt{Q(e_p)}, \tag{4.30}$$

A, E, B_s, C_s, and D_p have expansions

$$\begin{aligned} A(z,x) &= \sum_{\ell=1}^{n} z^{n-\ell} \sum_{a=0}^{2\ell} A_{a\ell}(x - x_0)^{-a}, \\ E(z,x) &= \sum_{\ell=1}^{n} z^{n-\ell} \sum_{a=1}^{2\ell-1} E_{a\ell}(x - x_0)^{-a}, \\ B_s(z) &= \sum_{\ell=1}^{n} z^{n-\ell} B_{s\ell}, \\ C_s(z) &= \sum_{\ell=1}^{n} z^{n-\ell} C_{s\ell}, \\ D_p(z) &= \sum_{\ell=1}^{n} z^{n-\ell} D_{p\ell}, \end{aligned} \tag{4.31}$$

and satisfy the constraint

$$E_{1\ell} + \sum_{s=1}^{r} C_{s\ell} + \sum_{p=1}^{q} D_{p\ell} = 0 \qquad \text{for all } \ell \tag{4.32}$$

(so that there is no pole at $x=\infty$). Demanding regularity at $x=x_0$ after making the change of coordinates

$$\begin{aligned} y &= \widetilde{y} - M\frac{P(x)}{(x-x_0)}, \\ z &= \widetilde{z} - 2M\frac{\widetilde{y}}{(x-x_0)} + M^2\frac{P(x)}{(x-x_0)^2}, \end{aligned} \tag{4.33}$$

which is just the inverse of (4.26), then determines all the $A_{a\ell}$ and $E_{a\ell}$ in terms of $A_{0\ell}$, $B_{s\ell}$, $C_{s\ell}$, and $D_{p\ell}$. This curve (4.29) lives on the Q_M surface whose equation (excluding the $x=x_0$ submanifold) we repeat here:

$$\begin{aligned} y^2 &= zP(x) + Q(x), \\ P &\equiv \prod_{i=1}^{4}(x-e_i), \\ Q &\equiv \sum_{i=1}^{4}\mu_i^2\prod_{j\neq i}[(x-e_j)(e_i-e_j)]. \end{aligned} \tag{4.34}$$

Finally, the x and y directions are compactified by two points at infinity, which can be realized by embedding all curves in $\mathbb{CP}^2_{(1,1,2)}$—*i.e.* by adding powers of a new variable w to each term to make all terms homogenous in (w,x,y) with weights assigned as $(1,1,2)$, respectively. This completes our construction of the general M5 brane curve in the Q_M background; this paragraph assembles all the relevant formulas.

The following sections will discuss explicit examples of these curves.

5. Consistency checks

In this section we perform some simple consistency checks on our curves. The number of such possible checks is immense: there should be an intricate web of relations among these curves and other curves known in the literature corresponding to the relations among the $N=2$ gauge theories as gauge factors are decoupled (by taking weak coupling limits), or Higgsed at weak coupling (by taking vevs large), or as matter hypermultiplets are decoupled (by taking masses large). We will only describe here a sample of a few of the simplest checks, though it will be apparent that many more can quickly and easily be performed. The main point of the examples will be to illustrate how to handle the weak coupling and large mass limits, and how to deduce the appropriate identifications of the parameters appearing in the curves with the classical masses, couplings, and vevs. These identifications are good semi-classically, but are necessarily ambiguous up to non-perturbative redefinitions [11]. They are deduced by taking various weak coupling limits of the theory in which various patterns of gauge group breakings and matter decouplings are reproduced.

Along the way we will also show that the background Q_M surface, when thought of as a curve at fixed z is equivalent to the curve describing the Coulomb branch of the scale invariant

SU(2) theory with four fundamental hypermultiplets. Also, in section 5.4 we will show how the curves for SU(k) theories with k odd can be derived by a simple breaking procedure.

5.1 Parameter counting

The most basic check is that the number of parameters entering into the curve (4.29)—(4.34) match the number of parameters describing the corresponding gauge theory on its Coulomb branch. The background surface Q_M has six parameters: M, the four μ_i, and the one SL$(2,\mathbb{C})$-invariant combination of the e_i. The curve (4.29) has r NS5 brane positions x_s as well as the independent coefficients of the A, B_s, C_s, and D_q functions. The regularity conditions at $x = x_0$ under the coordinate change (4.33) imply that the coefficients of inverse powers of $(x - x_0)$ can be expressed in terms of the leading coefficients (*i.e.*, those with no $x - x_0$ dependence). Since each of A, B_s, C_s and D_p are polynomials of order $n-1$ in z, they each have n independent leading coefficients, for a total of $n(1+2r+q)$. However (4.32) enforces n relations among them. Thus the total number of parameters is $6 + r + 2rn + qn$. Recalling that $k = 2n$ is the number of D4 branes, this matches precisely the counting of parameters given at the end of section 2 for the gauge theories. (The counting also works for the degenerate $q = r = 0$ case, as the reader can easily check.)

5.2 Some examples

Let us now concentrate on two simple cases to illustrate parameter matching. We will focus on the ($r = 0$, $q = 1$, $k = 2n$) and the ($r = 0$, $q = 2$, $k = 2n$) curves.

($r = 0$, $q = 1$, $k = 2n$) curve

According to the IIA brane correspondence described in section 2, this case describes the Coulomb branch of the Sp($2n$) gauge theory with 4 fundamental and one antisymmetric hypermultiplet. Putting the one stuck NS5 brane at $x = e_1$, and placing (for simplicity) the shift at $x_0 = 0$, the curve reads

$$0 = z^n + A(z,x) + yE(z,x) + \frac{y - y_1}{x - e_1} D(z) \tag{5.1}$$

where

$$\begin{aligned} A &= \sum_{\ell=1}^{n} z^{n-\ell} \sum_{a=0}^{2\ell} A_{a\ell} x^{-a}, \\ E &= \sum_{\ell=1}^{n} z^{n-\ell} \left(-D_\ell x^{-1} + \sum_{a=2}^{2\ell-1} E_{a\ell} x^{-a} \right), \\ D &= \sum_{\ell=1}^{n} z^{n-\ell} D_\ell, \\ y_1 &= \sqrt{Q(e_1)}, \end{aligned} \tag{5.2}$$

and the $A_{a\ell}$, $E_{a\ell}$ and D_ℓ coefficients must be chosen so that all poles at $x=0$ cancel upon changing to $(\widetilde{y},\widetilde{z})$ variables according to

$$y=\widetilde{y}-M\frac{P(x)}{x}, \qquad z=\widetilde{z}-2M\frac{\widetilde{y}}{x}+M^2\frac{P(x)}{x^2}, \tag{5.3}$$

where $P(x)$, $Q(x)$ are given in (4.34).

For example, when $n=2$, we have

$$\begin{aligned} A &= z\left(A_{01}+\frac{A_{11}}{x}+\frac{A_{21}}{x^2}\right)+A_{02}+\frac{A_{12}}{x}+\frac{A_{22}}{x^2}+\frac{A_{32}}{x^3}+\frac{A_{42}}{x^4}, \\ E &= -z\frac{D_1}{x}-\frac{D_2}{x}+\frac{E_{22}}{x^2}+\frac{E_{32}}{x^3}, \\ D &= zD_1+D_2, \end{aligned} \tag{5.4}$$

and we find, after a second of CPU time,

$$\begin{aligned} A_{11} &= 6M^2(s_3-e_1s_2) \\ A_{21} &= 6M^2e_1s_3 \\ A_{12} &= A_{01}M^2(s_3-e_1s_2)+4q_1M^2+8q_0M^2e_1^{-1}+4y_1M^3s_2 \\ &\qquad +4M^4\left[e_1^2(s_1s_2-s_3)-e_1s_2^2+s_3s_2\right] \\ A_{22} &= A_{01}M^2e_1s_3+4q_0M^2-4y_1M^3s_3+M^4\left[e_1^2(s_2^2-4s_3s_1)+2e_1s_3s_2-3s_3^2\right] \\ A_{32} &= 2M^4e_1s_3(s_3-e_1s_2) \\ A_{42} &= M^4e_1^2s_3^2 \\ D_1 &= -4M \\ D_2 &= -2A_{01}M+8M^2y_1e_1^{-1}+4M^3(2e_1s_1-s_2) \\ E_{22} &= 4M^3(2s_3-e_1s_2) \\ E_{32} &= 4M^3e_1s_3 \end{aligned} \tag{5.5}$$

where we have defined

$$\begin{aligned} &q_0=Q(0), \qquad q_1=Q'(0), \\ &s_1=e_2+e_3+e_4, \qquad s_2=e_2e_3+e_2e_4+e_3e_4, \qquad s_3=e_2e_3e_4, \end{aligned} \tag{5.6}$$

and A_{01} and A_{02} are arbitrary.

Note that when we send the shift parameter to zero, $M\to 0$, only the $A_{0\ell}$ coefficients remain non-zero, as in the curves we found in section 4.2. Thus, in this limit the curve reduces to

$$0=z^n+A(z)=z^n+\sum_{\ell=1}^{n}z^{n-\ell}A_{0\ell} \tag{5.7}$$

which is precisely the form of the $r=q=0$ curve. This bears out the interpretation of the $r=q=0$ curve given in section 2 as the $\mathrm{Sp}(2n)$ theory with four fundamentals and a massless antisymmetric.

Furthermore, the curve for this massless antisymmetric theory was found in [22, 23] where it was shown that the Coulomb branch is generically the tensor product of n copies—one for each independent $\mathrm{Sp}(2n)$ adjoint scalar vev—of the one-dimensional Coulomb branch of the $\mathrm{SU}(2)$ with four fundamental flavors theory. But that is precisely the content of (5.7): its solutions are just the n roots z_s, $s=1,\ldots,n$, of the polynomial $z^n+A(z)$. When plugged back into the equation (4.34) for the Q_0 surface this gives the n tori

$$y^2 = z_s P(x) + Q(x) \tag{5.8}$$

where $P(x)$ and $Q(x)$ are given as usual by (4.34). To show the equivalence with the known $\mathrm{Sp}(2n)$ curve, it therefore just remains to show the equivalence of the curve (5.8) (which is just the Q_0 surface (4.34) at fixed z) with the curve [3] for the $\mathrm{SU}(2)$ theory with four fundamentals. This requires matching the coupling, masses, and vev parameters, and will be done in section 5.3 below.

$(r=0,\ q=2,\ k=2n)$ curve

According to the IIA brane correspondence described in section 2, this case describes the Coulomb branch of the $\mathrm{SU}(2n)$ gauge theory with 4 fundamental and 2 antisymmetric hypermultiplets. Putting the two stuck NS5 branes at $x=e_2$ and $x=e_3$, and placing (for simplicity) the shift at $x_0=0$, the curve reads

$$0 = z^n + A(z,x) + yE(z,x) + \frac{y-y_2}{x-e_2}D_1(z) + \frac{y-y_3}{x-e_3}D_2(z) \tag{5.9}$$

where

$$\begin{aligned}
A &= \sum_{\ell=1}^{n} z^{n-\ell} \sum_{a=0}^{2\ell} A_{a\ell} x^{-a}, \\
E &= \sum_{\ell=1}^{n} z^{n-\ell} \left(-[D_{1\ell}+D_{2\ell}]x^{-1} + \sum_{a=2}^{2\ell-1} E_{a\ell} x^{-a} \right), \\
D_i &= \sum_{\ell=1}^{n} z^{n-\ell} D_{i\ell}, \\
y_i &= \sqrt{Q(e_i)},
\end{aligned} \tag{5.10}$$

and the $A_{a\ell}$, $E_{a\ell}$ and $D_{i\ell}$ coefficients must be chosen so that all poles at $x=0$ cancel upon changing to $(\widetilde{y},\widetilde{z})$ variables according to (5.3).

For example, when $n=2$, we have

$$\begin{aligned}
A &= z\left(A_{01} + \frac{A_{11}}{x} + \frac{A_{21}}{x^2}\right) + A_{02} + \frac{A_{12}}{x} + \frac{A_{22}}{x^2} + \frac{A_{32}}{x^3} + \frac{A_{42}}{x^4}, \\
E &= -z\frac{D_{11}+D_{21}}{x} - \frac{D_{12}+D_{22}}{x} + \frac{E_{22}}{x^2} + \frac{E_{32}}{x^3}, \\
D_1 &= zD_{11} + D_{12}, \\
D_2 &= zD_{21} + D_{22},
\end{aligned} \tag{5.11}$$

and we find, after a few more seconds of CPU time,

$$
\begin{aligned}
A_{11} &= 3D_{11}Me_1e_4(e_2-e_3) - 6M^2(e_1e_2e_3 - e_1e_2e_4 + e_1e_3e_4 + e_2e_3e_4)\\
A_{21} &= 6M^2e_1e_2e_3e_4\\
A_{12} &= 2D_{11}q_0M\left(\frac{1}{e_2}-\frac{1}{e_3}\right) + D_{22}Me_1e_4(e_2-e_3)\\
&\quad - A_{01}M^2(e_1e_2e_3 + e_1e_2e_4 - e_1e_3e_4 + e_2e_3e_4)\\
&\quad + 4q_1M^2 + 8q_0M^2e_3^{-1} - D_{11}M^2y_2(e_1e_3 + e_1e_4 + e_3e_4)\\
&\quad + D_{11}M^2y_3e_3^{-1}(e_1e_2e_3 + 2e_1e_2e_4 - e_1e_3e_4 + e_2e_3e_4)\\
&\quad + 4M^3y_3e_3^{-1}(e_1e_2e_3 + 2e_1e_2e_4 - e_1e_3e_4 + e_2e_3e_4)\\
&\quad - D_{11}M^3(e_2-e_3)[2e_2e_3(e_1^2+e_4^2)\\
&\qquad\qquad + e_1e_4(e_1e_2 - 2e_1e_3 - e_1e_4 + 2e_2e_3 + e_2e_4 - 2e_3e_4)]\\
&\quad + 4M^4\left[(e_1+e_4)(e_2^2e_3^2 + e_1e_3e_4(e_2+e_3)) - (e_1^2+e_4^2)e_2e_3(e_2-e_3)\right]\\
A_{22} &= A_{01}M^2e_1e_2e_3e_4 + 4q_0M^2 + D_{11}M^2e_1e_4(y_2e_3 - y_3e_2) - 4y_3M^3e_1e_2e_4\\
&\quad + D_{11}M^3e_1e_4(e_2-e_3)(e_1e_2e_3 - e_1e_2e_4 - e_1e_3e_4 + e_2e_3e_4)\\
&\quad + M^4[e_1^2e_2^2e_3^2 - e_1^2e_2^2e_4^2 + e_1^2e_3^2e_4^2 + e_2^2e_3^2e_4^2\\
&\qquad + 2e_1e_2e_3e_4(e_1e_2 - e_1e_3 + e_1e_4 - e_2e_3 + e_2e_4 - e_3e_4)]\\
A_{32} &= D_{11}M^3e_1^2e_2e_3e_4^2(e_2-e_3) - 2M^4e_1e_2e_3e_4(e_1e_2e_3 - e_1e_2e_4 + e_1e_3e_4 + e_2e_3e_4)\\
A_{42} &= M^4e_1^2e_2^2e_3^2e_4^2\\
D_{21} &= -4M - D_{11}\\
D_{22} &= -D_{12} - 2A_{01}M - 2D_{11}M\left(\frac{y_2}{e_2}-\frac{y_3}{e_3}\right) + 8M^2y_3e_3^{-1} - 3D_{11}M^2(e_1+e_4)(e_2-e_3)\\
&\quad - 4M^3(e_1e_2 - 2e_1e_3 + e_1e_4 - 2e_2e_3 + e_2e_4 - 2e_3e_4)\\
E_{22} &= 3D_{11}M^2e_1e_4(e_2-e_3) - 4M^3(e_1e_2e_3 - 2e_1e_2e_4 + e_1e_3e_4 + e_2e_3e_4)\\
E_{32} &= 4M^3e_1e_2e_3e_4
\end{aligned}
\tag{5.12}
$$

where we have defined $q_0 = Q(0)$, $q_1 = Q'(0)$, and A_{01}, A_{02}, D_{11}, and D_{12} are arbitrary.

Finally, when the shift parameter M vanishes, the curve dramatically simplifies to

$$0 = z^n + A(z) + \left(\frac{y-y_2}{x-e_2} - \frac{y-y_3}{x-e_3}\right)D(z) \tag{5.13}$$

where now A and D are arbitrary polynomials of degree $n-1$ in z.

5.3 Identifying coupling and mass parameters: Sp theory

The gauge coupling and theta angle of the scale invariant theory can be combined into a complex coupling which we denote

$$q = e^{2\pi i\tau}, \tag{5.14}$$

where $\tau = (\theta/2\pi) + i(4\pi/g^2)$. Likewise, denote the fundamental hypermultiplet masses[4] that appear in the weak coupling action by m_i. In this subsection we will find the relation between the gauge coupling q and masses m_i and the parameters e_i and μ_i of the Q_M surface. We will do this first for the Sp$(2n)$ theories (or, more generally, for theories with only one stuck NS5 brane), and leave the SU(k) case for the next subsection.

First, however, it will be useful in what follows to record the explicit change of variables on y and z needed to keep the form of the surface the same under SL$(2,\mathbb{C})$ transformations on w and x. (Recall the discussion after equation (3.13) in section 3.2). Write the surface in terms of the same variables with tildes, and redefine them by

$$\begin{aligned} \widetilde{x} &= Ax + Bw \\ \widetilde{w} &= Cx + Dw \\ \widetilde{y} &= \left[\prod_i (A - \widetilde{e}_i C)\right] y \end{aligned} \tag{5.15}$$

with $AD - BC = 1$. The inverse of this change of variables is

$$\begin{aligned} x &= D\widetilde{x} - B\widetilde{w} \\ w &= -C\widetilde{x} + A\widetilde{w} \\ y &= \left[\prod_i (Ce_i + D)\right] \widetilde{y}. \end{aligned} \tag{5.16}$$

Here e_i and $\widetilde{e}_i$ are related by

$$\widetilde{e}_i = \frac{Ae_i + B}{Ce_i + D}, \qquad e_i = \frac{D\widetilde{e}_i - B}{-C\widetilde{e}_i + A}, \tag{5.17}$$

which imply the useful identities

$$A - \widetilde{e}_i C = (Ce_i + D)^{-1}, \tag{5.18}$$

and

$$\widetilde{e}_i - \widetilde{e}_j = (e_i - e_j)(Ce_i + D)^{-1}(Ce_j + D)^{-1}. \tag{5.19}$$

Plugging this change of variables into the tilded surface gives back the original surface after some algebra and with z related to $\widetilde{z}$ by

$$z = \left[\prod_i (Ce_i + D)\right] \widetilde{z} + C \sum_j \frac{\mu_j^2}{Ce_j + D} \prod_{k \neq j} (e_j - e_k). \tag{5.20}$$

[4] We use an unconventional normalization of the hypermultiplet masses m so that they enter the classical action as $\mathcal{W} = \sqrt{2}\{\mathrm{tr}(Q\Phi\widetilde{Q}) + m\,\mathrm{tr}(Q\widetilde{Q})\}$ where $\mathcal{W}$ is the superpotential for the the $N=2$ theory written in $N=1$ notation with $(Q,\widetilde{Q})$ the two $N=1$ chiral superfields making up the $N=2$ hypermultiplet, and Φ the $N=1$ adjoint chiral superfield part of the $N=2$ vector multiplet. Canonically normalized masses $\widetilde{m}$ are then given by $\widetilde{m} = \sqrt{2}m$.

We can now identify the gauge coupling parameter q in terms of the parameters of the Q_M background. General arguments [11] imply that q can be a function only of dimensionless parameters, so we can set the mass parameters $\mu_i = 0$. Then the background geometry is $y^2 = z\prod_i(x - e_i)$, which at fixed z describes a torus with (SL$(2,\mathbb{C})$-invariant) complex structure τ_t. In the IIA brane picture the imaginary part of τ_t measures the circumference of the x_6-circle. Since the inverse-squared gauge coupling is the distance between the O6$^-$ planes in the brane picture, we identify $\tau = \frac{1}{2}\tau_t$, or

$$q = 4\sqrt{q_t} \tag{5.21}$$

where $q_t \equiv e^{2\pi i\tau_t}$. The factor of 4 in (5.21) is for later convenience; it represents a modification to the definition of the coupling by some one-loop threshold correction [11]. In general, the above identification of q could be modified by adding terms with higher powers of $\sqrt{q_t}$ to the right hand side: these would just represent a non-perturbative redefinition of the coupling which does not affect the weak coupling limit $q \to 0$. In this section matching parameters in the weak coupling limit is all we will be interested in, since that is sufficient to take decoupling limits. More detailed checks against instanton expansions require a more refined matching of parameters, as in [11].

According to [3], q_t is related to the e_i, in the $q_t \to 0$ limit, by

$$\begin{aligned}
e_1 &= -(1/3) + 8\sqrt{q_t} + \mathcal{O}(q_t)\\
e_2 &= -(1/3) - 8\sqrt{q_t} + \mathcal{O}(q_t)\\
e_3 &= +(2/3) + \mathcal{O}(q_t)\\
e_4 &= \infty.
\end{aligned} \tag{5.22}$$

It will be convenient in what follows to make an SL$(2,\mathbb{C})$ transformation, $e_i \to (Ae_i + B)/(Ce_i + D)$, which takes the average of e_1 and e_2 to 0, $e_3 \to 1$, and $e_4 \to -1$. The transformation which does this is (to order q_t)

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{pmatrix} -3 & -1\\ 3 & -5\end{pmatrix}, \tag{5.23}$$

and gives the new e_i as

$$\begin{aligned}
e_1 &= -4\sqrt{q_t} = -q,\\
e_2 &= +4\sqrt{q_t} = +q,\\
e_3 &= +1,\\
e_4 &= -1.
\end{aligned} \tag{5.24}$$

Then the weak coupling limit corresponds to the limit in which the torus degenerates by having the e_1 and e_2 branch points collide. In terms of the IIA brane picture, we thus see that e_1 and e_2 correspond to the lift of one O6$^-$ plane, while e_3 and e_4 are the lift of the other one.

We now turn to identifying the μ_i deformation parameters with the fundamental hypermultiplet masses m_i. The masses and deformation parameters, by dimensional analysis and the fact that the residues of the Seiberg-Witten one-form can only depend linearly on the masses with no q-dependence, must therefore be related by

$$\mu_i = a_{ij} m_j \tag{5.25}$$

for some matrix of constants a_{ij}.[5] We can determine this matrix by the following symmetry argument. On the world-volume of 2 D6 branes in the presence of an O6$^-$ plane we expect an enhanced SO(4) gauge symmetry when they all coincide. Therefore the global flavor symmetry which should be manifest in our background—which is found by deforming two pairs of D6 branes away from two O6$^-$ planes—should be SO(4)$\times$SO(4). (The full theory will have an SO(8) flavor symmetry; but this need not be manifest in our particular way of writing the background surface.) Therefore the μ_i^2 parameters that appear in the background surface should be the SO(4) $\times$ SO(4) adjoint invariants, thinking of the masses as the eigenvalues of a mass matrix m^i_j in the adjoint. For one SO(4) factor such invariants are $\mathrm{tr}(m^2)$ and $\mathrm{Pf}(m)$ (the Pfaffian of m^i_j), which in terms of its eigenvalues m_1 and m_2 are $m_1^2 + m_2^2$ and $m_1 m_2$ respectively. A similar story applies to the second SO(4) as well (with mass eigenvalues m_3 and m_4). Since μ_1 and μ_2 are associated with the resolution of one O6$^-$ plane (corresponding to the branch points at e_1 and e_2, say) they are therefore functions of the m_1 and m_2 masses only; likewise μ_3 and μ_4 are functions of m_3 and m_4 only. Thus we expect

$$\mu_1^2 = a(m_1^2 + m_2^2) + b(m_1 m_2), \qquad \mu_2^2 = c(m_1^2 + m_2^2) + d(m_1 m_2), \tag{5.26}$$

for some numbers a, b, c, and d, and similarly for m_3 and m_4. For this to be consistent with (5.25), and using the symmetry between μ_1 and μ_2, and the fact that the BPS masses m_i should be integer linear combinations of the residues μ_i (corresponding to the number of times their cycles encircle the poles) implies that

$$\mu_1 = \tfrac{1}{2}(m_1{+}m_2), \qquad \mu_2 = \tfrac{1}{2}(m_1{-}m_2), \qquad \mu_3 = \tfrac{1}{2}(m_3{+}m_4), \qquad \mu_4 = \tfrac{1}{2}(m_3{-}m_4). \tag{5.27}$$

So, with (5.24) and (5.27) we have an explicit parameterization of the Q_M surface (4.34):

$$\begin{aligned} y^2 &= z(x^2 - q^2 w^2)(x^2 - w^2) \\ &\quad + (1 - q^2)\Big\{-q^2(m_1^2 + m_2^2)w^2(x^2 - w^2) + 2q(m_1 m_2)wx(x^2 - w^2) \\ &\qquad\qquad + (m_3^2 + m_4^2)w^2(x^2 - q^2 w^2) + 2(m_3 m_4)wx(x^2 - q^2 w^2)\Big\}. \end{aligned} \tag{5.28}$$

The asymmetrical way in which the four fundamental masses enter in (5.28) is the inevitable consequence of choosing a specific weak coupling parametrization as we have done above. Reparametrizations of the surface under global holomorphic coordinate changes on $\mathbb{CP}^2_{(1,1,2)}$

[5] Since the Seiberg-Witten one-form found in section 3.3 above could be multiplied by an arbitrary function of q, the a_{ij} constants may actually be functions of q. We only determine them in the $q \to 0$ limit here.

of the form (5.15) together with appropriate shifts of z can be found which leave the form of (5.28) unchanged except for permutations of the m_i; see section 5.5 below.

So far we have implicitly set the shift parameter M to zero. As described in section 2, the shift parameter M is the mass m_A of the antisymmetric hypermultiplet. From the IIA brane picture m_A will enter along with the m_i in the identification of the μ_i parameters given in (5.27) above. The reason is that the existence of an NS5 brane stuck at one of the O6$^-$ planes breaks the manifest SO(4) flavor symmetry at that orientifold to $\mathrm{U}(1)^2$. This means that the m_A mass is allowed to enter as a common shift in the m_1, m_2 masses (if we put the stuck 5 brane at e_1 or e_2). Thus the μ_1 and μ_2 identifications of (5.27) will be modified to

$$\mu_1 = m_A - \tfrac{1}{2}(m_1 + m_2), \qquad \mu_2 = \tfrac{1}{2}(m_2 - m_1). \tag{5.29}$$

The rest of the parameters in the curve for the 5 brane in the Q_M background describe the vevs for the $\mathrm{Sp}(2n) \to \mathrm{U}(1)^n$ breaking on the Coulomb branch. For example, from the IIA brane picture in the simple case where the antisymmetric mass vanishes, $M = 0$, the parameters of the resulting curve (5.7) are related to the eigenvalues $\pm\phi_\ell$, $\ell = 1, \dots, n$ of the complex adjoint scalar field in the vector multiplet by

$$A_{0k} = \sum_{\ell_1 < \cdots < \ell_k} \phi_{\ell_1}^2 \cdots \phi_{\ell_k}^2 \tag{5.30}$$

so that the roots of the curve are at

$$z_\ell = -\phi_\ell^2. \tag{5.31}$$

Note that, as discussed above for the coupling and masses, these identifications may be modified by terms with higher powers of q. In addition, these identifications are also valid only for large ϕ_ℓ: they may be modified even at "tree level" by shifts proportional to powers of the masses [11]. As we will compute in the $M = 0$ case momentariy, such a shift indeed occurs, and the more accurate identification of parameters implies that the ϕ_ℓ^2's in (5.30) are replaced by $\phi_\ell^2 \to \phi_\ell^2 - m_3^2 - m_4^2$, or equivalently, z in the 5 brane curve (5.7) is shifted by

$$z \to z - m_3^2 - m_4^2. \tag{5.32}$$

Finally, we are now in a position to check the equivalence of the Q_0 curve (5.28) at fixed $z = \phi^2$ with the curve [24]

$$y^2 = (x^2 - \widetilde{\phi}^2)^2 - \widetilde{q} \prod_{i=1}^{4} (x + \widetilde{m}_i) \tag{5.33}$$

of the SU(2) theory with four fundamental flavors. An explicit $\mathrm{SL}(2, \mathbb{C})$ transformation relating the two curves is very hard to find, so we will make a less direct argument. Consider the leading q behavior of the discriminants in x of the right hand sides of the two curves. For the background curve (5.28) it is proportional to

$$q^2(z - m_3^2 - m_4^2)^2(z - m_3^2 - m_4^2 + m_1^2)(z - m_3^2 - m_4^2 + m_2^2)(z - m_4^2)(z - m_3^2) \tag{5.34}$$

while for the SU(2) curve it is proportional to

$$\widetilde{q}^2\widetilde{\phi}^4(\widetilde{\phi}^2-\widetilde{m}_1^2)(\widetilde{\phi}^2-\widetilde{m}_2^2)(\widetilde{\phi}^2-\widetilde{m}_3^2)(\widetilde{\phi}^2-\widetilde{m}_4^2), \tag{5.35}$$

which match with the identifications (already derived above)

$$q \propto \widetilde{q}, \qquad z = -\widetilde{\phi}^2 + m_3^2 + m_4^2, \qquad m_i = \widetilde{m}_i. \tag{5.36}$$

Since we only kept the leading terms in q in the discriminants, they miss some of the structure of the singularities of the curves at values of $|z| \ll m_i^2$ which should also match. We can check that this structure also matches by comparing the structure of the curves in the limit of large masses. We will perform this check in section 5.5 below.

5.4 Identifying coupling and mass parameters: SU theory

The matching analysis for the SU(k) theory—or for theories with $q=2$ stuck NS5 branes, more generally—is much the same as for the Sp ($q=1$) theories given above. The main difference is the appearance of NS5 branes stuck on the two O6$^-$ planes which implies that the manifest SO(4) flavor symmetry at each O6$^-$ is broken to U(1)^2. This means in practice that the relation between the deformation parameters μ_i and the fundamental masses m_i derived in the last subsection can be shifted by SO(4) singlet masses. From the IIA brane picture, summarized in figure 3, it is easy to see that the background parameters are related to the hypermultiplet masses by

$$\begin{aligned}
\mu_1 &= m_{A1} - \tfrac{1}{2}(m_1+m_2),\\
\mu_2 &= \tfrac{1}{2}(m_2-m_1),\\
\mu_3 &= m_{A2} - \tfrac{1}{2}(m_3+m_4),\\
\mu_4 &= \tfrac{1}{2}(m_4-m_3),\\
M &= m_{A1} - m_{A2},
\end{aligned} \tag{5.37}$$

where m_{Ai} denote the two antisymmetric hypermultiplet masses. Plugging these identifications along with the coupling constant identification (5.24) into (4.34) gives an explicit parametrization of the Q_M background surface for the SU(k) theories.

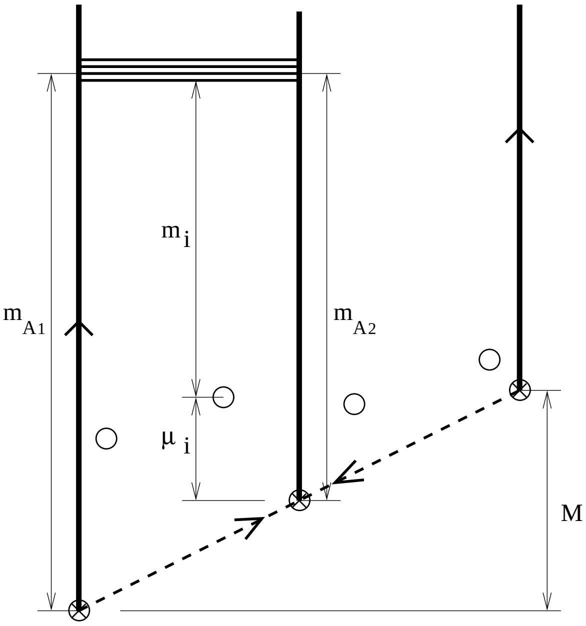


Figure 3: The brane arrangement (in the notation of figure 1) for the SU(4) theory with two antisymmetrics and four fundamentals, showing the brane separations corresponding to the various hypermultiplet masses, as well as the background parameters μ_i and M.

The rest of the parameters in the 5 brane curve describe the eigenvalues ϕ_ℓ, $\ell = 1, \ldots, 2n$, of the SU$(2n)$ adjoint scalar vevs on the Coulomb branch, as well as the average antisymmetric mass

$$m_A \equiv \tfrac{1}{2}(m_{A1} + m_{A2}). \tag{5.38}$$

At least in the case where the shift parameter M vanishes (so that the two antisymmetrics have equal mass), the resulting SU$(2n)$ curve (5.13) is easy to parameterize explicitly:

$$0 = \sum_{a=0}^{n} r_{2a}(z - m_3^2 - m_4^2)^{n-a} + \mathcal{P}(w,x,y)\sum_{a=1}^{n} r_{2a-1}(z - m_3^2 - m_4^2)^{n-a} \tag{5.39}$$

where the r_a coefficients are related (at weak coupling) to the eigenvalues ϕ_ℓ by

$$r_a = \sum_{\ell_1<\cdots<\ell_a} (-\phi_{\ell_1} - m_A)\cdots(-\phi_{\ell_a} - m_A), \tag{5.40}$$

(and $r_0 \equiv 1$) where the ϕ_ℓ satisfy $\sum_\ell \phi_\ell = 0$, and

$$\begin{aligned}\mathcal{P}(w,x,y) &= \frac{y - y_2 w^2}{w(x-qw)} - \frac{y - y_3 w^2}{w(x-w)} - (1-q^2)(m_3+m_4) \\ &= -\frac{(1-q)y}{(x-qw)(x-w)} - (1-q^2)\left[\frac{q(m_2-m_1)w}{x-qw} - \frac{2m_A w - (m_3+m_4)x}{x-w}\right].\end{aligned} \tag{5.41}$$

Note that in (5.39) we have shifted z by (5.32) and also shifted $\mathcal{P}$ by the constant $-(1-q^2)(m_3+m_4)$ relative to its value given in (5.13). Both these shifts could be undone by appropriate redefinitions of the ϕ_ℓ, which is just to say that, as in the Sp case, these shifts reflect the appropriate matching of the r_a parameters with the weak couplings vevs ϕ_ℓ. Unfortunately, we know of no simpler way of deducing these shifts than actually performing the matching to some known Coulomb branch physics, *e.g.*, at weak coupling.[6] A simple check is to take the curve for $n = 1$ and $m_A = 0$. This should give the known curve for the SU(2) theory with four fundamentals. But in this case (5.39) reduces to $z = \phi_1^2 + m_3^2 + m_4^2$. Plugging this in the background curve (5.28) then indeed gives precisely the SU(2) curve as we saw above in (5.36).

We are now in a position to answer the question raised at the end of section 4.2 as to what are the curves for the $q = 2$ theories with an odd number k of D4 branes, corresponding to the SU(k) theories with k odd. The SU$(2n-1)$ theory can be found from the SU$(2n)$ theory by going out on the Coulomb branch in a direction corresponding to one ϕ_ℓ getting large, while all the others remain close together. In terms of the eigenvalues ϕ_ℓ and the antisymmetric masses m_A this corresponds to taking the limit $\phi_1 + m_A \equiv \mathcal{M} \to \infty$ keeping all the other $\phi_i + m_A$ fixed (for $i \neq 1$). To leading order in $\mathcal{M}$ this has the effect on the r_a coefficients of $r_a \to \mathcal{M}\widetilde{r}_{a-1}$ where $\widetilde{r}_a$, $a = 1, \ldots, 2n-1$ are the corresponding coefficients for the SU$(2n-1)$

[6] The $-(m_3+m_4)$ shift in (5.41) was determined only to leading order in q; the $(1-q^2)$ factor in the shift was inserted only to make the second line in (5.41) prettier.

theory. Plugging these into the curve (5.39) and rescaling by an overall factor of $\mathcal{M}$ gives, in the $\mathcal{M} \to \infty$ limit the new curve

$$0 = \sum_{a=1}^{n} \widetilde{r}_{2a-1}(z - m_3^2 - m_4^2)^{n-a} + \mathcal{P}(x, y, z) \sum_{a=1}^{n} \widetilde{r}_{2a-2}(z - m_3^2 - m_4^2)^{n-a}. \tag{5.42}$$

This is thus the curve for the SU$(2n-1)$ theory; it differs from the SU$(2n)$ curve essentially by deleting the leading z^n term.

5.5 Decoupling fundamental hypermultiplets

We now examine the decoupling of the fundamental flavors. To decouple a hypermultiplet we should send its mass m_i to infinity while taking the gauge coupling $q \to 0$ keeping the strong coupling scale $\Lambda = qm_i$ of the resulting asymptotically free theory fixed.

This works with no other tunings for m_1 and m_2 in (5.28) but not so obviously for m_3 or m_4. However, by an SL$(2, \mathbb{C})$ transformation with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & i\sqrt{q} \\ i/\sqrt{q} & 0 \end{pmatrix}, \tag{5.43}$$

we can exchange $e_{1,2}$ with $e_{3,4}$, or equivalently exchange $m_{1,2}$ with $m_{3,4}$ in (5.28). Thus we can figure out the appropriate rescalings and shifts of w, x, y, and z from the changes of variables (5.16) and (5.20), which are, for the above SL$(2, \mathbb{C})$ transformation,

$$\begin{aligned} w &= -i\widetilde{x}/\sqrt{q} \\ x &= -i\sqrt{q}\widetilde{w} \\ y &= \widetilde{y} \\ z &= \widetilde{z} + (1 - q^2)(-m_1^2 - m_2^2 + m_3^2 + m_4^2). \end{aligned} \tag{5.44}$$

(This applies to the Sp or SU cases with the antisymmetric masses set to zero; for non-zero antisymmetric masses the m_i should be shifted according to (5.29) or (5.37).) Since the decoupling limit of, say, m_1 keeps w, x, y, and z fixed, so the decoupling limit of say m_3 should keep $\widetilde{w}$, $\widetilde{x}$, $\widetilde{y}$, and $\widetilde{z}$ fixed. This implies, by (5.44) that in doing the decoupling of m_3:

$$m_3 = \Lambda/q, \qquad q \to 0, \tag{5.45}$$

we should scale or shift the w, x, y and z coordinates by

$$\begin{aligned} w &\to (1/\sqrt{q})w \\ x &\to \sqrt{q}x \\ y &\to y \\ z &\to z + (1 - q^2)(-m_1^2 - m_2^2 + m_3^2 + m_4^2). \end{aligned} \tag{5.46}$$

Plugging the shifts (5.46) into our surface and taking the limits (5.45) indeed gives a well-defined surface with m_3 decoupled. In this way it is easy to find changes of variables to successively decouple all the masses. (And once you have this, you can forget all the above $\mathrm{SL}(2,\mathbb{C})$-ing.)

So, explicitly: Start with our surface (5.28) with z shifted by

$$z \to z + (1-q^2)(m_3^2 + m_4^2) \tag{5.47}$$

so that it reads

$$\begin{aligned} y^2 = & \; z(x^2 - q^2 w^2)(x^2 - w^2) \\ & + (1-q^2)\Big\{ -q^2(m_1^2 + m_2^2)w^2(x^2 - w^2) + 2q(m_1 m_2)wx(x^2 - w^2) \\ & \qquad + (m_3^2 + m_4^2)x^2(x^2 - q^2w^2) + 2(m_3m_4)wx(x^2 - q^2w^2)\Big\}. \end{aligned} \tag{5.48}$$

Decouple m_1 by taking $m_1 = \Lambda_1/q$ and $q \to 0$ (and no other rescalings) giving the surface

$$\begin{aligned} y^2 = & \; zx^2(x^2 - w^2) + \Big\{ -\Lambda_1^2 w^2(x^2 - w^2) + 2\Lambda_1 m_2 wx(x^2 - w^2) \\ & \qquad + (m_3^2 + m_4^2)x^4 + 2m_3m_4wx^3 \Big\}. \end{aligned} \tag{5.49}$$

Next decouple m_2 by taking $m_2 = \Lambda_2^2/\Lambda_1$ and $\Lambda_1 \to 0$ (and no other rescalings) giving the surface

$$y^2 = zx^2(x^2 - w^2) + \left\{ 2\Lambda_2^2 wx(x^2 - w^2) + (m_3^2 + m_4^2)x^4 + 2m_3m_4wx^3 \right\}. \tag{5.50}$$

Next decouple m_3 by taking $m_3 = \Lambda_3^3/\Lambda_2^2$ and $\Lambda_2 \to 0$ and rescale $w = \Lambda_2^{-1}\hat{w}$, $x = \Lambda_2 \hat{x}$, giving the surface

$$y^2 = -z\hat{x}^2\hat{w}^2 + \left(-2\hat{w}^3\hat{x} + \Lambda_3^6\hat{x}^4 + 2\Lambda_3^3 m_4 \hat{w}\hat{x}^3 \right). \tag{5.51}$$

Finally, decouple m_4 by taking $m_4 = \Lambda_4^4/\Lambda_3^3$ and $\Lambda_3 \to 0$ (and no other rescalings) giving the surface

$$y^2 = -z\hat{x}^2\hat{w}^2 - 2\hat{w}^3\hat{x} + 2\Lambda_4^4\hat{w}\hat{x}^3. \tag{5.52}$$

These five surfaces, (5.48)–(5.52), give the background geometry for the $\mathrm{Sp}(2n)$ or $\mathrm{SU}(k)$ curves with 4, 3, 2, 1, and 0 fundamentals, respectively. Note that all surfaces (for generic z) are non-degenerate tori. Furthermore, by the arguments of sections 5.2 and 5.3, these curves should be equivalent to those of the $\mathrm{SU}(2)$ theory with 4, 3, 2, 1, and 0 fundamentals, respectively. For the lower numbers of flavors this is easy to check simply by matching discriminants.

The 5 brane curves with decoupled fundamentals are found by taking the same scaling limits as above. For example, the reader can easily check that the $\mathrm{SU}(2n)$ curve with equal mass m_A antisymmetrics (*i.e.* no shift parameter) and no fundamentals is

$$0 = \sum_{a=0}^{n} r_{2a} z^{n-a} + \left(\frac{y}{\hat{x}\hat{w}} - 2m_A \right) \sum_{a=1}^{n} r_{2a-1} z^{n-a} \tag{5.53}$$

with r_a's given by (5.40), as found by taking the appropriate shifts, rescalings, and limits of (5.39).

6. Deriving S duality for the N=4 SU(n) theory

As an application of the curves we have found, we will use them in this section to derive part of the S duality identifications of the $N = 4$ supersymmetric SU(n) theory following the argument of [10]. The basic idea is to realize the $N = 4$ theory as an infrared fixed point in an asymptotically free $N = 2$ theory found by appropriately tuning parameters and vevs in the asymptotically free theory. Then by scaling to the infrared the full scale-invariant $N = 4$ theory can be recovered. In particular we can find the geometry of the coupling space of the embedded $N = 4$ theory from the Coulomb branch of the asymptotically free $N = 2$ theory. The complex geometry of the space of couplings is precisely the information encoded in the S duality group [10].

The particular embedding we will use is the following. Start with the asymptotically free $N = 2$ theory with gauge group SU$(2n)$, 2 massless antisymmetric hypermultiplets, and strong coupling scale Λ. By appropriately tuning vevs on the Coulomb branch we can higgs this theory at a scale μ to SU$(n)\times$SU$(n)\times$U(1) with two massless bifundamental hypermultiplets and with equal dimensionless couplings in the two SU(n) factors. The U(1) factor decouples, leaving the scale-invariant SU$(n)\times$ SU(n) elliptic model of [1]. Finally, by turning on a vev for one of the bifundamentals (on its Higgs branch) we higgs the theory to the diagonal SU(n) with a massless adjoint hypermultiplet—the $N = 4$ theory.

Classically we can break SU$(2n) \to$ SU$(n) \times$ SU$(n) \times$ U(1) on the Coulomb branch by giving the adjoint scalar Φ a vev of the form

$$\Phi = \mu \begin{pmatrix} 1\!\!1_n & 0 \\ 0 & -1\!\!1_n \end{pmatrix} \tag{6.1}$$

where $1\!\!1_n$ is the $n \times n$ identity matrix and μ is the scale of the vev. Under this breaking each antisymmetric hypermultiplet $(A, \widetilde{A})$ decomposes as

$$A = \begin{pmatrix} A_1 & B \\ -B & A_2 \end{pmatrix} \tag{6.2}$$

and similarly for $\widetilde{A}$, where A_1 and A_2 are antisymmetrics under the first and second SU(n) factors respectively, while B is a bifundamental. Writing the $N = 2$ action in $N = 1$ superfield notation, each antisymmetric enters the superpotential as

$$\begin{aligned} \mathcal{W}_A &= \mathrm{tr}(A\Phi\widetilde{A}) + m_A \mathrm{tr}(A\widetilde{A}) = \mathrm{tr}(A[\Phi + m_A 1\!\!1_{2n}]\widetilde{A}) \\ &= \mathrm{tr}\begin{pmatrix} A_1 & B \\ -B & A_2 \end{pmatrix}\begin{pmatrix} m_A+\mu & 0 \\ 0 & m_A-\mu \end{pmatrix}\begin{pmatrix} \widetilde{A}_1 & \widetilde{B} \\ -\widetilde{B} & \widetilde{A}_2 \end{pmatrix} \\ &= (m_A+\mu)\mathrm{tr}(A_1\widetilde{A}_1) + (m_A-\mu)\mathrm{tr}(A_2\widetilde{A}_2) - 2m_A\mathrm{tr}(B\widetilde{B}). \end{aligned} \tag{6.3}$$

Thus, if we keep the antisymmetric in the SU$(2n)$ theory massless we see that the antisymmetrics in the SU(n) factors become massive while the bifundamentals remain massless, and we indeed get the scale invariant SU$(n) \times$ SU(n) elliptic model. Furthermore, a one-loop renormalization group matching tells us that the couplings, q_{e1}, q_{e2}, of the two SU(n) factors are given by (at weak coupling)

$$q_{e1} = q_{e2} \sim \frac{\Lambda^2}{\mu^2} \tag{6.4}$$

and the equality between the two q's is enforced by the symmetry between the two SU(n) factors. (The proportionality factor can be found from a detailed one loop matching; but for the purposes of this argument this factor is unimportant.)

Finally, upon giving one of the bifundamentals a vev we break the SU(n) $\times$ SU(n) theory to the diagonal SU(n) and the remaining bifundamental decomposes as a massless adjoint hypermultiplet and a singlet (which is therefore decoupled). This is precisely the $N = 4$ SU(n) theory. By a tree-level matching its coupling q is just

$$q = q_{e1} q_{e2} \sim \frac{\Lambda^4}{\mu^4}. \tag{6.5}$$

The curve describing the Coulomb branch of the SU($2n$) theory with two massless antisymmetrics is given by (5.52), (5.53), and (5.40) with $m_A = 0$. In the $\hat{x} = 1$ $\mathbb{CP}^2_{(1,1,2)}$ coordinate patch, and renaming $\hat{w} \to x$, these curves read

$$\begin{aligned} y^2 &= -2x(x^2 + \tfrac{1}{2}zx - \Lambda^4), \\ 0 &= \sum_{a=0}^{n} r_{2a} z^{n-a} + \frac{y}{x} \sum_{a=1}^{n} r_{2a-1} z^{n-a}, \end{aligned} \tag{6.6}$$

where

$$r_a = (-)^a \sum_{\ell_1 < \cdots < \ell_a} \phi_{\ell_1} \cdots \phi_{\ell_a}, \qquad r_0 = 1, \qquad \sum_{\ell=1}^{2n} \phi_\ell \ = \ 0. \tag{6.7}$$

To implement the higgsing (6.1) we should take in the curve

$$\phi_\ell = \begin{cases} +\mu + \varphi_{1,\ell} & \ell = 1, \ldots, n \\ -\mu + \varphi_{2,(\ell-n)} & \ell = n+1, \ldots, 2n \end{cases} \tag{6.8}$$

with $|\varphi_{i\ell}| \ll |\mu|$ and $\sum_{\ell=1}^{n} \varphi_{i\ell} = 0$. Classically the $\varphi_{i\ell}$ are the eigenvalues of the adjoint scalars for each SU(n) factor, and the scale invariant theory is the "origin" of their Coulomb branch: $\varphi_{i\ell} = 0$ for $i = 1, 2$ and $\ell = 1, \ldots, n$. In fact, this is true quantum mechanically as well. For if we set $\varphi_{i\ell} = 0$ in the curve (6.6), then it follows that $r_{2a-1} = 0$ and $r_{2a} = \binom{n}{a}(-\mu^2)^a$, so the 5 brane curve becomes simply

$$0 = (z - \mu^2)^n \tag{6.9}$$

which has the singularity expected of a scale invariant theory at $z = \mu^2$.

We have therefore shown that the space of scale invariant SU(n) $\times$ SU(n) elliptic models with equal couplings in the two factors is described by a one-complex dimensional subspace of the Coulomb branch of the SU($2n$) theory given by (6.1). Finally, by the breaking on the Higgs branch of one of the bifundamentals we obtain the $N = 4$ theory; furthermore, by an

$N=2$ non-remormalization theorem [25] the Coulomb branch geometry can not depend on hypermultiplet vevs, so the μ plane remains an image of the space of couplings of the $N=4$ theory.

Actually, we have seen that at weak coupling the $N=4$ coupling $q \sim \mu^{-4}$, so at weak coupling at least, the μ-plane is in fact a four-fold cover of the coupling space. However, the asymptotically free SU$(2n)$ with two antisymmetrics theory has a classical $\mathrm{U}(1)_R$ R-symmetry which is broken by instantons down to a non-anomalous $\mathbb{Z}_4$ symmetry. This $\mathbb{Z}_4$ global symmetry acts on Φ by a phase rotation $\Phi \to e^{i\pi/2}\Phi$, and therefore acts on μ in the same way. This means that the coupling space is really the μ-plane modded out by this $\mathbb{Z}_4$ action, or, equivalently, the μ^4-plane instead, matching the weak coupling result (6.4).

By looking at the effective theory on the μ^4 subspace we can explore some features of the space of $N=4$ theories. The curve describing the Coulomb branch geometry is just (6.6) at $z=\mu^2$. By rescaling $x \to \mu^2 x$ and $y \to \mu^3 y$ the curve becomes the torus

$$y^2 = -2x\left(x^2 + \frac{1}{2}x - \frac{\Lambda^4}{\mu^4}\right). \tag{6.10}$$

This is regular everywhere except at three points where the torus degenerates: $\mu^4 = \infty$, $-16\Lambda^4$, and 0. The $\mu^4 = \infty$ point is just the weak coupling singularity. The other two are apparently "infinitely strongly" coupled singularities. This result shows that there must be S duality identifications on the naive classical coupling space of the $N=4$ theory, since classically the $N=4$ theory has a whole line of infinitely strongly coupled singularities ($\mathrm{Im}\tau = 0$ or $|q|=1$). It should be emphasized that these S duality identifications thus derived are exact equivalences of the whole $N=4$ quantum theory, and not just of low-energy or supersymmetric quantities, since they were derived from a definition of the scale invariant theory as an infrared fixed point of an asymptotically free theory [10].

Note, however, that the above derived S duality identifications need not be the complete set of S dualities of the $N=4$ theory [10, 26]. Indeed, the $N=4$ theory is believed to have an exact $SL(2,\mathbb{Z})$ group of identifications of its coupling τ. The resulting coupling space—a fundamental domain of $\mathrm{SL}(2,\mathbb{C})$—has only one singularity which corresponds to weak coupling. It is easy to see that these further S duality identifications are consistent with our curve. Indeed, since the complex structure of any torus, and our low-energy torus (6.10) in particular, is invariant under $\mathrm{SL}(2,\mathbb{C})$ transformations, there will be a further set of identifications of the coupling parameter Λ^4/μ^4 which identify the two ultra-strong coupling points with the weak coupling point. From the point of view of the specific asymptotically free $N=2$ theory in which we have embedded the $N=4$ theory, these extra identifications just look like accidental equivalences of the low energy effective actions, and are not necessarily exact equivalences enforced by a global (microscopic) symmetry. It would be interesting to see whether there are other embeddings of the SU(n) $N=4$ theory which, together with the embedding shown here, allow one to derive the full S duality group of the $N=4$ theory.

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